The role of Hardy's inequality in the theory of function spaces

Batsheva de Rothschild Seminar – in honor of Moshe Marcus

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- If $1 \le p < \infty$, $0 < r < \infty$ and $g : (0,\infty) \to [0,\infty]$ measurable, then

$$\int_0^\infty t^{-r-1}\left(\int_0^t g(u)\,du\right)^p dt \le \left(\frac{p}{r}\right)^p \int_0^\infty u^{-r+p-1}(g(u))^p\,du,$$

$$\int_0^\infty t^{r-1}\left(\int_t^\infty g(u)\,du\right)^p dt \le \left(\frac{p}{r}\right)^p \int_0^\infty u^{r+p-1}(g(u))^p du$$

- "Usual" choice: $p > 1$, $r = p-1$, $g = |f'|$, $f \in W^{1,p}((0,\infty))$, $f(0) = 0$, version "at 0":

$$\int_0^\infty \frac{|f(t)|^p}{t^p}\,dt \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f'(u)|^p\,du$$

- If $1 \le p < \infty$, $0 < \lambda < p$, $\lambda \ne 1$, $f \in C_c^\infty([0,\infty))$ and $f(0) = 0$ if $\lambda > 1$, then

$$\int_0^\infty \frac{|f(t)|^p}{t^\lambda}\, dt \le C_{p,\lambda} \int_0^\infty \int_0^\infty \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\, dxdy$$

- This does not hold when $\lambda = 1$
- If $1 \le p < \infty$, $0 < \lambda < 1$ and $f : J = (a,b) \to \mathbb{R}$ measurable, then

$$\int_J \frac{|f(t)|^p}{\mathrm{dist}(t,\{a,b\})^\lambda}\, dt \le C'_{p,\lambda} \int_J \int_J \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}}\, dxdy$$

under any reasonable hypothesis that "kills non zero constants", e.g.:

  - $\int_J f(x)\, dx = 0$ if J is bounded
  - $f(x) = O(x^{-2})$ at infinity if J is unbounded

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- Given function spaces X and Y, find (the) functions Φ such that the superposition T_Φ, $f \xrightarrow{T_\Phi} \Phi \circ f$ maps (continuously) X into Y (necessary, sufficient, necessary and sufficient conditions on Φ)
- If $0 < s \leq 1$, $1 \leq p < \infty$, if Φ is Lipschitz and $\Phi(0) = 0$, then T_Φ maps continuously $W^{s,p}(\mathbb{R}^n)$ into $W^{s,p}(\mathbb{R}^n)$ · Igari 1965 Φ Lipschitz is necessary · Marcus, Mizel 1975, 1979 Continuity of T_Φ when $s = 1$
- Dahlberg 1979 If $s \in \mathbb{N}$, $s \geq 2$, $1 < p < \infty$ and $sp < n$, and if $T_\Phi(W^{s,p}(\mathbb{R}^n)) \subset W^{s,p}(\mathbb{R}^n)$, then $\Phi(t) = ct$ · Sickel 1989, 1997 in $W^{s,p}$ for non integer s, when $1 + 1/p < s < n/p$
- If $s > 0$, $1 \leq p < \infty$, $sp > n$, ℓ integer $\geq s$, $\Phi \in C^\ell(\mathbb{R})$, $\Phi(0) = 0$, then T_Φ maps continuously $W^{s,p}(\mathbb{R}^n)$ into $W^{s,p}(\mathbb{R}^n)$

- $\Phi(t) = |t|$ Bourdaud, Meyer 1991 If $1 \le p < \infty$, $1 \le q \le \infty$, $1 < s < 1 + 1/p$ and $\Phi(t) := |t|$, then T_Φ maps continuously the Besov space $B^s_{p,q}(\mathbb{R}^n)$ into $B^s_{p,q}(\mathbb{R}^n)$
- Proof by nonlinear interpolation starting from the special case $p = q$: $B^s_{p,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$
- In $W^{s,p}$: reduction to the case $n = 1$
- Key ingredient in $W^{s,p}(\mathbb{R})$: the fractional Hardy's inequality
- If $q > p$, it is not possible to perform directly a dimensional reduction in $B^s_{p,q}$: "non restriction property" in $B^s_{p,q}(\mathbb{R}^n)$ if $q > p$ · M, Russ, Sire 2017, Brasseur 2017

