All automatic sequences satisfy the full Sarnak conjecture

Clemens Müllner

23. February 2016

Complexity of a sequence

Definition

A bounded complex valued sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is said to be **deterministic** if for every $\varepsilon > 0$ the set $\{(u_{n+1}, \ldots, u_{n+m}) : n \in \mathbb{N}\}$ can be covered by $O(exp(o(m)))$ balls of radius ε (as $m \to \infty$).

Example

Let

$$u_n = f(T^n x)$$

for a minimal topological dynamical system (X, T) with **zero topological entropy** (and a continuous function f), then $(u_n)_{n\geq 0}$ is deterministic.

Complexity of a sequence

Definition

A bounded complex valued sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is said to be **deterministic** if for every $\varepsilon > 0$ the set $\{(u_{n+1}, \ldots, u_{n+m}) : n \in \mathbb{N}\}$ can be covered by $O(exp(o(m)))$ balls of radius ε (as $m \to \infty$).

Example

Let

$$u_n = f(T^n x)$$

for a minimal topological dynamical system (X, T) with **zero topological entropy** (and a continuous function f), then $(u_n)_{n\geq 0}$ is deterministic.

Sarnak Conjecture

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

A sequence $\mathbf{u}$ is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n \le N} \mu(n) u_n = o(N) \qquad (N \to \infty).$$

Conjecture (Sarnak conjecture)

Every deterministic bounded complex valued sequence $\mathbf{u} = (u_n)_{n>0}$ is orthogonal to the Möbius function $\mu(n)$.

Sarnak Conjecture

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

A sequence **u** is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n \le N} \mu(n) u_n = o(N) \qquad (N \to \infty).$$

Conjecture (Sarnak conjecture)

Every deterministic bounded complex valued sequence $\mathbf{u} = (u_n)_{n>0}$ is orthogonal to the Möbius function $\mu(n)$.

Sarnak Conjecture

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

A sequence $\mathbf{u}$ is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n \leq N} \mu(n) u_n = o(N) \qquad (N \to \infty).$$

Conjecture (Sarnak conjecture)

Every deterministic bounded complex valued sequence $\mathbf{u} = (u_n)_{n>0}$ is orthogonal to the Möbius function $\mu(n)$.

„Full“ Sarnak Conjecture

Dynamical System (X, T) related to $\mathbf{u}$

$\mathbf{u} = (u_n)_{n\geq 0} \ldots$ bounded complex sequence

$T\mathbf{u} = (u_{n+1})_{n\geq 0} \ldots$ shift operator

$X = \overline{\{T^k(\mathbf{u}) : k \geq 0\}}$

We say that $\mathbf{u}$ satisfies the **„Full“ Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n\geq 0} \in X$ are orthogonal to $\mu(n)$.

„Full“ Sarnak Conjecture

Dynamical System (X, T) related to $\mathbf{u}$

$\mathbf{u} = (u_n)_{n\geq 0} \ldots$ bounded complex sequence

$T\mathbf{u} = (u_{n+1})_{n\geq 0} \ldots$ shift operator

$X = \overline{\{T^k(\mathbf{u}) : k \geq 0\}}$

We say that $\mathbf{u}$ satisfies the **„Full“ Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n\geq 0} \in X$ are orthogonal to $\mu(n)$.

„Full“ Sarnak Conjecture

Dynamical System (X, T) related to $\mathbf{u}$

$\mathbf{u} = (u_n)_{n\geq 0} \dots$ bounded complex sequence

$T\mathbf{u} = (u_{n+1})_{n\geq 0} \dots$ shift operator

$X = \overline{\{T^k(\mathbf{u}) : k \geq 0\}}$

We say that $\mathbf{u}$ satisfies the **„Full“ Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n\geq 0} \in X$ are orthogonal to $\mu(n)$.

„Full“ Sarnak Conjecture

Dynamical System (X, T) related to $\mathbf{u}$

$\mathbf{u} = (u_n)_{n\geq 0} \ldots$ bounded complex sequence

$T\mathbf{u} = (u_{n+1})_{n\geq 0} \ldots$ shift operator

$X = \overline{\{T^k(\mathbf{u}) : k \geq 0\}}$

We say that $\mathbf{u}$ satisfies the **„Full“ Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n\geq 0} \in X$ are orthogonal to $\mu(n)$.

Deterministic Finite Automata

Definition (Automaton)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

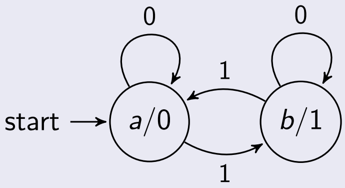
$$\mathbf{u} = (u_n)_{n \geq 0} = 01101001100101101001011001101001\ldots$$

Deterministic Finite Automata

Definition (Automaton)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

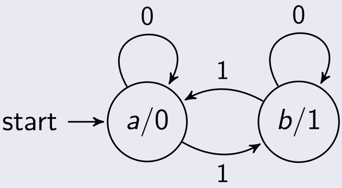
$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110 1001 \ldots$$

Deterministic Finite Automata

Definition (Automaton)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

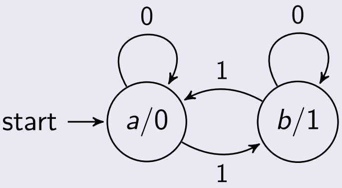
$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110 1001 \dots$$

Deterministic Finite Automata

Definition (Automaton)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110 1001 \ldots$$

Properties

- For every automatic sequence $\mathbf{u}$ there exists the logarithmic density
$$logdens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{log(N)} \sum_{1\leq n\leq N} \frac{1}{n}\mathbf{1}_{[u_n=a]}.$$
- The subword complexity p_k of an automatic sequence is (at most) linear.The dynamical system (X, T) related to an automatic sequence has zero topological entropy.
- Every subsequence $(u_{an+b})_{n\geq 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n\geq 0}$ is again automatic.
- Let $u^{(1)}(n), \dots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \dots, u^{(j)}(n))$ is again automatic.

Properties

- For every automatic sequence $\mathbf{u}$ there exists the logarithmic density

$$logdens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{log(N)} \sum_{1\leq n\leq N} \frac{1}{n} \mathbf{1}_{[u_n=a]}.$$

- The subword complexity p_k of an automatic sequence is (at most) linear. The dynamical system (X, T) related to an automatic sequence has zero topological entropy.
- Every subsequence $(u_{an+b})_{n\geq 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n\geq 0}$ is again automatic.
- Let $u^{(1)}(n), \ldots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \ldots, u^{(j)}(n))$ is again automatic.

Properties

- For every automatic sequence $\mathbf{u}$ there exists the logarithmic density

$$logdens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{log(N)} \sum_{1\leq n\leq N} \frac{1}{n}\mathbf{1}_{[u_n=a]}.$$

- The subword complexity p_k of an automatic sequence is (at most) linear. The dynamical system (X, T) related to an automatic sequence has zero topological entropy.
- Every subsequence $(u_{an+b})_{n\geq 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n\geq 0}$ is again automatic.
- Let $u^{(1)}(n), \ldots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \ldots, u^{(j)}(n))$ is again automatic.

Properties

- For every automatic sequence $\mathbf{u}$ there exists the logarithmic density
$$logdens(\mathbf{u}, a) = \lim_{N \to \infty} \frac{1}{log(N)} \sum_{1 \le n \le N} \frac{1}{n} \mathbf{1}_{[u_n = a]}.$$
- The subword complexity p_k of an automatic sequence is (at most) linear. The dynamical system (X, T) related to an automatic sequence has zero topological entropy.
- Every subsequence $(u_{an+b})_{n \ge 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n \ge 0}$ is again automatic.
- Let $u^{(1)}(n), \ldots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \ldots, u^{(j)}(n))$ is again automatic.

Properties

- For every automatic sequence $\mathbf{u}$ there exists the logarithmic density
$$logdens(\mathbf{u}, a) = \lim_{N \to \infty} \frac{1}{log(N)} \sum_{1 \leq n \leq N} \frac{1}{n} \mathbf{1}_{[u_n = a]}.$$
- The subword complexity p_k of an automatic sequence is (at most) linear.The dynamical system (X, T) related to an automatic sequence has zero topological entropy.
- Every subsequence $(u_{an+b})_{n \geq 0}$ along an arithmetic progression of an automatic sequence $(u_n)_{n \geq 0}$ is again automatic.
- Let $u^{(1)}(n), \dots, u^{(j)}(n)$ be automatic sequences. Then $u(n) = f(u^{(1)}(n), \dots, u^{(j)}(n))$ is again automatic.

Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example

$\mathbf{w}_0 = 010.$

Theorem (Deshouillers + Drmota + M.)

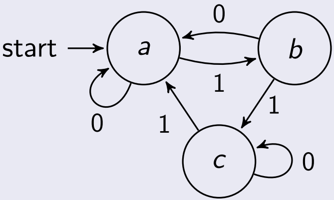
Let $\mathbf{u} = (u_n)n > 0$ be generated by a synchronizing automaton. Then $\mathbf{u} = (u_n)_{n>0}$ satisfies the full Sarnak conjecture.

Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example



$\mathbf{w}_0 = 010.$

Theorem (Deshouillers + Drmota + M.)

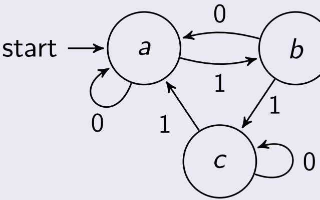
Let $\mathbf{u} = (u_n)n > 0$ be generated by a synchronizing automaton. Then $\mathbf{u} = (u_n)_{n>0}$ satisfies the full Sarnak conjecture.

Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example



$\mathbf{w}_0 = 010.$

Theorem (Deshouillers + Drmota + M.)

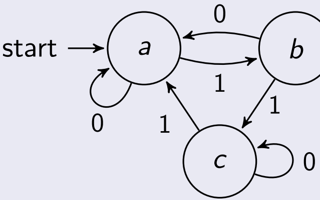
Let $\mathbf{u} = (u_n)n > 0$ be generated by a synchronizing automaton. Then $\mathbf{u} = (u_n)_{n>0}$ satisfies the full Sarnak conjecture.

Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

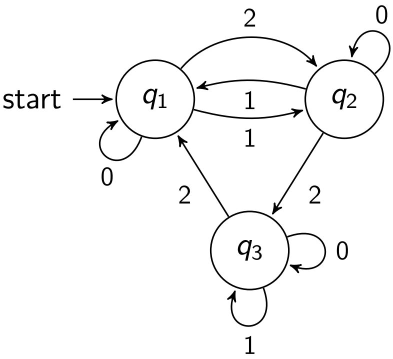
Example

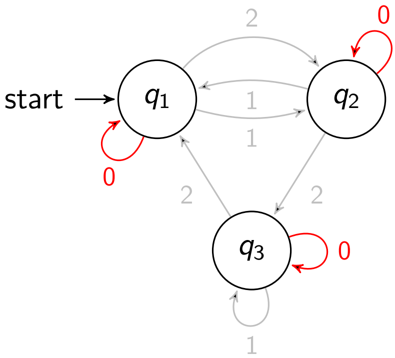


$\mathbf{w}_0 = 010.$

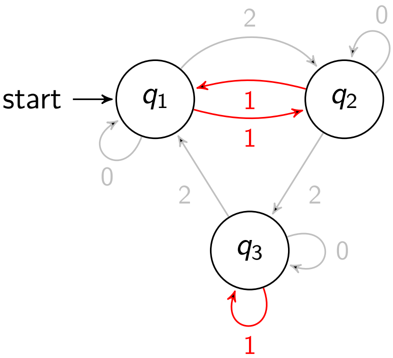
Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ be generated by a synchronizing automaton. Then $\mathbf{u} = (u_n)_{n>0}$ satisfies the full Sarnak conjecture.

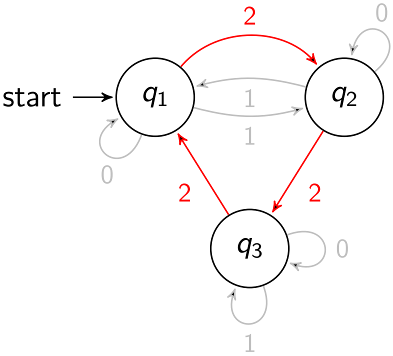

start
q_1
q_2
q_3
2
0
1
1
0
2
2
0
1



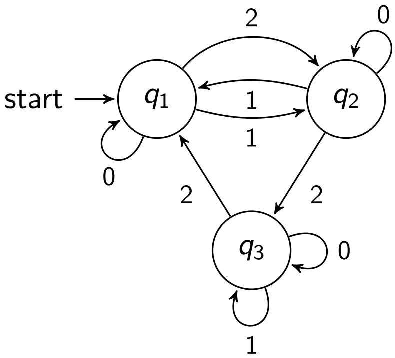
$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

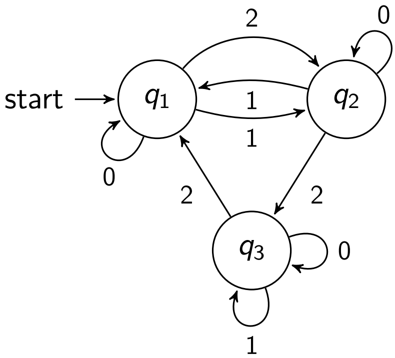


$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$11 = (102)_3 : \qquad M_2 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Remark:

If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the densities

$$dens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} \mathbf{1}_{[u_n=a]}$$

exist and coincide with the logarithmic densities.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton. Then $\mathbf{u}$ is orthogonal to $\mu(n)$.

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Remark:

If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the densities

$$dens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} \mathbf{1}_{[u_n=a]}$$

exist and coincide with the logarithmic densities.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton. Then $\mathbf{u}$ is orthogonal to $\mu(n)$.

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Remark:
If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the densities

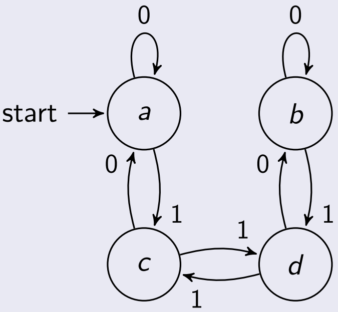
$$dens(\mathbf{u}, a) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mathbf{1}_{[u_n = a]}$$

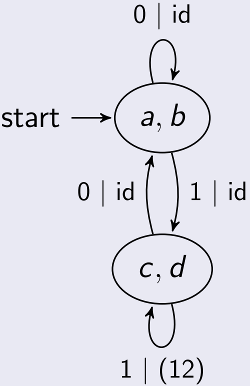
exist and coincide with the logarithmic densities.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n \geq 0}$ is generated by an invertible automaton. Then $\mathbf{u}$ is orthogonal to $\mu(n)$.

Example (Rudin-Shapiro)

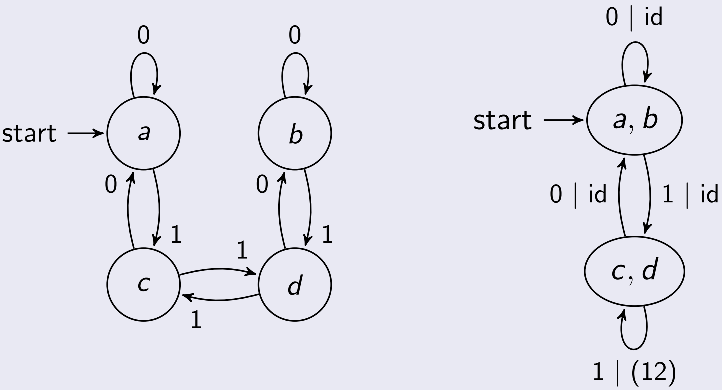




Theorem [Mauduit + Rivat, Tao]

The Rudin-Shapiro Sequence is orthogonal to the Möbius function.

Example (Rudin-Shapiro)



Theorem [Mauduit + Rivat, Tao]

The Rudin-Shapiro Sequence is orthogonal to the Möbius function.

Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. some technical conditions
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. $\mathcal{T}_A$ is synchronizing
5. some minimality conditions

Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. some technical conditions
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. $\mathcal{T}_A$ is synchronizing
5. some minimality conditions

Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. some technical conditions
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. $\mathcal{T}_A$ is synchronizing
5. some minimality conditions

Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q'_0)$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. some technical conditions
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. $\mathcal{T}_A$ is synchronizing
5. some minimality conditions

Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

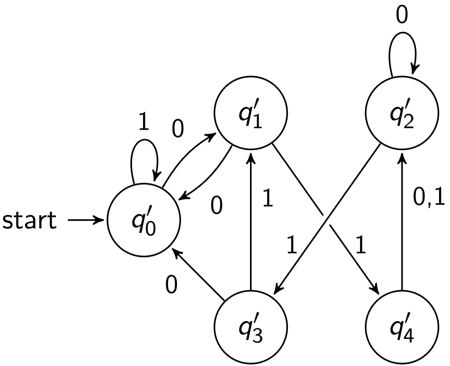
1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. some technical conditions
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. $\mathcal{T}_A$ is synchronizing
5. some minimality conditions

Definition (Naturally Induced Transducer)

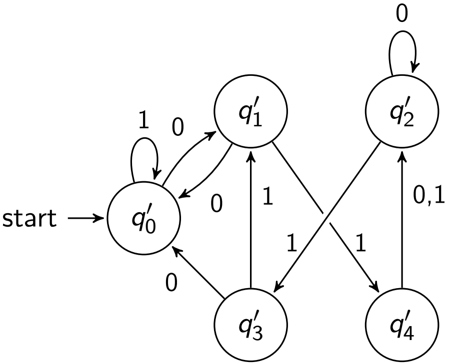
Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

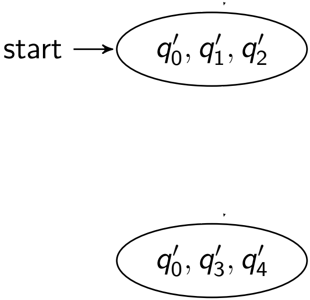
1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. some technical conditions
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. $\mathcal{T}_A$ is synchronizing
5. some minimality conditions

Example

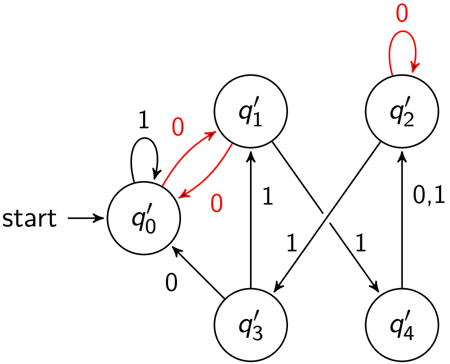


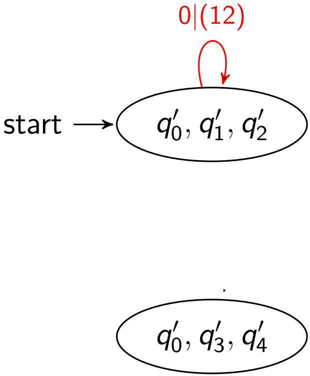
Example



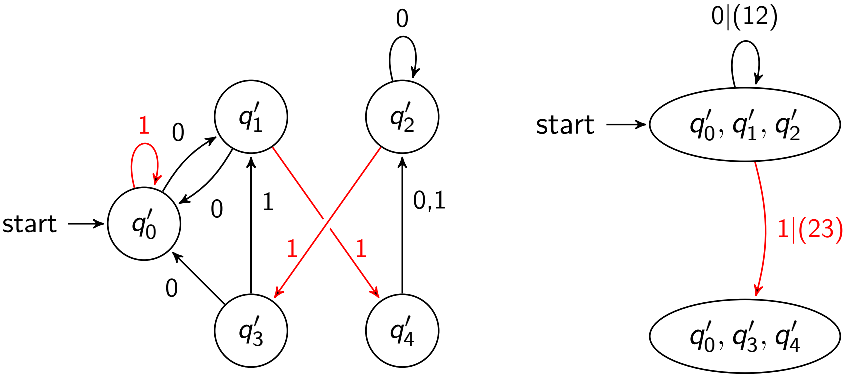


Example

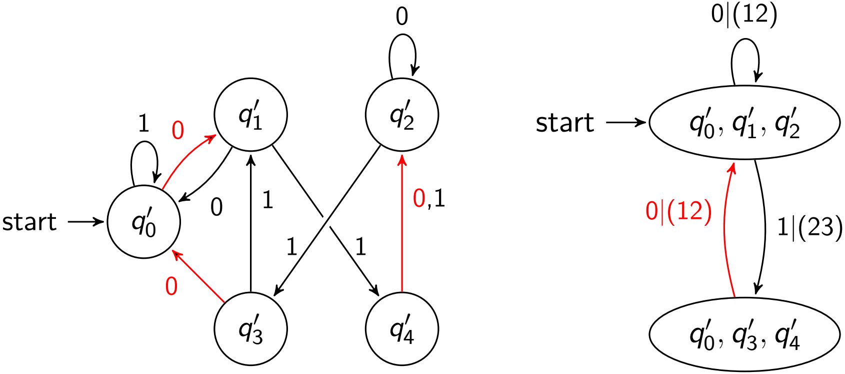




Example

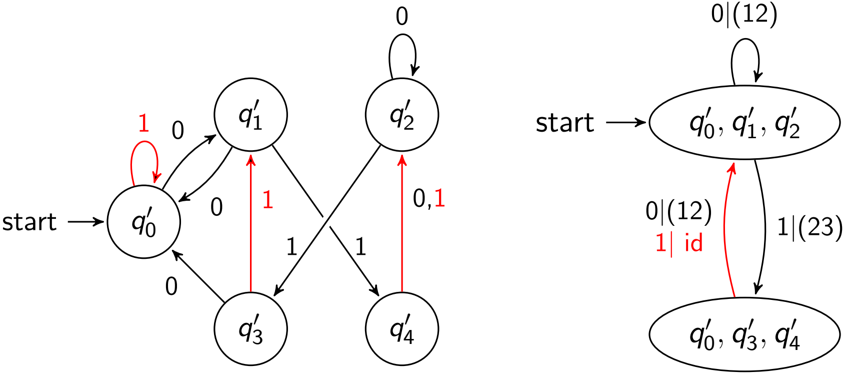


Example



start
q'_0
q'_1
q'_2
q'_3
q'_4
1
0
0
1
0
1
1
0,1
0
0
start
q'_0, q'_1, q'_2
q'_0, q'_3, q'_4
0|(12)
0|(12)
1|(23)

Example


0
1
0
start
q'_1
q'_2
q'_0
0
1
0,1
1
1
0
q'_3
q'_4
0|(12)
start
q'_0, q'_1, q'_2
0|(12)
1| id
1|(23)
q'_0, q'_3, q'_4

Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Proof (first part of the Theorem):

Define

$$n_0 := \min\{\#\delta'(Q', \mathbf{w}) : \mathbf{w} \in \Sigma^*\}$$
$$S(A) := \{M \subseteq Q' : \#M = n_0, \exists \mathbf{w}_M \in \Sigma^*, \delta'(Q', \mathbf{w}_M) = M\}$$

Define n_0-tuple q_M corresponding to $M \in S(A)$.

- $\delta'(M, a) \in S(A) \Rightarrow \delta(q_M, a) := q_{\delta'(M,a)}$
- choose λ accordingly.
- synchronizing:

$$\forall q : \delta(q, \mathbf{w}_M) = q_M.$$

Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Proof (first part of the Theorem):

Define

$$n_0 := \min\{\#\delta'(Q', \mathbf{w}) : \mathbf{w} \in \Sigma^*\}$$
$$S(A) := \{M \subseteq Q' : \#M = n_0, \exists \mathbf{w}_M \in \Sigma^*, \delta'(Q', \mathbf{w}_M) = M\}$$

Define n_0-tuple q_M corresponding to $M \in S(A)$.

- $\delta'(M, a) \in S(A) \Rightarrow \delta(q_M, a) := q_{\delta'(M,a)}$
- choose λ accordingly.
- synchronizing:

$$\forall q : \delta(q, \mathbf{w}_M) = q_M.$$

Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Proof (first part of the Theorem):
Define

$$n_0 := \min\{\#\delta'(Q', \mathbf{w}) : \mathbf{w} \in \Sigma^*\}$$

$$S(A) := \{M \subseteq Q' : \#M = n_0, \exists \mathbf{w}_M \in \Sigma^*, \delta'(Q', \mathbf{w}_M) = M\}$$

Define n_0-tuple q_M corresponding to $M \in S(A)$.

