Fully Packed Loop Configurations in a triangle

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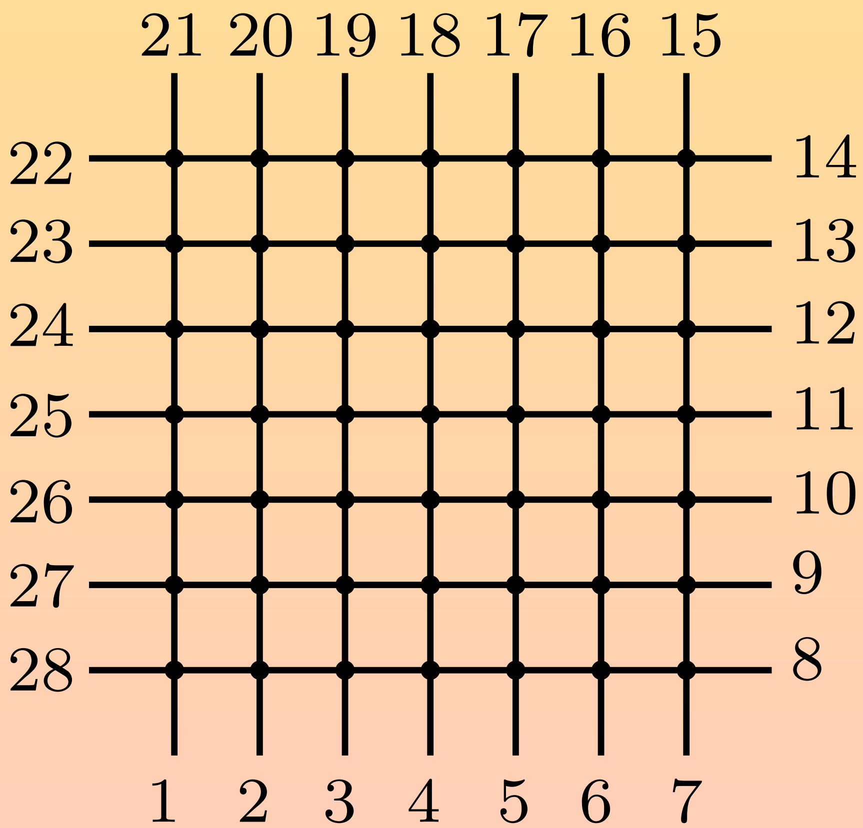
FPL configurations : Definition

Start with the square grid G_n with n^2 vertices and $4n$ external edges.In the example, we have $n = 7$.



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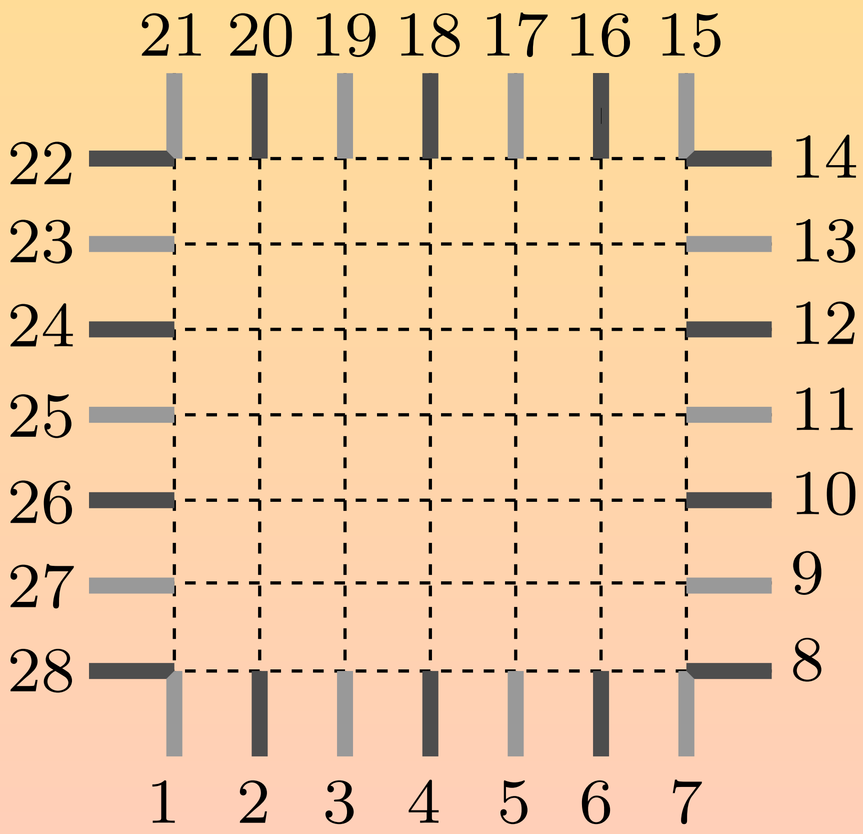

A FPL configuration of size n is a subgraph of the grid G_n

(1) containing every other external edge,i.e. contains either all odd edges or all even edges.

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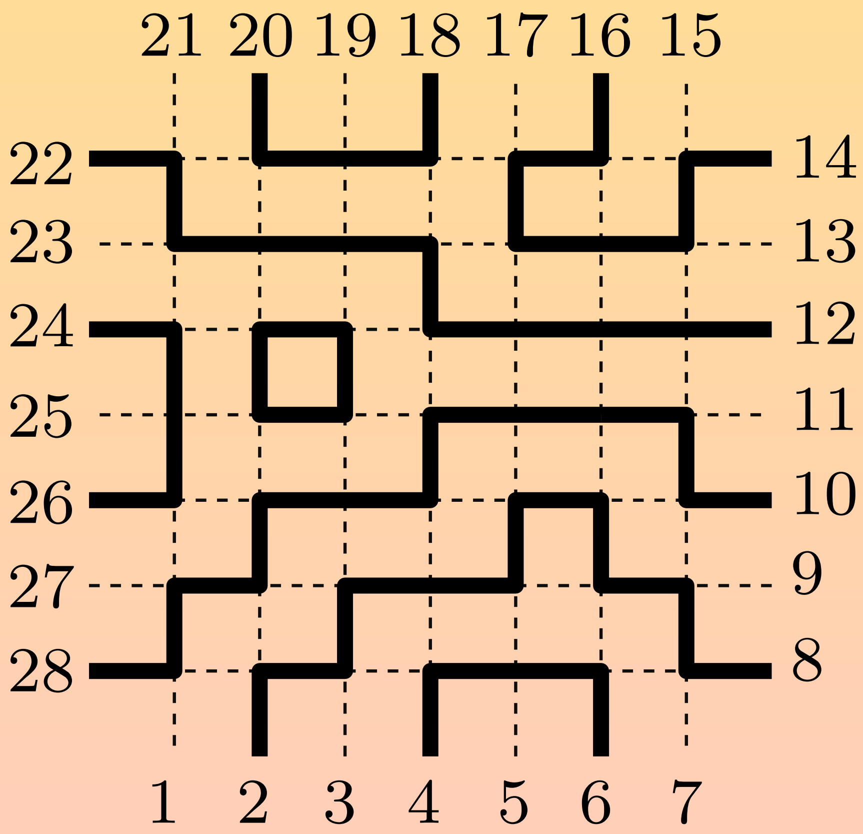


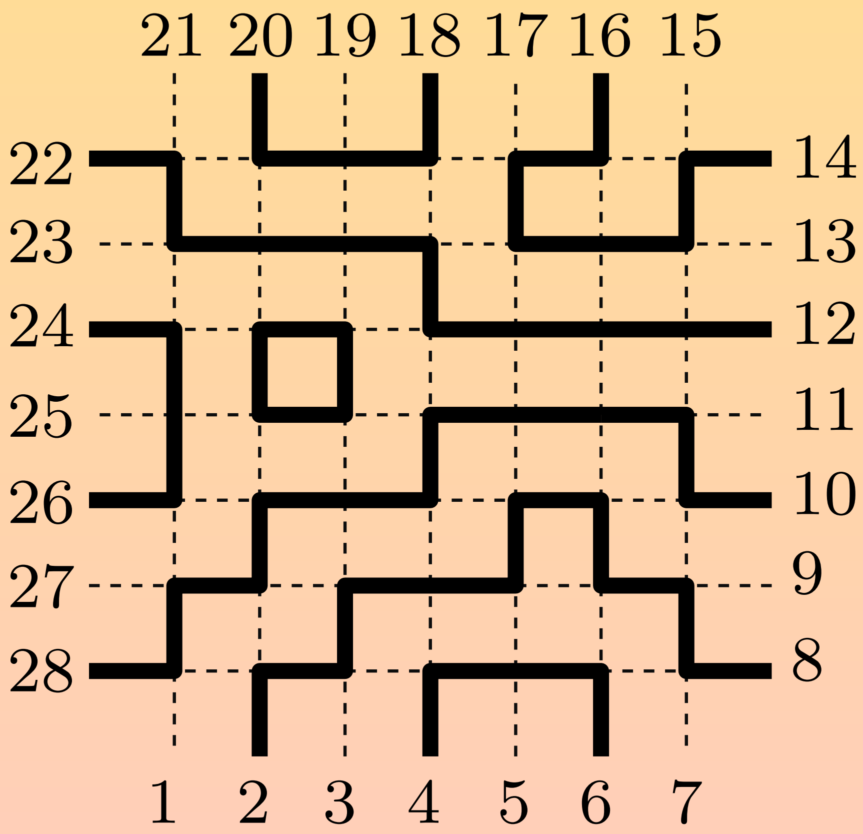
A FPL configuration of size n is a sub-graph of the grid G_n

(1) containing every other external edge ;

(2) such that around each vertex of G_n, 2 edges out of 4 are selected.

FPL configurations : Enumeration

Such FPL configurations are in simple bijection with numerous objects : alternating sign matrices, height matrices, configurations of the six vertex model, Gog triangles,...



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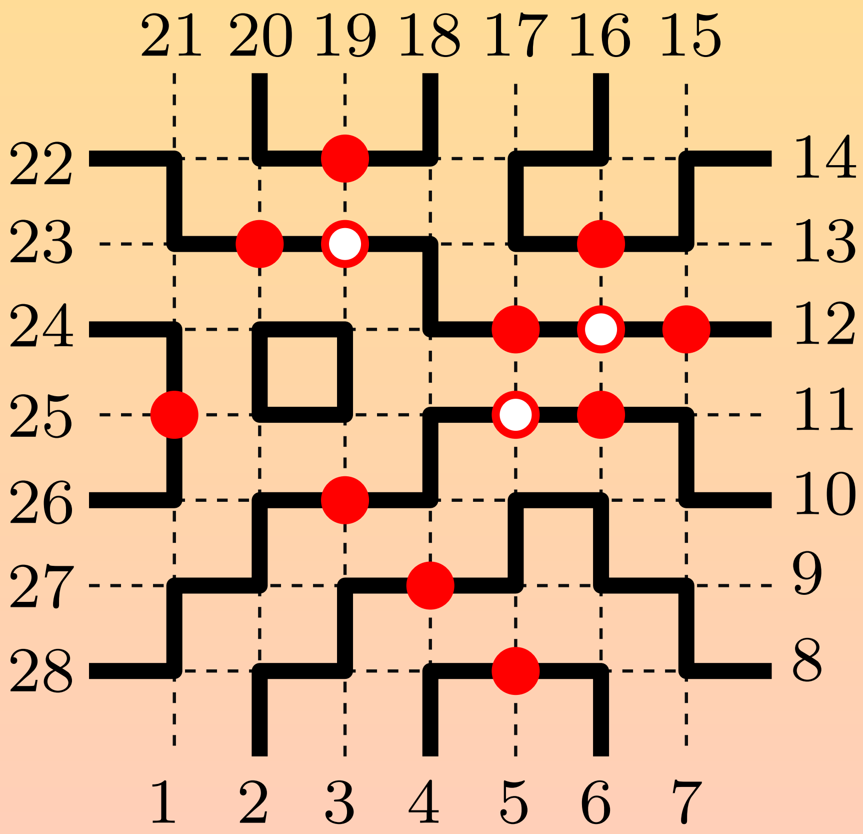

FPL of size n with even boundary

↕

Alternating sign matrices of size n

[ASM $=$ matrix with coefficients in $\{1, 0, -1\}$ such that on each row or column 1 and -1 alternate, and the sum is 1.]

Here $1 \longrightarrow$ ● and -1 $\longrightarrow$ ○

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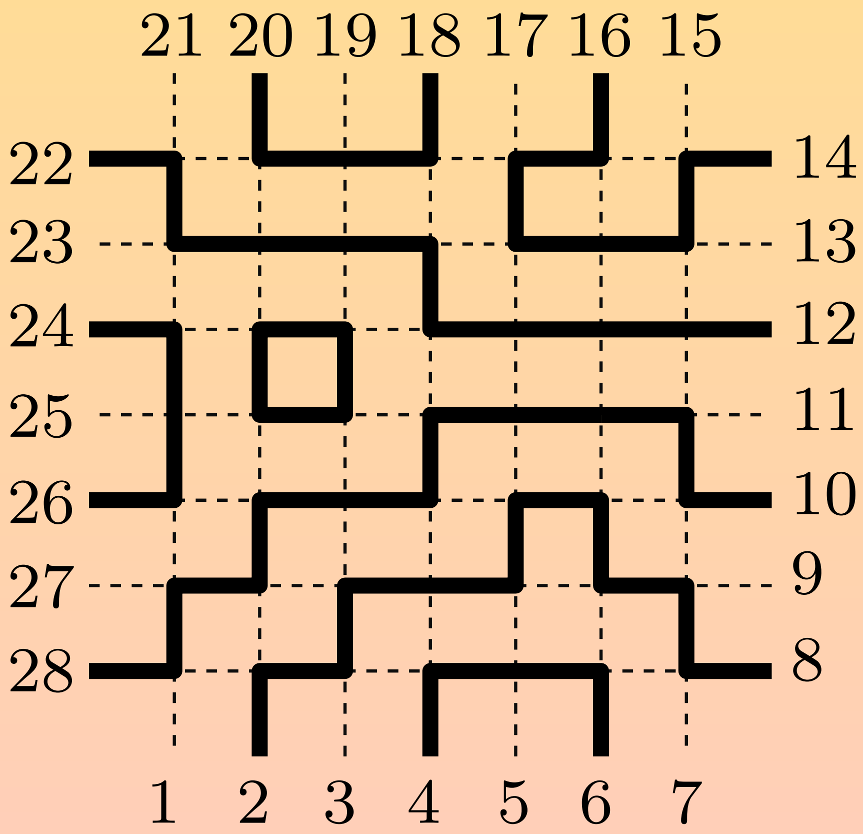

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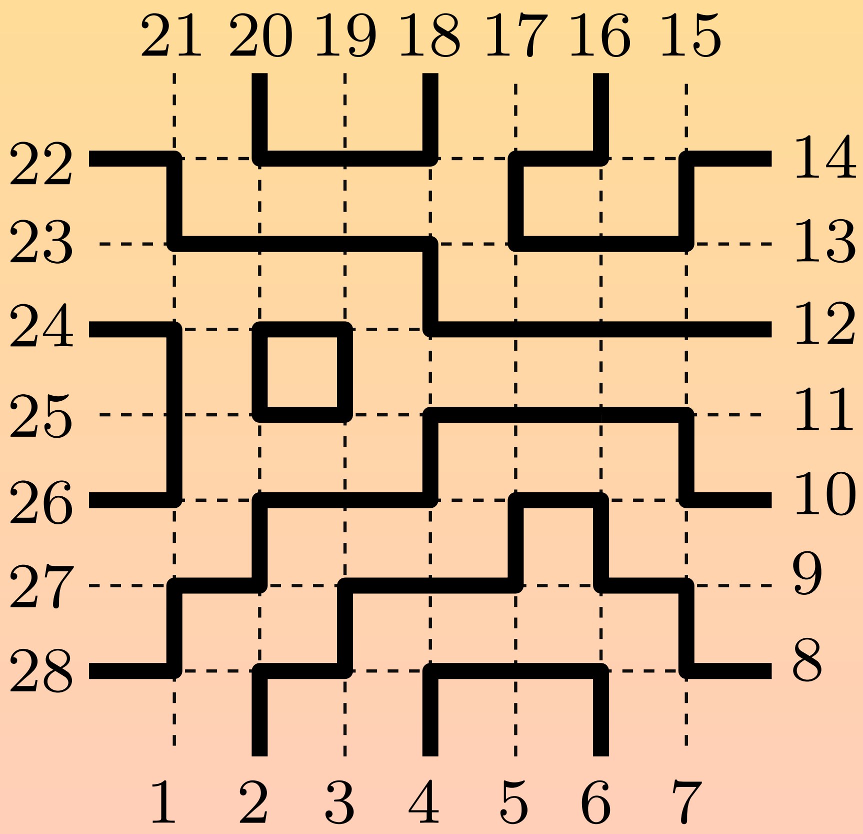
Alternating sign matrices of size n

