All automatic sequences satisfy the Sarnak conjecture

Clemens Müllner

5. September 2016

Sarnak Conjecture

A (bounded complex) sequence $\mathbf{u}$ is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n \le N} \mu(n) u_n = o(N) \qquad (N \to \infty).$$

Conjecture (Sarnak conjecture, 2010)

Every bounded complex sequence $\mathbf{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system is orthogonal to the Möbius function.

Let $\mathbf{u}$ be a bounded complex sequence and (X, S) the symbolic dynamical system associated with $\mathbf{u}$. We say that $\mathbf{u}$ satisfies the **Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n\ge0} \in X$ are orthogonal to $\mu(n)$.

Sarnak Conjecture

A (bounded complex) sequence $\mathbf{u}$ is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n \leq N} \mu(n) u_n = o(N) \qquad (N \to \infty).$$

Conjecture (Sarnak conjecture, 2010)

Every bounded complex sequence $\mathbf{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system is orthogonal to the Möbius function.

Let $\mathbf{u}$ be a bounded complex sequence and (X, S) the symbolic dynamical system associated with $\mathbf{u}$. We say that $\mathbf{u}$ satisfies the **Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n \geq 0} \in X$ are orthogonal to $\mu(n)$.

Sarnak Conjecture

A (bounded complex) sequence **u** is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n \leq N} \mu(n) u_n = o(N) \qquad (N \to \infty).$$

Conjecture (Sarnak conjecture, 2010)

Every bounded complex sequence $\mathbf{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system is orthogonal to the Möbius function.

Let **u** be a bounded complex sequence and (X, S) the symbolic dynamical system associated with **u**. We say that **u** satisfies the **Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n \geq 0} \in X$ are orthogonal to $\mu(n)$.

Automatic Sequence

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

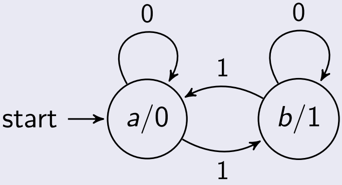
$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110100 1 \ldots$$

Automatic Sequence

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

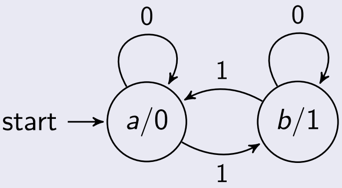
$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110100 1\dots$$

Automatic Sequence

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \dots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

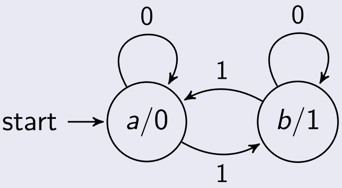
$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110 1001\dots$$

Automatic Sequence

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

$$\mathbf{u} = (u_n)_{n \geq 0} = 0110100110010110100101100110100\ldots$$

Results

Theorem 1

Every automatic sequence $(a_n)_{n\geq 0}$ satisfies the Sarnak Conjecture

Theorem 2

Let $A = (Q', \Sigma, \delta', q_0', \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k-1\}$ and $\delta'(q_0', 0) = q_0'$. Then the frequencies of the letters for the prime-subsequence $(a_p)_{p\in\mathcal{P}}$ exist, i.e.

$$dens_{\mathcal{P}}(\mathbf{u}, \alpha) = \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{1\leq p\leq N} \mathbf{1}_{[u_p=\alpha]}.$$

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they distribute as expected.

Results

Theorem 1

Every automatic sequence $(a_n)_{n\geq 0}$ satisfies the Sarnak Conjecture

Theorem 2

Let $A = (Q', \Sigma, \delta', q_0', \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k-1\}$ and $\delta'(q_0', 0) = q_0'$. Then the frequencies of the letters for the prime-subsequence $(a_p)_{p\in\mathcal{P}}$ exist, i.e.

$$dens_{\mathcal{P}}(\mathbf{u}, \alpha) = \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{1\leq p\leq N} \mathbf{1}_{[u_p=\alpha]}.$$

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they distribute as expected.

Results

Theorem 1

Every automatic sequence $(a_n)_{n\geq 0}$ satisfies the Sarnak Conjecture

Theorem 2

Let $A = (Q', \Sigma, \delta', q_0', \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k-1\}$ and $\delta'(q_0', 0) = q_0'$. Then the frequencies of the letters for the prime-subsequence $(a_p)_{p\in\mathcal{P}}$ exist, i.e.

$$dens_{\mathcal{P}}(\mathbf{u}, \alpha) = \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{1\leq p\leq N} \mathbf{1}_{[u_p=\alpha]}.$$

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they distribute as expected.

Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example

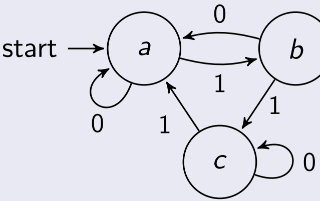
$\mathbf{w}_0 = 010.$

Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example



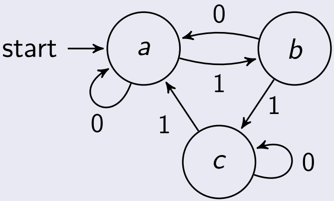
$\mathbf{w}_0 = 010.$

Synchronizing Automata

Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example



$\mathbf{w}_0 = 010.$

Synchronizing Automata

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ be generated by a synchronizing automaton. Then for every α the density

$$dens(\mathbf{u}, \alpha) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mathbf{1}_{[u_n = \alpha]}$$

exists. Furthermore, the densities for the prime - subsequences $(u_p)_{p \in \mathcal{P}}$ exists.

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ be generated by a synchronizing automaton. Then $\mathbf{u} = (u_n)_{n>0}$ is orthogonal to the Möbius function $\mu(n)$.

Synchronizing Automata

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then for every* α *the density*

$$dens(\mathbf{u}, \alpha) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mathbf{1}_{[u_n = \alpha]}$$

exists. Furthermore, the densities for the prime - subsequences $(u_p)_{p \in \mathcal{P}}$ *exists.*

Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then* $\mathbf{u} = (u_n)_{n>0}$ *is orthogonal to the Möbius function* $\mu(n)$.

Synchronizing Automata

Theorem (Deshouillers + Drmota + M.)

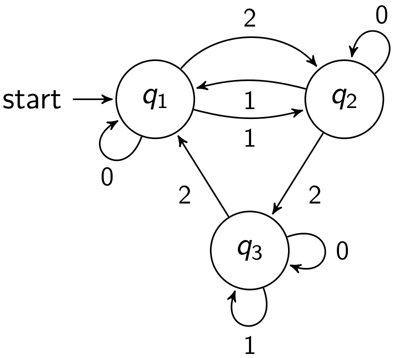
Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then for every* α *the density*

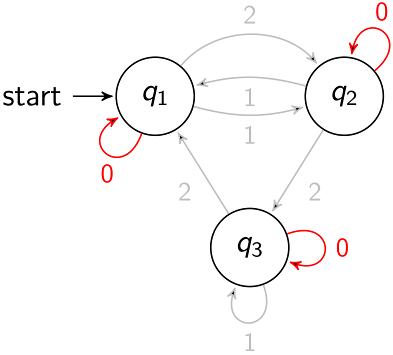
$$dens(\mathbf{u}, \alpha) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mathbf{1}_{[u_n = \alpha]}$$

exists. Furthermore, the densities for the prime - subsequences $(u_p)_{p \in \mathcal{P}}$ *exists.*

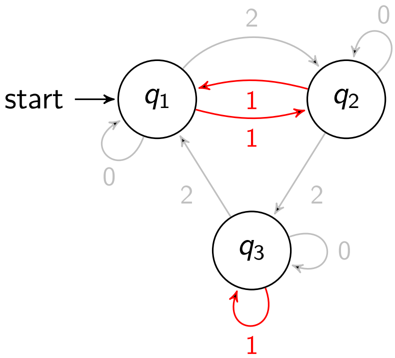
Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then* $\mathbf{u} = (u_n)_{n>0}$ *is orthogonal to the Möbius function* $\mu(n)$*.*

