Lecture # 3
REGULARITY THEORY

(a) Warnings

[A] **Exercise (Weierstrass' counterexample to Dirichlet's principle)** Let $0 < \alpha < 1$ and set

$$v(x,y) := (x^2 - y^2)\,(-\ln(x^2+y^2))^{\alpha},\ \forall\,(x,y) \in \mathbb{D}.$$

Prove that:

(a) $v \notin C^2(\mathbb{D})$.

(b) The distributional Laplacian $f := \Delta v$ is continuous on $\mathbb{D}$.

(c) The equation $\Delta u = f$ has no classical (i.e., C^2) solution near the origin.

[B] **Exercise** Let $N \geq 2$ and $u \in C^1(\mathbb{R}^N \setminus \{0\})$ be such that $\partial_1 u \in L^1_{loc}(\mathbb{R}^N)$. Prove that $u \in L^1_{loc}(\mathbb{R}^N)$ and that $\partial_1 u$ is the distributional derivative of u. What about $N = 1$?

[C] Useful reference for items [C] and [D]: [10]

Exercise Let $\alpha \in \mathbb{R} \setminus \{-1, 1-N\}$ and set

$$u(x) := x_1\,|x|^{\alpha},\ \forall\,x \in \mathbb{R}^N \setminus \{0\},\ \beta := -\frac{\alpha(\alpha+N)}{(\alpha+1)(\alpha+N-1)}.$$

Then

$$\sum_{1\leq i\leq N} \partial_i \left(\sum_{1\leq j\leq N} (\delta_{ij} + \beta\,x_i x_j\,|x|^{-2})\partial_j u \right) = 0 \text{ in } \mathbb{R}^N \setminus \{0\}.$$

[D] **Theorem** (Serrin) A homogeneous uniformly elliptic equation in divergence form may have locally unbounded $W^{1,1}_{loc}(\Omega)$ weak solutions.

More specifically, if $N \geq 2$ and $0 < \varepsilon < 1$, and we set

$$u(x) := \frac{x_1}{|x|^{N-1+\varepsilon}},\ x \in \mathbb{R}^N,$$

and

$$A(x) := \mathrm{Id}_N + \frac{b}{|x|^2}(x_i\,x_j)_{1\leq i,j\leq N},\ x \in \mathbb{R}^N,\ \text{with } b := \frac{N-1}{\varepsilon(\varepsilon+N-2)} - 1,$$

then $u \in W^{1,1}_{loc}(\mathbb{R}^N) \setminus L^{\infty}_{loc}(\mathbb{R}^N)$, A is uniformly elliptic in $\mathbb{R}^N$, and

$$\mathrm{div}\,(A\nabla u) = 0 \text{ in } \mathscr{D}'(\mathbb{R}^N).$$

(b) Singular integrals

General reference: [8, Section 3]

A **Exercise.** Let ω_N be the area of $\mathbb{S}^{N-1}$. Let E be "the" fundamental solution of $-\Delta$ in $\mathbb{R}^N$,

$$E(x) := \begin{cases} -(1/\omega_2)\ \ln|x|, & \text{if } N = 2 \\ (1/[(N-2)\omega_N])\,|x|^{2-N}, & \text{if } N \geq 3 \end{cases}.$$

(a) Prove that, in the distributions sense,

$$\partial_j E = g_j, \text{ where } g_j(x) := -\frac{1}{\omega_N}\frac{x_j}{|x|^N}.$$

(b) If $1 \leq p \leq \infty$ and $f \in L^p_c(\mathbb{R}^N)$, then, in the distributions sense,

$$\partial_j(f * E) = h_j, \text{where } h_j(x) := \int_{\mathbb{R}^N} f(y)\, g_j(x-y)\, dy.$$

Exercise. Let $K \in \mathscr{D}'(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N \setminus \{0\})$. Let $f \in C^\infty_c(\mathbb{R}^N)$ and set $L := \operatorname{supp} f$. Then:

$$(K * f)(x) = \int_{\mathbb{R}^N} f(y)K(x-y)\, dy = \int_L f(y)K(x-y)\, dy,\ \forall\, x \notin \operatorname{supp} L. \tag{1}$$

B **Proposition.** With the above notation, let $K := \partial_k g_j = \partial_k \partial_j E$ (in the distributions sense). Then:

(a) $K \in \mathscr{D}'(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$, and in particular (1) holds.

