Automatic sequences are orthogonal to aperiodic multiplicative functions

Clemens Müllner

Tuesday, November 27, 2018

Multiplicative functions

Definition (multiplicative function)

A function $f : \mathbb{N} \to \mathbb{C}$ is called *multiplicative* if $f(nm) = f(n)f(m)$ for all n, m that are coprime.

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

Two bounded sequences $\mathbf{u}, \mathbf{v}$ are *orthogonal* if

$$\sum_{n \leq N} u_n \overline{v_n} = o(N) \qquad (N \to \infty).$$

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Sarnak Conjecture

Definition

A dynamical system is said to be determinist, if its topological entropy is 0.

Conjecture (Sarnak conjecture, 2010)

Every bounded complex sequence $\mathbf{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system is orthogonal to the Möbius function $\mu(n)$.

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Deterministic Finite Automata

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

$$\mathbf{u} = (u_n)_{n \geq 0} = 01101001100101101001011001101001\dots$$

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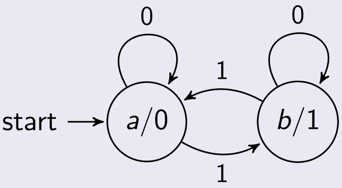

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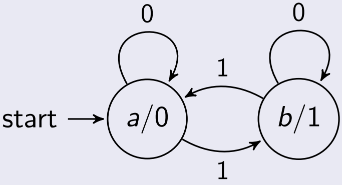

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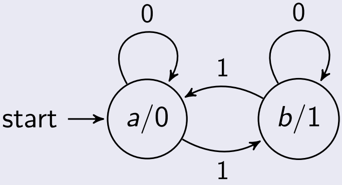

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Coding of the fixpoint of a substitution:

$$a \to ab \qquad a \mapsto 0$$
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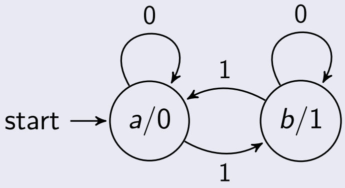

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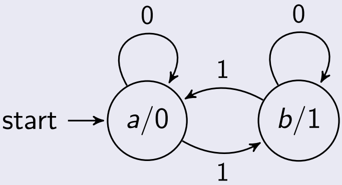

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Substitutions

Definition (Substitution of constant length)

Let $\mathbb{A}$ be a finite set. Then we call $\theta : \mathbb{A} \to \mathbb{A}^\lambda$ a *substitution of length* λ.

We write $\theta(a) = \theta(a)_0 \ldots \theta(a)_{\lambda-1}$. We extend θ to finite blocks and infinite sequences by concatenation.

$$\theta(b_1 \ldots b_r) = \theta(b_1) \ldots \theta(b_r).$$

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We call a substitution *primitive* if there exists k such that for all $a, b \in \mathbb{A}$ there exists j with

$$\theta^k(a)_j = b.$$

Example

$$a \xrightarrow{\theta} a\,b\,a \xrightarrow{\theta} a\,b\,a\,b\,a\,c\,a\,b\,a$$

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Earlier and new Results

Theorem (M., 2017)

Every automatic sequence $(a_n)_{n\geq 0}$ fulfills the Sarnak Conjecture

Theorem (Ferenczi, Kulaga-Przymus, Lemanczyk, Mauduit, 2016)

Every bijective automatic sequence is orthogonal to every aperiodic bounded multiplicative function.

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Every primitive automatic sequence is orthogonal to any bounded aperiodic multiplicative function.

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Naive Question

Why not all bounded multiplicative functions?

Trivial counter-example: periodic sequences.
Non-trivial counter-example: $a(n) = (-1)^{\nu_2(n)}$.

Definition (aperiodic sequence)

We call a sequence $\mathbf{u}$ aperiodic if for all $k, \ell \in \mathbb{N}$

$$\frac{1}{N} \sum_{n \leq N} u(kn + \ell) \to 0.$$

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Dynamical Systems

$$X_\theta := \{x \in \mathbb{A}^{\mathbb{Z}} : \forall r < s \; \exists a \in \mathbb{A}, k, j \in \mathbb{N} : x[r,s] = \theta^k(a)[j, j+s-r+1]\}$$

Shift $T : \mathbb{A}^{\mathbb{Z}} \to \mathbb{A}^{\mathbb{Z}}$, $T(x)[n] = x[n+1]$.
This gives a dynamical system (X_θ, T).

Proposition (Michel; Dekking)

Let θ be a primitive substitution of length λ. Then there exists a unique measure μ_θ such that $(X_\theta, T, \mu_\theta)$ is ergodic. Furthermore, $(X_\theta, T, \mu_\theta)$ is minimal.

