Automatic sequences satisfy Sarnak's conjecture

Clemens Müllner

Wednesday, May 3, 2017

Möbius function

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise} \end{cases}$$

A sequence **u** is **orthogonal to the Möbius function** $\mu(n)$ if

$$\sum_{n\leq N} \mu(n) u_n = o(\sum_{n\leq N} |u_n|) \qquad (N \to \infty).$$

Old Heuristic - Mobius Randomness Law

Any "reasonably defined" bounded sequence independent of μ is orthogonal to μ.

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Orthogonality to μ

Results

- Constant sequences $\Leftrightarrow$ PNT
- Periodic sequences $\Leftrightarrow$ PNT in arithmetic Progressions
- Quasiperiodic sequences $f(n) = F(\alpha n \bmod 1)$ - Davenport
- Nilsequences - Green and Tao
- Horocycle Flows - Bourgain, Sarnak and Ziegler
- Bounded depth circuits - Green
- Some special examples/classes of automatic sequences

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Sarnak Conjecture

Definition

A dynamical system is said to be determinist, if its topological entropy is 0.

Conjecture (Sarnak conjecture, 2010)

Every bounded complex sequence $\mathbf{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system is orthogonal to the Möbius function $\mu(n)$.

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Let $0 \leq a_1 < a_2 < \ldots < a_t$ and $k_1, k_2, \ldots, k_t$ in $\{1, 2\}$ not all even, then as $N \to \infty$

$$\sum_{n \leq N} \mu^{k_1}(n + a_1)\mu^{k_2}(n + a_2) \cdots \mu^{k_t}(n + a_t) = o(N).$$

Theorem (Sarnak)

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Sarnak Conjecture

Dynamical System (X, T) related to $\mathbf{u}$

$\mathbf{u} = (u_n)_{n\geq 0} \ldots$ bounded complex sequence

$T\mathbf{u} = (u_{n+1})_{n\geq 0} \ldots$ shift operator

$X = \overline{\{T^k(\mathbf{u}) : k \geq 0\}}$

We say that $\mathbf{u}$ satisfies the **Sarnak conjecture** if all sequences $\mathbf{a} = (a_n)_{n\geq 0} \in X$ are orthogonal to $\mu(n)$.

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Results

Theorem 1 (M., 2016)

Every automatic sequence $(a_n)_{n\geq 0}$ fulfills the Sarnak Conjecture

Theorem 2 (M., 2016)

Let $A = (Q', \Sigma, \delta', q_0', \tau)$ be a strongly connected DFAO such that $\Sigma = \{0, \ldots, k-1\}$ and $\delta'(q_0', 0) = q_0'$. Then the frequencies of the letters for the prime-subsequence $(a_p)_{p\in\mathcal{P}}$ exist, i.e.

$$dens_{\mathcal{P}}(\mathbf{u}, \alpha) = \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{1\leq p\leq N} \mathbf{1}_{[u_p=\alpha]}.$$

Remark: All block-additive (i.e. digital) functions are covered by Theorem 2 and they are "usually" uniformly distributed.

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Definition (Synchronizing Automaton / Word)

$\exists \mathbf{w}_0 : \delta(q, \mathbf{w}_0) = a \quad \forall q.$

Example

$\mathbf{w}_0 = 010.$

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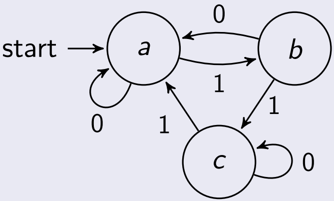

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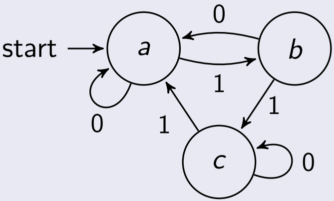

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Theorem (Deshouillers + Drmota + M.)