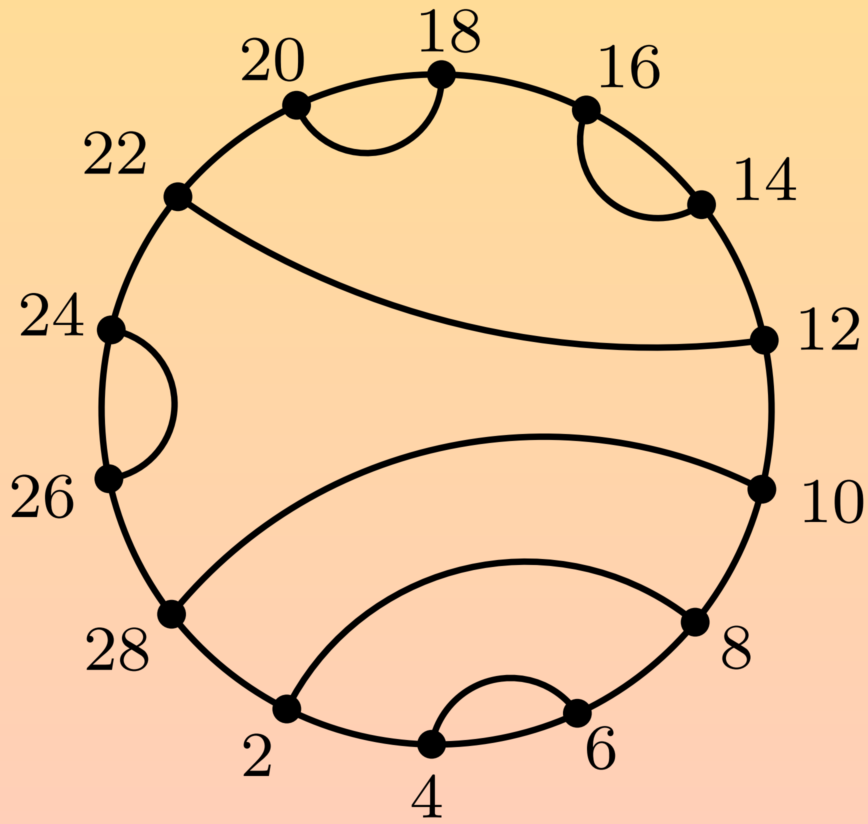
$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

[Zeilberger '96, Kuperberg '96]

FPL configurations : Refined enumeration

Every FPL configuration determines a link pattern on the odd or even external edges of the grid G_n.





FPL configurations : Refined enumeration

Now if we are given a pairing X of odd (or even) external edges, our main question will be : how many FPL configurations respect the link pattern X ?

Definition We note A_X the number of FPL configurations which induce the link pattern X.

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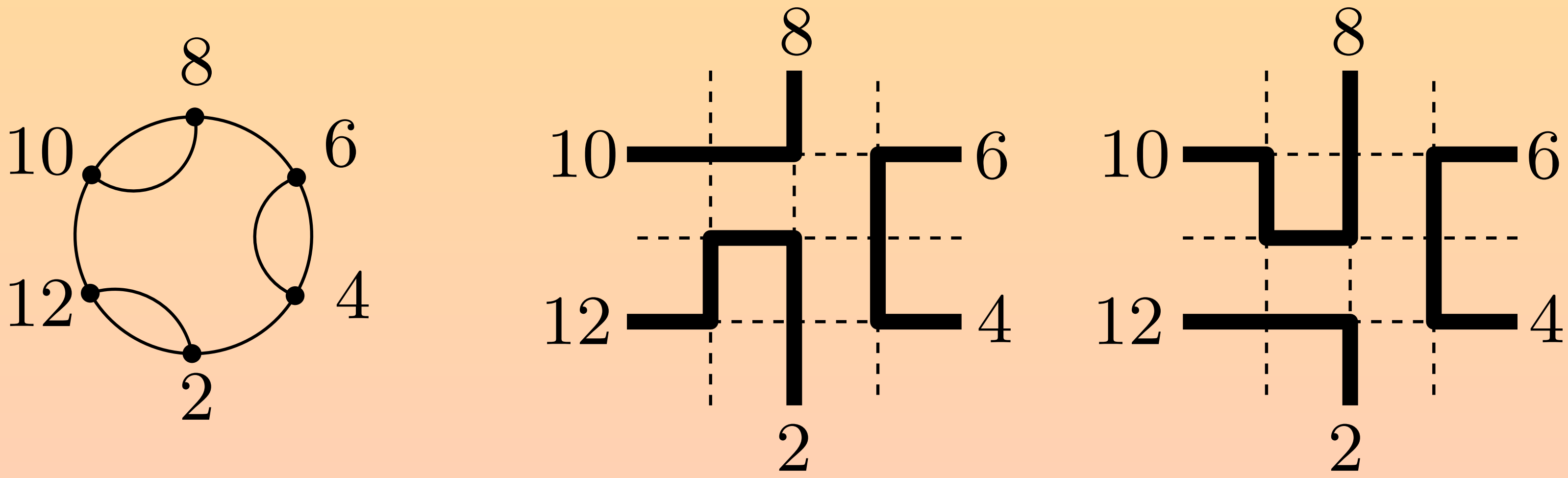
If $X =$



For this link pattern we have $A_X = 2$.

FPL configurations : Refined enumeration

Now given a link pattern X, let X' be defined by

$$(i,j) \in X' \Leftrightarrow (i-1, j-1) \in X$$

Theorem [Wieland '00]

$$A_X = A_{X'}$$

This means that "rotating the link pattern" does not change the number of FPL configurations attached to it.

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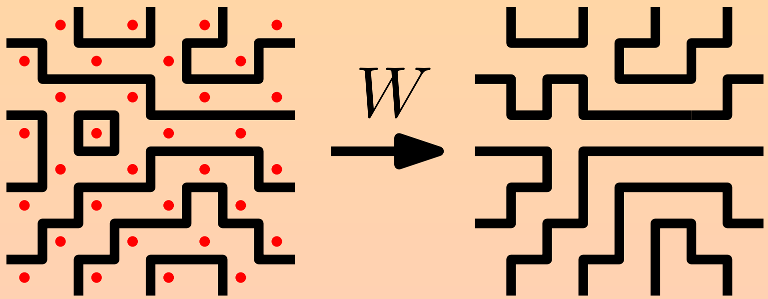

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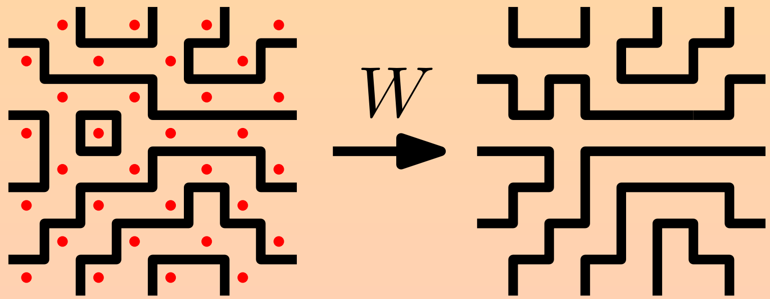
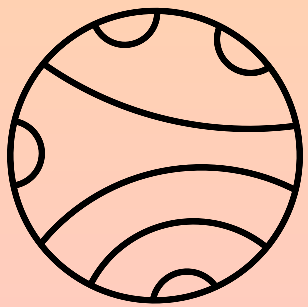
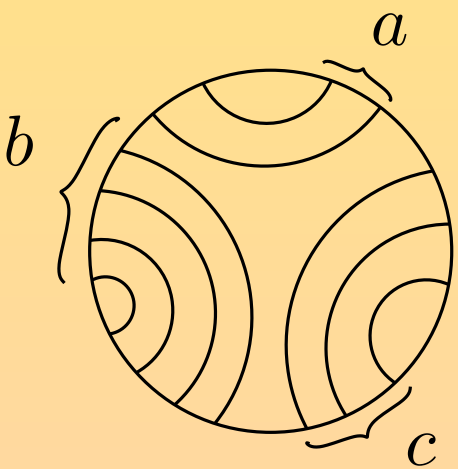


For enumeration purposes, we can then use unlabeled link patterns :

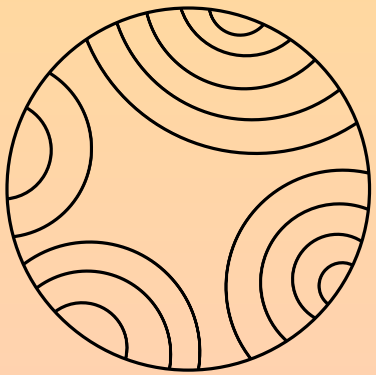


Outline of the talk

Known enumerations for the numbers A_X are



$$= \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$$



= Complicated determinant formulas

+ certain variants of these.

These results are due to Zinn-Justin, Zuber, Di Francesco, Caselli, Krattenthaler,...

Outline of the talk

For a given link pattern X of size n, there exist numerous instances in [Zuber '04] of conjectured identities of the form

$$A_X = \sum_{c_{XX'} \in \mathbb{Z}} c_{XX'} A_{X'} \text{ where } X' \text{ are link patterns of size } n-1.$$

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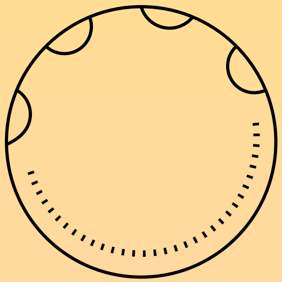
Example  $= \sum_X A_X = A_{n-1}$

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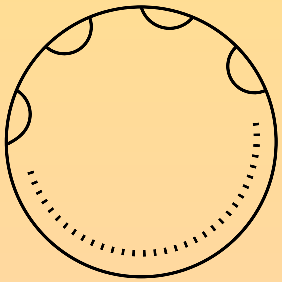
Example  $= \sum_X A_X = A_{n-1}$

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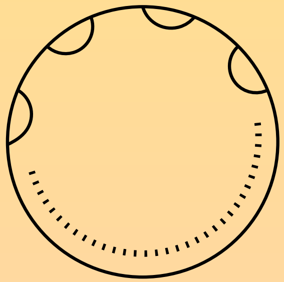
Example  $= \sum_X A_X = A_{n-1}$

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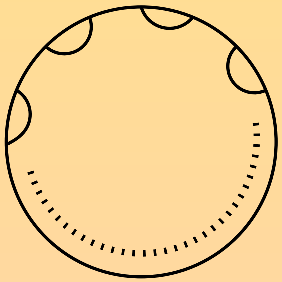
Example  $= \sum_X A_X = A_{n-1}$

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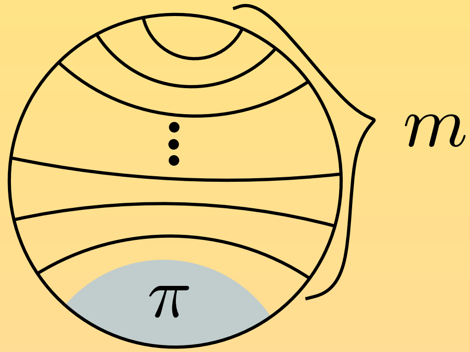
In one such case, we will show that the answer is given by the famous Littlewood-Richardson coefficients.

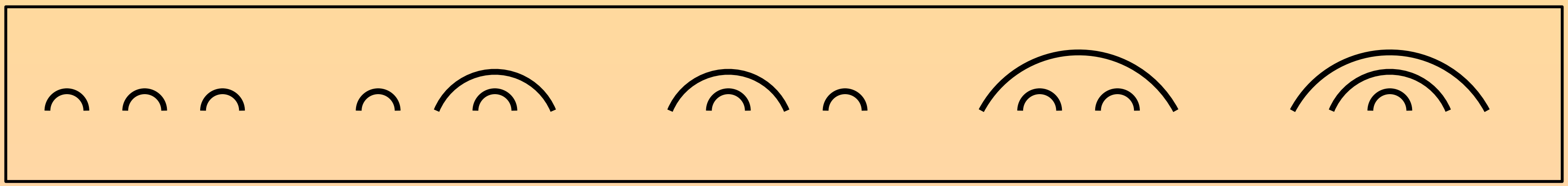
Link patterns with nested arches



$\pi \cup m$

We consider now integers $n, m \geq 0$, and link patterns with m nested arches, and π is a noncrossing matching with n arches.

For instance if $n = 3$, there are 5 possible π :



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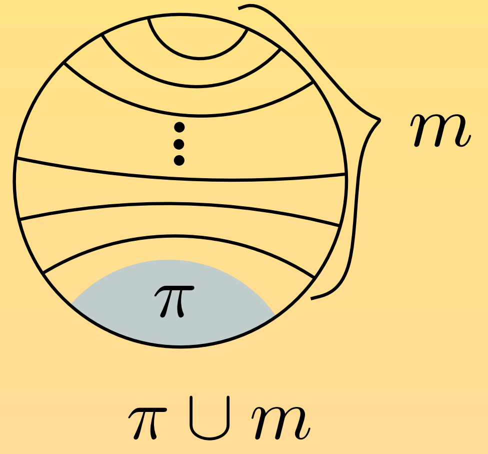

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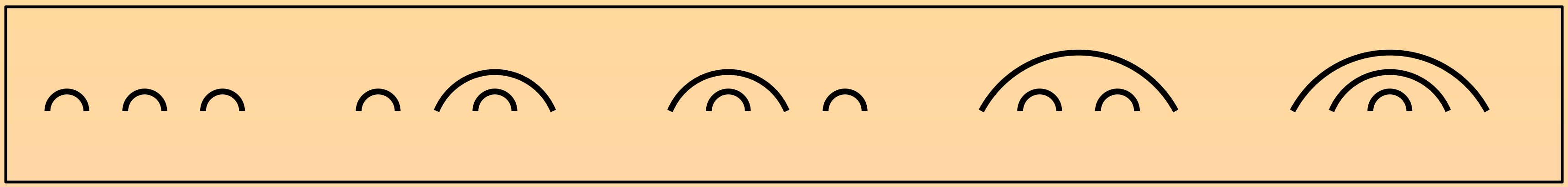

We consider such link patterns as functions of m (π being fixed), so we write the number $A_{\pi \cup m}$ as $A_\pi(m)$.