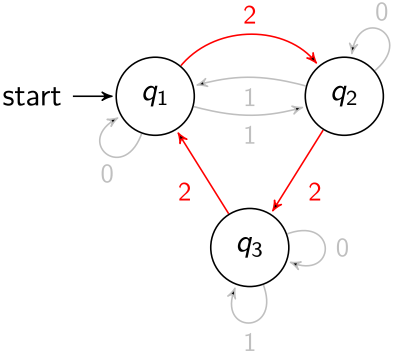

start
q_1
q_2
q_3
2
0
1
1
0
2
2
0
1



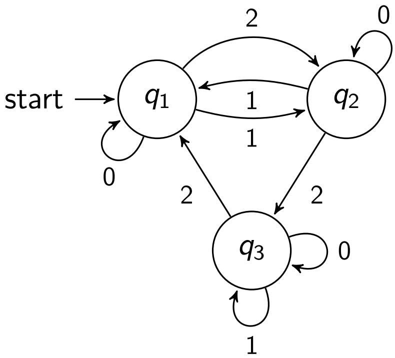
$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

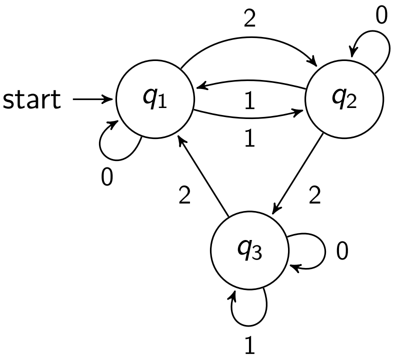


$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$11 = (102)_3 : \qquad M_2 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Remark:

If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the densities

$$dens(\mathbf{u}, a) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \mathbf{1}_{[u_n = a]}$$

exist.

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

Remark:

If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the densities

$$dens(\mathbf{u}, a) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} \mathbf{1}_{[u_n=a]}$$

exist.

Results for Invertible Automata

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n \geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

Theorem[Drmota]

The frequency of each letter of the subsequence $(u_p)_{p \in \mathcal{P}}$ exists.

Results for Invertible Automata

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

Theorem[Drmota]

The frequency of each letter of the subsequence $(u_p)_{p\in\mathcal{P}}$ exists.

Results for Invertible Automata

Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton.

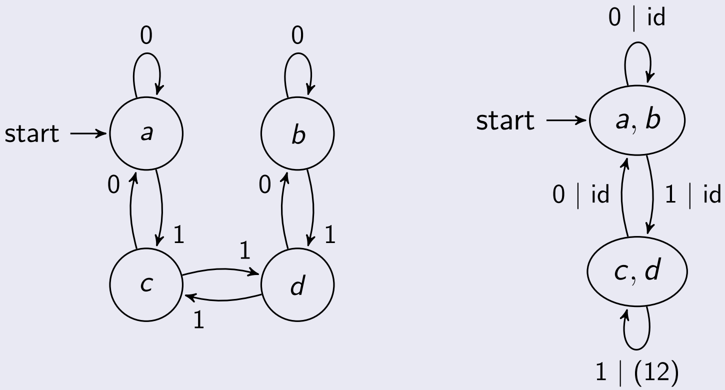
Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

Theorem[Drmota]

The frequency of each letter of the subsequence $(u_p)_{p\in\mathcal{P}}$ exists.

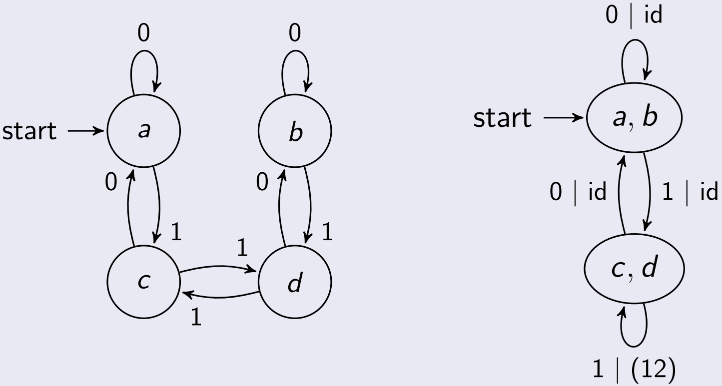
Example (Rudin-Shapiro)



Theorem [Mauduit + Rivat, Tao]

The Rudin-Shapiro Sequence is orthogonal to the Möbius function.

Example (Rudin-Shapiro)



Theorem [Mauduit + Rivat, Tao]

The Rudin-Shapiro Sequence is orthogonal to the Möbius function.

Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q'_0)$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. $\mathcal{T}_A$ is synchronizing
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. some technical conditions

Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. $\mathcal{T}_A$ is synchronizing
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. some technical conditions

Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. $\mathcal{T}_A$ is synchronizing
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. some technical conditions

Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. $\mathcal{T}_A$ is synchronizing
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. some technical conditions

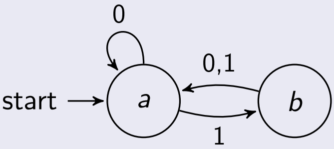
Definition (Naturally Induced Transducer)

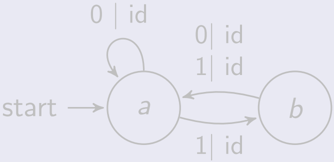
Let $A = (Q', \Sigma, \delta', q'_0)$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
2. $\mathcal{T}_A$ is synchronizing
3. $\delta'(q, a) = \lambda(q, a) \cdot \delta(q, a)$
4. some technical conditions

Examples

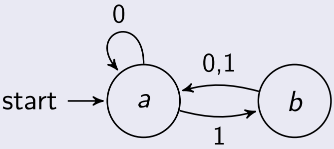
Example (Synchronizing Automaton)

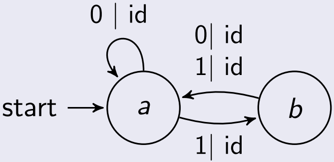




Examples

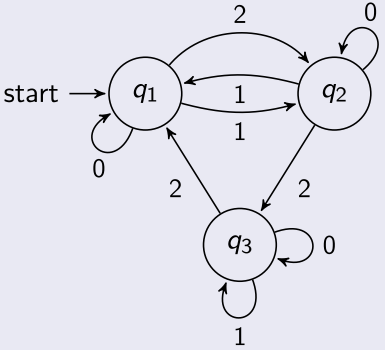
Example (Synchronizing Automaton)

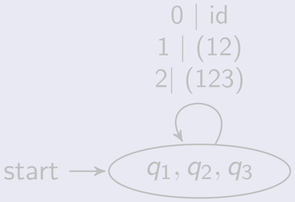




Examples

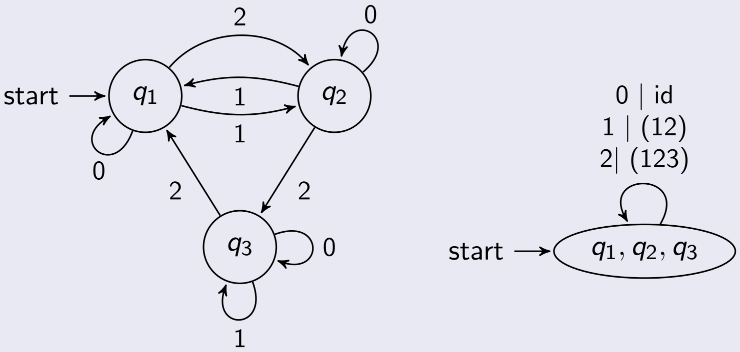
Example (Invertible Automaton)





Examples

Example (Invertible Automaton)



Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Example:

Theorem

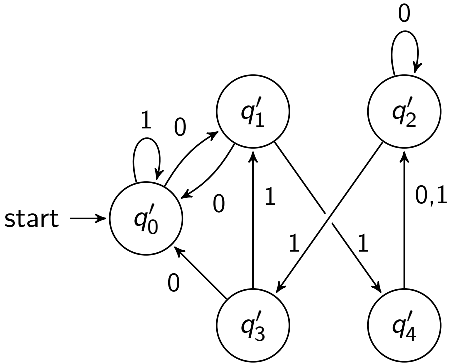
For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Example:

Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

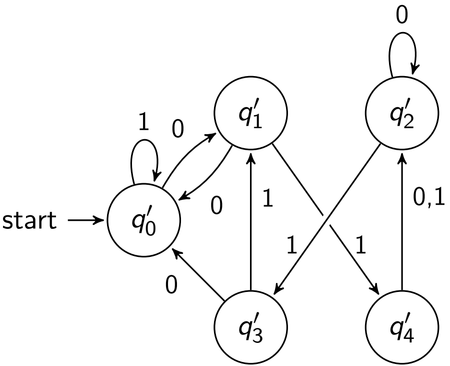
Example:

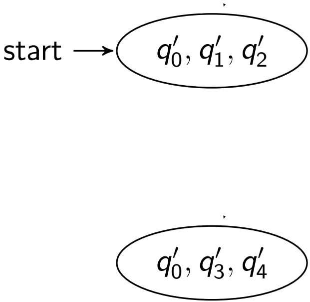


Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Example:

