Automatic sequences satisfy Sarnak's conjecture II

Clemens Müllner

4. Dec 2016

Automatic Sequences

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

$$\mathbf{u} = (u_n)_{n\geq 0} = 0110100110010110100101100110100 1\ldots$$

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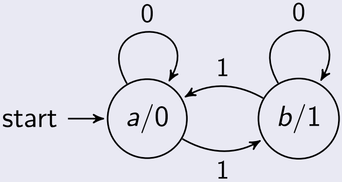

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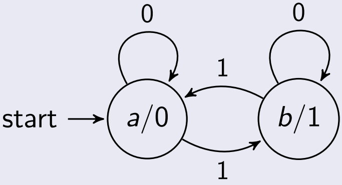

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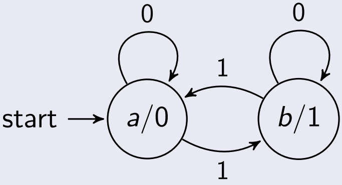

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Dynamical System (X, T) related to $\mathbf{u}$

$\mathbf{u} = (u_n)_{n\geq 0}$ ... bounded complex sequence

$T\mathbf{u} = (u_{n+1})_{n\geq 0}$ ... shift operator

$X = \overline{\{T^k(\mathbf{u}) : k \geq 0\}}$

We say that $\mathbf{u}$ satisfies the **Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n\geq 0} \in X$ are orthogonal to $\mu(n)$.

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Let $A = (Q', \Sigma, \delta', q_0', \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k-1\}$ and $\delta'(q_0', 0) = q_0'$. Then the frequencies of the letters for the prime-subsequence $(a_p)_{p\in\mathcal{P}}$ exist, i.e.

$$dens_{\mathcal{P}}(\mathbf{u}, \alpha) = \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{1\leq p\leq N} \mathbf{1}_{[u_p=\alpha]}.$$

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Lemma

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence that takes values in Δ. Suppose that for every $j \geq 1$ and for every function $g : \Delta^j \to \mathbb{C}$ we have

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Suppose that for every automatic sequence $(a_n)_{n\in\mathbb{N}}$ with values in $\mathbb{C}$

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$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

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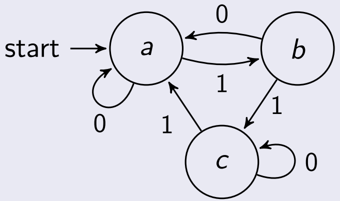

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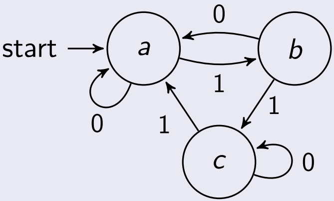

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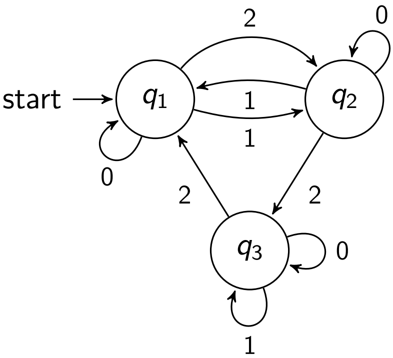
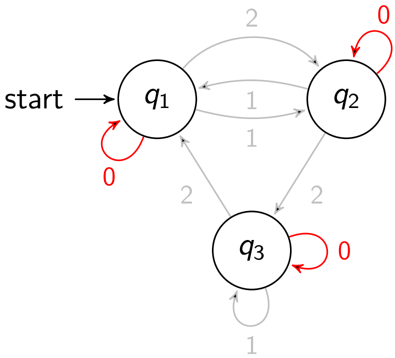
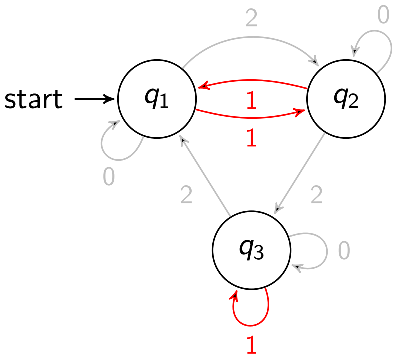

start
q_1
q_2
q_3
2
0
1
1
0
2
2
0
1



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



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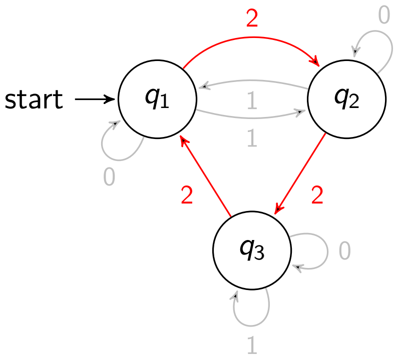