- $\delta'(M, a) \in S(A) \Rightarrow \delta(q_M, a) := q_{\delta'(M,a)}$
- choose λ accordingly.
- synchronizing:

$$\forall q : \delta(q, \mathbf{w}_M) = q_M.$$

Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Proof (first part of the Theorem):
Define

$$n_0 := \min\{\#\delta'(Q', \mathbf{w}) : \mathbf{w} \in \Sigma^*\}$$
$$S(A) := \{M \subseteq Q' : \#M = n_0, \exists \mathbf{w}_M \in \Sigma^*, \delta'(Q', \mathbf{w}_M) = M\}$$

Define n_0-tuple q_M corresponding to $M \in S(A)$.

- $\delta'(M, a) \in S(A) \Rightarrow \delta(q_M, a) := q_{\delta'(M,a)}$
- choose λ accordingly.
- synchronizing:

$$\forall q : \delta(q, \mathbf{w}_M) = q_M.$$

Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Proof (first part of the Theorem):
Define

$$n_0 := \min\{\#\delta'(Q', \mathbf{w}) : \mathbf{w} \in \Sigma^*\}$$
$$S(A) := \{M \subseteq Q' : \#M = n_0, \exists \mathbf{w}_M \in \Sigma^*, \delta'(Q', \mathbf{w}_M) = M\}$$

Define n_0-tuple q_M corresponding to $M \in S(A)$.

- $\delta'(M, a) \in S(A) \Rightarrow \delta(q_M, a) := q_{\delta'(M,a)}$
- choose λ accordingly.
- synchronizing:

$$\forall q : \delta(q, \mathbf{w}_M) = q_M.$$

Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Proof (first part of the Theorem):
Define

$$n_0 := \min\{\#\delta'(Q', \mathbf{w}) : \mathbf{w} \in \Sigma^*\}$$
$$S(A) := \{M \subseteq Q' : \#M = n_0, \exists \mathbf{w}_M \in \Sigma^*, \delta'(Q', \mathbf{w}_M) = M\}$$

Define n_0-tuple q_M corresponding to $M \in S(A)$.

- $\delta'(M, a) \in S(A) \Rightarrow \delta(q_M, a) := q_{\delta'(M,a)}$
- choose λ accordingly.
- synchronizing:

$$\forall q : \delta(q, \mathbf{w}_M) = q_M.$$

Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Proof (first part of the Theorem):
Define

$$n_0 := \min\{\#\delta'(Q', \mathbf{w}) : \mathbf{w} \in \Sigma^*\}$$
$$S(A) := \{M \subseteq Q' : \#M = n_0, \exists \mathbf{w}_M \in \Sigma^*, \delta'(Q', \mathbf{w}_M) = M\}$$

Define n_0-tuple q_M corresponding to $M \in S(A)$.

- $\delta'(M, a) \in S(A) \Rightarrow \delta(q_M, a) := q_{\delta'(M,a)}$
- choose λ accordingly.
- synchronizing:

$$\forall q : \delta(q, \mathbf{w}_M) = q_M.$$

Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Proof (first part of the Theorem):
Define

$$n_0 := \min\{\#\delta'(Q', \mathbf{w}) : \mathbf{w} \in \Sigma^*\}$$
$$S(A) := \{M \subseteq Q' : \#M = n_0, \exists \mathbf{w}_M \in \Sigma^*, \delta'(Q', \mathbf{w}_M) = M\}$$

Define n_0-tuple q_M corresponding to $M \in S(A)$.

- $\delta'(M, a) \in S(A) \Rightarrow \delta(q_M, a) := q_{\delta'(M,a)}$
- choose λ accordingly.
- synchronizing:

$$\forall q : \delta(q, \mathbf{w}_M) = q_M.$$

Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Proof (first part of the Theorem):

Define

$$n_0 := \min\{\#\delta'(Q', \mathbf{w}) : \mathbf{w} \in \Sigma^*\}$$
$$S(A) := \{M \subseteq Q' : \#M = n_0, \exists \mathbf{w}_M \in \Sigma^*, \delta'(Q', \mathbf{w}_M) = M\}$$

Define n_0-tuple q_M corresponding to $M \in S(A)$.

- $\delta'(M, a) \in S(A) \Rightarrow \delta(q_M, a) := q_{\delta'(M,a)}$
- choose λ accordingly.
- synchronizing:

$$\forall q : \delta(q, \mathbf{w}_M) = q_M.$$

Definition

Denote by

$$T(q, w_1 \dots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \dots \circ \lambda(\delta(q, w_1 \dots w_{r-1}), w_r).$$

Lemma

Let A be a strongly connected automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

$$\delta'(q_0', \mathbf{w}) = \pi_1(T(q_0, \mathbf{w}) \cdot \delta(q_0, \mathbf{w}))$$

holds for all $\mathbf{w} \in \Sigma^*$.

Definition

Denote by

$$T(q, w_1 \dots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \dots \\ \circ \lambda(\delta(q, w_1 \dots w_{r-1}), w_r).$$

Lemma

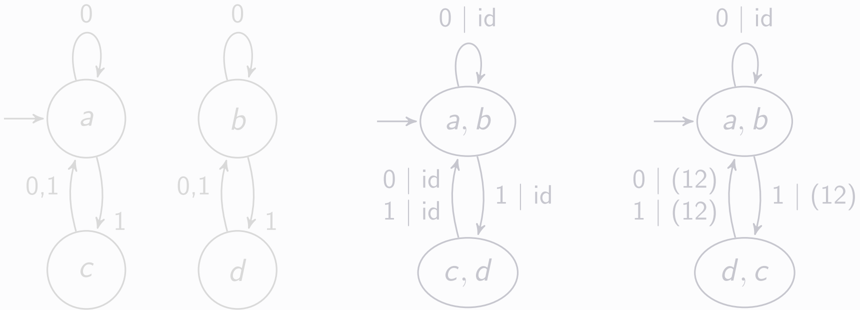
Let A be a strongly connected automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

$$\delta'(q_0', \mathbf{w}) = \pi_1(T(q_0, \mathbf{w}) \cdot \delta(q_0, \mathbf{w}))$$

holds for all $\mathbf{w} \in \Sigma^*$.

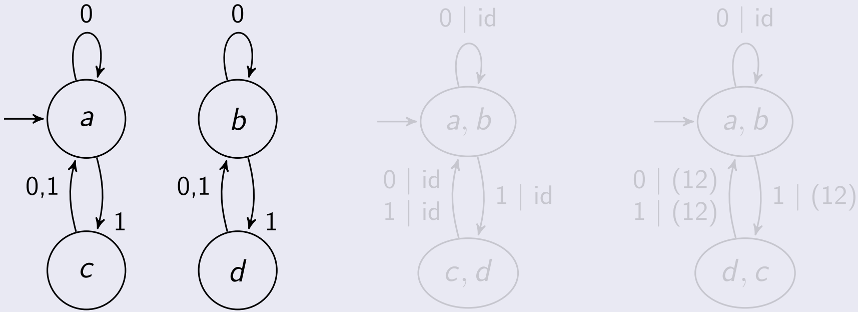
Are some naturally induced transducers better than others?

(Oversimplified) Example



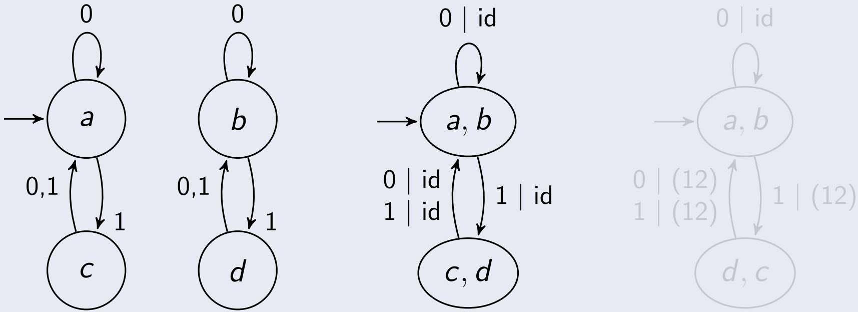
Are some naturally induced transducers better than others?