Let $\mathbf{u} = (u_n)n > 0$ *be generated by a synchronizing automaton. Then for every* α *the density*

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exists. Furthermore, the densities for the following subsequences exist

- $(u_p)_{p\in\mathcal{P}}$
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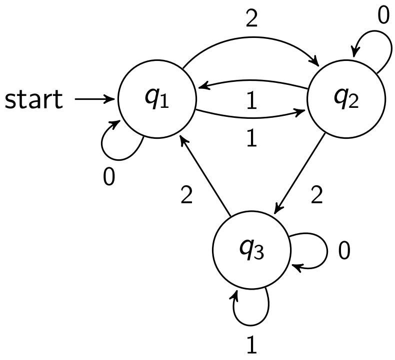
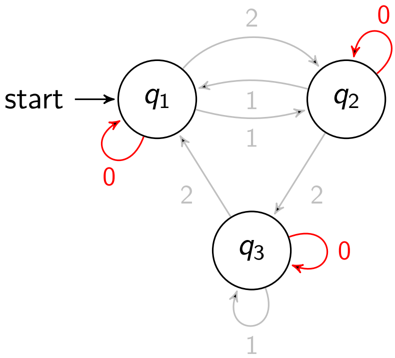
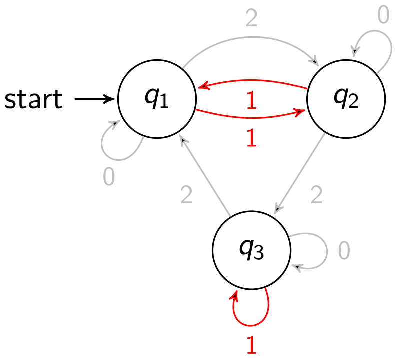


start
q_1
q_2
q_3
2
0
1
1
0
2
2
0
1



$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



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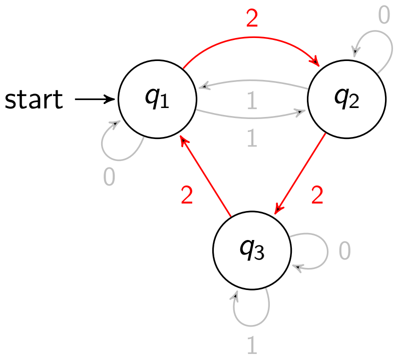

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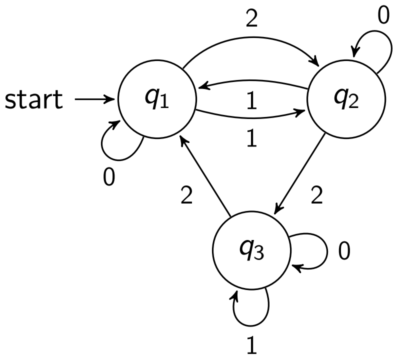

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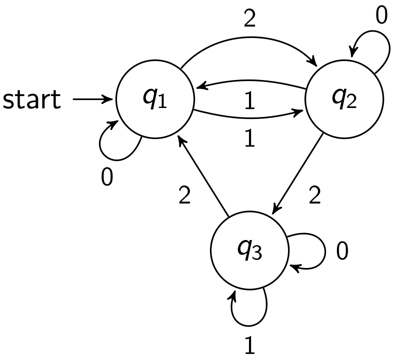
$$11 = (102)_3 : \qquad M_2 \circ M_0 \circ M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



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$$T(n) := M_{\varepsilon_0(n)} M_{\varepsilon_1(n)} \cdots M_{\varepsilon_{\ell-1}(n)}$$

$$u(n) = f(T(n)\mathbf{e}_1) \qquad \mathbf{e}_1 = (1 \quad 0 \quad 0)^T$$

Definition

An automaton is called invertible if all transition matrices $M_0, \ldots, M_{k-1}$ are invertible and if $M = M_0 + \ldots + M_{k-1}$ is primitive.

M is primitive iff there exists $m \geq 0$ such that for every $a, b \in Q$ exists $\mathbf{w} \in \Sigma^m$ such that $\delta(a, \mathbf{w}) = b$.