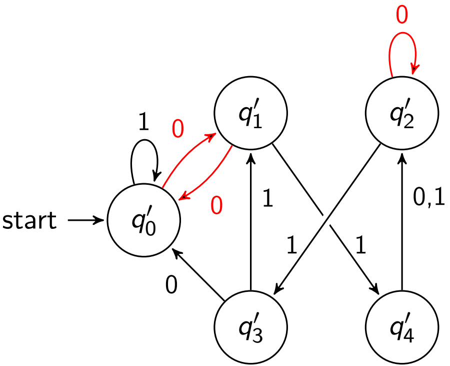


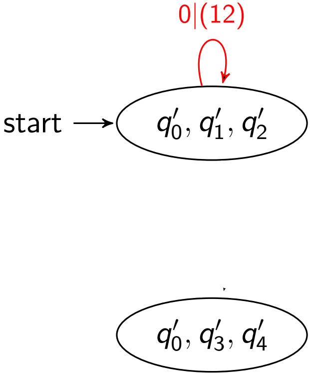


Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Example:

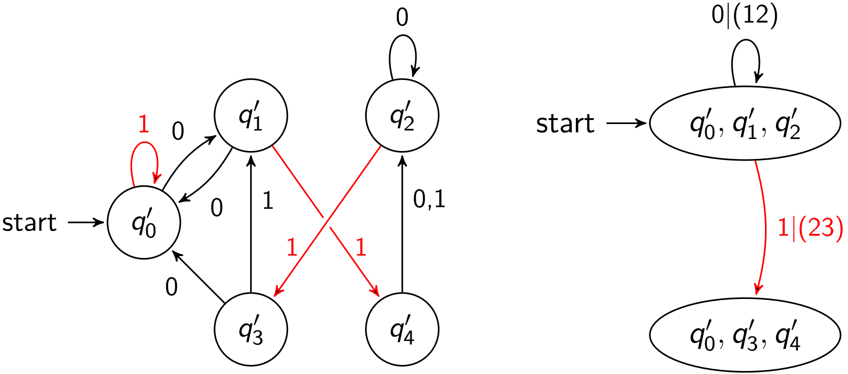




Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

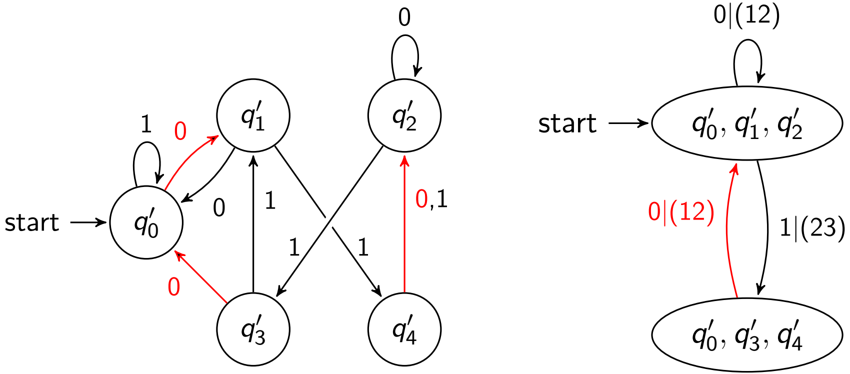
Example:



Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

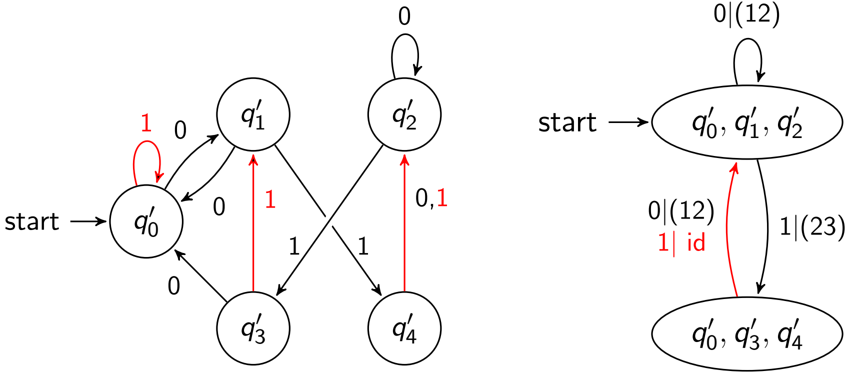
Example:



Theorem

For every strongly connected automaton A, there exists a naturally induced transducer $\mathcal{T}_A$. All other naturally induced transducers can be obtained by changing the order on the elements of Q.