(Oversimplified) Example



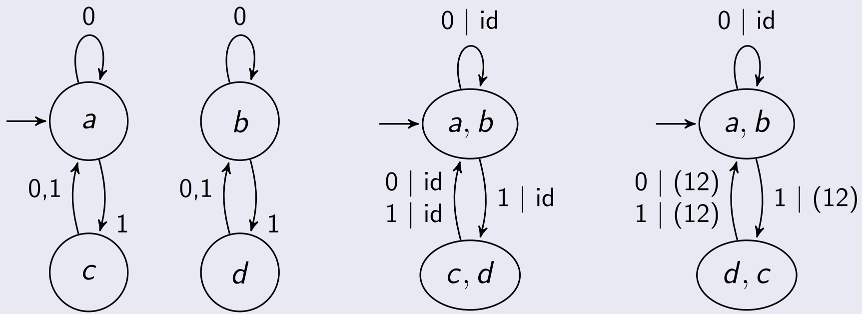
Are some naturally induced transducers better than others?

(Oversimplified) Example



Are some naturally induced transducers better than others?

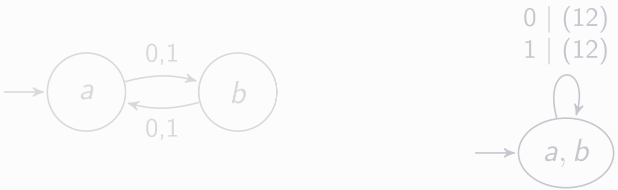
(Oversimplified) Example



All elements of Δ appear as values of $T(q_0, .)$ for „good“ naturally induced transducer.

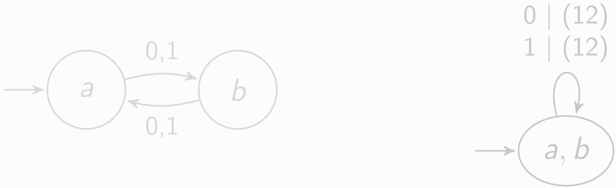
Do all elements of Δ appear as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$, where n is large?

Example



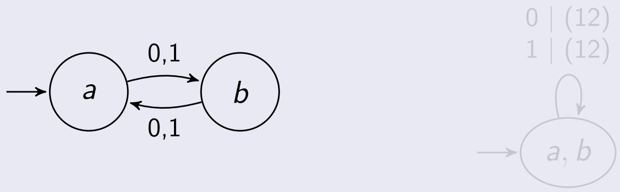
All elements of Δ appear as values of $T(q_0, .)$ for „good" naturally induced transducer.
Do all elements of Δ appear as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$, where n is large?

Example



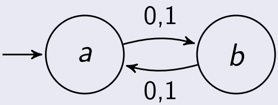
All elements of Δ appear as values of $T(q_0, .)$ for „good“ naturally induced transducer.
Do all elements of Δ appear as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$, where n is large?

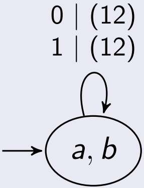
Example



All elements of Δ appear as values of $T(q_0, .)$ for „good" naturally induced transducer.
Do all elements of Δ appear as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$, where n is large?

Example

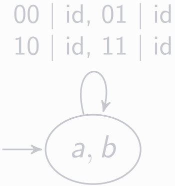




Do all elements of Δ appear as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$, where n is large?

The key point is to avoid periodic behavior.

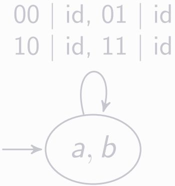
Example



Do all elements of Δ appear as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$, where n is large?

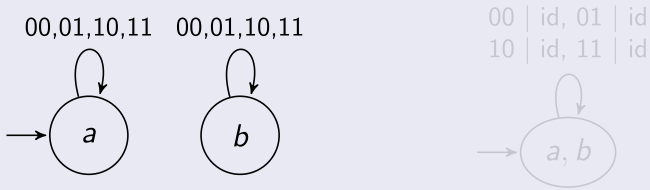
The key point is to avoid periodic behavior.

Example



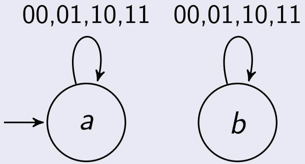
Do all elements of Δ appear as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$, where n is large?
The key point is to avoid periodic behavior.

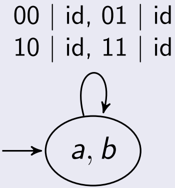
Example



Do all elements of Δ appear as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$, where n is large?
The key point is to avoid periodic behavior.

Example





Theorem 1

Every automatic sequence $(a_n)_{n\geq 0}$ fulfills the full Sarnak conjecture.

Theorem 2

Let $A = (Q', \Sigma, \delta', q_0', \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k-1\}$ and $\delta'(q_0', 0) = q_0'$. Then the frequencies of the letters for the subsequence $(a_p)_{p\in\mathcal{P}}$ exist.

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they are equally distributed under reasonable conditions.

Theorem 1

Every automatic sequence $(a_n)_{n\geq 0}$ fulfills the full Sarnak conjecture.

Theorem 2

Let $A = (Q', \Sigma, \delta', q_0', \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k-1\}$ and $\delta'(q_0', 0) = q_0'$. Then the frequencies of the letters for the subsequence $(a_p)_{p\in\mathcal{P}}$ exist.

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they are equally distributed under reasonable conditions.

Theorem 1

Every automatic sequence $(a_n)_{n\geq 0}$ fulfills the full Sarnak conjecture.

Theorem 2

Let $A = (Q', \Sigma, \delta', q_0', \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \dots, k-1\}$ and $\delta'(q_0', 0) = q_0'$. Then the frequencies of the letters for the subsequence $(a_p)_{p\in\mathcal{P}}$ exist.

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they are equally distributed under reasonable conditions.

Ideas for the proof of Theorem 1

We assume that the automaton is strongly connected and $\delta'(q_0', 0) = q_0'$ and proof only

$$\sum_{n<N} \mu(n) a_n = o(N).$$

Fix $\varepsilon > 0$. We need to show

$$\left| \sum_{n<N} \mu(n) a_n \right| \leq \varepsilon N.$$

$$\sum_{n<N} \mu(n) a_n = \sum_{n<N} \mu(n) \cdot \pi_1(T(q_0, n) \cdot \delta(q_0, n))$$

Ideas for the proof of Theorem 1

We assume that the automaton is strongly connected and $\delta'(q_0', 0) = q_0'$ and proof only

$$\sum_{n<N} \mu(n) a_n = o(N).$$

Fix $\varepsilon > 0$. We need to show

$$\left| \sum_{n<N} \mu(n) a_n \right| \leq \varepsilon N.$$

$$\sum_{n<N} \mu(n) a_n = \sum_{n<N} \mu(n) \cdot \pi_1(T(q_0, n) \cdot \delta(q_0, n))$$

Ideas for the proof of Theorem 1

We assume that the automaton is strongly connected and $\delta'(q_0', 0) = q_0'$ and proof only

$$\sum_{n<N} \mu(n) a_n = o(N).$$

Fix $\varepsilon > 0$. We need to show

$$\left| \sum_{n<N} \mu(n) a_n \right| \leq \varepsilon N.$$

$$\sum_{n<N} \mu(n) a_n = \sum_{n<N} \mu(n) \cdot \pi_1(T(q_0, n) \cdot \delta(q_0, n))$$

$$\sum_{n<N} \mu(n) \cdot \pi_1(T(q_0, n) \cdot \delta(q_0, n))$$
$$= \sum_{m<k^\lambda} \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot \pi_1(T(q_0, n) \cdot \delta(q_0, n))$$
$$\approx \sum_{m<k^\lambda} \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot \pi_1(T(q_0, n) \cdot \delta(q_0, m))$$
$$= \sum_{m<k^\lambda} \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot f_{\delta(q_0,m)}(T(q_0, n))$$

$$\left| \sum_{n<N} \mu(n) a_n \right| \lesssim k^\lambda \max_{m<k^\lambda} \max_{q \in Q} \left| \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot f_q(T(q_0, n)) \right|$$

$$\begin{aligned}\sum_{n<N}&\mu(n)\cdot\pi_1(T(q_0,n)\cdot\delta(q_0,n))\\ &=\sum_{m<k^\lambda}\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}}\mu(n)\cdot\pi_1(T(q_0,n)\cdot\delta(q_0,n))\\ &\approx\sum_{m<k^\lambda}\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}}\mu(n)\cdot\pi_1(T(q_0,n)\cdot\delta(q_0,m))\\ &=\sum_{m<k^\lambda}\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}}\mu(n)\cdot f_{\delta(q_0,m)}(T(q_0,n))\end{aligned}$$

$$\left|\sum_{n<N}\mu(n)a_n\right|\lessapprox k^\lambda\max_{m<k^\lambda}\max_{q\in Q}\left|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}}\mu(n)\cdot f_q(T(q_0,n))\right|$$