Remark:

If the matrix $M = M_0 + \ldots + M_{k-1}$ is primitive then the frequencies

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Suppose that an automatic sequence $\mathbf{u} = (u_n)_{n\geq 0}$ is generated by an invertible automaton.

Theorem [Drmota, Ferenczi + Kulaga-Przymus+Lemanczyk+Mauduit]

$\mathbf{u}$ is orthogonal to $\mu(n)$.

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We call a sequence $(a_n)_{n\geq 0}$ *digital* if there exists $m \geq 1$ and $F : \{0, \ldots, k-1\}^m \to \mathbb{C}$ such that

$$a_n = \sum_{i\geq 0} F(\varepsilon_{i+m-1}(n), \ldots, \varepsilon_i(n)).$$

Lemma

Let $(a_n)_{n\geq 0}$ be a digital sequence. Then $(a_n \bmod m')_{n\geq 0}$ is an automatic sequence for every $m' \in \mathbb{N}$.

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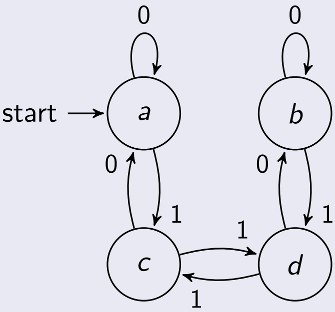

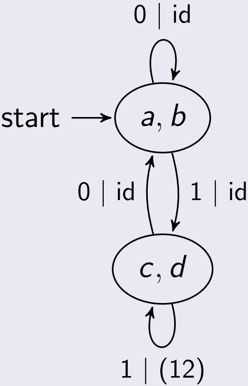

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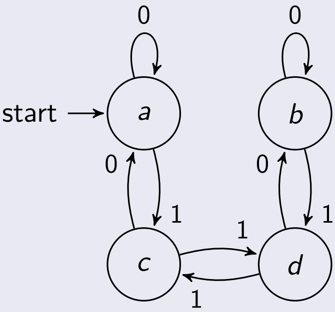

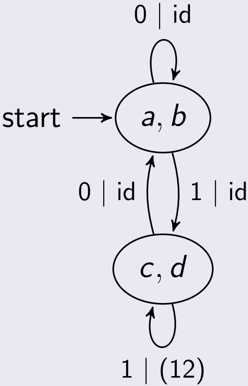

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Definition (Naturally Induced Transducer)

Let $A = (Q', \Sigma, \delta', q'_0)$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

1. $\exists n_0 \in \mathbb{N} : Q \subseteq (Q')^{n_0}$
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5. some minimality/technical conditions

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Let $A = (Q', \Sigma, \delta', q_0')$ be a strongly connected automata. We call $\mathcal{T}_A = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$ a **naturally induced transducer** iff

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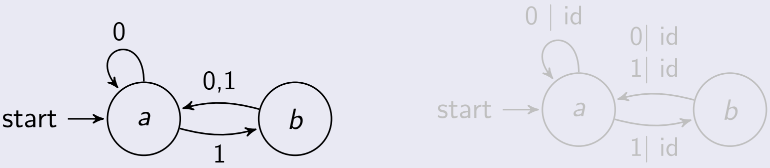

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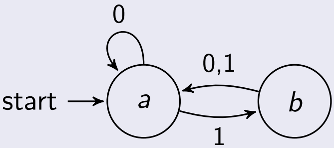

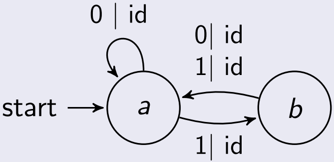

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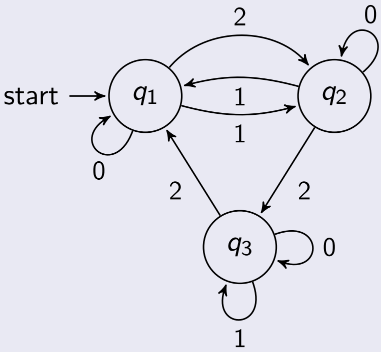