- $\Phi(t)=|t|^a$ M 2015 If $1<p<\infty$, $0<a<1$ and $\Phi(t):=|t|^a$, then T_Φ maps continuously $W^{1,p}(\mathbb{R}^n)$ into $W^{a,p/a}(\mathbb{R}^n)$
- Wrong when $p=1$
- Key ingredient: Hardy's inequality
- More generally, if $1<p<\infty$, Φ even, $\Phi:[0,\infty)\to[0,\infty)$ concave increasing bijective, set $\Psi:[0,\infty)\to[0,\infty)$, $\Psi:=\Phi^{-1}$ and

$$F(t):=\int_0^t\left(\int_0^s[\Psi'(\tau)]^{1-1/p}[\Psi''(\tau)]^{1/p}\,d\tau\right)^p ds,\ \forall\, t\geq 0$$

Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{F(|\Phi(u(y))-\Phi(u(x))|)}{|y-x|^{n+p}}\,dxdy\leq C_{p,\Phi,n}\int_{\mathbb{R}^n}|\nabla u(x)|^p\,dx$$

- Gagliardo 1957 If $1 \le p < \infty$ and $U \in W^{1,p}(\mathbb{R}^{n+1}_+)$, then $u := \operatorname{tr} U \in W^{1-1/p,p}(\mathbb{R}^n)$ and $|u|_{W^{1-1/p,p}} \le C_{p,n} \|\nabla U\|_{L^p}$ · Direct trace theorem
- Hardy's inequality is a key ingredient in the proof of the direct theorem
- Gagliardo 1957 If $u \in W^{1-1/p,p}(\mathbb{R}^n)$, then there exists $U \in W^{1,p}(\mathbb{R}^{n+1}_+)$ such that $u = \operatorname{tr} U$ and $\|\nabla U\|_{L^p} \le C'_{p,n}|u|_{W^{1-1/p,p}}$ · Inverse trace theorem
- We may take $U(x,\varepsilon) := u * \rho_\varepsilon(x)$ when $1 < p < \infty$
- Peetre 1979 If $p = 1$, then the operator $u \mapsto U$ cannot be linear + continuous

- Uspenskii 1961 If $s > 0$, $1 \le p < \infty$, ℓ integer $> s$ and $U \in C_c^\infty(\mathbb{R}_+^{n+1})$, then $u := \operatorname{tr} U$ satisfies

$$|u|^p_{B^s_{p,p}} \le C_{s,p,\ell,n} \sum_{|\alpha|=\ell} \int_0^\infty \varepsilon^{p(\ell-s)-1} \|\partial^\alpha U(\cdot,\varepsilon)\|^p_{L^p(\mathbb{R}^n)}\, d\varepsilon$$ · Direct theorem

- Uspenskii 1961 If $s > 0$, $1 \le p < \infty$, ℓ integer $> s$ and $u \in B^s_{p,p}(\mathbb{R}^n)$, then there exists some U such that $u = \operatorname{tr} U$ and

$$\sum_{|\alpha|=\ell} \int_0^\infty \varepsilon^{p(\ell-s)-1} \|\partial^\alpha U(\cdot,\varepsilon)\|^p_{L^p(\mathbb{R}^n)}\, d\varepsilon \le C'_{s,p,\ell,n} |u|^p_{B^s_{p,p}}$$ · Inverse theorem

- We may always take $u \mapsto U$ linear (e.g. $U(x,\varepsilon) = u * \rho_\varepsilon(x)$)
- M, Russ 2015 Simplified arguments, valid in $B^s_{p,q}$, $1 \le q < \infty$. Not all derivatives required in the direct theorem. Extra derivatives controlled in the inverse theorem. Applications to functional calculus