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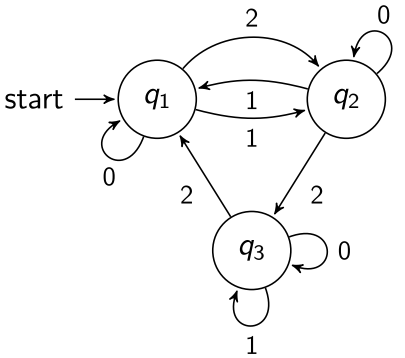

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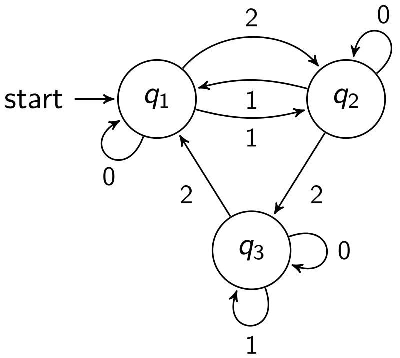
$11 = (102)_3$: $$M_2 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



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$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

M is primitive iff there exists $m \geq 0$ such that for every $a, b \in Q$ exists $\mathbf{w} \in \Sigma^m$ such that $\delta(a, \mathbf{w}) = b$.

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If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the frequencies

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Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

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We call a sequence $(a_n)_{n\geq 0}$ *digital* if there exists $m \geq 1$ and $F : \{0, \ldots, k-1\}^m \to \mathbb{C}$ such that

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Lemma

Let $(a_n)_{n\geq 0}$ be a digital sequence. Then $(a_n \bmod m')_{n\geq 0}$ is an automatic sequence for every $m' \in \mathbb{N}$.

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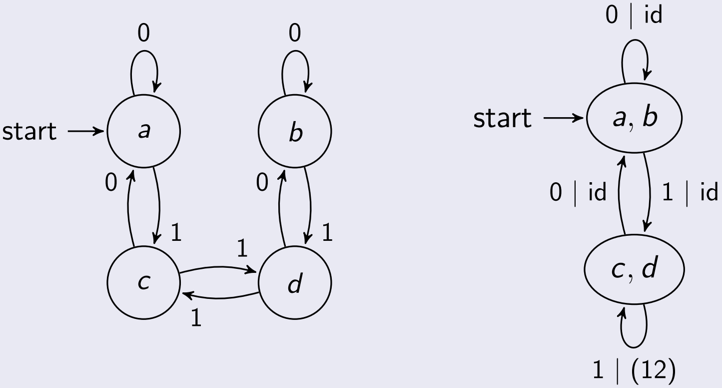

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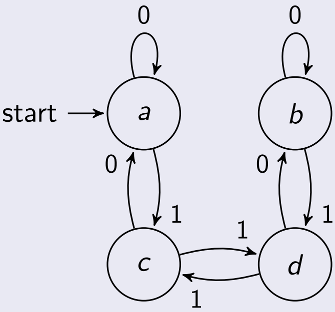

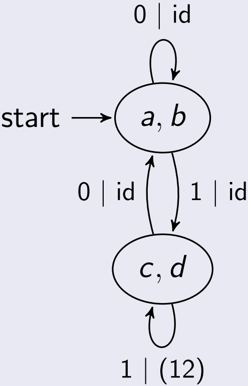

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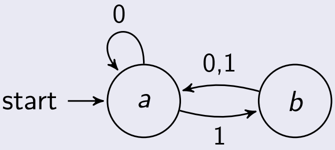

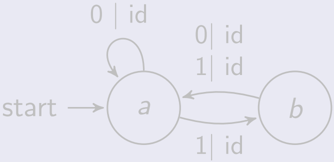

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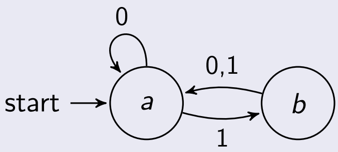