(b) $K \in \mathscr{S}'$ and, in the distributions sense,

$$\widehat{K}(\xi) = \ell_{j,k}, \text{ where } \ell_{j,k}(\xi) := -\frac{\xi_j \xi_k}{|\xi|^2}.$$

(c) For some finite C, we have $|\nabla K(x)| \leq C/|x|^{N+1}$, $\forall\, x \in \mathbb{R}^N \setminus \{0\}$.

Useful reference: [6, Theorem 2.3.4]

C **Exercise.** Let $(X, \mathscr{T}, \mu)$ be a measured space. (Warning: μ is not supposed σ-finite.) If $f : X \to \mathbb{R}$ is measurable and $1 \leq p < \infty$, then

$$\|f\|_p^p = p\int_0^\infty t^{p-1} \underbrace{\mu([|f| > t])}_{:=F_f(t)}\, dt.$$

D **Marcinkiewicz interpolation theorem** (special case) Let $(X, \mathscr{T}, \mu)$ be a measured space. Let $1 < r < \infty$ and let T be a linear operator on $L^1 \cap L^r(X)$ such that, for every $f \in L^1 \cap L^r(X)$, Tf is a measurable function on X and, for some $K_1, K_r < \infty$, we have

$$\mu([|Tf| > t]) \leq K_1 \frac{\|f\|_1}{t},\ \forall\, f \in L^1 \cap L^r(X),\ \forall\, t > 0,$$

$$\mu([|Tf| > t]) \leq K_r \frac{\|f\|_r^r}{t^r},\ \forall\, f \in L^1 \cap L^r(X),\ \forall\, t > 0.$$

Then, for every $1 < p < r$ and some $C_p < \infty$,

$$\|Tf\|_p \leq C_p \|f\|_p, \ \forall\, f \in L^1 \cap L^r(X),$$

and in particular T admits a unique linear continuous extension from $L^p(X)$ into $L^p(X)$.

In the special case where μ is a Radon measure in $\mathbb{R}^N$, the same holds if T is initially defined on $L^r_c(\mathbb{R}^N)$.

F **Calderón-Zygmund decomposition, second form** Let $f \in C_c(\mathbb{R}^N)$ and $t > 0$. Then, with finite constants independent of f and t there exist: a family of disjoint cubes $C_n \subset \mathbb{R}^N$ and functions $g, h_n \in L^\infty_c(\mathbb{R}^N)$ (depending on f and t) such that

(a) $g = f$ in $\mathbb{R}^N \setminus \cup_n C_n$.

(b) $|g| \leq Ct$.

(c) $\text{supp } h_n \subset C_n, \forall\, n$.

(d) $\displaystyle\int h_n = 0, \forall\, n$.

(e) $\displaystyle⨍ |h_n| \leq Ct, \forall\, n$.

(f) $\displaystyle f = g + \sum_n h_n$ (pointwise).

(g) $\displaystyle\sum_n |C_n| \leq C\frac{\|f\|_1}{t}$.

(h) $\displaystyle\|g\|_1 + \sum_n \|h_n\|_1 \leq C\|f\|_1$.

G **Calderón-Zygmund theorem adapted to the Laplace equation** Let $K \in \mathscr{S}'(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$ satisfy

(i) $\widehat{K}$ is a bounded *real* function.

(ii) $|\nabla K(x)| \leq C/|x|^{N+1}, \forall\, x \in \mathbb{R}^N \setminus \{0\}$, for some finite C.