$$\sum_{n<N} \mu(n) \cdot \pi_1(T(q_0, n) \cdot \delta(q_0, n))$$

$$= \sum_{m<k^\lambda} \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot \pi_1(T(q_0, n) \cdot \delta(q_0, n))$$

$$\approx \sum_{m<k^\lambda} \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot \pi_1(T(q_0, n) \cdot \delta(q_0, m))$$

$$= \sum_{m<k^\lambda} \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot f_{\delta(q_0, m)}(T(q_0, n))$$

$$\left| \sum_{n<N} \mu(n) a_n \right| \lesssim k^\lambda \max_{m<k^\lambda} \max_{q \in Q} \left| \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot f_q(T(q_0, n)) \right|$$

$$\sum_{n<N} \mu(n) \cdot \pi_1(T(q_0,n) \cdot \delta(q_0,n))$$
$$= \sum_{m<k^\lambda} \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot \pi_1(T(q_0,n) \cdot \delta(q_0,n))$$
$$\approx \sum_{m<k^\lambda} \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot \pi_1(T(q_0,n) \cdot \delta(q_0,m))$$
$$= \sum_{m<k^\lambda} \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot f_{\delta(q_0,m)}(T(q_0,n))$$

$$\left| \sum_{n<N} \mu(n) a_n \right| \lessapprox k^\lambda \max_{m<k^\lambda} \max_{q \in Q} \left| \sum_{\substack{n<N \\ n \equiv m \bmod k^\lambda}} \mu(n) \cdot f_q(T(q_0,n)) \right|$$

Continuous functions from a compact group to $\mathbb{C}$

Definition (Representation)

Let G be a compact group and $k \in \mathbb{N}$. A **Representation** of rank k is a continuous homomorphism $D : G \to \mathbb{C}^{k \times k}$.

Lemma

Let f be a continuous function from G to $\mathbb{C}$ and $\varepsilon > 0$. There exists $r \in \mathbb{N}$ and unitary, irreducible representations $D^{(\ell)} = (d_{i,j}^{(\ell)})_{i,j<k_\ell}$ along with $c_\ell \in \mathbb{C}$ such that

$$\left| f(g) - \sum_{\ell < r} c_\ell d_{i_\ell, j_\ell}^{(\ell)}(g) \right| \leq \varepsilon$$

holds for all $g \in G$.

Continuous functions from a compact group to $\mathbb{C}$

Definition (Representation)

Let G be a compact group and $k \in \mathbb{N}$. A **Representation** of rank k is a continuous homomorphism $D : G \to \mathbb{C}^{k \times k}$.

Lemma

Let f be a continuous function from G to $\mathbb{C}$ and $\varepsilon > 0$. There exists $r \in \mathbb{N}$ and unitary, irreducible representations $D^{(\ell)} = (d_{i,j}^{(\ell)})_{i,j<k_\ell}$ along with $c_\ell \in \mathbb{C}$ such that

$$\left| f(g) - \sum_{\ell<r} c_\ell d_{i_\ell,j_\ell}^{(\ell)}(g) \right| \leq \varepsilon$$

holds for all $g \in G$.

$$\begin{aligned}
\left|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) f(T(q_0,n))\right|
&\approx \left|\sum_{n<N} \mu(n) \sum_{\ell<r} c_\ell d^{(\ell)}_{i_\ell,j_\ell}(T(q_0,n))\right|\\
&\leq \sum_{\ell<r} |c_\ell| \left|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) d^{(\ell)}_{i_\ell,j_\ell}(T(q_0,n))\right|\\
&\leq \sum_{\ell<r} |c_\ell| \left\|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) D^{(\ell)}(T(q_0,n))\right\|_F
\end{aligned}$$

$$\left|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) f(T(q_0,n))\right|$$

$$\approx \left|\sum_{n<N} \mu(n) \sum_{\ell<r} c_\ell d^{(\ell)}_{i_\ell,j_\ell}(T(q_0,n))\right|$$

$$\leq \sum_{\ell<r} |c_\ell| \left|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) d^{(\ell)}_{i_\ell,j_\ell}(T(q_0,n))\right|$$

$$\leq \sum_{\ell<r} |c_\ell| \left\|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) D^{(\ell)}(T(q_0,n))\right\|_F$$

$$\left|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) f(T(q_0,n))\right|$$

$$\approx \left|\sum_{n<N} \mu(n) \sum_{\ell<r} c_\ell d^{(\ell)}_{i_\ell,j_\ell}(T(q_0,n))\right|$$

$$\leq \sum_{\ell<r} |c_\ell| \left|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) d^{(\ell)}_{i_\ell,j_\ell}(T(q_0,n))\right|$$

$$\leq \sum_{\ell<r} |c_\ell| \left\|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) D^{(\ell)}(T(q_0,n))\right\|_F$$

$$\left|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) f(T(q_0,n))\right|$$

$$\approx \left|\sum_{n<N} \mu(n) \sum_{\ell<r} c_\ell d^{(\ell)}_{i_\ell,j_\ell}(T(q_0,n))\right|$$

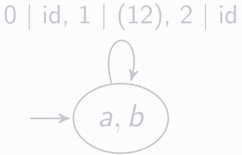
$$\leq \sum_{\ell<r} |c_\ell| \left|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) d^{(\ell)}_{i_\ell,j_\ell}(T(q_0,n))\right|$$

$$\leq \sum_{\ell<r} |c_\ell| \left\|\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) D^{(\ell)}(T(q_0,n))\right\|_F$$

Special Representations

There exist representations that correspond to arithmetic properties of the automatic sequence.

Example



$$T(q_0, n) = id \Leftrightarrow s_3(n) \equiv 0 \bmod 2 \Leftrightarrow n \equiv 0 \bmod 2$$

$$D(id) = 1, D((12)) = -1$$

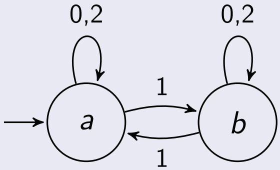
$$D(T(q_0, n)) = (-1)^n$$

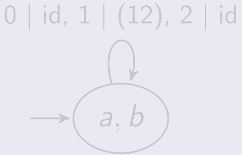
$$D(T(q_0, n)) = \exp\left(2\pi i \frac{j}{k-1}\right)$$

Special Representations

There exist representations that correspond to arithmetic properties of the automatic sequence.

Example





$$T(q_0, n) = id \Leftrightarrow s_3(n) \equiv 0 \bmod 2 \Leftrightarrow n \equiv 0 \bmod 2$$

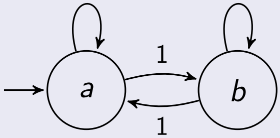
$$D(id) = 1, D((12)) = -1$$
$$D(T(q_0, n)) = (-1)^n$$

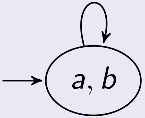
$$D(T(q_0, n)) = \exp\left(2\pi i \frac{j}{k-1}\right)$$

Special Representations

There exist representations that correspond to arithmetic properties of the automatic sequence.

Example





$$T(q_0, n) = id \Leftrightarrow s_3(n) \equiv 0 \bmod 2 \Leftrightarrow n \equiv 0 \bmod 2$$

$$D(id) = 1, D((12)) = -1$$

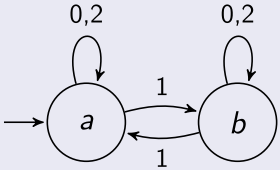
$$D(T(q_0, n)) = (-1)^n$$

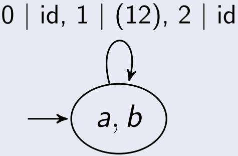
$$D(T(q_0, n)) = \exp\left(2\pi i \frac{j}{k-1}\right)$$

Special Representations

There exist representations that correspond to arithmetic properties of the automatic sequence.

Example





$$T(q_0, n) = id \Leftrightarrow s_3(n) \equiv 0 \bmod 2 \Leftrightarrow n \equiv 0 \bmod 2$$

$$D(id) = 1, D((12)) = -1$$
$$D(T(q_0, n)) = (-1)^n$$

$$D(T(q_0, n)) = \exp\left(2\pi i \frac{j}{k-1}\right)$$

Special Representations

There exist representations that correspond to arithmetic properties of the automatic sequence.