- M, Russ 2015 One can use the theory of weighted Sobolev spaces as a "black box" for recovering with little technology the properties of functional calculus in Besov spaces
- If $s > 0$, $1 \le p < \infty$, $1 \le q < \infty$ and $\Phi \in C^\infty(\mathbb{R})$, with $\Phi(0) = 0$, then T_Φ maps continuously $W^{s,p} \cap L^\infty(\mathbb{R}^n)$ into $W^{s,p}(\mathbb{R}^n)$ · Meyer 1981 using paraproducts (for non integer s)
- If $0 < s < 2$, $1 < p < \infty$, and $\Phi \in C^2_c(\mathbb{R})$, $\Phi(0) = 0$, then T_Φ maps continuously the positive cone of $B^s_{p,q}(\mathbb{R}^n)$ into $B^s_{p,q}(\mathbb{R}^n)$ · Maz'ya 1972 in $W^{2,p}$ · Bourdaud, Meyer 1991 by nonlinear interpolation in $B^s_{p,q}$
- If $s > 1$, $1 \le p < \infty$, ℓ integer $\ge s$, $\Phi \in C^\ell_c(\mathbb{R})$, $\Phi(0) = 0$, then T_Φ maps continuously $W^{s,p} \cap W^{1,sp}(\mathbb{R}^n)$ into $W^{s,p}(\mathbb{R}^n)$ · Brezis, M 2001 using paraproducts and ℓ^q-maximal inequalities · Maz'ya, Shaposhnikova 2002 using maximal inequalities in fractional Sobolev spaces

- Frame Let $N \subset \mathbb{R}^m$ be a compact connected (boundaryless) embedded manifold, $\pi : E \to N$ a covering of N, B the unit ball in $\mathbb{R}^n$, X a function space. Is it possible to lift any $u \in X(B;N)$ as $u = \pi \circ \varphi$, with $\varphi \in X(B;E)$?
- Standard examples

  $N = \mathbb{S}^1$, $E = \mathbb{R}$, $\pi(t) = e^{\imath t}$: φ is a phase of u

  $N = \mathbb{RP}^2$, $E = \mathbb{S}^2$, $\pi(x) = \widehat{x} = \{x, -x\}$: φ is an orientation of u

  $N = \mathbb{S}^1$, $E = \mathbb{S}^1$, $\pi(z) = z^2$: φ is a square root of u
- If E is the universal covering and $X = W^{s,p}(B;N)$ · $N = \mathbb{S}^1$: Bourgain, Brezis, M 2000, full answer · Arbitrary N: Bethuel, Chiron 2007 partial answer M, Van Schaftingen in progress full answer
- Square root problem, $X = W^{s,p}(B;\mathbb{S}^1)$: · Weak version: M 2008 · Strong version M 2010, M, Van Schaftingen in progres
- Besov frame: partial answers in $B^s_{p,q}(B;\mathbb{S}^1)$ · M, Russ, Sire 2017

- If $s > 0$, $1 \le p < \infty$, $1 \le q < \infty$ and $sp = n$, then maps $u \in B^s_{p,q}(B;\mathbb{S}^1)$ lift as $u = e^{\imath\varphi}$, with $\varphi \in B^s_{p,q}(B;\mathbb{R})$
- Main idea: Move from B to $B \times (0,1)$ using a "good" extension U of u: If $|u| = 1$ and $u = e^{\imath\varphi}$, there is an explicit formula for $\nabla\varphi$, not for φ
- Then $|U| \ge 1/2$ on $B \times (0,\delta)$ · Schoen, Uhlenbeck 1982, Boutet de Monvel, Gabber 1991, Brezis, Nirenberg 1995
- The smooth map $w := U/|U|$ has a smooth lifting ψ on $B \times (0,\delta)$
- Estimate ψ in terms of w, U, u and rely on the theory of weighted Sobolev space in order to estimate $\varphi := \operatorname{tr}\psi$

- Factorization is a substitute to lifting in $W^{s,p}(B;\mathbb{S}^1)$ when not all $W^{s,p}(B;\mathbb{S}^1)$ lift
- Lifting theorem If $s>0$ and $1\le p<\infty$, then any map $u\in W^{s,p}(B;\mathbb{S}^1)$ can be factorized as $u=e^{\imath\psi}v$, with $\psi\in W^{s,p}$, $v\in B^{sp}_{1,1}$ + control · Bourgain, Brezis 2003, Bourgain, Brezis, M 2004, Nguyen 2008, M 2010, M, Molnar 2015
- Steps of the proof
  - Guess explicit expressions for ψ and v, formulas relying on a "good" extension U of u
  - Guess good extensions Ψ and V of ψ and v
  - Use the geometric information that $|u|=1$ via the following estimate: if

    $$d(x):=\inf\{\varepsilon>0;\,|U(x,\varepsilon)|\le 1/2\},$$

    then

    $$\int_B \frac{1}{[d(x)]^{sp}}\,dx \le C_{s,p,n}|u|^p_{W^{s,p}}$$

  - Estimate Ψ and V in terms of U and $d(x)$. The theory of weighted Sobolev spaces is not sufficient!