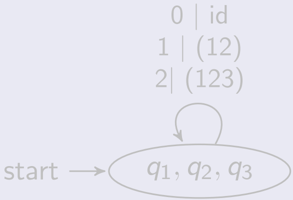

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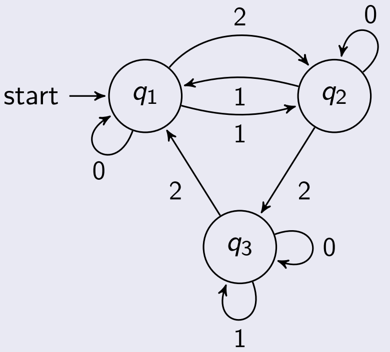

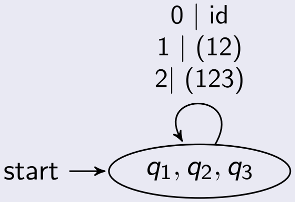

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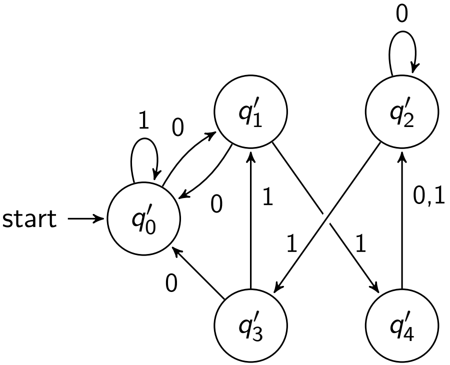

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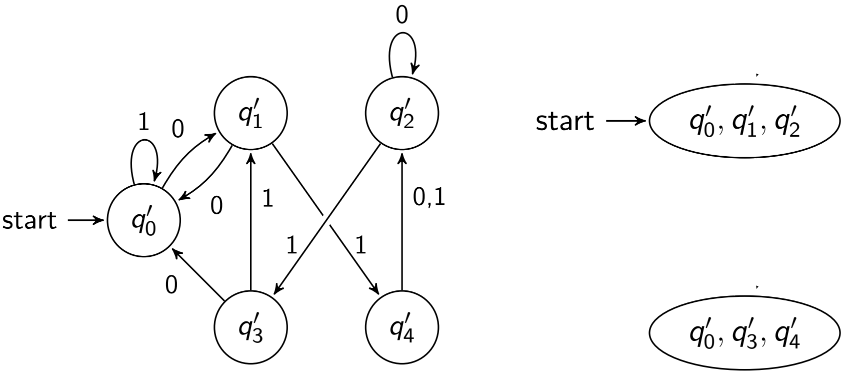

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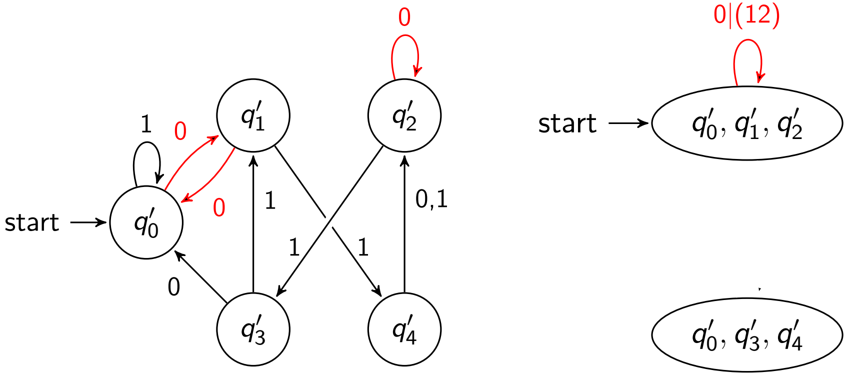