Let $Tf := K * f, \forall\, f \in C^\infty_c(\mathbb{R}^N)$. Then

$$\|Tf\|_p \leq C_{p,N}\|f\|_p, \ \forall\, 1 < p < \infty, \ \forall\, f \in C^\infty_c(\mathbb{R}^N).$$

In particular, for $1 < p < \infty$, T admits a unique linear continuous extension from $L^p(\mathbb{R}^N)$ into itself.

Corollary. Let $1 < p < \infty$ and $f \in L^p_c(\mathbb{R}^N)$, and set $u := E * f$. Then

$$\|\partial_j \partial_k\|_p \leq C_{p,N}\|f\|_p, \ \forall\, 1 \leq j,k \leq N.$$

H **A standard "elliptic estimate"** Let $1 < p < \infty$, $K \subset \Omega \subset \mathbb{R}^N$, with K compact and Ω open. If $-\Delta u = f \in L^p(\Omega)$, then $u \in W^{2,p}_{loc}(\Omega)$ and, for some finite $C = C_{p,N,\Omega,K}$,

$$\|u\|_{W^{2,p}(K)} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}).$$

Exercise. Let $u \in H^1_0(\Omega)$ be an eigenfunction of $-\Delta$. Prove that $u \in C^\infty(\Omega)$.

(c) L^p **regularity theory**

Useful references: [4, Chapter 9] for the regularity theory, [8, Section 1.5] for trace theory

A **Main regularity theorem** (**Calderón**, **Zygmund**, **Koselev**, Greco, Agmon, Douglis, Nirenberg, ...) Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$-domain. Let $1 < p < \infty$ and $f \in L^p(\Omega)$. Then the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

has a unique (generalized) solution $u \in W^{2,p}(\Omega)$. In addition, for some finite C independent of f, $\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_p$.

Exercise. The above u is not only a distributional solution, but also a *strong solution*, in the sense that for a.e. $x \in \Omega$ we have

$$-\sum_{j=1}^{N} \partial_{jj} u(x) = f(x).$$

B **Toolbox** In what follows, $\omega, \Omega \subset \mathbb{R}^N$ are bounded open sets.

For the record.

Rademacher's theorem A Lipschitz function $f : \Omega \to \mathbb{R}$ is differentiable a.e., and its distributional gradient and point gradient coincide.

Useful reference: [3, Section 3.1.2]

Exercise. Let $\Phi : \omega \to \Omega$ be a bi-Lipschitz homeomorphism. Prove that, with constants $0 < C_{1,p} < C_{2,p} < \infty$ depending only on $1 \leq p < \infty$ and on the Lipschitz constants of Φ and Φ^{-1}, we have

$$C_1\|f \circ \Phi\|_{L^p(\omega)} \leq \|f\|_{L^p(\Omega)} \leq C_2\|f \circ \Phi\|_{L^p(\omega)}, \forall \text{ measurable function } f : \Omega \to \mathbb{R}.$$

Exercise. Let $\Phi : \omega \to \Omega$ be a C^1-diffeomorphism. If $f \in W^{1,1}_{loc}(\Omega)$, prove that $f \circ \Phi \in W^{1,1}_{loc}(\omega)$ and that the chain rule holds, i.e.,

$$\partial_i (f \circ \Phi) = \sum_{j=1}^{N} [(\partial_j f) \circ \Phi]\, [\partial_i \Phi_j],\ \forall\, 1 \leq i \leq N.$$

Exercise. Let $\Phi : \overline{\omega} \to \overline{\Omega}$ be a $C^{1,1}$-diffeomorphism and $1 \leq p \leq \infty$. Prove that $f \mapsto \|f \circ \Phi\|_{W^{2,p}(\omega)}$ is equivalent to the usual norm on $W^{2,p}(\Omega)$.