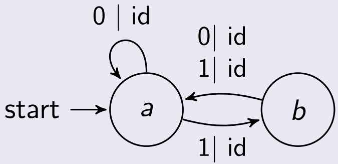

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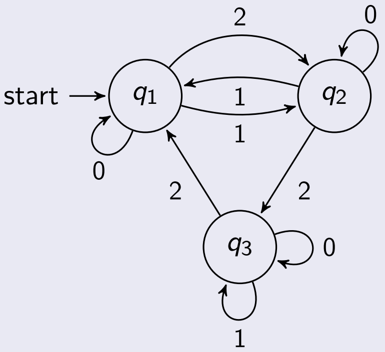

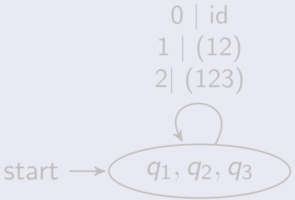

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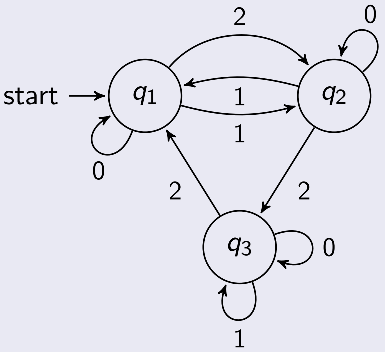

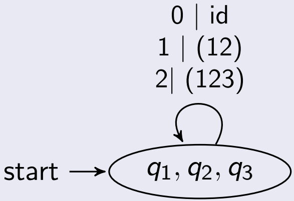

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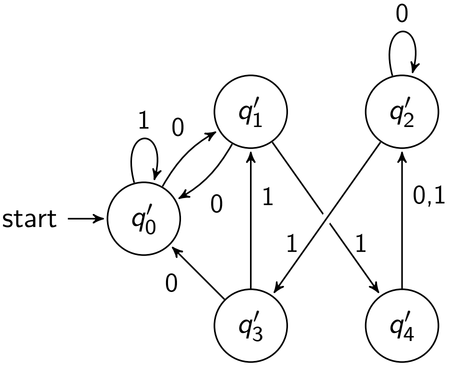

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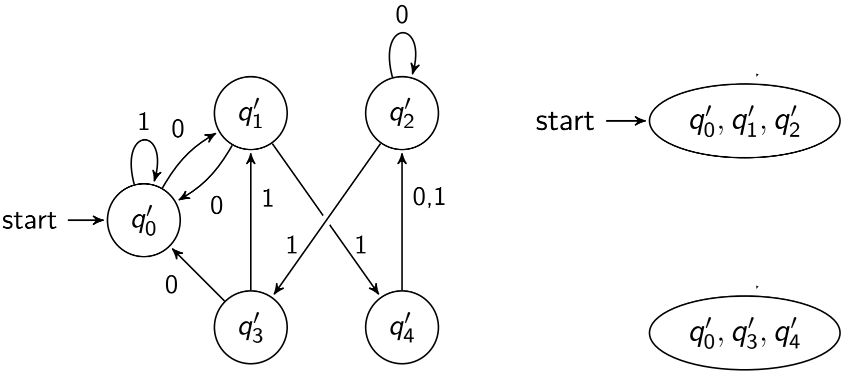

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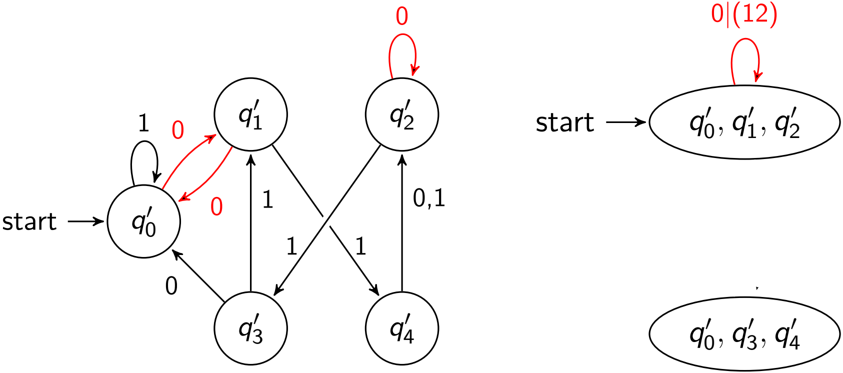