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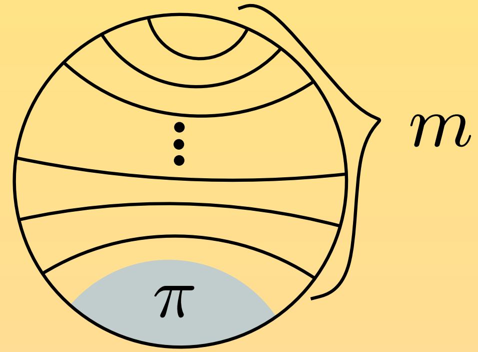

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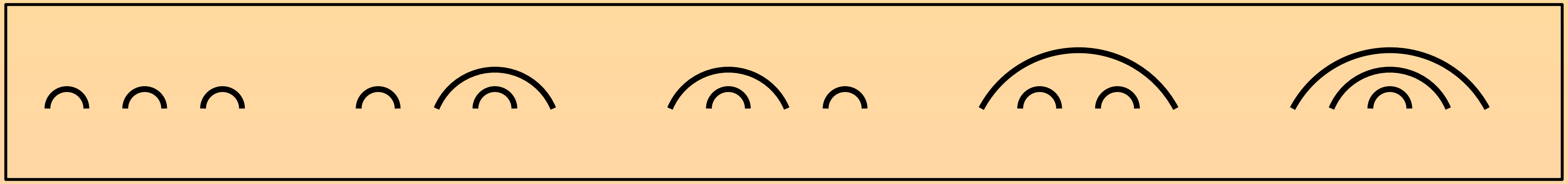

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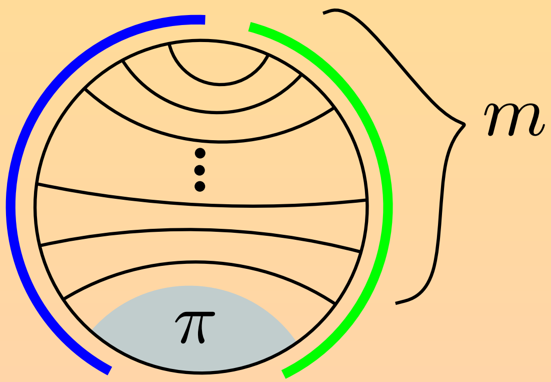
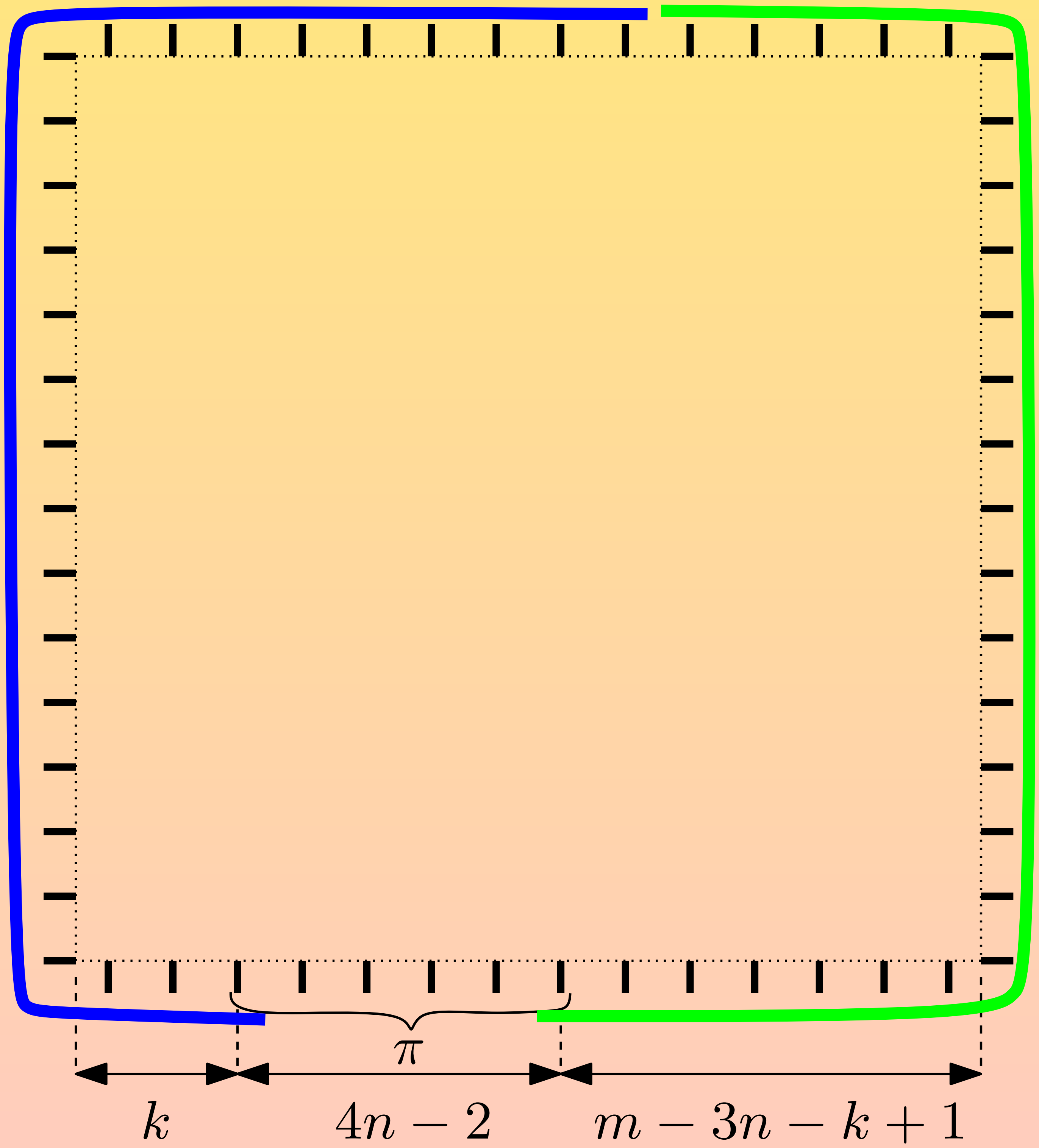
Theorem [Caselli,Krattenthaler,Lass,N. '05]

$A_\pi(m)$ is a polynomial function of m.

We suppose $m \geq 3n-1$, and choose k such that $0 \leq k \leq m-(3n-1)$.





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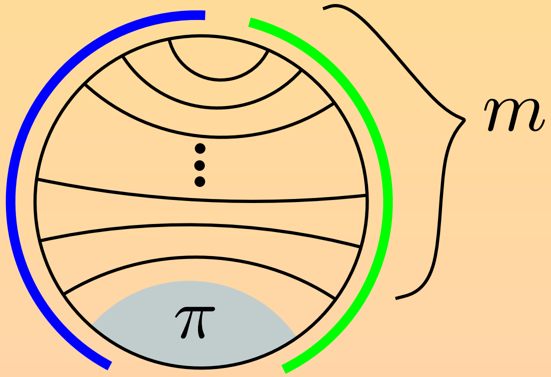
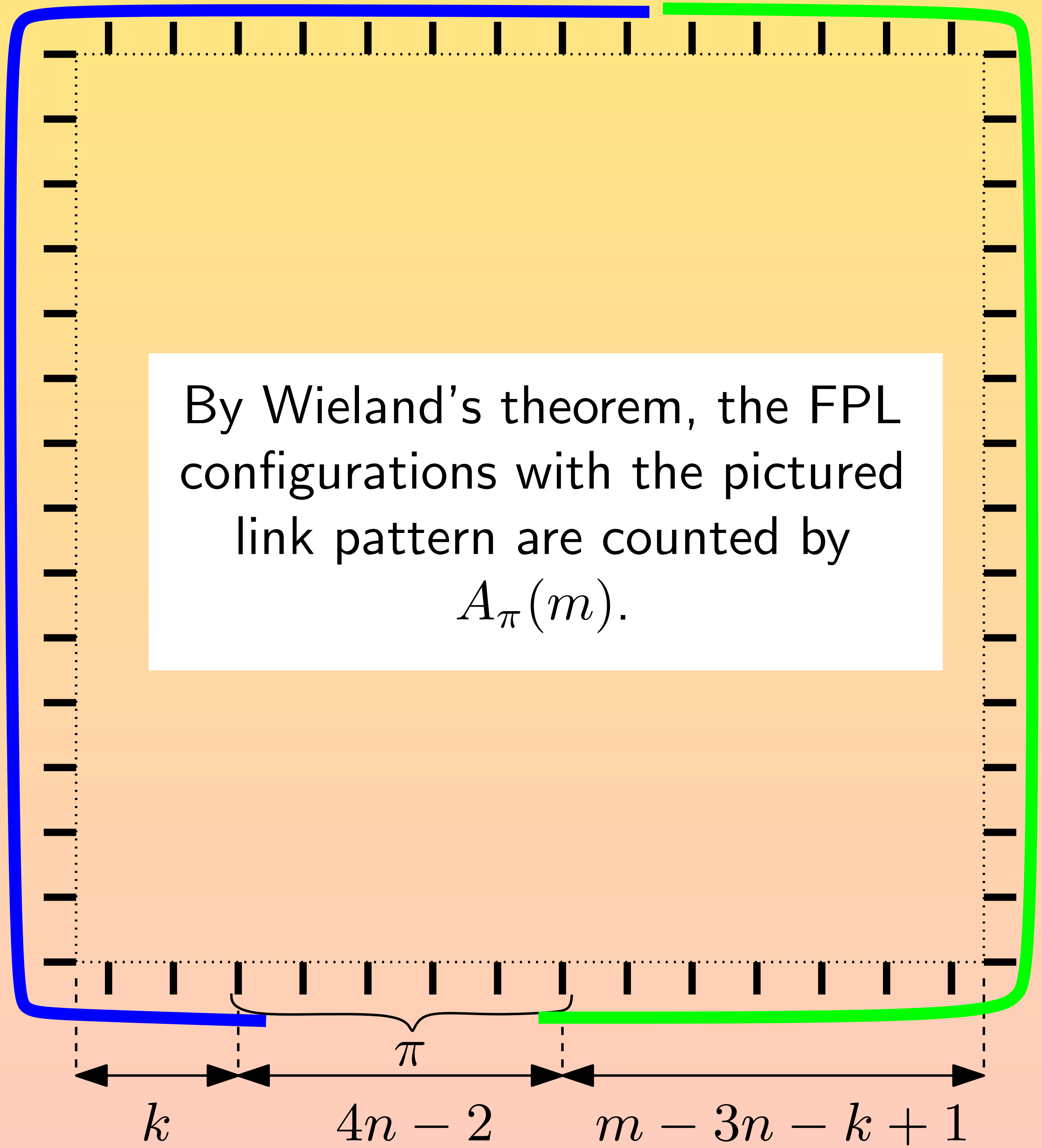


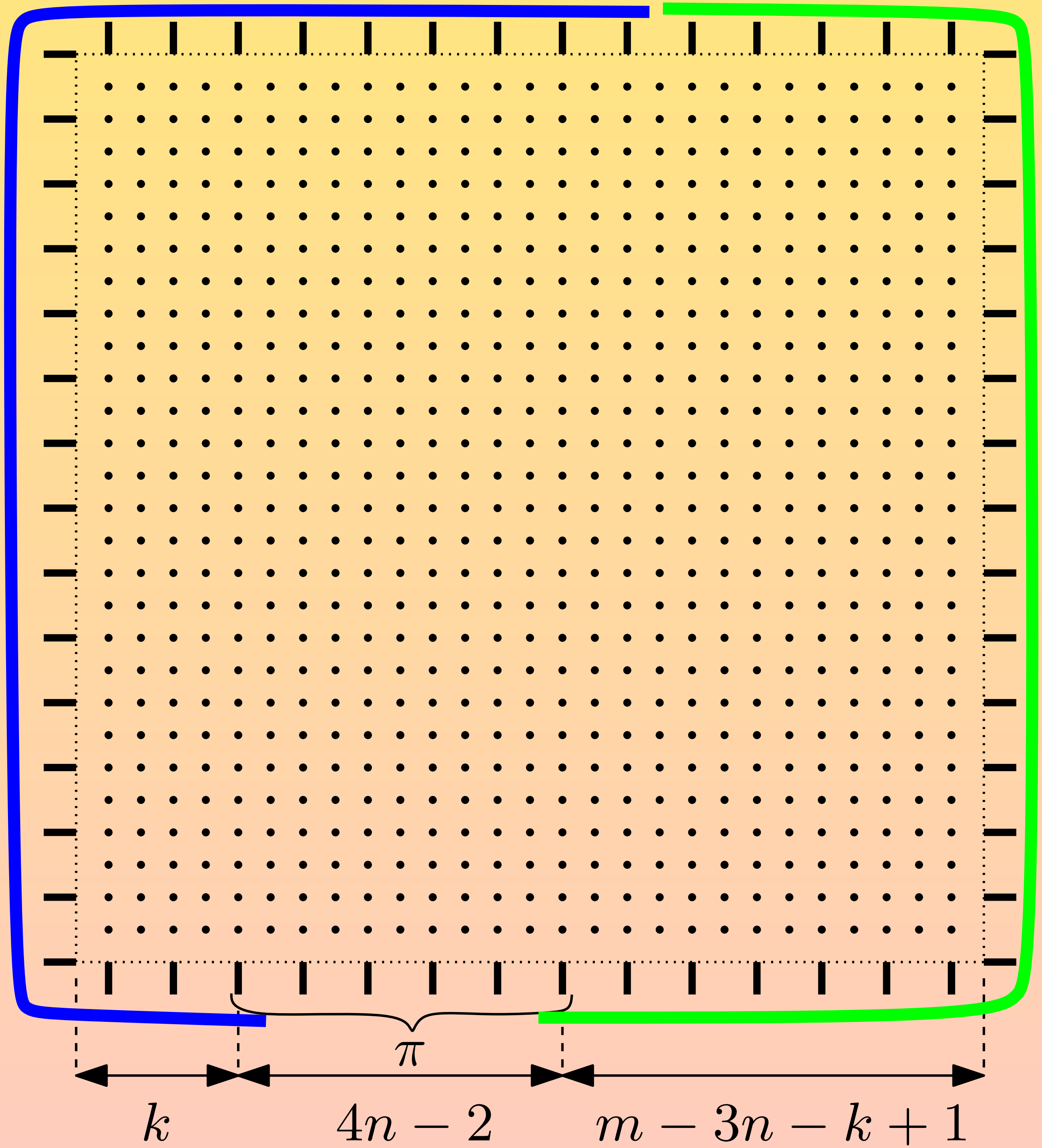


By Wieland's theorem, the FPL configurations with the pictured link pattern are counted by $A_\pi(m)$.

Many edges of the grid belong to every FPL configuration respecting the link pattern.

$\Rightarrow$"Fixed edges"

To find them, the main tool is a lemma proved in [de Gier, '02].



