Partial examination
October 27, 2023 – two hours

Exercise # 1. Let:

(i) $N \geq 3$.

(ii) $\Omega \subset \mathbb{R}^N$ a bounded Lipschitz open set.

(iii) $a \in L^{(2N)/(N+2)}(\Omega)$.

(iv) $f \in C^1(\mathbb{R}, \mathbb{R})$ a Lipschitz function.

Set

$$F(u) := \int_\Omega a(x) f(u(x))\, dx,\ \forall\, u \in H^1(\Omega).$$

Prove that $F \in C^1(H^1(\Omega), \mathbb{R})$, and that

$$F'(u)\varphi = \int_\Omega a(x) f'(u(x))\varphi(x)\, dx,\ \forall\, u \in H^1(\Omega),\ \forall\, \varphi \in H^1(\Omega).$$

Exercise # 2. Recall the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

a) Preliminary question. Prove that, for each $\varepsilon > 0$, there exists some $C_\varepsilon < \infty$ such that

$$t \leq \varepsilon t^3 + C_\varepsilon,\ \forall\, t \geq 0.$$

b) Let $\Omega \subset \mathbb{R}^3$ be a bounded open set. Let $f \in L^{6/5}(\Omega)$ and $\lambda \in \mathbb{R}$. Prove that the equation

$$-\Delta u + u^5 = \lambda u + f \text{ in } \mathscr{D}'(\Omega)$$

has a distributional solution $u \in H^1_0(\Omega)$.

Exercise # 3. Let $\Omega \subset \mathbb{R}^N$ be a bounded *connected* open set, and let $\lambda_1(\Omega)$ be the best constant in Poincaré's inequality, i.e., the largest constant C such that

$$C \int_\Omega u^2 \leq \int_\Omega |\nabla u|^2,\ \forall\, u \in H^1_0(\Omega).$$

We take for granted the following result: if $u \in H^1_0(\Omega)$ is such that

$$\lambda_1(\Omega) \int_\Omega u^2 = \int_\Omega |\nabla u|^2,$$

then u is smooth and either $u \equiv 0$, or $u(x) \neq 0, \forall\, x \in \Omega$.

Let $a \in C(\overline{\Omega})$ be such that $a(x) \leq 1, \forall\, x \in \Omega$, and $a \not\equiv 1$. Prove that there exists some $\varepsilon > 0$ such that

$$(\lambda_1(\Omega) + \varepsilon) \int_\Omega a u^2 \leq \int_\Omega |\nabla u|^2,\ \forall\, u \in H^1_0(\Omega).$$

Exercise # 4. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let p be such that $\dfrac{2N}{N+1} < p < N$. Set $r := \dfrac{Np}{2N-p}$. Sketch a proof of the following results.

a) If $u, v \in W^{1,p}_0(\Omega)$, then $uv \in W^{1,r}_0(\Omega)$.

b) If $u_j \rightharpoonup u$ and $v_j \rightharpoonup v$ in $W^{1,p}_0(\Omega)$, then $u_j v_j \rightharpoonup uv$ in $W^{1,r}_0(\Omega)$.