Lemma. Let $\Phi : \omega \to \Omega$ be a C^1-diffeomorphism. Let $u \in W^{1,1}_{loc}(\Omega)$ satisfy $-\Delta u = f \in L^1_{loc}(\Omega)$ in the distributions sense. Set $v := u \circ \Phi \in W^{1,1}_{loc}(\omega)$. Then, in the distributions sense, we have

$$-\operatorname{div}(A\nabla v) = g \in L^1_{loc}(\omega), \tag{3}$$

where

$$A = A(x) := |J\,\Phi|\,[(J\Phi)^{-1}]\,[{}^t[(J\Phi)^{-1}]],\ g := |J\,\Phi|\,f \circ \Phi.$$

Lemma. Let $A \in \mathrm{Lip}_{loc}(\omega)$, $v \in W^{1,1}_{loc}(\omega)$, $\zeta \in \mathrm{Lip}_{loc}(\omega)$, and $g \in L^1_{loc}(\omega)$. If (3) holds, then

$$-\operatorname{div}(A\nabla(\zeta v)) = \zeta\, g - v\, \operatorname{div}(A\nabla\zeta) - (A\nabla v)\cdot\nabla\zeta - (A\nabla\zeta)\cdot\nabla v.$$

Exercise. Let $u \in W^{1,1}(\Omega)$ and $\varphi \in C^1(\overline{\Omega})$. Prove that $\mathrm{tr}(\varphi u) = \varphi_{|\partial\Omega}\ \mathrm{tr}\ u$.

Exercise. Let Ω be a C^1-domain, and let $\Psi : U \to \mathbb{R}^N$ be a C^1-diffeomorphism from an open set $U \subset \mathbb{R}^N$ into its image. Set $\Xi := \Psi_{|U} : U \to \Psi(U)$ and $\Phi := \Xi^{-1}$. Set also $\Sigma := \partial\Omega \cap U$ and $\Lambda := \Psi(\Sigma)$. Let $u \in W^{1,1}(U)$ and set $v := u \circ \Phi$. Give a meaning to and prove the equality $\mathrm{tr}_{\Psi(\Sigma)}\, v = (\mathrm{tr}_\Sigma(u)) \circ [(\Psi)^{-1}_{|\Lambda}]$.

Exercise. Set $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N;\ x_N > 0\}$. Let $u \in W^{1,1}(\mathbb{R}^N_+)$. Let $h \in \mathbb{R}^{N-1} \times \{0\}$. Give a meaning to and prove the equality $\mathrm{tr}\ u(\cdot + h) = (\mathrm{tr}\ u)(\cdot + h)$.

Lemma. Let p, q be conjugated exponents, $g \in L^q(\mathbb{R}^N_+)$, $w \in W^{1,p}(\mathbb{R}^N_+)$, $h \in \mathbb{R}^{N-1}\times\{0\}$. Then

$$\left|\int_{\mathbb{R}^N_+} (g(x+h) - g(x))\, w(x)\, dx\right| \le |h|\, \|g\|_q\, \|\nabla w\|_p.$$

Exercise. Let $f \in L^1_{loc}(\mathbb{R}^N_+)$. Then

$$\lim_{t\to 0} \frac{f(\cdot + te_j) - f}{t} = \partial_j f \text{ in } \mathscr{D}'(\mathbb{R}^N_+),\ \forall\, 1 \le j \le N-1.$$

Exercise. Let $u \in W^{2,1}(\mathbb{R}^N_+)$.

(a) Let $\Sigma := \mathbb{R}^{N-1} \times \{0\}$, that we identify with $\mathbb{R}^{N-1}$. When $\varphi \in C^2_c(\overline{\mathbb{R}^N_+})$, prove the generalized (second) Green formula

$$\int_{\mathbb{R}^N_+} (-\Delta u)\,\varphi = \int_{\mathbb{R}^{N-1}} [\mathrm{tr}_{|\Sigma}\,\partial_N u]\,\varphi - \int_{\mathbb{R}^{N-1}} [\mathrm{tr}_{|\Sigma}\, u]\,\partial_N\varphi + \int_{\mathbb{R}^N_+} u\,(-\Delta\varphi).$$

(b) If $F : \mathbb{R}^n_+ \to \mathbb{R}$, set

$$F^*(x) = F^*(x_1, \ldots, x_N) := \begin{cases} F(x), & \text{if } x_N > 0 \\ -F(x_1, \ldots, x_{N-1}, -x_N), & \text{if } x_N < 0 \end{cases}.$$

Let $u \in W^{2,1}(\mathbb{R}^N_+)$ satisfy $\mathrm{tr}_{|\Sigma}\, u = 0$. Prove that $-\Delta(u^*) = (-\Delta u)^*$.