Example:



Definition

Denote by

$$T(q, w_1 \ldots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \ldots \circ \lambda(\delta(q, w_1 \ldots w_{r-1}), w_r).$$

Lemma

Let A be a strongly connected automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

$$\delta'(q'_0, \mathbf{w}) = \pi_1(T(q_0, \mathbf{w}) \cdot \delta(q_0, \mathbf{w}))$$

holds for all $\mathbf{w} \in \Sigma^*$.

Definition

Denote by

$$T(q, w_1 \dots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \dots \circ \lambda(\delta(q, w_1 \dots w_{r-1}), w_r).$$

Lemma

Let A be a strongly connected automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

$$\delta'(q_0', \mathbf{w}) = \pi_1(T(q_0, \mathbf{w}) \cdot \delta(q_0, \mathbf{w}))$$

holds for all $\mathbf{w} \in \Sigma^*$.

Lemma

Suppose that

$$\sum_{\substack{n<N \\ \dots}} D(T(n))\mu(n) = o(N)$$

holds for all irreducible unitary representations D of G. Then $\mathbf{u} = (u_n)_{n\geq 0}$ is orthogonal to $\mu(n)$.

We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.

(Adopted) Definition

Let $U(n)$ be a sequence of unitary matrices. We say that U has the **Fourier property** if there exists $\eta > 0$ and c such that for all λ, α and t

$$\left\| \frac{1}{k^\lambda} \sum_{m<k^\lambda} U(mk^\alpha) e(mt) \right\| \leq ck^{-\eta\lambda}.$$

We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.

(Adopted) Definition

Let $U(n)$ be a sequence of unitary matrices. We say that U has the **Fourier property** if there exists $\eta > 0$ and c such that for all λ, α and t

$$\left\| \frac{1}{k^\lambda} \sum_{m < k^\lambda} U(mk^\alpha) e(mt) \right\| \leq c k^{-\eta\lambda}.$$

Let D be a unitary and irreducible representation of G.

(Adopted) Theorem

Suppose that $D \circ T$ has the Fourier property. Then we have for any real θ

$$\left\| \sum_{n<N} \mu(n) D(T(n)) e(\theta n) \right\| \ll c_1(k)(\log N)^{c_2(k)} N^{1-\eta'}$$

(Adopted) Theorem

Suppose that $D \circ T$ has the Fourier property. Then we have for any real θ

$$\left\| \sum_{n<N} \Lambda(n) D(T(n)) e(\theta n) \right\| \ll c_1(k)(\log N)^{c_3(k)} N^{1-\eta'}$$

Let D be a unitary and irreducible representation of G.

(Adopted) Theorem

Suppose that $D \circ T$ has the Fourier property. Then we have for any real θ

$$\left\| \sum_{n<N} \mu(n) D(T(n)) e(\theta n) \right\| \ll c_1(k) (\log N)^{c_2(k)} N^{1-\eta'}$$

(Adopted) Theorem

Suppose that $D \circ T$ has the Fourier property. Then we have for any real θ

$$\left\| \sum_{n<N} \Lambda(n) D(T(n)) e(\theta n) \right\| \ll c_1(k) (\log N)^{c_3(k)} N^{1-\eta'}$$

Automatic Sequences along Primes

The treatment is very similar to the orthogonality to the Möbius function.

One has to work more carefully to extract the main term.

The actual frequencies can be made explicit and are what one expects - i.e. they are determined by how the automatic sequence behaves along arithmetic progressions.

Automatic Sequences along Primes

The treatment is very similar to the orthogonality to the Möbius function.

One has to work more carefully to extract the main term.

The actual frequencies can be made explicit and are what one expects - i.e. they are determined by how the automatic sequence behaves along arithmetic progressions.

Automatic Sequences along Primes

The treatment is very similar to the orthogonality to the Möbius function.
One has to work more carefully to extract the main term.
The actual frequencies can be made explicit and are what one expects - i.e. they are determined by how the automatic sequence behaves along arithmetic progressions.