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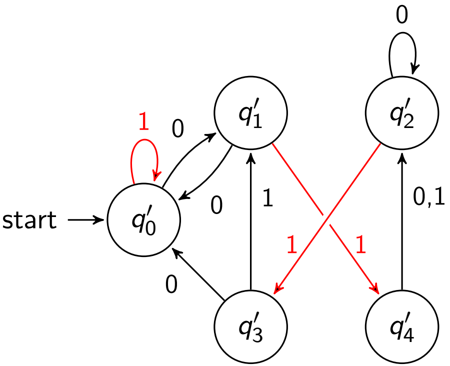

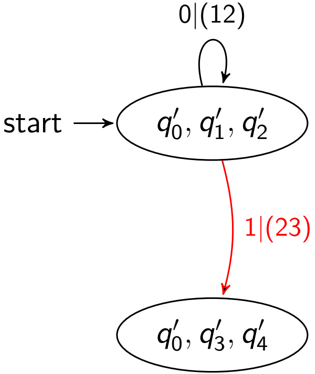

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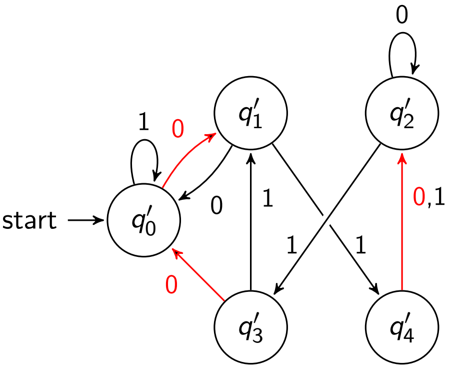

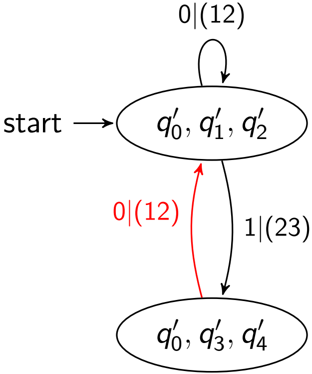

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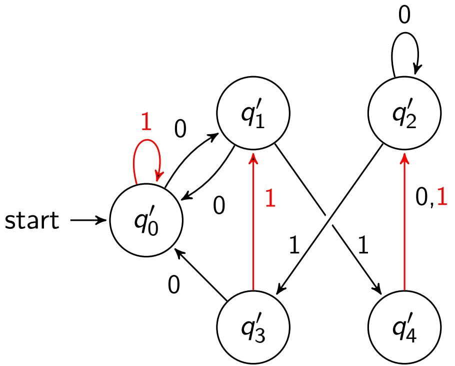
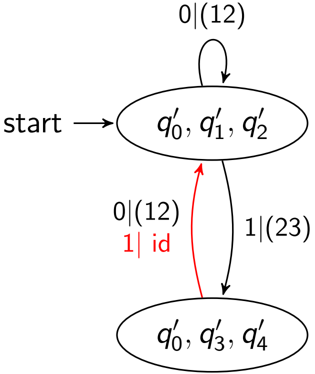


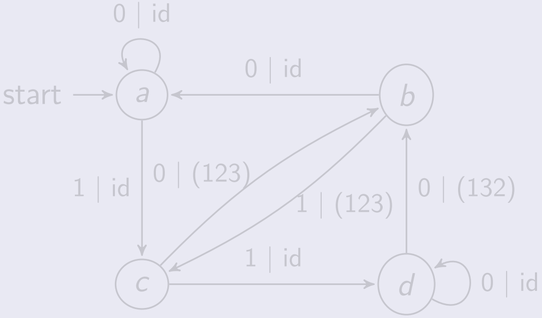


Motivation

Example (Digital Sequence)