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Definition (column number)

Let $\theta : \mathbb{A} \to \mathbb{A}^\lambda$. We define the *column number*

$$c(\theta) := \min_{k,j} \#\{\theta^k(a)_j : a \in \mathbb{A}\}.$$

$$a \xrightarrow{\theta} a\,b\,a \xrightarrow{\theta} a\,b\,a\,b\,a\,c\,a\,b\,a$$

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Definition (height)

Let u be a one-sided fixed-point of θ. We define the height to be the maximal h, coprime to λ such that we can partition $\mathbb{A}$ into h classes, so that the resulting sequence is h-periodic.

$$a\,b\,a\,b\,a\,c\,a\,b\,a\,b\,a\,c\,a\,b\,a\,c\,a\,b\,a\,b\,a\,b\,a\,c\,a\,b\,a\ldots$$

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Lemma (Lemanczyk, M.)

Let θ be a primitive substitution of constant length. Then $h(\theta) | c(\theta)$.

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Structure of (X_θ, T)

Theorem

Let θ be a primitive substitution of length λ and X_θ be infinite. Then for each $k, \in \mathbb{N}, x \in X_\theta$ there exists a unique $j < \lambda^k, y \in X_\theta$ such that

$$x = T^j \theta^k(y).$$

Example:

$$\dots b)(ba)(ba)(ab)(ba)(ab)(a.b)(ba)(ab)(ba)(ba)(ab)(ba)(a \dots$$

This way we can assign to each $x \in X_\theta$ a sequence $(j_k)_{k \in \mathbb{N}}$, where $j_k \in \{0, \dots, \lambda^k - 1\}$ and $j_{k+1} \equiv j_k \bmod \lambda^k$.

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The λ Odometer

Definition (λ Odometer)

$$H_\lambda := \mathit{liminv}\, \mathbb{Z} / \lambda^k \mathbb{Z} .$$

$$H_\lambda \ni x = (j_k)_{k\in\mathbb{N}}, \text{ where } j_{k+1} \equiv j_k \bmod \lambda^k.$$

$R : H_\lambda \to H_\lambda$

$$R((j_k)_{k\in\mathbb{N}}) = (j_k + 1)_{k\in\mathbb{N}}.$$

(H_λ, R) is a uniquely ergodic system with discrete spectrum, i.e. $L^2(H_\lambda, R, \mu)$ is spanned by eigenfunctions of the unitary operator $U_R : f \to f \circ R$ on $L^2(H_\lambda, R, \mu)$.

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(H_λ, R) is a uniquely ergodic system with discrete spectrum, i.e. $L^2(H_\lambda, R, \mu)$ is spanned by eigenfunctions of the unitary operator $U_R : f \to f \circ R$ on $L^2(H_\lambda, R, \mu)$.

The λ Odometer

Definition (λ Odometer)

$$H_\lambda := \mathit{liminv}\, \mathbb{Z} / \lambda^k \mathbb{Z}.$$

$$H_\lambda \ni x = (j_k)_{k \in \mathbb{N}}, \text{ where } j_{k+1} \equiv j_k \bmod \lambda^k.$$

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H_λ as a factor of X_θ

We see that

$$\pi_\theta : x \mapsto (j_k)_{k\in\mathbb{N}},$$

settles a map from (X_θ, T) to (H_λ, R).

Theorem (Dekking)

Let θ be primitive and $h(\theta) = 1$. The map π_θ is $c(\theta)$-to-1 almost everywhere.
In this case we have that H_λ is the Kronecker factor of X_θ, i.e. the largest factor with discrete spectrum.

Corollary

If $c(\theta) = 1$, then $(X_\theta, T) \cong (H_\lambda, R)$ measure-theoretically.

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The synchronizing part of a substitution

Let $\theta : \mathbb{A} \to \mathbb{A}^\lambda$.
We denote by $\mathcal{X}$ the set of „minimal columns“,

$$\mathcal{X} := \{M \subset \mathbb{A} : |M| = c(\theta), \exists k, j : \theta^k(\mathbb{A})_j = M\}.$$

Example:

$$a \xrightarrow{\theta} a\,b\,a \xrightarrow{\theta} a\,b\,a\,b\,a\,c\,a\,b\,a$$

$$b \xrightarrow{\theta} b\,a\,c \xrightarrow{\theta} b\,a\,c\,a\,b\,a\,c\,a\,b$$

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Joining θ and $\widetilde{\theta}$

We consider $\mathcal{A} \subset \mathbb{A} \times \mathcal{X}$:

$$\mathcal{A} := \{(a, M) \in \mathbb{A} \times \mathcal{X} : a \in M\}.$$

$\Theta : \mathcal{A} \to \mathcal{A}^\lambda$,

$$\Theta((a, M))_j := (\theta(a)_j, \tilde{\theta}(M)_j).$$

We consider now X_Θ and find projections $\pi_1 : X_\Theta \to X_\theta, \pi_2 : X_\Theta \to X_{\widetilde{\theta}}$:

$$\pi_1((x[n], M[n])_{n\in\mathbb{Z}}) = (x[n])_{n\in\mathbb{Z}},$$
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How different are X_θ and X_Θ?