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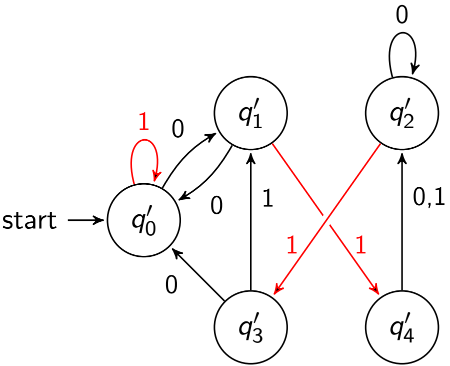

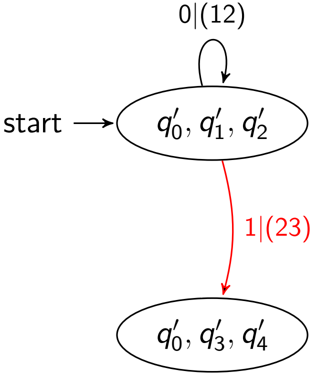

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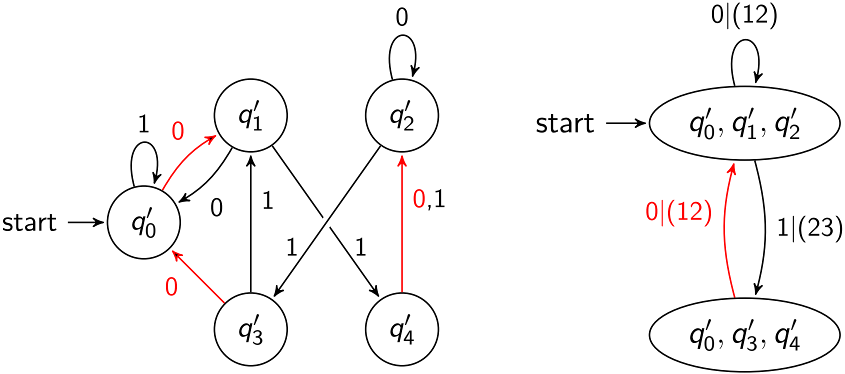

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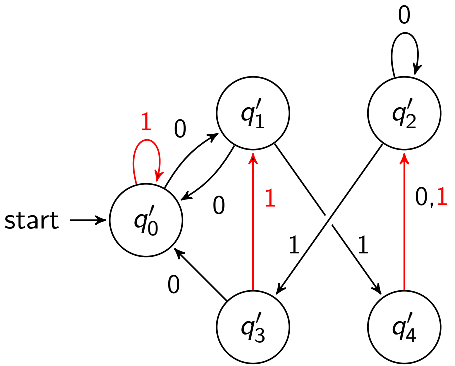
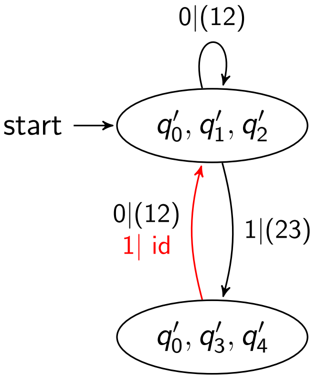


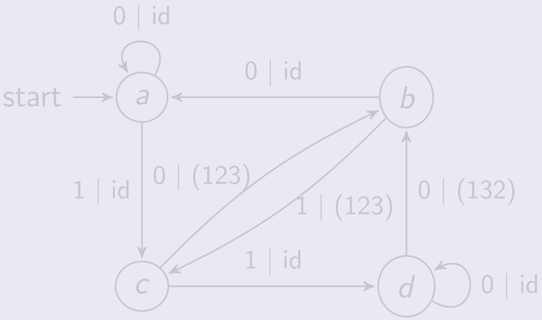


Motivation

Example (Digital Sequence)

„Generic Example": $k = 2, m = 3, m' = 3$

$F(010) = 1, F(110) = 2, F(101) = 1$



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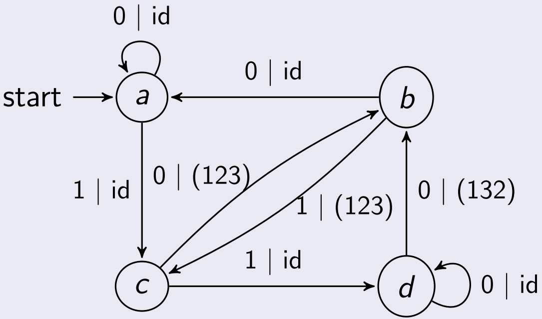