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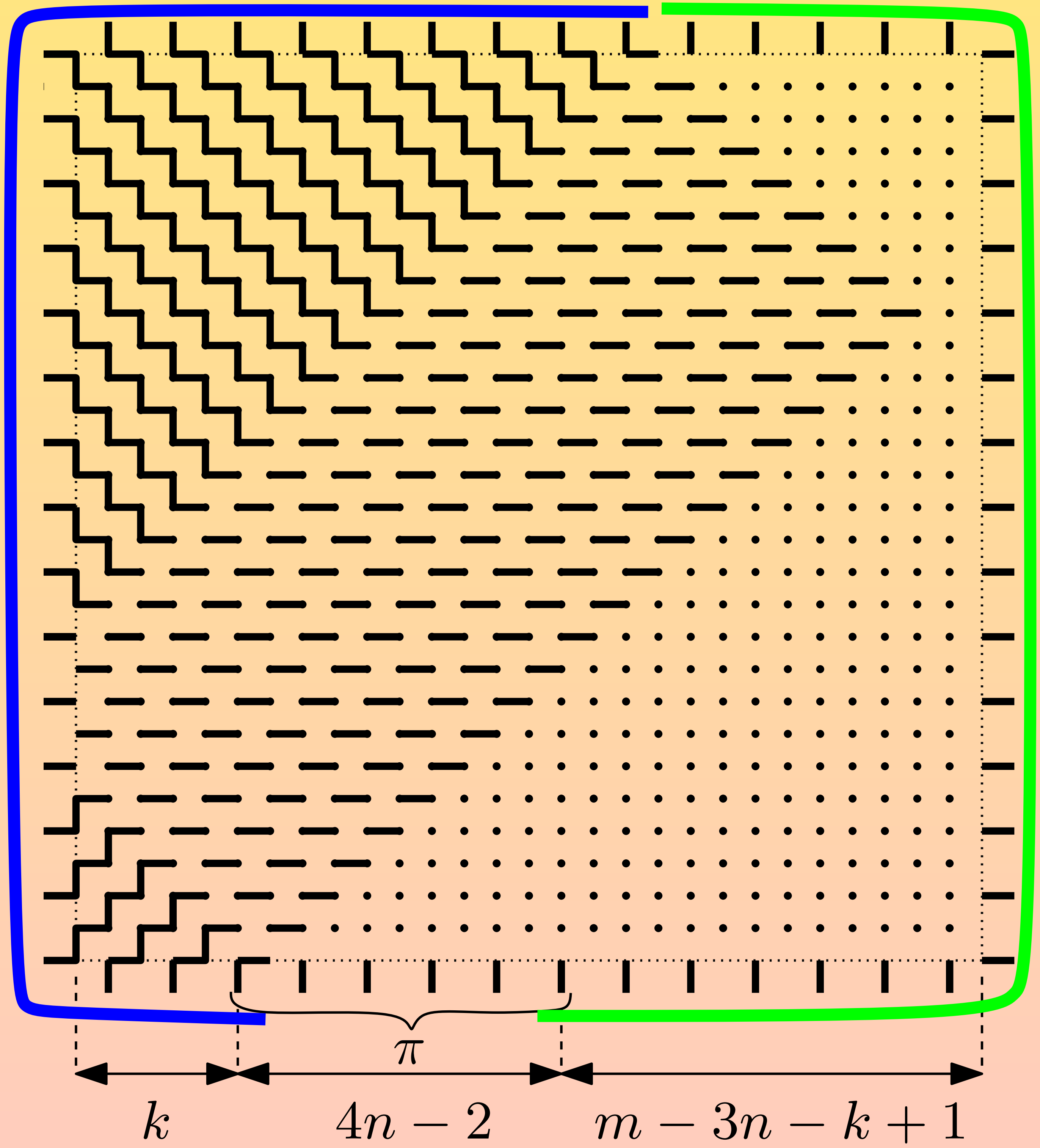

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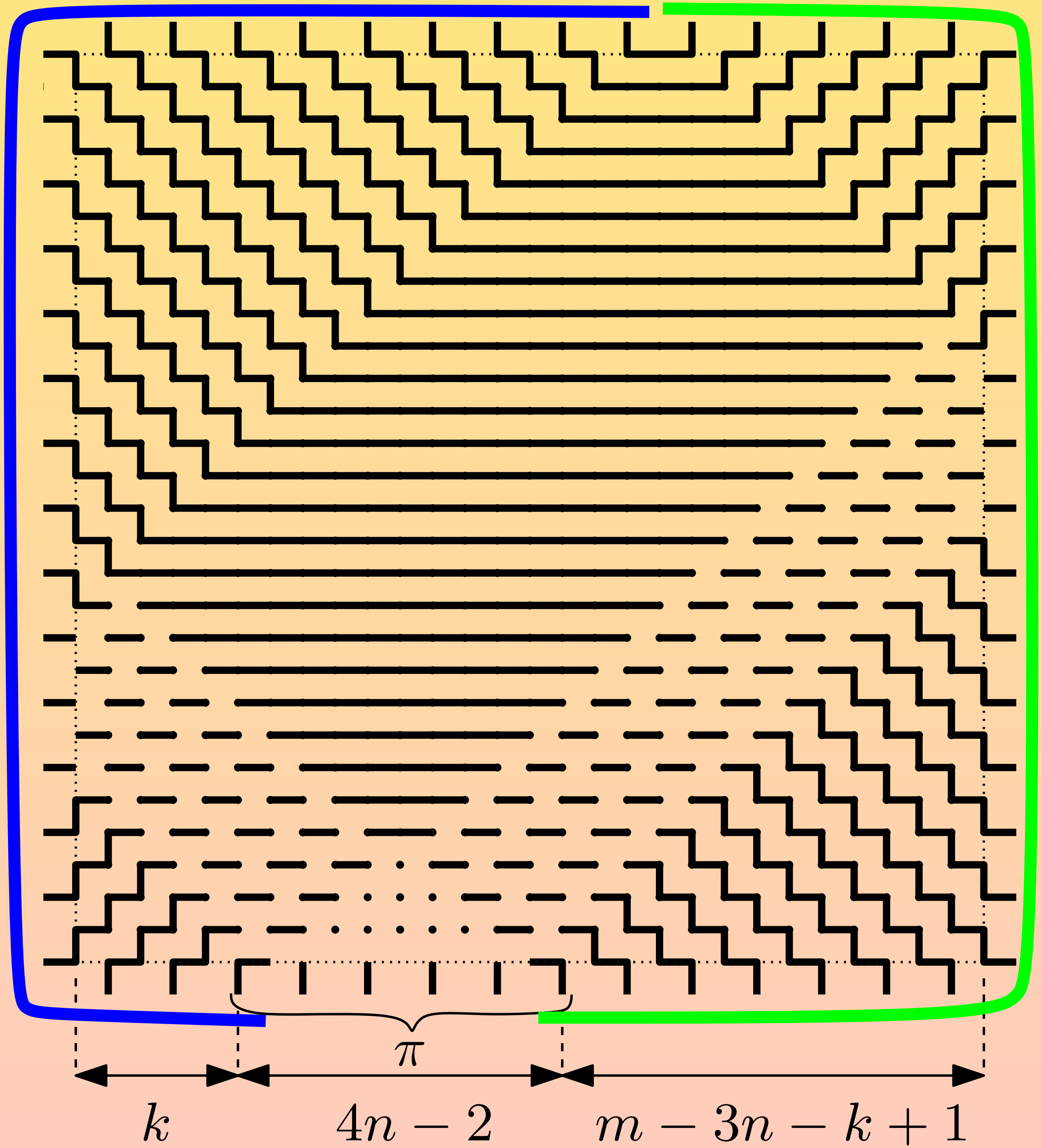

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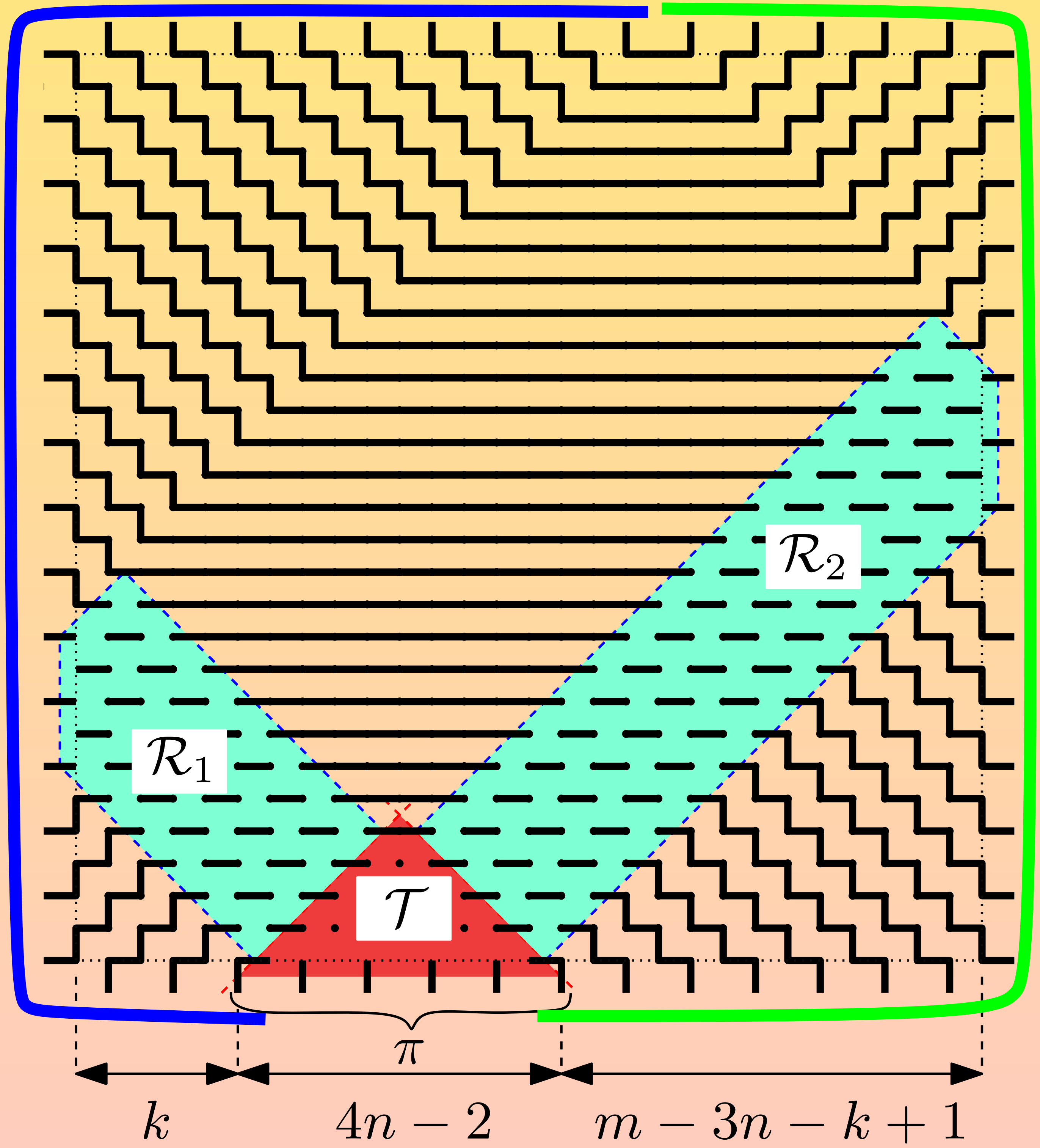
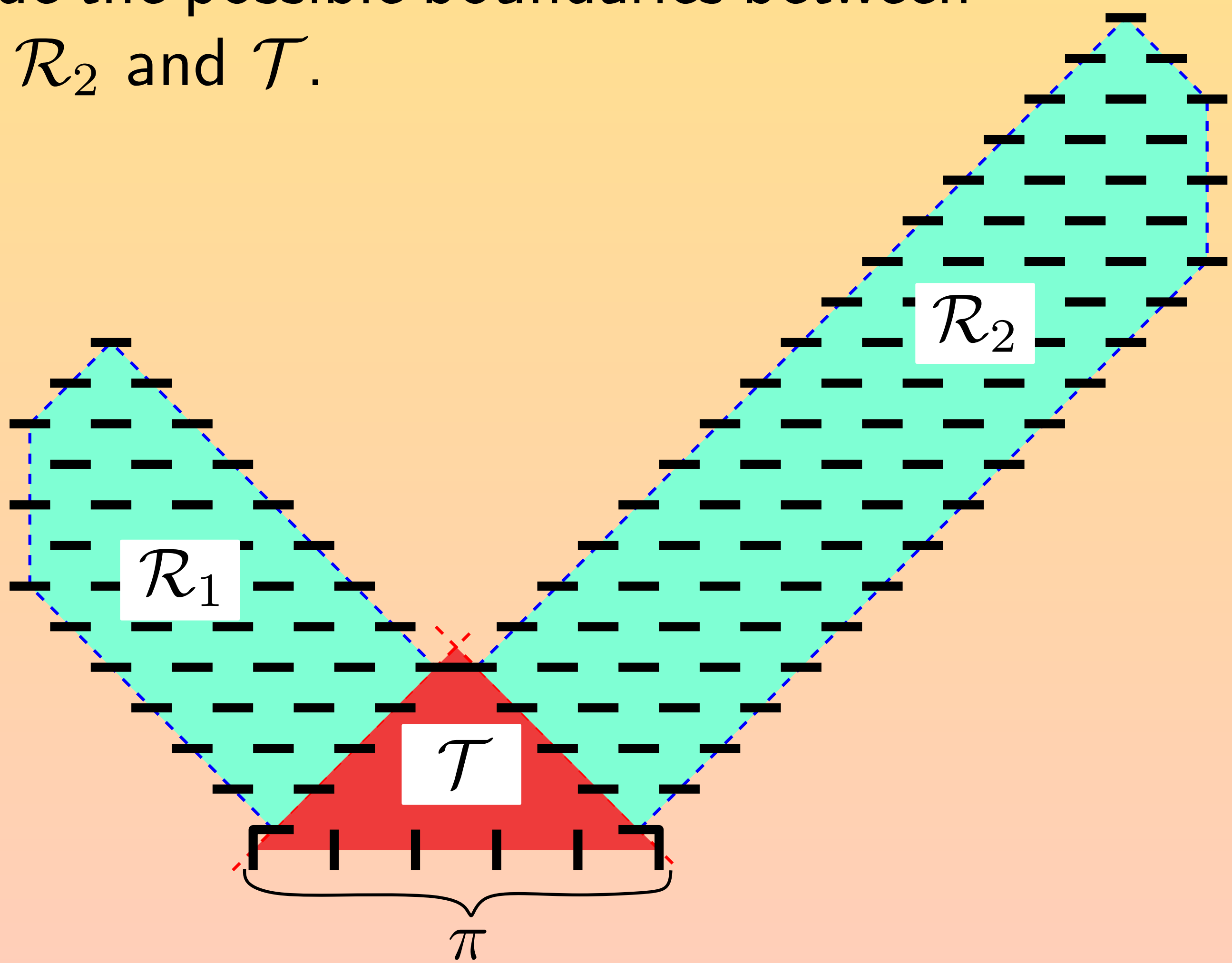


To compute the numbers $A_\pi(m)$, we will count FPL configurations separately in $\mathcal{R}_1, \mathcal{R}_2, \mathcal{T}$.

For this, we need to encode the possible boundaries between $\mathcal{R}_1$ and $\mathcal{T}$, and between $\mathcal{R}_2$ and $\mathcal{T}$.



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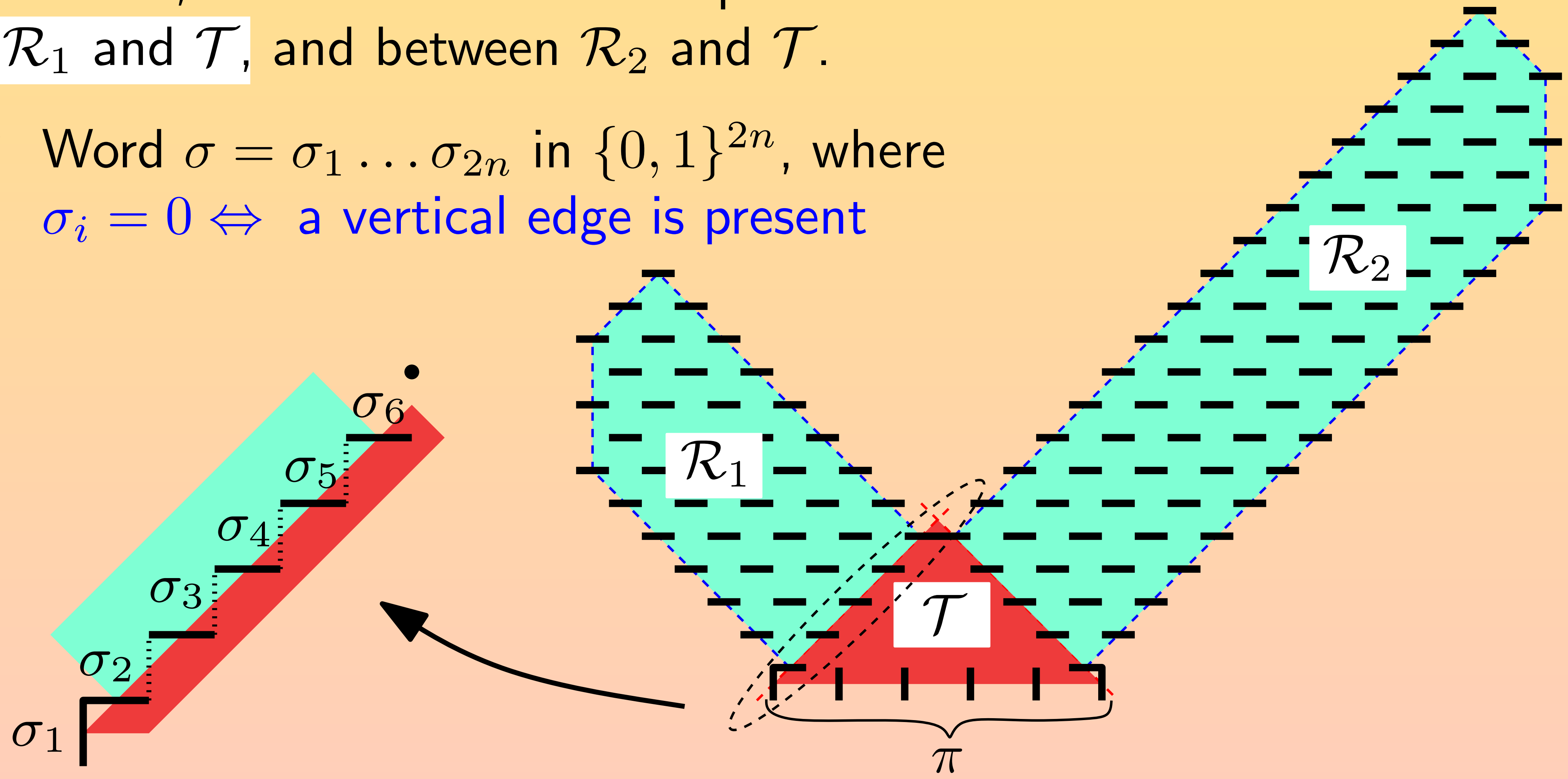
Word $\sigma = \sigma_1 \ldots \sigma_{2n}$ in $\{0,1\}^{2n}$, where
$\sigma_i = 0 \Leftrightarrow$ a vertical edge is present