„Generic Example“: $k = 2, m = 3, m' = 3$

$F(010) = 1, F(110) = 2, F(101) = 1$



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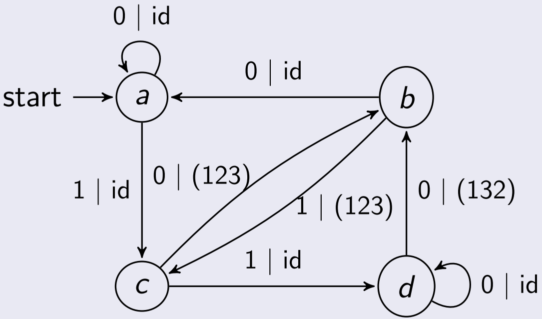

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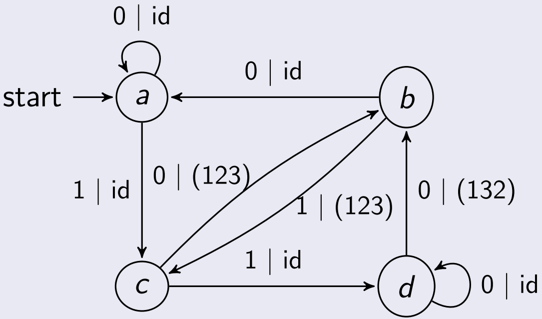

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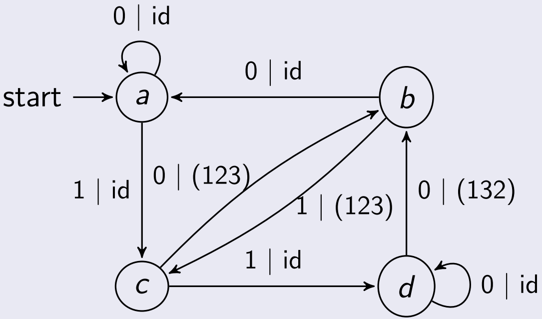

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Denote by

$$T(q, w_1 \dots w_r) := \lambda(q, w_1) \circ \lambda(\delta(q, w_1), w_2) \circ \dots \circ \lambda(\delta(q, w_1 \dots w_{r-1}), w_r).$$

Lemma

Let A be a strongly connected automaton and $\mathcal{T}_A$ a naturally induced transducer. Then,

$$\delta'(q_0', \mathbf{w}) = \pi_1(T(q_0, \mathbf{w}) \cdot \delta(q_0, \mathbf{w}))$$

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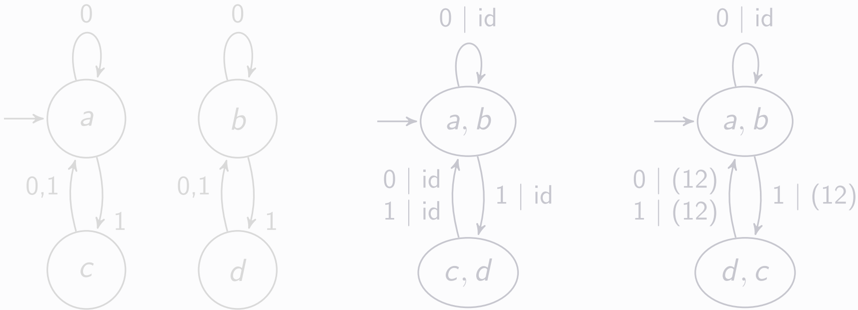

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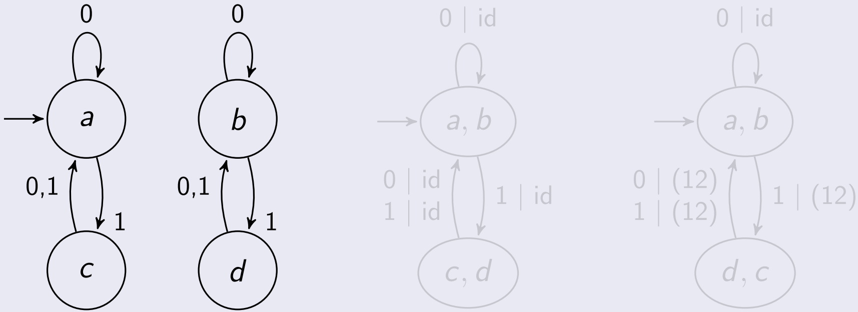