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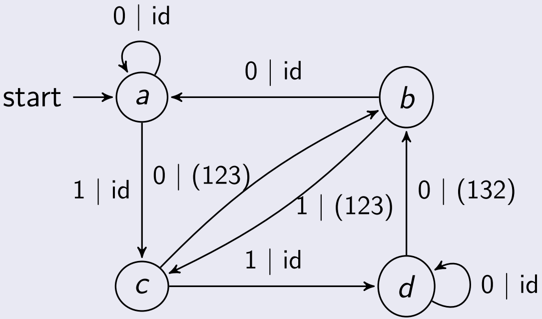

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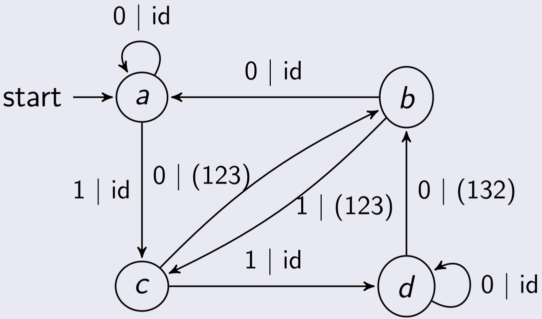

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Definition

Denote by

$$T(q, w_1 \dots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \dots \\ \circ \lambda(\delta(q, w_1 \dots w_{r-1}), w_r).$$

Lemma

Let A be a strongly connected automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

$$\delta'(q'_0, \mathbf{w}) = \pi_1(T(q_0, \mathbf{w}) \cdot \delta(q_0, \mathbf{w}))$$

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Are some naturally induced transducers better than others?

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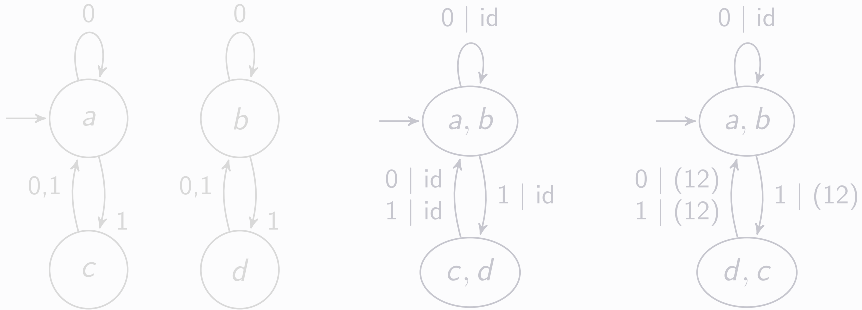

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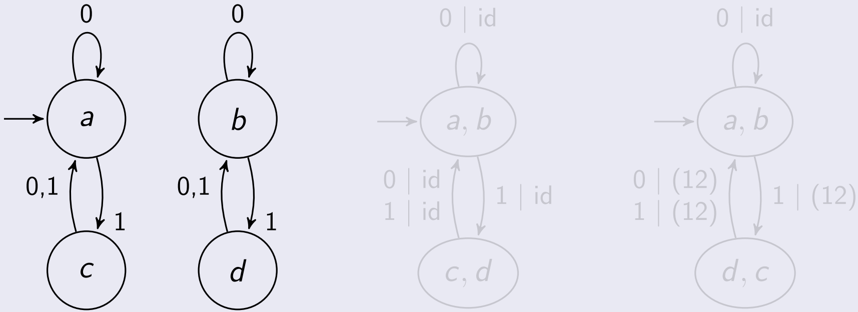

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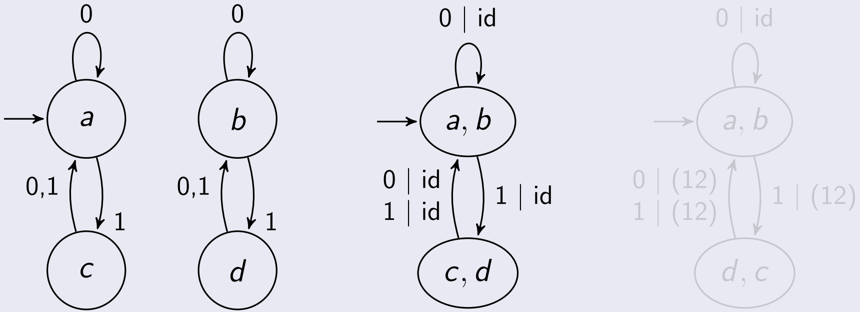