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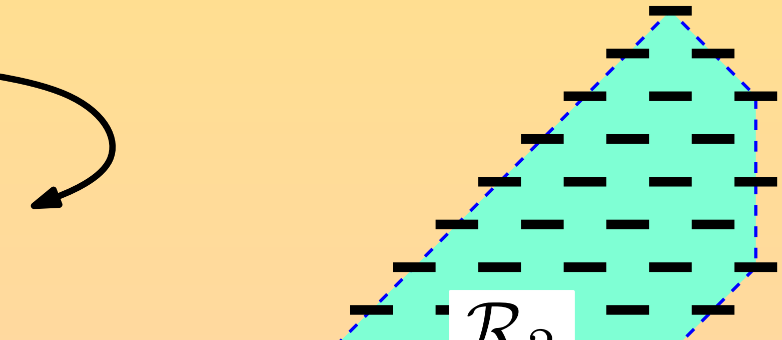
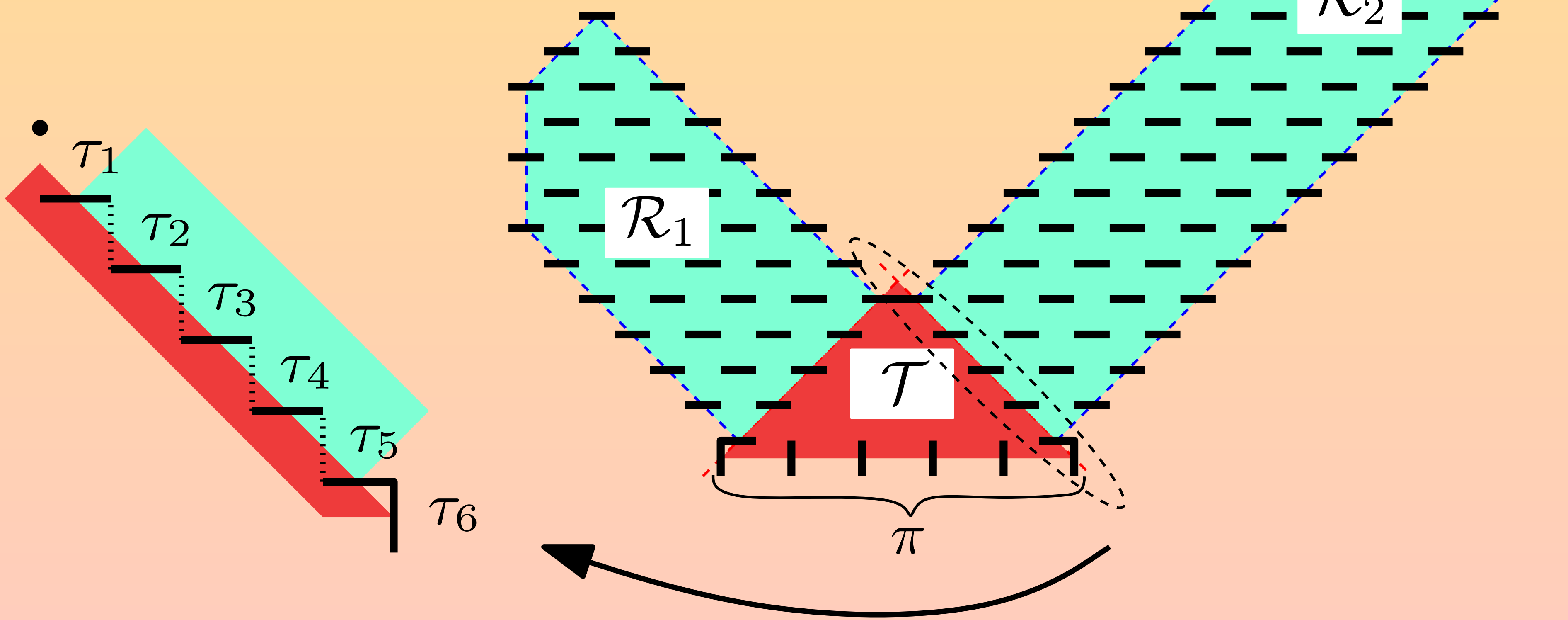




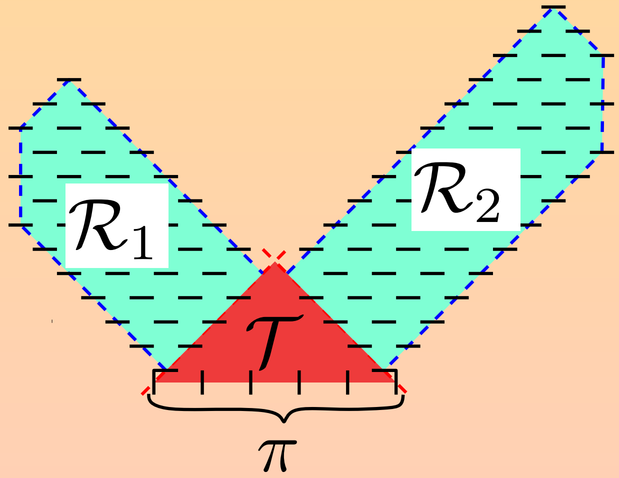
Putting things together

We can then write, for $m \geq 3n-1$ and $0 \leq k \leq m-(3n-1)$

$$A_\pi(m) = \sum_{\sigma,\tau} |\mathcal{R}_1(\sigma,k)| \times t^{\pi}_{\sigma,\tau} \times |\mathcal{R}_2(\tau, m-3n-k+1)|$$

where

- σ, τ are words of length $2n$ on $\{0,1\}$;
- $\mathcal{R}_1(\sigma, .), \mathcal{R}_2(\tau, .)$ are the sets of FPL configurations in the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ with boundaries σ, τ respectively ;
- $t^{\pi}_{\sigma,\tau}$ is the number of FPL configurations in the triangle $\mathcal{T}$ with boundary data $\{\sigma, \pi, \tau\}$.



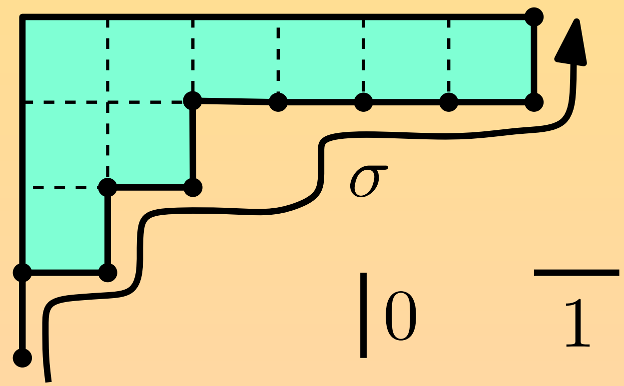
Words and Shapes

Let $\sigma = \sigma_1 \ldots \sigma_p$ be a word in $\{0,1\}^p$; we write $|\sigma| := p$.

We will identify words and Ferrers shapes in a box.

$\sigma = 0101011110$

$|\sigma| = 10, |\sigma|_0 = 4, |\sigma|_1 = 6$



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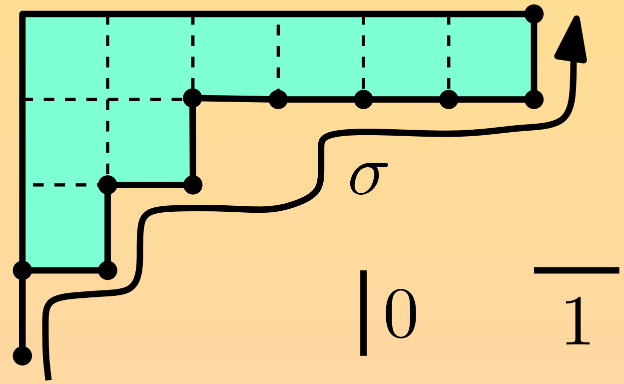

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Length $\ell(\sigma)$:= the number of boxes in the diagram σ.

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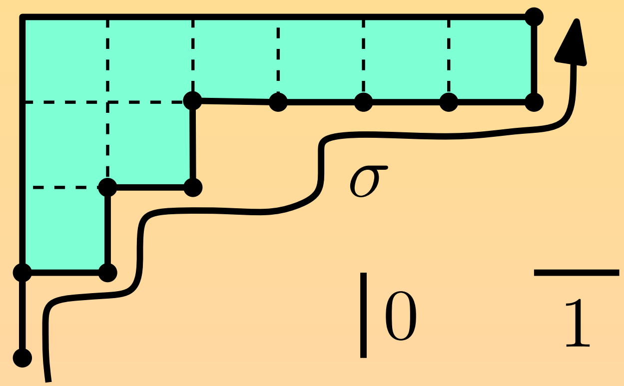

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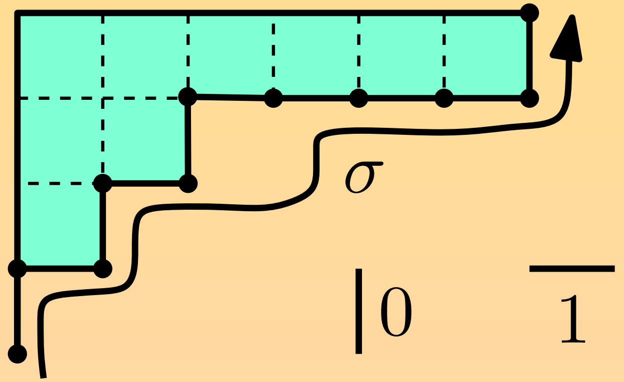

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For two words σ, σ' with $|\sigma|_0 = |\sigma'|_0$ and $|\sigma|_1 = |\sigma'|_1$ we define :

- $\sigma \leq \sigma'$ if, as shapes, σ is included in σ'.
- $\sigma \to \sigma'$ if $\sigma \leq \sigma'$, and σ' has at most one more box in each column ; σ, σ' form a horizontal strip.

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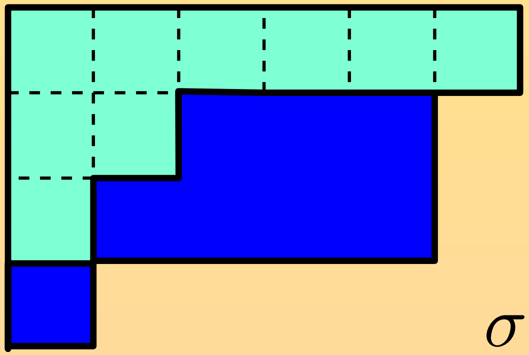

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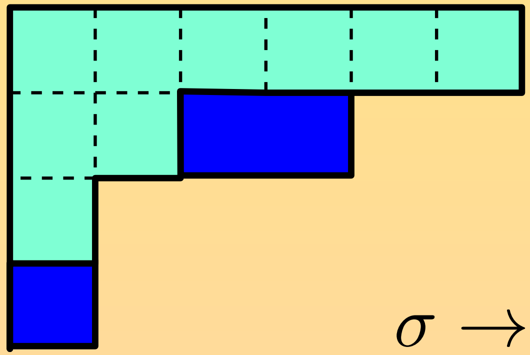
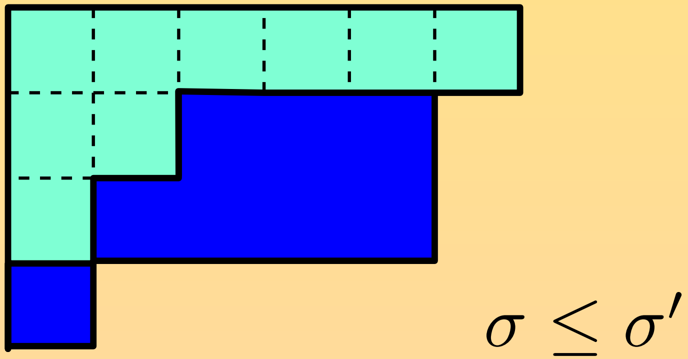
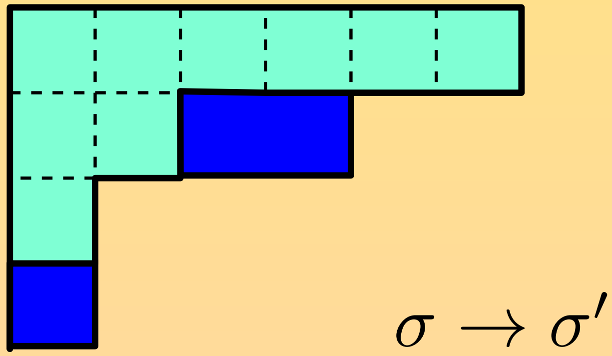


$\sigma \rightarrow \sigma'$

Words and Shapes





Definition

> A semi standard Young tableau of shape σ and entries bounded by N is a filling of the shape σ by integers in $\{1, \ldots, N\}$ such that entries are strictly increasing in columns and weakly increasing in rows.

Such a tableau can be equivalently defined by a sequence of shapes

$$\emptyset = \sigma_0 \to \sigma_1 \to \ldots \to \sigma_N = \sigma$$

Words and Shapes

Given a box u in a Ferrers diagram, in the ith row from the top and jth column form the left, we define

- the content $c(u) := j - i$;

0	1	2
-1	0	

- the hook-length $h(u)$ as the number of boxes below it, or to its right, including the u itself.