Lemma

If θ is primitive, then so is Θ and furthermore:

$$c(\theta) = c(\Theta)$$
$$h(\theta) = h(\Theta).$$

Let us consider a generic point $z \in H_\lambda$:

$$\pi_\theta^{-1}(z) = \{(x^{(i)}[n])_{n\in\mathbb{Z}} : i = 1, \ldots, c\}$$
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Renaming the alphabet of Θ

We see that $\Theta(., M)_j$ is a bijection from M to $\tilde{\theta}(M)_j$.

We rename our alphabet: $(a, M) \to (i, M)$ where $i \in \{1, \ldots, c\}$.

Example:

$$(a, R) \mapsto (1, R) \qquad (b, R) \mapsto (2, R)$$
$$(a, S) \mapsto (1, S) \qquad (c, S) \mapsto (2, S)$$

$$(1, R) \xrightarrow{\tilde{\Theta}} (1, R)\,(2, R)\,(1, S)$$
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Towards a group extension

$\tilde{\Theta}(., M)_j$ is a bijection from $\{1, \dots, c\}$ to itself.
$\sigma_{M,j} \in S_c, \sigma_{M,j}(m) = n$ iff $\tilde{\Theta}(n, M)_j = (m, \tilde{\theta}(M)_j)$.
$G :=< \sigma_{M,j} : M \in \mathcal{X}, j \in \{1, \dots, c\} >$.
$\widehat{\Theta} : (G \times \mathcal{X}) \to (G \times \mathcal{X})^\lambda$,

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$G = S_3$,

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How close are X_θ and $X_{\widehat{\Theta}}$

Lemma

If θ is primitive, then so is $\widehat{\Theta}$. X_θ is a topological factor of $X_{\widehat{\Theta}}$.

$c(\widehat{\Theta}) = |G|$

$h(\widehat{\Theta}) = ?$.

BUT, we have a group structure!

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If θ is primitive, then so is $\widehat{\Theta}$. X_θ is a topological factor of $X_{\widehat{\Theta}}$.
$c(\widehat{\Theta}) = |G|$
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Group extensions

Definition (compact group extension)

Given (X, T, μ) and $\phi : X \to G$ measurable. Then we call $(X \times G, T_\phi, \mu \otimes m_G)$ a compact group extension, where

$$T_\phi(x, g) = (T(x), \phi(x)g).$$

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We have for each $\tau \in G$ an automorphism $V_\tau : X_{\widehat{\Theta}} \to X_{\widehat{\Theta}}$,

$$V_\tau(\sigma[n], M[n])_{n\in\mathbb{Z}} = (\tau \circ \sigma[n], M[n])_{n\in\mathbb{Z}}.$$

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General Plan

We build successively bigger dynamical systems

$$X_\theta \hookrightarrow X_\Theta = X_{\tilde{\Theta}} \hookrightarrow X_{\widehat{\Theta}}.$$

We want to show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} F(T^n(x))m(n) = 0,$$

where $x \in X_{\widehat{\Theta}}, F \in C(\widehat{\Theta}, \mathbb{C})$.
We write $F = F_d + F_c$ where $F_d \in L^2(H_\lambda, R)$ and continuous, and F_c is orthogonal to $L^2(H_\lambda, R)$.
F_d can be approximated by periodic functions and is hence orthogonal to aperiodic functions.
For F_c, we use the Katai Criterion.

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Key Tool

Katai Criterion

Suppose that for all but finitely many $p, q \in \mathbb{P}$ we have

$$\sum_{n \leq N} a(pn)\overline{a(qn)} = o(N).$$

Then we have for all multiplicative bounded functions $m : \mathbb{N} \to \mathbb{N}$,

$$\sum_{n \leq N} a(n)m(n) = o(N).$$

So we need to control

$$\frac{1}{N}\sum_{n \leq N} F_c(T^{pn}(x))\overline{F_c(T^{qn}(x))}.$$

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where ρ is a joining of $(X_{\widehat{\Theta}}, T^p)$ and $(X_{\widehat{\Theta}}, T^q)$.

By a series of results on group extensions of H_λ we find that the only such joinings are relatively independent extensions over the isomorphism W between R^p and R^q, i.e.

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