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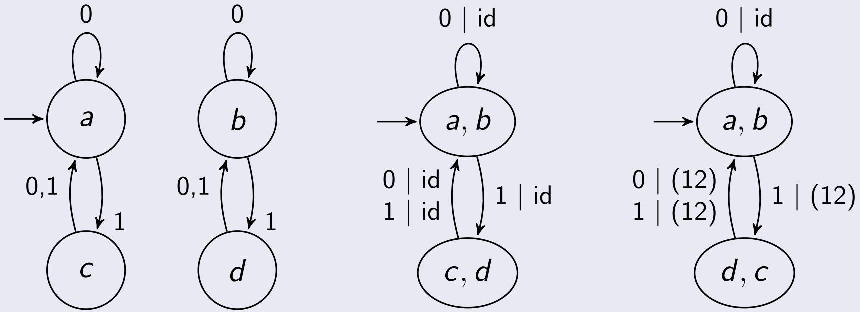

All elements of Δ appear as values of $T(q_0, .)$ for „good“ naturally induced transducer.

Do all elements of Δ appear simultaneously as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$ for a single n, where n is large?

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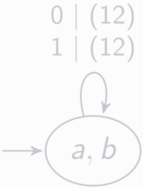

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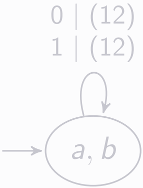

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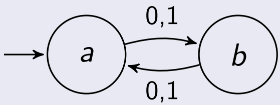

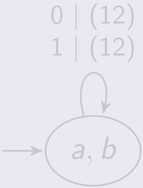

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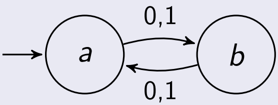

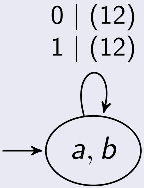

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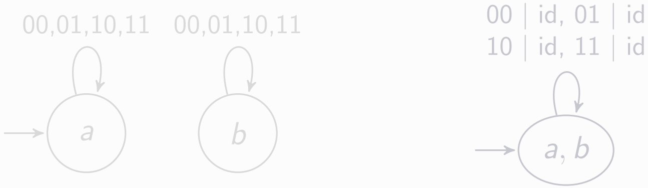

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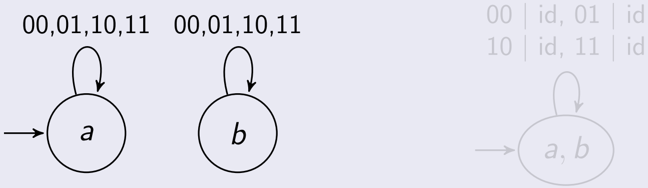

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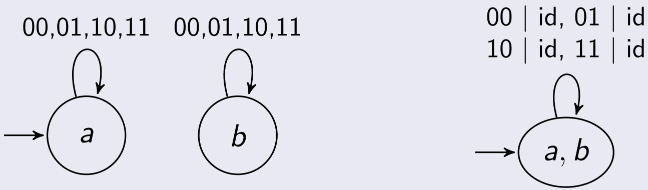

Continuous functions from a compact group to $\mathbb{C}$

Definition (Representation)

Let G be a finite group and $k \in \mathbb{N}$. A **Representation** of rank k is a continuous homomorphism $D : G \to \mathbb{C}^{k\times k}$.

Lemma

Let f be a continuous function from G to $\mathbb{C}$. There exists $r \in \mathbb{N}$ and unitary, irreducible representations $D^{(\ell)} = (d_{i,j}^{(\ell)})_{i,j<k_\ell}$ along with $c_\ell \in \mathbb{C}$ such that

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Lemma

Suppose that

$$\sum_{\substack{n<N \\ \dots}} D(T(n))\mu(n) = o(N)$$

holds for all irreducible unitary representations of G. Then $\mathbf{u} = (u_n)_{n\geq 0}$ is orthogonal to $\mu(n)$.

We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.

(Adopted) Definition

Let $U(n)$ be a sequence of unitary matrices. We say that U has the **Fourier property** if there exists $\eta > 0$ and c such that for all λ, α and t

$$\left\| \frac{1}{k^\lambda} \sum_{m<k^\lambda} U(mk^\alpha) e(mt) \right\| \leq ck^{-\eta\lambda}.$$

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