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Theorem [Stanley]

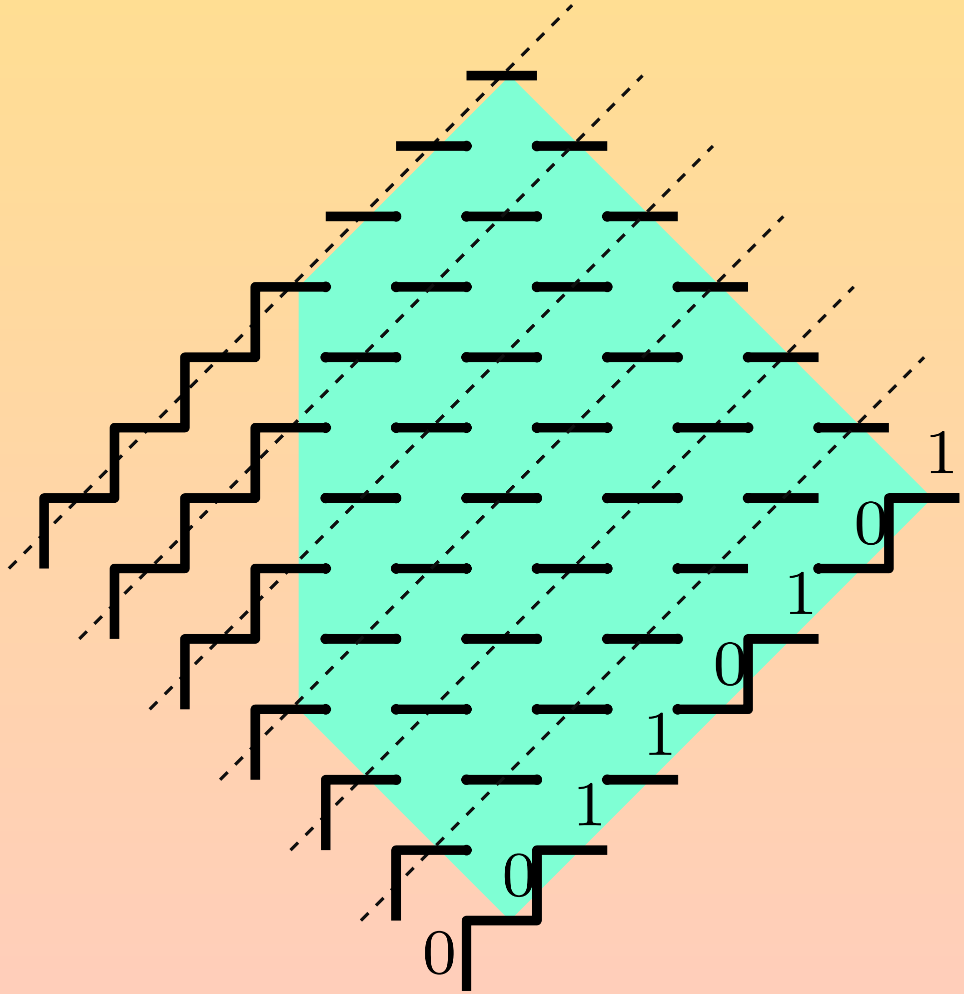
The number of semistandard Young tableaux of shape λ and entries bounded by N is given by

$$SSYT(\lambda, N) = \prod_{u \in \lambda} \frac{N + c(u)}{h(u)}$$

Polynomial of with leading term $\frac{1}{h(\lambda)} N^{\ell(\lambda)}$

Regions $\mathcal{R}_1$ and $\mathcal{R}_2$

Consider the region $\mathcal{R}_1$ with border σ, and extend it to make it rectangular.



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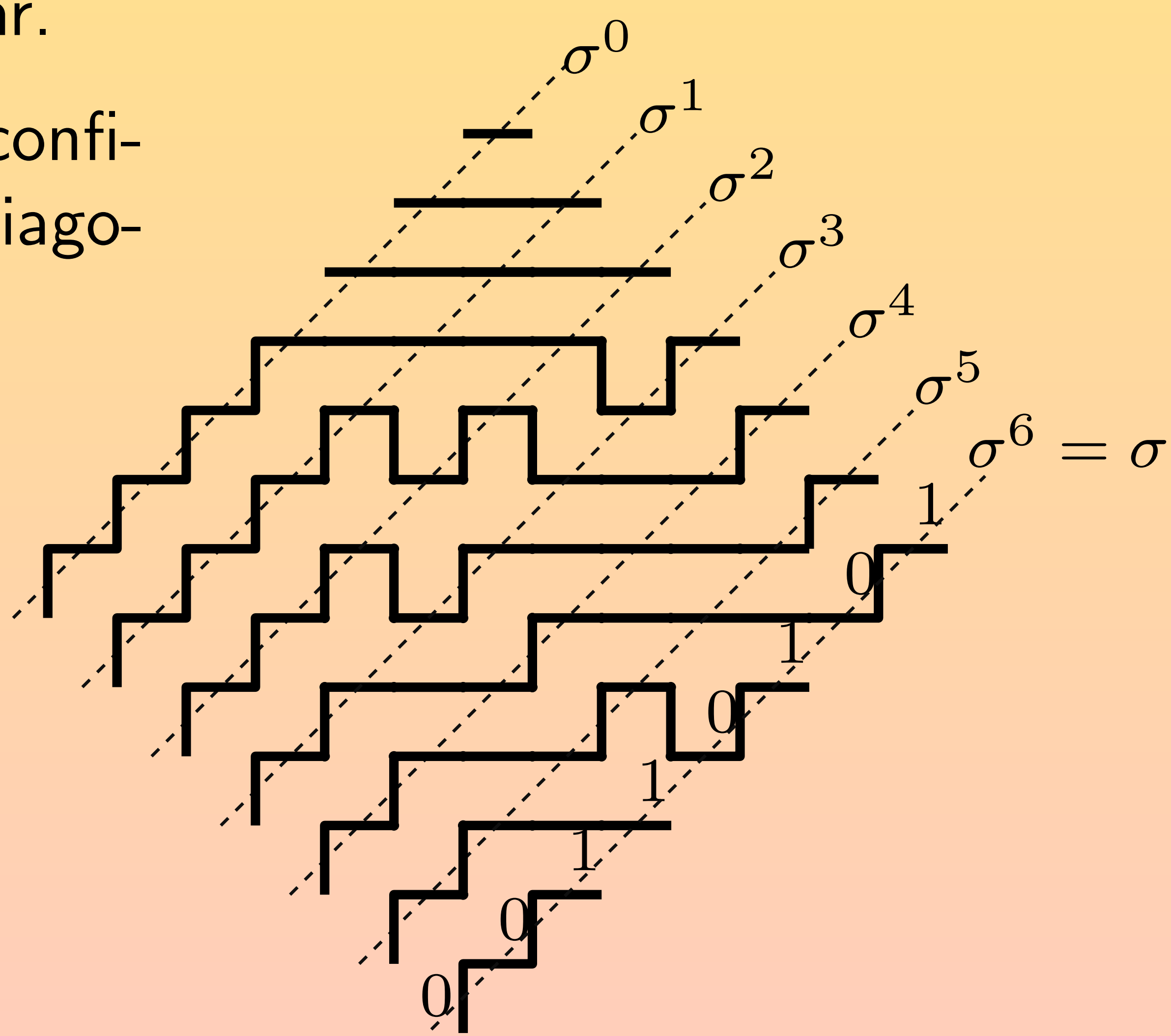
Given the restriction of a FPL configuration in $\mathcal{R}_1$, encode each diagonal by a word σ^i on $\{0, 1\}$.



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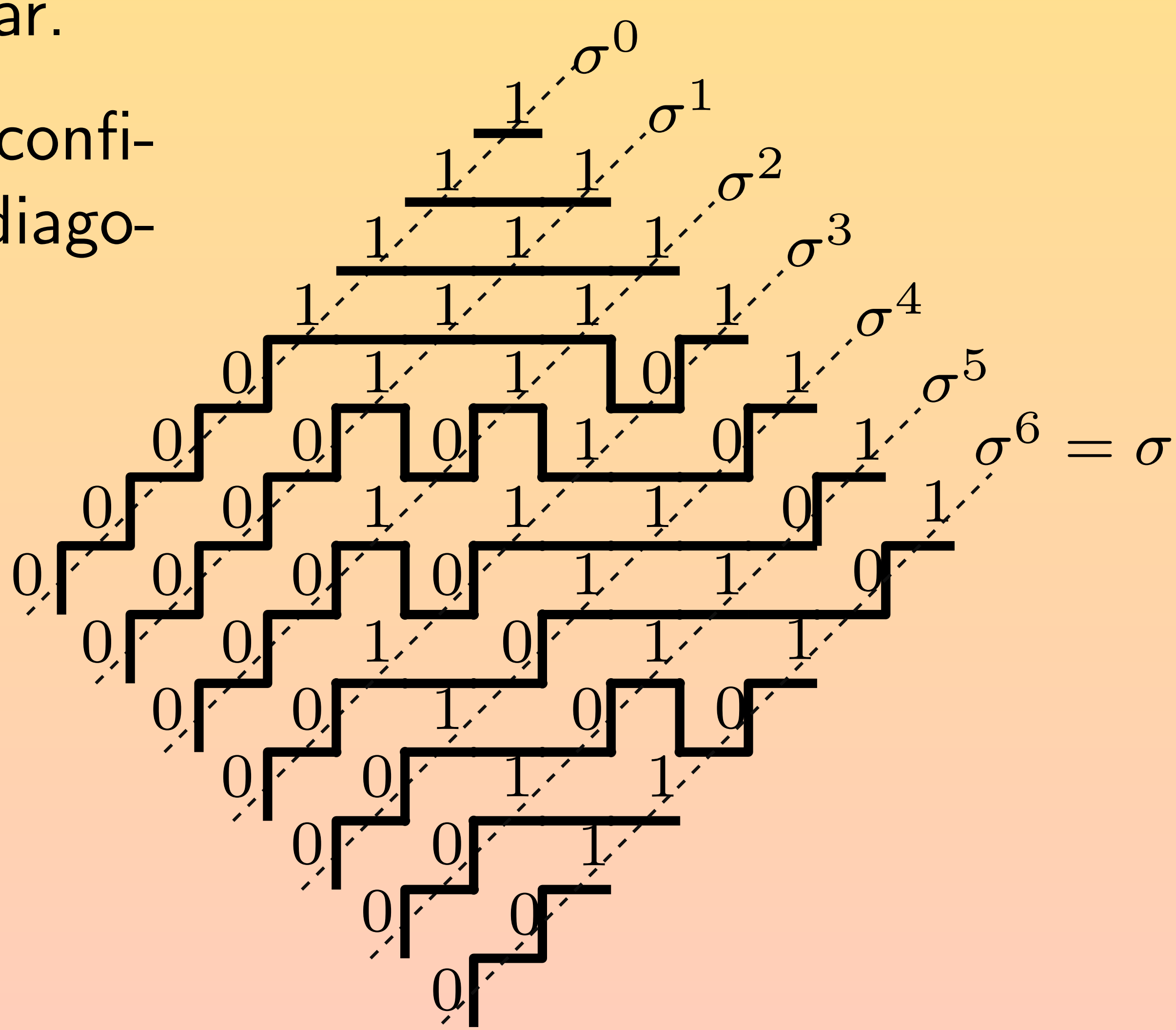

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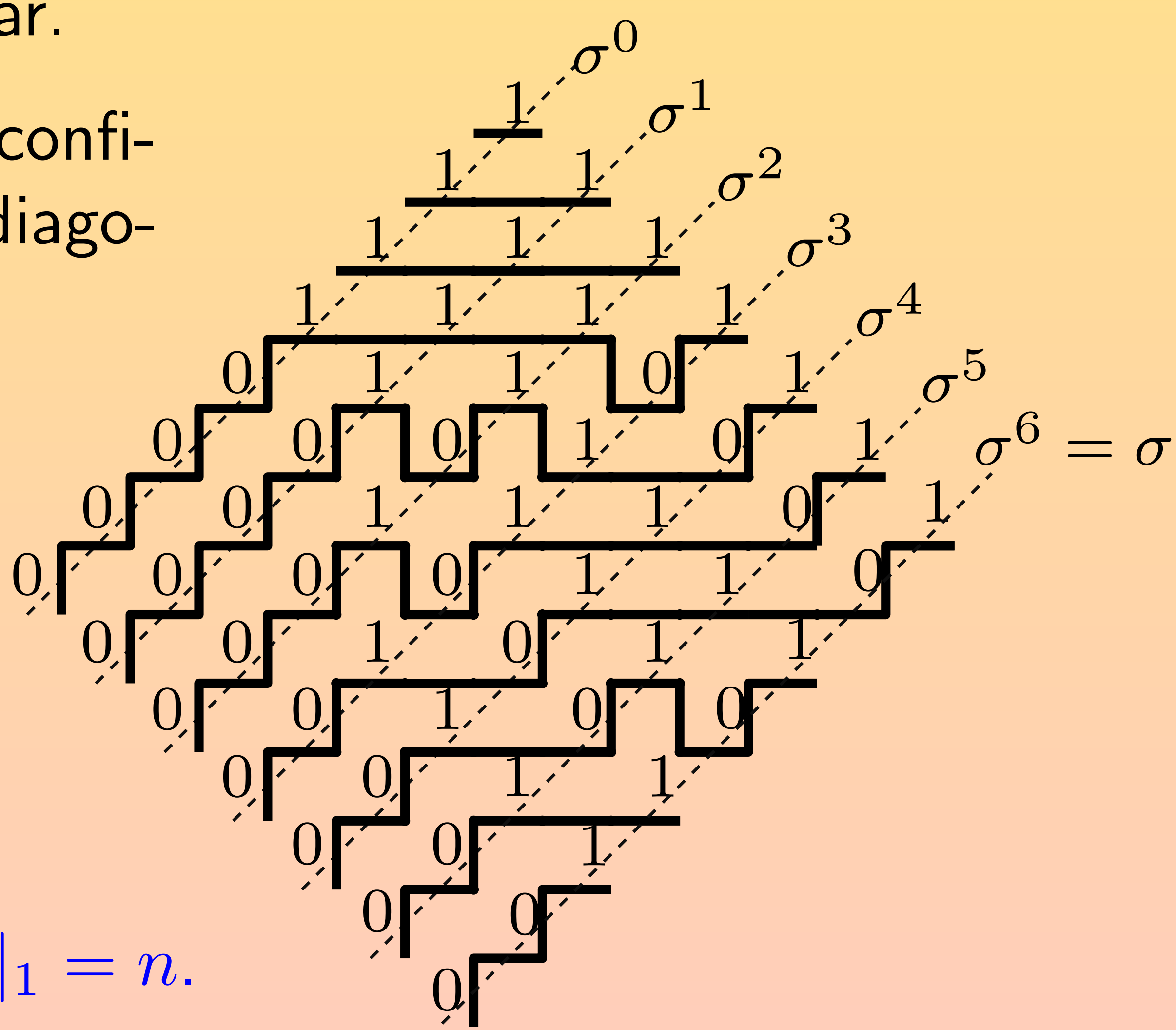

All words σ^i verify $|\sigma^i|_0 = |\sigma^i|_1 = n$.

Regions $\mathcal{R}_1$ and $\mathcal{R}_2$

Proposition [CKLN '05]

> For any FPL configuration in $\mathcal{R}^1$, the sequence of shapes $\sigma^0, \sigma^1, \ldots, \sigma^{n+k}$ form a semistandard Young tableau.
> This is a bijection between $\mathcal{R}_1(\sigma, k)$ and tableaux of shape σ and length $n + k$.

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So
$$A_\pi(m) = \sum_{\sigma,\tau} |\mathcal{R}_1(\sigma, 0)| \cdot t^{\pi}_{\sigma,\tau} \cdot |\mathcal{R}_2(\tau, m-3n+1)|$$
$$= \sum_{\sigma,\tau} SSYT(\sigma, n) \cdot t^{\pi}_{\sigma,\tau} \cdot SSYT(\tau^*, m-2n+1)$$

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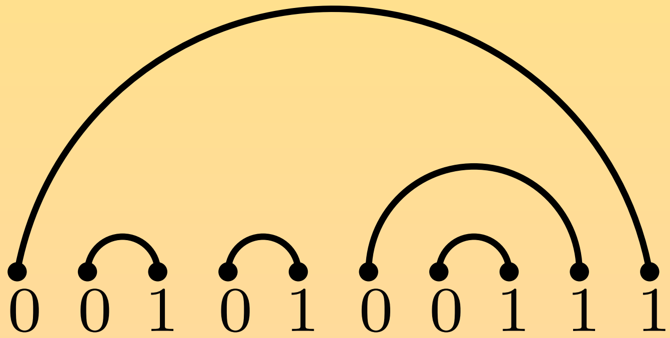
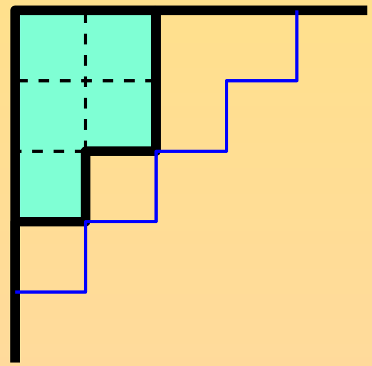
In fact, $A_\pi(m)$ is given by the same polynomial for $m < 3n-1$ [CKLN '05].