C **Theorem (Higher order regularity)** Let $k \ge 0$, $\Omega \in C^{k+2,1}$, and $1 < p < \infty$. If $f \in W^{k,p}(\Omega)$, then the solution u of (2) satisfies $u \in W^{k+2,p}(\Omega)$ and, for some finite C independent of f, $\|u\|_{W^{k+2,p}(\Omega)} \le C\|f\|_{W^{k,p}(\Omega)}$.

D For the record, we mentions some results in lower order regularity theory.

Theorem. Let $\Omega \in C^{1,1}$ and $1 < p < \infty$. For $F \in L^p(\Omega; \mathbb{R}^N)$, the equation

$$-\Delta u = \operatorname{div} F \text{ in } \mathscr{D}'(\Omega)$$

has a unique solution $u \in W_0^{1,p}(\Omega)$. In addition, with some finite constant C independent of F, we have the estimate $\|\nabla u\|_p \leq C\|F\|_p$.

Theorem (Stampacchia) Let $\Omega \in C^{1,1}$. For $f \in L^1(\Omega)$, the equation

$$-\Delta u = f \text{ in } \mathscr{D}'(\Omega)$$

has a unique solution $u \in W_0^{1,1}(\Omega)$. Moreover, this u satisfies $u \in \cap_{1 \leq p < N/(N-1)} W_0^{1,p}(\Omega)$ and, with finite constants C_p independent of f,

$$\|\nabla u\|_p \leq C_p\|f\|_1, \; \forall \, 1 \leq p < \frac{N}{N-1}.$$

Useful reference: [9, Section 4.1]

(d) A glimpse of the C^α regularity theory

Useful reference: [4, Lemma 4.4, Theorem 6.14, Theorem 6.19]. For the record:

Theorem (C^α regularity) (Kellogg) Let $0 < \alpha < 1$, $k \geq 0$, $\Omega \in C^{k+2,\alpha}$. If $f \in C^{k,\alpha}(\overline{\Omega})$, then the solution of (2) satisfies $u \in C^{k+2,\alpha}(\overline{\Omega})$. In addition, for some finite C independent of f, $\|u\|_{C^{k+2,\alpha}(\overline{\Omega})} \leq C\|f\|_{C^{k,\alpha}(\overline{\Omega})}$.

Lemma (Hölder estimates for the Newtonian potential) (Korn) Let $0 < \alpha < 1$. If $f \in C_c^\alpha(\mathbb{R}^N)$ and $u := E * f$, then, for some finite C independent of f,

$$\left|D^2 u\right|_{C^\alpha(\mathbb{R}^N)} \leq C|f|_{C^\alpha(\mathbb{R}^N)}.$$

(e) Power growth nonlinearities. Bootstrap

Useful reference: [8, Section 3.3.2]. In what follows, we assume that $N \geq 3$.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying

$$|f(x,t) \leq C(1 + |t|^p), \; \forall \, x \in \Omega, \; \forall \, t \in \mathbb{R}.$$

Let u satisfy

$$u \in H^1_{loc}(\Omega), \; x \mapsto f(x, u(x)) \in L^1_{loc}(\Omega)$$
$$-\Delta u = f(x, u(x)) \text{ in } \mathscr{D}'(\Omega).$$

A **Exercise.** Assume that $p < \dfrac{N+2}{N-2}$. Then $u \in W^{2,r}_{loc}(\Omega)$, $\forall \, r < \infty$.

B **Proposition.** The same holds when $p = \dfrac{N+2}{N-2}$.

Moreover, if $u \in H^1_{loc}(\Omega)$ satisfies

$$-\Delta u = a(x)u + b(x), \text{ with } a \in L^{N/2}_{loc}(\Omega),\ b \in L^\infty_{loc}(\Omega),$$

then $u \in L^r_{loc}(\Omega)$, $\forall\, r < \infty$.