Example

$$T(q_0, n) = id \Leftrightarrow s_3(n) \equiv 0 \bmod 2 \Leftrightarrow n \equiv 0 \bmod 2$$

$$D(id) = 1, D((12)) = -1$$

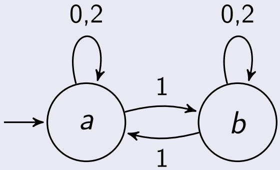
$$D(T(q_0, n)) = (-1)^n$$

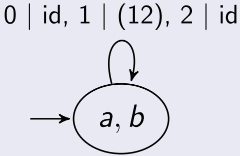
$$D(T(q_0, n)) = \exp\left(2\pi i \frac{j}{k-1}\right)$$

Special Representations

There exist representations that correspond to arithmetic properties of the automatic sequence.

Example





$$T(q_0, n) = id \Leftrightarrow s_3(n) \equiv 0 \bmod 2 \Leftrightarrow n \equiv 0 \bmod 2$$

$$D(id) = 1, D((12)) = -1$$
$$D(T(q_0, n)) = (-1)^n$$

$$D(T(q_0, n)) = \exp\left(2\pi i \frac{j}{k-1}\right)$$

Special Representations

$$\sum_{\substack{n<N \\ n\equiv m \bmod k^\lambda}} \mu(n) D(T(q_0,n)) = \sum_{\substack{n<N \\ n\equiv m \bmod k^\lambda}} \mu(n) \exp\left(2\pi i \frac{j}{k-1}\right)$$

Möbius function in arithmetic progressions.

Lemma

$$\sum_{h<k^\lambda} \exp\left(2\pi i \frac{n}{k^\lambda}\right) = \mathbf{1}_{[n\equiv 0 \bmod k^\lambda]}.$$

$$\sum_{\substack{n<N \\ n\equiv m \bmod k^\lambda}} \mu(n) D(T(q_0,n))$$

$$= \frac{1}{k^\lambda} \sum_{h<k^\lambda} \sum_{n<N} \exp\left(2\pi i \frac{h(n-m)}{k^\lambda}\right) \mu(n) D(T(q_0,n))$$

Special Representations

$$\sum_{\substack{n<N \\ n\equiv m \bmod k^\lambda}} \mu(n) D(T(q_0,n)) = \sum_{\substack{n<N \\ n\equiv m \bmod k^\lambda}} \mu(n) \exp\left(2\pi i \frac{j}{k-1}\right)$$

Möbius function in arithmetic progressions.

Lemma

$$\sum_{h<k^\lambda} \exp\left(2\pi i \frac{n}{k^\lambda}\right) = \mathbf{1}_{[n\equiv 0 \bmod k^\lambda]}.$$

$$\sum_{\substack{n<N \\ n\equiv m \bmod k^\lambda}} \mu(n) D(T(q_0,n))$$

$$= \frac{1}{k^\lambda} \sum_{h<k^\lambda} \sum_{n<N} \exp\left(2\pi i \frac{h(n-m)}{k^\lambda}\right) \mu(n) D(T(q_0,n))$$

Special Representations

$$\sum_{\substack{n<N \\ n\equiv m \bmod k^\lambda}} \mu(n) D(T(q_0,n)) = \sum_{\substack{n<N \\ n\equiv m \bmod k^\lambda}} \mu(n) \exp\left(2\pi i \frac{j}{k-1}\right)$$

Möbius function in arithmetic progressions.

Lemma

$$\sum_{h<k^\lambda} \exp\left(2\pi i \frac{n}{k^\lambda}\right) = \mathbf{1}_{[n\equiv 0 \bmod k^\lambda]}.$$

$$\sum_{\substack{n<N \\ n\equiv m \bmod k^\lambda}} \mu(n) D(T(q_0,n))$$

$$= \frac{1}{k^\lambda} \sum_{h<k^\lambda} \sum_{n<N} \exp\left(2\pi i \frac{h(n-m)}{k^\lambda}\right) \mu(n) D(T(q_0,n))$$

Special Representations

$$\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) D(T(q_0,n)) = \sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) \exp\left(2\pi i \frac{j}{k-1}\right)$$

Möbius function in arithmetic progressions.

Lemma

$$\sum_{h<k^\lambda} \exp\left(2\pi i \frac{n}{k^\lambda}\right) = \mathbf{1}_{[n\equiv 0 \bmod k^\lambda]}.$$

$$\sum_{\substack{n<N\\ n\equiv m \bmod k^\lambda}} \mu(n) D(T(q_0,n))$$
$$= \frac{1}{k^\lambda} \sum_{h<k^\lambda} \sum_{n<N} \exp\left(2\pi i \frac{h(n-m)}{k^\lambda}\right) \mu(n) D(T(q_0,n))$$

We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.
$f(n)$ ... complex sequence with $|f(n)| = 1$.
$f_\lambda(n) = f(n \bmod k^\lambda)$ ... periodic with period k^λ

Definition

We say that f has the **carry property** if, uniformly for $\lambda, \kappa, \rho > 0$ with $\rho < \lambda$, the number of integers $0 \leq \ell < k^\lambda$ such that there exists $k_1, k_2 \in \{0, 1, \ldots, k^\kappa - 1\}$ with

$$f(\ell k^\kappa + k_1 + k_2)\overline{f(\ell k^\kappa + k_1)} \neq f_{\kappa+\rho}(\ell k^\kappa + k_1 + k_2)\overline{f_{\kappa+\rho}(\ell k^\kappa + k_1)}$$

is at most $O(k^{\lambda-\rho})$, where the implied constant may depend on k and f.

We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.
$f(n)$ ... complex sequence with $|f(n)| = 1$.
$f_\lambda(n) = f(n \bmod k^\lambda)$ ... periodic with period k^λ

Definition

We say that f has the **carry property** if, uniformly for $\lambda, \kappa, \rho > 0$ with $\rho < \lambda$, the number of integers $0 \leq \ell < k^\lambda$ such that there exists $k_1, k_2 \in \{0, 1, \ldots, k^\kappa - 1\}$ with

$$f(\ell k^\kappa + k_1 + k_2)\overline{f(\ell k^\kappa + k_1)} \neq f_{\kappa+\rho}(\ell k^\kappa + k_1 + k_2)\overline{f_{\kappa+\rho}(\ell k^\kappa + k_1)}$$

is at most $O(k^{\lambda-\rho})$, where the implied constant may depend on k and f.

Definition

We say that f has the **Fourier property** if there exists a non-decreasing real function γ with $\lim_{\lambda\to\infty}\gamma(\lambda) = +\infty$ and a constant c such that for all non-negativ integers $\lambda, \alpha \geq 0$ with $\alpha \leq c\lambda$ and real t

$$\left|\frac{1}{k^\lambda}\sum_{m<k^\lambda} f(mk^\alpha)e(mt)\right| \leq k^{-\gamma(\lambda)}$$

Theorem (Mauduit + Rivat)

Suppose that f has the carry and the Fourier property (for some $c \geq 10$). Then we have for any real θ

$$\left| \sum_{n<N} \mu(n) f(n) e(\theta n) \right| \ll c_1(k) (\log N)^{c_2(k)} N k^{-\gamma(2\lfloor \log N/(80 \log k) \rfloor)/20}$$

Theorem (Mauduit + Rivat)

Suppose that f has the carry and the Fourier property (for some $c \geq 10$). Then we have for any real θ

$$\left| \sum_{n<N} \Lambda(n) f(n) e(\theta n) \right| \ll c_1(k) (\log N)^{c_3(k)} N k^{-\gamma(2\lfloor \log N/(80 \log k) \rfloor)/20}$$

Theorem (Mauduit + Rivat)

Suppose that f has the carry and the Fourier property (for some $c \geq 10$). Then we have for any real θ

$$\left| \sum_{n<N} \mu(n) f(n) e(\theta n) \right| \ll c_1(k) (\log N)^{c_2(k)} N k^{-\gamma(2\lfloor \log N/(80 \log k) \rfloor)/20}$$

Theorem (Mauduit + Rivat)

Suppose that f has the carry and the Fourier property (for some $c \geq 10$). Then we have for any real θ

$$\left| \sum_{n<N} \Lambda(n) f(n) e(\theta n) \right| \ll c_1(k) (\log N)^{c_3(k)} N k^{-\gamma(2\lfloor \log N/(80 \log k) \rfloor)/20}$$

Remark: If $\gamma(\lambda)$ grows faster than $\log\log\lambda$ then the right hand side is $o(N)$.