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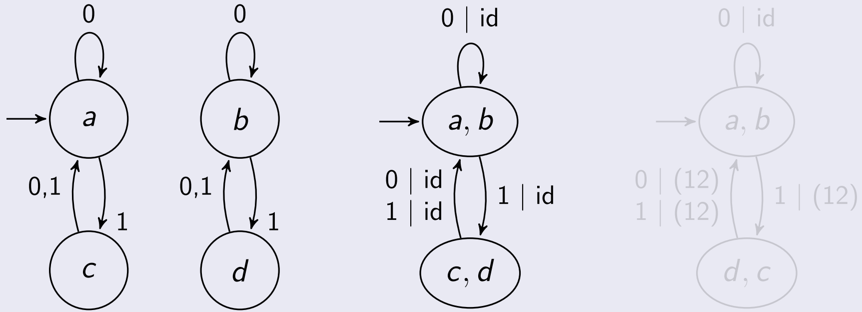

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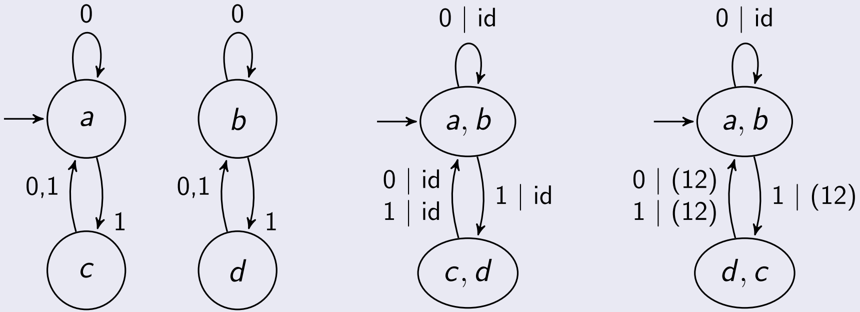

Let Δ be the group generated by $im(\lambda)$.

All elements of Δ appear as values of $T(q_0, .)$ for „good“ naturally induced transducer.

Do all elements of Δ appear simultaneously as values of $T(q_0, \mathbf{w})$ for $\mathbf{w} \in \Sigma^n$ for a single n, where n is large?

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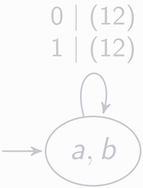

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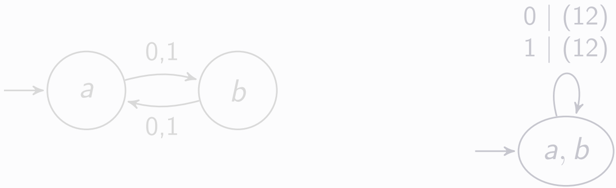

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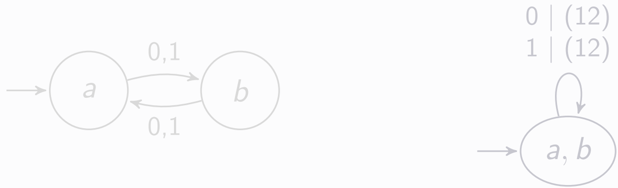

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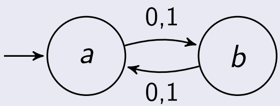

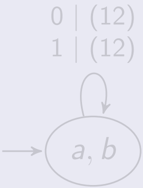

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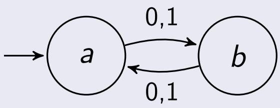