C **Exercise.** Let $p > \dfrac{N+2}{N-2}$. Prove that the equation $-\Delta u = |u|^p$ has a locally unbounded solution $u \in H^1 \cap L^p(B_1(0))$, of the form $u(x) = \lambda |x|^{-\alpha}$, for appropriate constants $\lambda, \alpha > 0$.

(f) A glimpse of the De Giorgi regularity theory

Useful references: [4, Sections 8.5–8.9], [5, Chapter 4]. For the record:

A **Theorem** (local boundedness; **Stampacchia**, Ladyzhenskaya, Uraltseva, Trudinger,...) Let $A = A(x)$ be uniformly elliptic in $\Omega := B_1(0)$. Let $u \in H^1(\Omega)$ satisfy $-\operatorname{div}(A\nabla u) = f \in L^p(\Omega)$, where $p > \dfrac{N}{2}$. Then $u \in L^\infty_{loc}(\Omega)$ and, with a finite constant depending only on $0 < R < 1$ and p,

$$\|u\|_{L^\infty(B_R(0))} \le C(\|f\|_{L^q(\Omega)} + \|u\|_{L^1(\Omega)}).$$

B **Theorem** (local C^α regularity; **De Giorgi**, Nash, Ladyzhenskaya, Uraltseva, **Moser**,...) There exists some $0 < \alpha < 1$ depending only on p and the ellipticity constants of A such that the above u belongs to $C^\alpha_{loc}(\Omega)$ and satisfies, with a finite constant C depending only on R and p

$$|u(x) - u(y)| \le C(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)}),\ \forall\, x, y \in B_R(0).$$

(g) Wente estimates. Compensation phenomena

Useful references: [1], [2], [7, Section 10.3]

A **Theorem** (Wente) Let $\Omega \in C^{1,1}$ be a bounded domain in $\mathbb{R}^N$, and let $F \in H^1(\Omega; \mathbb{R}^2)$. Then the problem

$$\begin{cases} -\Delta u = \det(JF) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a (unique) weak solution $u \in H^1_0(\Omega)$. In addition, we have $u \in C(\overline{\Omega})$ and, for some finite constant independent of F, we have the *Wente estimates*

$$\|u\|_\infty + \|\nabla u\|_2 \le C\|\nabla F\|_2.$$

B For the record:

Theorem (Fefferman, Stein, Coifman, Lions, Meyer, Semmes) If $F \in W^{1,N}_c(\mathbb{R}^N; \mathbb{R}^N)$, and we set $u := E * [\det\ JF)]$, then $D^2 u \in L^1(\mathbb{R}^N)$ and, with a finite constant independent of F and of its support,

$$\left\|D^2 u\right\|_1 \le C\|\nabla F\|_N.$$

References

[1] Haïm Brezis and Jean-Michel Coron. Multiple solutions of H-systems and Rellich's conjecture. *Comm. Pure Appl. Math.*, 37(2):149–187, 1984.

[2] Ronald Coifman, Pierre-Louis Lions, Yves Meyer, and Stephen Semmes. Compensated compactness and Hardy spaces. *J. Math. Pures Appl. (9)*, 72(3):247–286, 1993.

[3] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[4] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[5] Qing Han and Fanghua Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[6] Lars Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

[7] Petru Mironescu. *Fine properties of functions: an introduction*. `http://math.univ-lyon1.fr/~mironescu/resources/introduction_fine_properties_functions_2005.pdf`, 2005.

[8] Petru Mironescu. *Sobolev spaces. Elliptic equations* (Partial lecture notes). `http://math.univ-lyon1.fr/~mironescu/resources/sobolev_spaces_elliptic_equations_2010.pdf`, 2010.

[9] Augusto C. Ponce. *Elliptic PDEs, measures and capacities*, volume 23 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2016. From the Poisson equations to nonlinear Thomas-Fermi problems.

[10] James Serrin. Pathological solutions of elliptic differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 18:385–387, 1964.