Given a noncrossing matching π of size n, we can associate to it a word, and thus a Ferrers shape :





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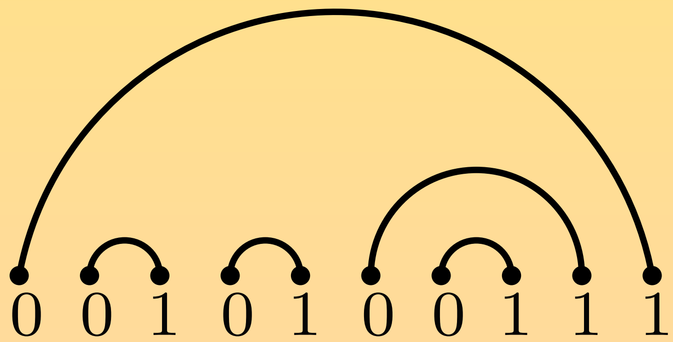

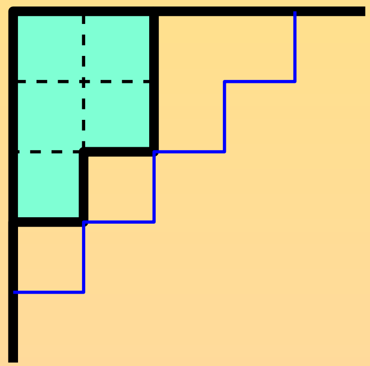

Definition We note $\mathcal{D}_n$ the words w such that $|w|_0 = |w|_1 = n$ and which are smaller than $(01)^n$.

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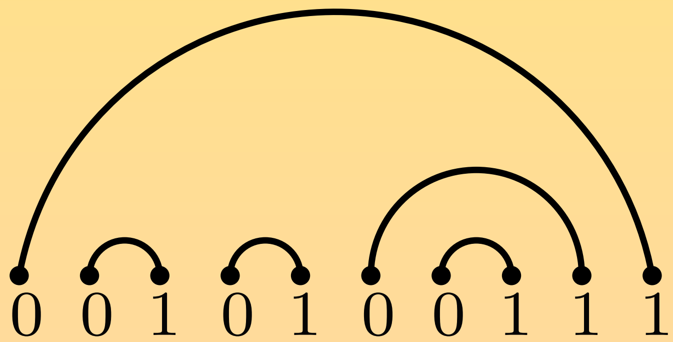

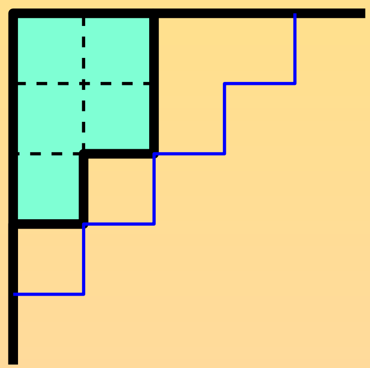

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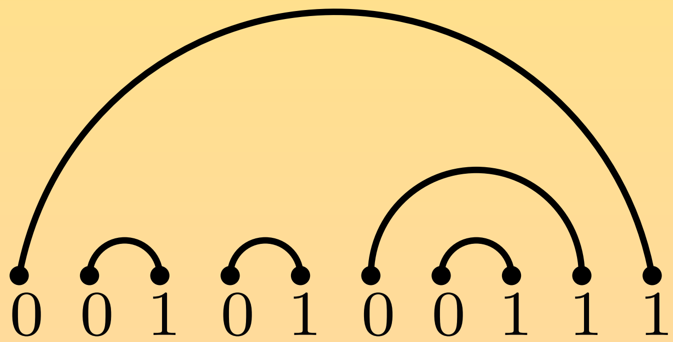

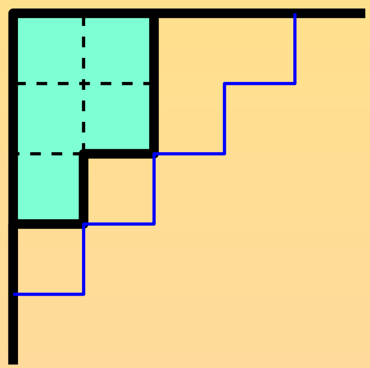

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Corollary

- The formula for $A_\pi(m)$ can be restricted to words $\sigma, \tau \in \mathcal{D}_n$,
- The polynomial $A_\pi(m)$ has leading term $\frac{1}{h(\pi)} t^{\ell(\pi)}$.

The decomposition formula

We want to write $A_\pi(m)$ as a $\mathbb{Z}$-linear combination of polynomials $A_\alpha(m-1)$, where α, π are in $\mathcal{D}_n$.

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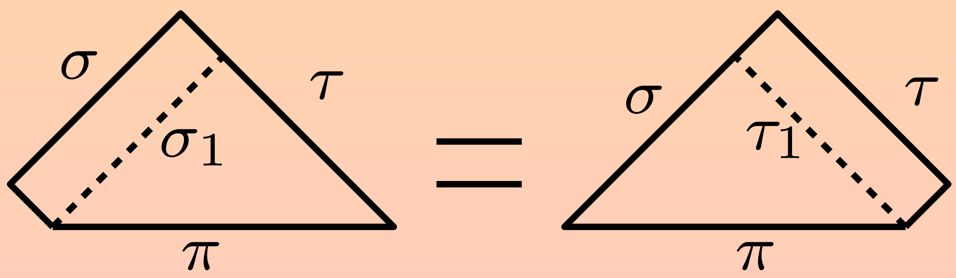
Theorem [N. '09] (conjectured in [Thapper '07]).

Let σ, τ, π be elements of $\mathcal{D}_n$. Then we have the equality :

$$\sum_{\substack{\sigma_1 \in \mathcal{D}_n \\ \sigma \to \sigma_1}} t^{\pi}_{\sigma_1, \tau} = \sum_{\substack{\tau_1 \in \mathcal{D}_n \\ \tau^* \to \tau_1^*}} t^{\pi}_{\sigma, \tau_1}.$$

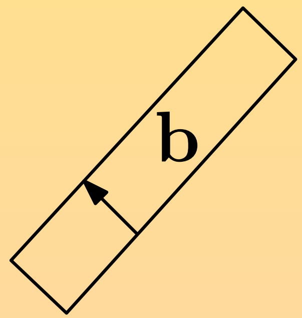
In terms of diagrams, this means precisely that



The proof is an application of Wieland's rotation.

The decomposition formula

We now define certain matrices endomorphisms $\mathbf{b}, \widetilde{\mathbf{b}}, \mathbf{t}^{\pi}$ acting on the complex vector space with distinguished basis $\mathcal{D}_n$.



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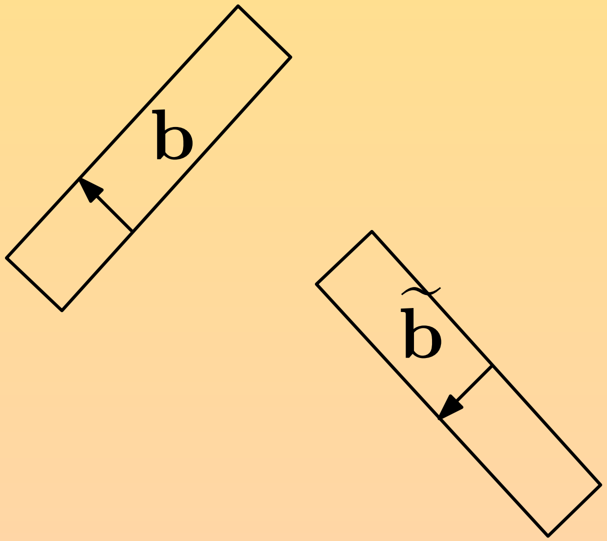

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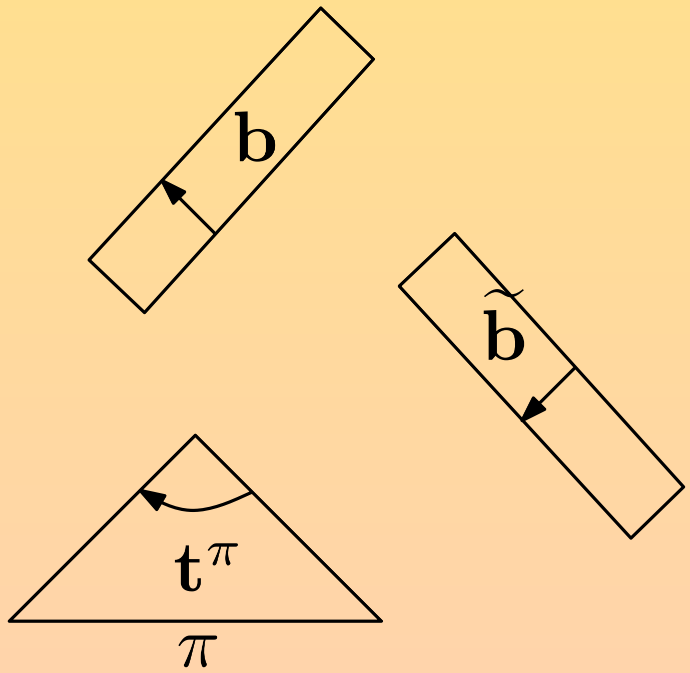

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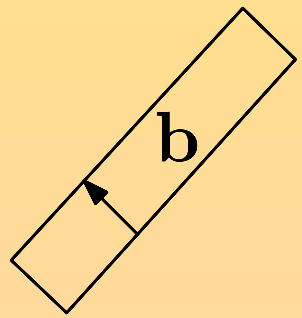

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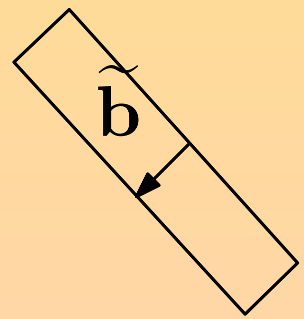

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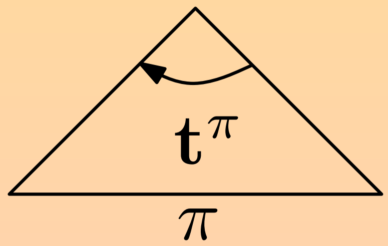
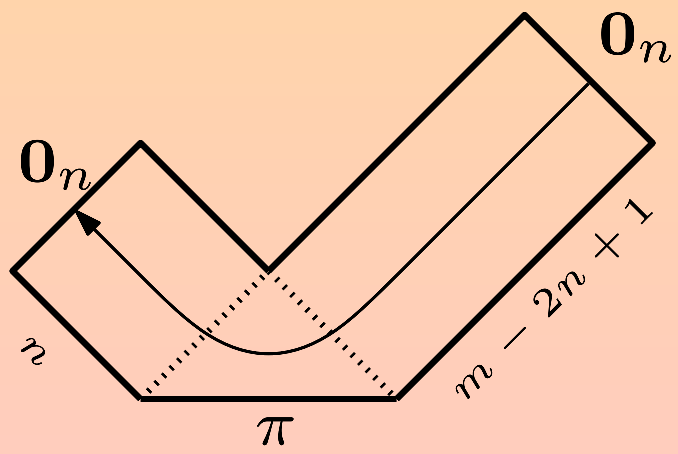


Putting these pieces together we get

$$A_\pi(m) = \left(\mathbf{b}^n \mathbf{t}^\pi \widetilde{\mathbf{b}}^{m-2n+1}\right)_{\mathbf{0}_n \mathbf{0}_n}$$



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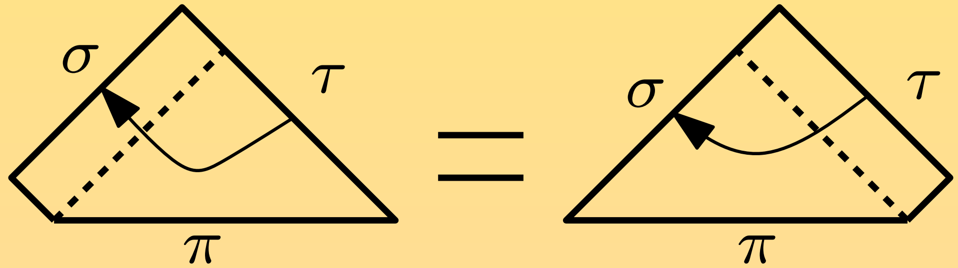
Now, the relation



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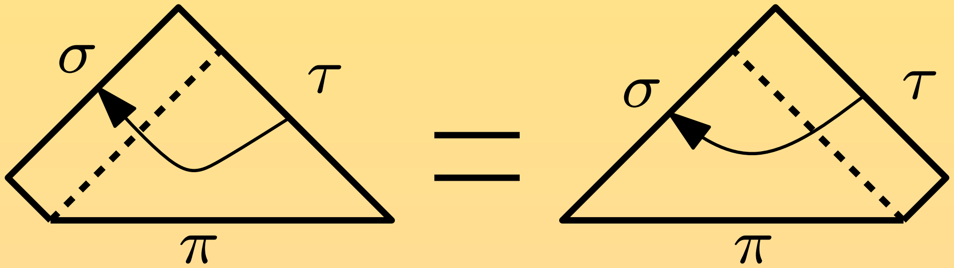

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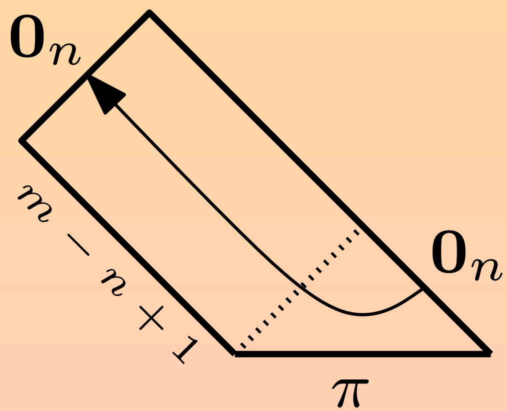
$$A_{\pi}(m) = \left(\mathbf{b}^{m-n+1}\mathbf{t}^{\pi}\right)_{\mathbf{0}_n\mathbf{0}_n}$$



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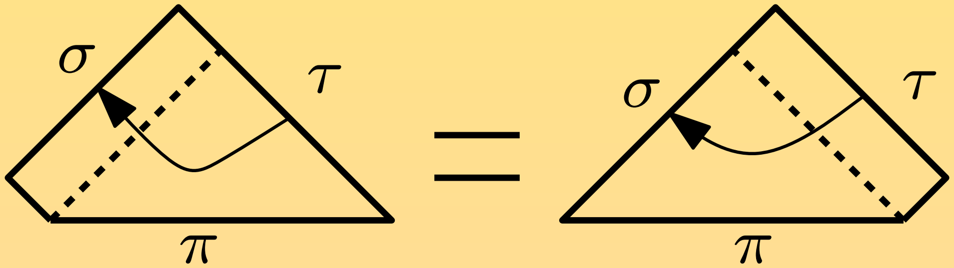

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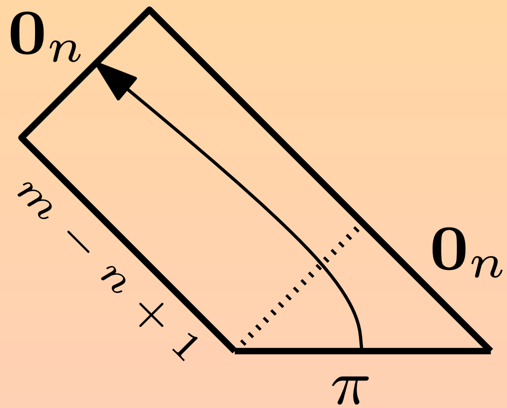
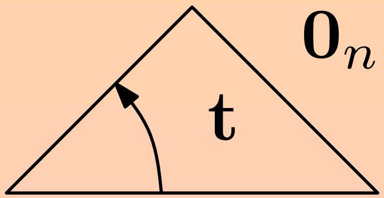


Defining $(\mathbf{t})_{\sigma\pi} := t^\pi_{\sigma,\mathbf{0}_n}$



we can rewrite this as $A_\pi(m) = \left(\mathbf{b}^{m-n+1}\mathbf{t}\right)_{\mathbf{0}_n \pi}$

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Proposition [N. '09]