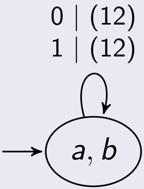

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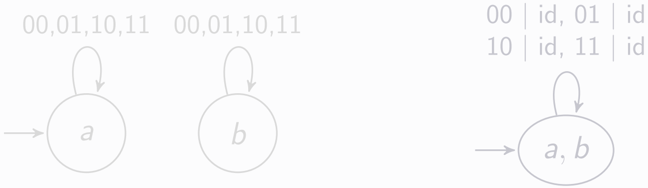

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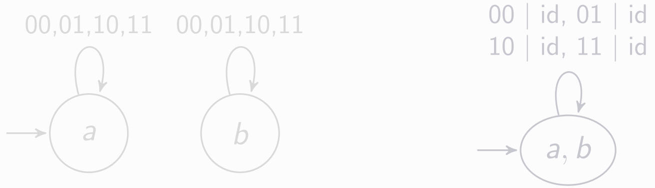

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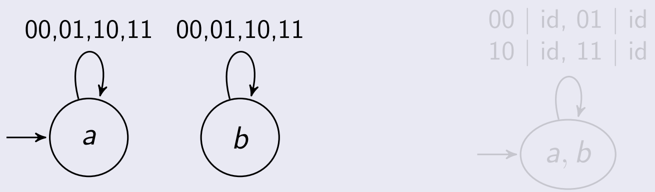

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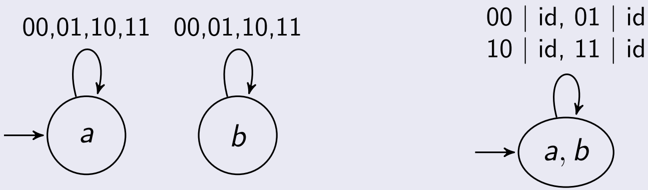

Continuous functions from a compact group to $\mathbb{C}$

Definition (Representation)

Let G be a finite group and $k \in \mathbb{N}$. A **Representation** of rank k is a continuous homomorphism $D : G \to \mathbb{C}^{k \times k}$.

Lemma

Let f be a continuous function from G to $\mathbb{C}$. There exists $r \in \mathbb{N}$ and unitary, irreducible representations $D^{(\ell)} = (d_{i,j}^{(\ell)})_{i,j<k_\ell}$ along with $c_\ell \in \mathbb{C}$ such that

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Suppose that

$$\sum_{\substack{n<N \\ \dots}} D(T(n))\mu(n) = o(N)$$

holds for all irreducible unitary representations of G. Then $\mathbf{u} = (u_n)_{n\geq 0}$ is orthogonal to $\mu(n)$.

We follow the method of Mauduit and Rivat that they use for studying the Rudin-Shapiro sequence.

(Adopted) Definition

Let $U(n)$ be a sequence of unitary matrices. We say that U has the **Fourier property** if there exists $\eta > 0$ and c such that for all λ, α and t

$$\left\| \frac{1}{k^\lambda} \sum_{m<k^\lambda} U(mk^\alpha) e(mt) \right\| \leq ck^{-\eta\lambda}.$$

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Suppose that $D \circ T$ has the Fourier property. Then we have for any real θ

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Vaughan method:
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provides estimates for

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Lemma

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Problem: Distinguish representations D that fulfill the Fourier Property.

Lemma

Let A be a DFA and $\mathcal{T}_A$ a naturally induced transducer. There exists d' and representations $D_0, \ldots, D_{d'-1}$ such that

$$D_\ell(T(q, (n)_k)) = \mathrm{e}\left(\frac{n\ell}{d'}\right).$$

Theorem

Let D be a unitary and irreducible representation different from $D_0, \ldots, D_{d'-1}$. Then $D(T(.))$ has the Fourier Property.

Automatic Sequences along Primes

The treatment is very similar to the orthogonality to the Möbius function.

One has to work more carefully to extract the main term.

The actual frequencies can be made explicit and are determined by the behavior of the automatic sequence along arithmetic progressions.

Primes vs all natural Numbers

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