$U(n) \ldots$ sequence of unitary matrices

$U_\lambda(n) = U(n \bmod k^\lambda) \ldots$ periodic with period k^λ

(Adopted) Definition

We say that U has the **carry property** if, uniformly for $\lambda, \kappa, \rho \geq 0$ with $\rho < \lambda$, the number of integers $0 \leq \ell < k^\lambda$ such that there exists $k_1, k_2 \in \{0, 1, \ldots, k^\kappa - 1\}$ with

$$U(\ell k^\kappa + k_1 + k_2) U(\ell k^\kappa + k_1)^H \neq U_{\kappa+\rho}(\ell k^\kappa + k_1 + k_2) U_{\kappa+\rho}(\ell k^\kappa + k_1)^H$$

is at most $O(k^{\lambda - \eta\rho})$ for some $\eta > 0$, where the implied constants may depend on k and U.

Remark: If $\gamma(\lambda)$ grows faster than $\log\log\lambda$ then the right hand side is $o(N)$.
$U(n)\dots$ sequence of unitary matrices
$U_\lambda(n) = U(n \bmod k^\lambda)\dots$ periodic with period k^λ

(Adopted) Definition

We say that U has the **carry property** if, uniformly for $\lambda, \kappa, \rho \geq 0$ with $\rho < \lambda$, the number of integers $0 \leq \ell < k^\lambda$ such that there exists $k_1, k_2 \in \{0, 1, \dots, k^\kappa - 1\}$ with

$$U(\ell k^\kappa + k_1 + k_2) U(\ell k^\kappa + k_1)^H \neq U_{\kappa+\rho}(\ell k^\kappa + k_1 + k_2) U_{\kappa+\rho}(\ell k^\kappa + k_1)^H$$

is at most $O(k^{\lambda - \eta\rho})$ for some $\eta > 0$, where the implied constants may depend on k and U.

Remark: If $\gamma(\lambda)$ grows faster than $\log\log\lambda$ then the right hand side is $o(N)$.
$U(n)\ldots$ sequence of unitary matrices
$U_\lambda(n) = U(n \bmod k^\lambda)\ldots$ periodic with period k^λ

(Adopted) Definition

We say that U has the **carry property** if, uniformly for $\lambda, \kappa, \rho \geq 0$ with $\rho < \lambda$, the number of integers $0 \leq \ell < k^\lambda$ such that there exists $k_1, k_2 \in \{0, 1, \ldots, k^\kappa - 1\}$ with

$$U(\ell k^\kappa + k_1 + k_2) U(\ell k^\kappa + k_1)^H \neq U_{\kappa+\rho}(\ell k^\kappa + k_1 + k_2) U_{\kappa+\rho}(\ell k^\kappa + k_1)^H$$

is at most $O(k^{\lambda-\eta\rho})$ for some $\eta > 0$, where the implied constants may depend on k and U.

(Adopted) Definition

We say that U has the **Fourier property** if there exists a non-decreasing real function γ with $\lim_{\lambda\to\infty}\gamma(\lambda) = +\infty$ and a constant c such that for all non-negativ integers $\lambda, \alpha \geq 0$ with $\alpha \leq c\lambda$ and real t

$$\left\| \frac{1}{k^\lambda} \sum_{m<k^\lambda} U(mk^\alpha) e(mt) \right\| \leq k^{-\gamma(\lambda)}.$$

Remark: The Fourier property is very hard to prove (compared to the carry property).

Remark: The carry property holds for all $U(n) = D(T(n))$ where D is a unitary representation, but the fourier property holds only for unitary, irreducible (and non-special) representations.

(Adopted) Definition

We say that U has the **Fourier property** if there exists a non-decreasing real function γ with $\lim_{\lambda\to\infty} \gamma(\lambda) = +\infty$ and a constant c such that for all non-negativ integers $\lambda, \alpha \geq 0$ with $\alpha \leq c\lambda$ and real t

$$\left\| \frac{1}{k^\lambda} \sum_{m<k^\lambda} U(mk^\alpha) e(mt) \right\| \leq k^{-\gamma(\lambda)}.$$

Remark: The Fourier property is very hard to prove (compared to the carry property).

Remark: The carry property holds for all $U(n) = D(T(n))$ where D is a unitary representation, but the fourier property holds only for unitary, irreducible (and non-special) representations.

(Adopted) Definition

We say that U has the **Fourier property** if there exists a non-decreasing real function γ with $\lim_{\lambda\to\infty}\gamma(\lambda) = +\infty$ and a constant c such that for all non-negativ integers $\lambda, \alpha \geq 0$ with $\alpha \leq c\lambda$ and real t

$$\left\| \frac{1}{k^\lambda} \sum_{m<k^\lambda} U(mk^\alpha)e(mt) \right\| \leq k^{-\gamma(\lambda)}.$$

Remark: The Fourier property is very hard to prove (compared to the carry property).
Remark: The carry property holds for all $U(n) = D(T(n))$ where D is a unitary representation, but the fourier property holds only for unitary, irreducible (and non-special) representations.

(Adopted) Theorem

Suppose that U has the carry property for some $\eta > 0$ and the Fourier property (for some $c \geq 10$). Then we have for any real θ

$$\left\| \sum_{n<N} \mu(n) U(n) e(\theta n) \right\| \ll c_1(k)(\log N)^{c_2(k)} N k^{-\eta\gamma(2\lfloor \log N/(80 \log k)\rfloor)/20}$$

(Adopted) Theorem

Suppose that U has the carry property for some $\eta >$ and the Fourier property (for some $c \geq 10$). Then we have for any real θ

$$\left\| \sum_{n<N} \Lambda(n) U(n) e(\theta n) \right\| \ll c_1(k)(\log N)^{c_3(k)} N k^{-\eta\gamma(2\lfloor \log N/(80 \log k)\rfloor)/20}$$

(Adopted) Theorem

Suppose that U has the carry property for some $\eta > 0$ and the Fourier property (for some $c \geq 10$). Then we have for any real θ

$$\left\| \sum_{n<N} \mu(n) U(n) e(\theta n) \right\| \ll c_1(k) (\log N)^{c_2(k)} N k^{-\eta\gamma(2\lfloor \log N/(80 \log k) \rfloor)/20}$$

(Adopted) Theorem

Suppose that U has the carry property for some $\eta >$ and the Fourier property (for some $c \geq 10$). Then we have for any real θ

$$\left\| \sum_{n<N} \Lambda(n) U(n) e(\theta n) \right\| \ll c_1(k) (\log N)^{c_3(k)} N k^{-\eta\gamma(2\lfloor \log N/(80 \log k) \rfloor)/20}$$

Automatic Sequences along Primes

The treatment is very similar to the orthogonality to the Möbius function.

One has to work more carefully to extract the main term.

The actual frequencies can be made explicit.

Primes vs all natural Numbers

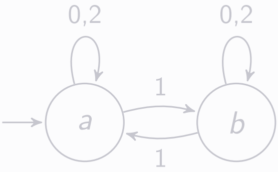
Automatic Sequences along Primes

The treatment is very similar to the orthogonality to the Möbius function.

One has to work more carefully to extract the main term.

The actual frequencies can be made explicit.

Primes vs all natural Numbers



Automatic Sequences along Primes

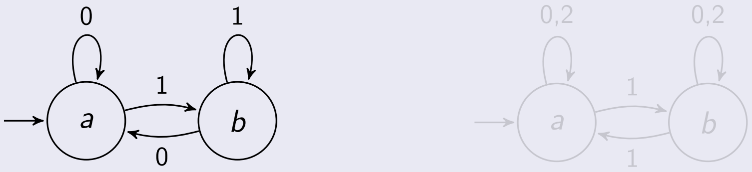
The treatment is very similar to the orthogonality to the Möbius function.
One has to work more carefully to extract the main term.
The actual frequencies can be made explicit.

Primes vs all natural Numbers

Automatic Sequences along Primes

The treatment is very similar to the orthogonality to the Möbius function.
One has to work more carefully to extract the main term.
The actual frequencies can be made explicit.

Primes vs all natural Numbers



Automatic Sequences along Primes

The treatment is very similar to the orthogonality to the Möbius function.
One has to work more carefully to extract the main term.
The actual frequencies can be made explicit.

Primes vs all natural Numbers