For $\pi \in \mathcal{D}_n$, and m an integer

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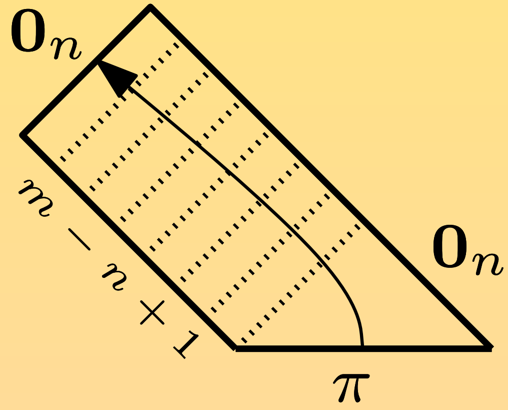

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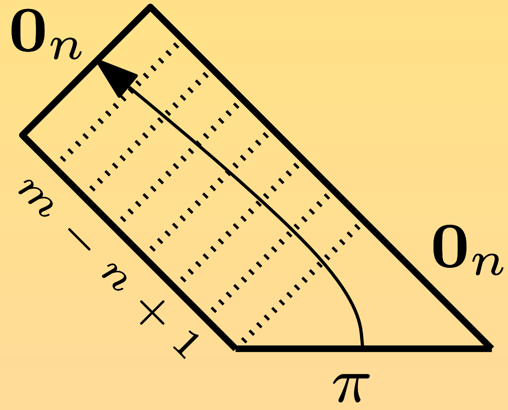

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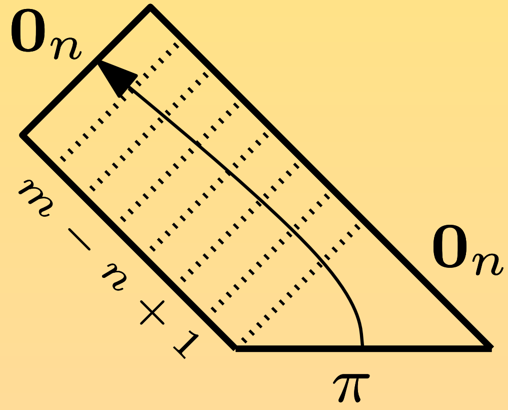

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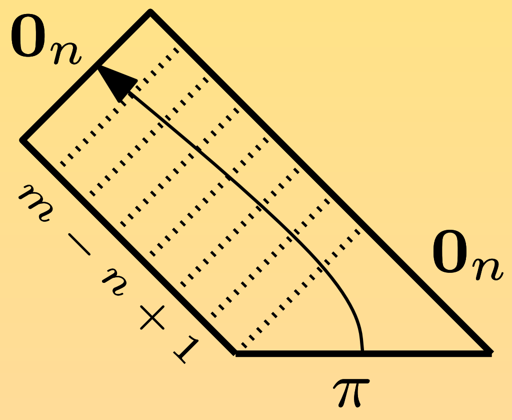

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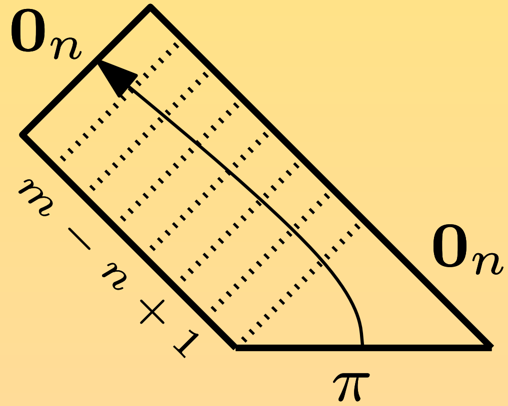

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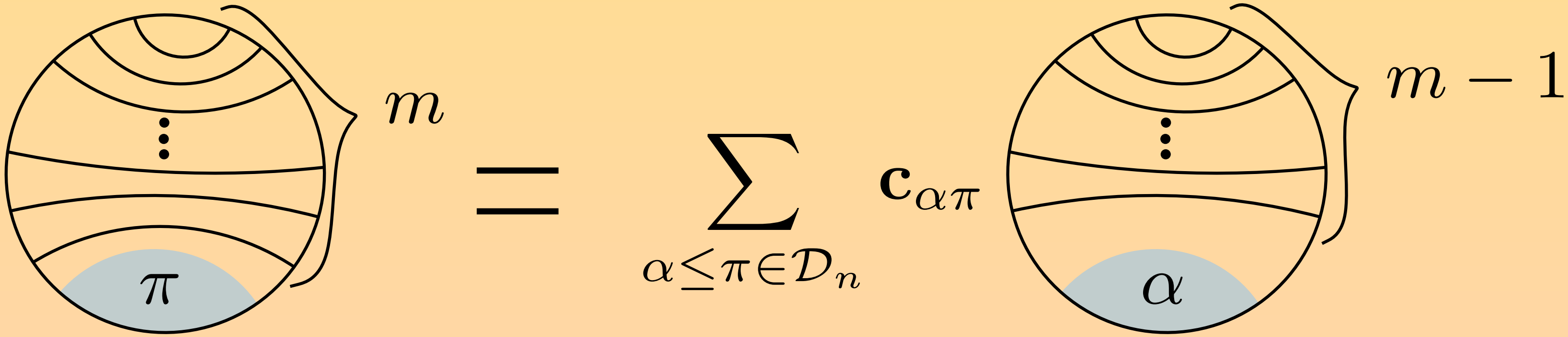

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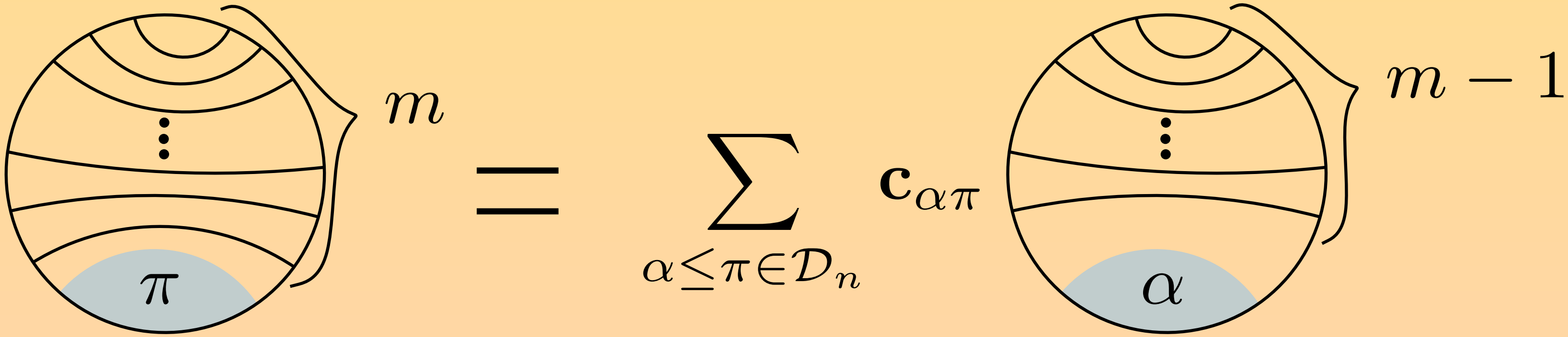
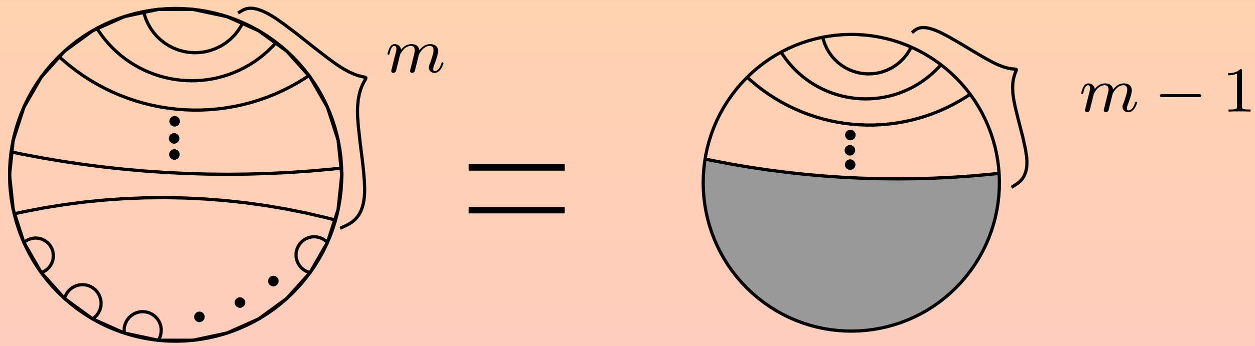


Conjecture [Thapper]

If $\pi = \mathbf{1}_n$, then $\mathbf{c}_{\alpha \mathbf{1}_n} = 1$ for any $\alpha \in \mathcal{D}_n$.

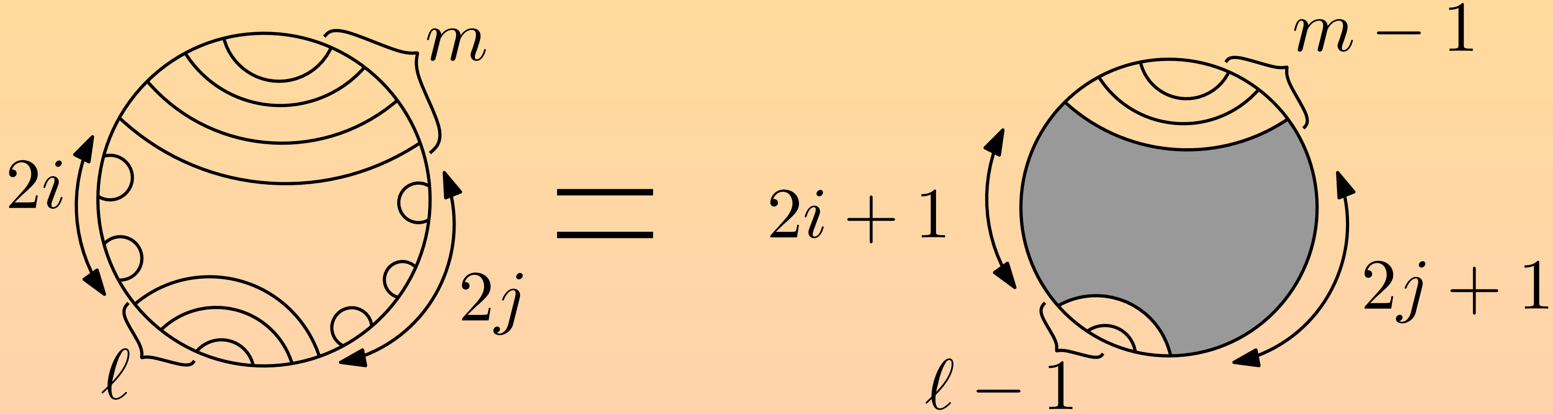
This implies



The decomposition formula

Some remarks on the coefficients $c_{\alpha\pi}$: there are of course not the unique numbers such that the previous theorem holds. But, based on data for small n, these numbers conjecturally :

- give nice decomposition formulas, for instance :

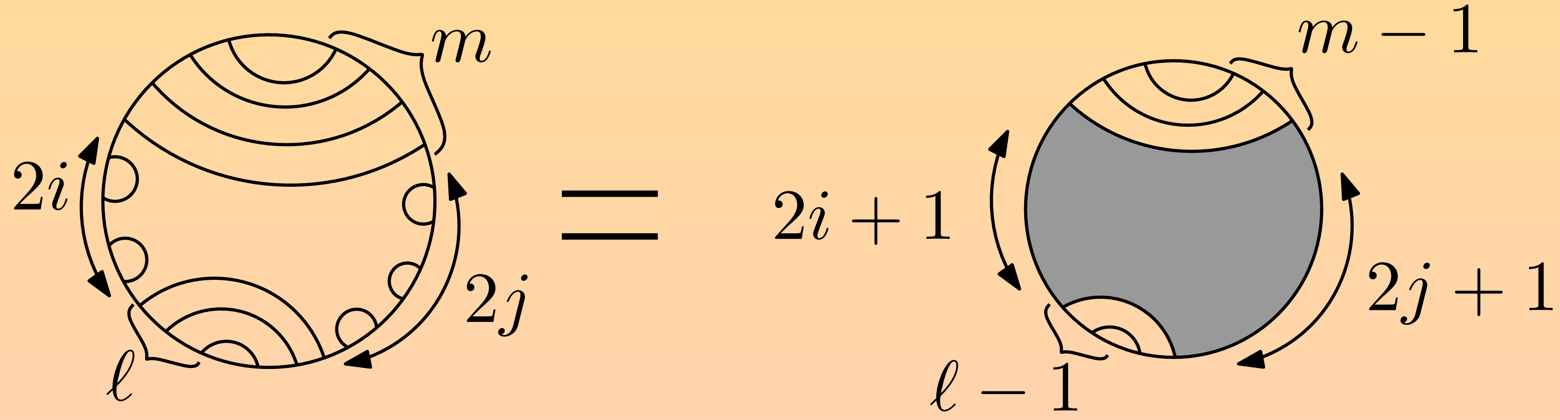


- verify $c_{\alpha\pi} = c_{\alpha^*\pi^*}$ and $c_{0\alpha 1, 0\pi 1} = c_{\alpha\pi}$.

The decomposition formula

Some remarks on the coefficients $c_{\alpha\pi}$: there are of course not the unique numbers such that the previous theorem holds. But, based on data for small n, these numbers conjecturally :

- give nice decomposition formulas, for instance :

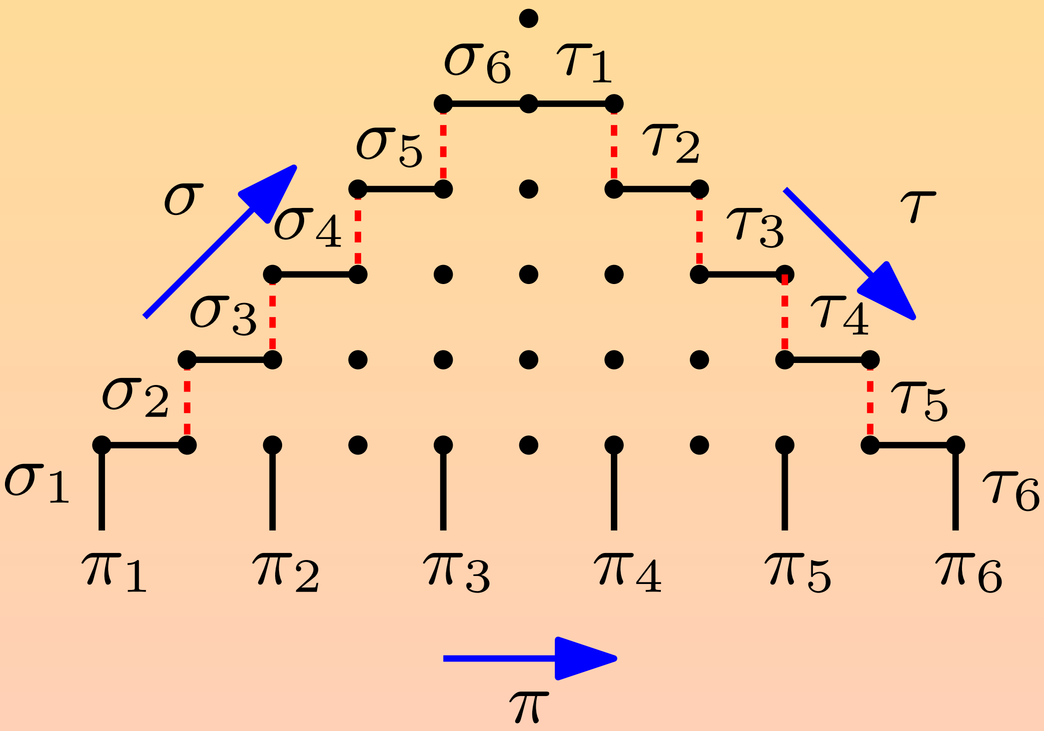


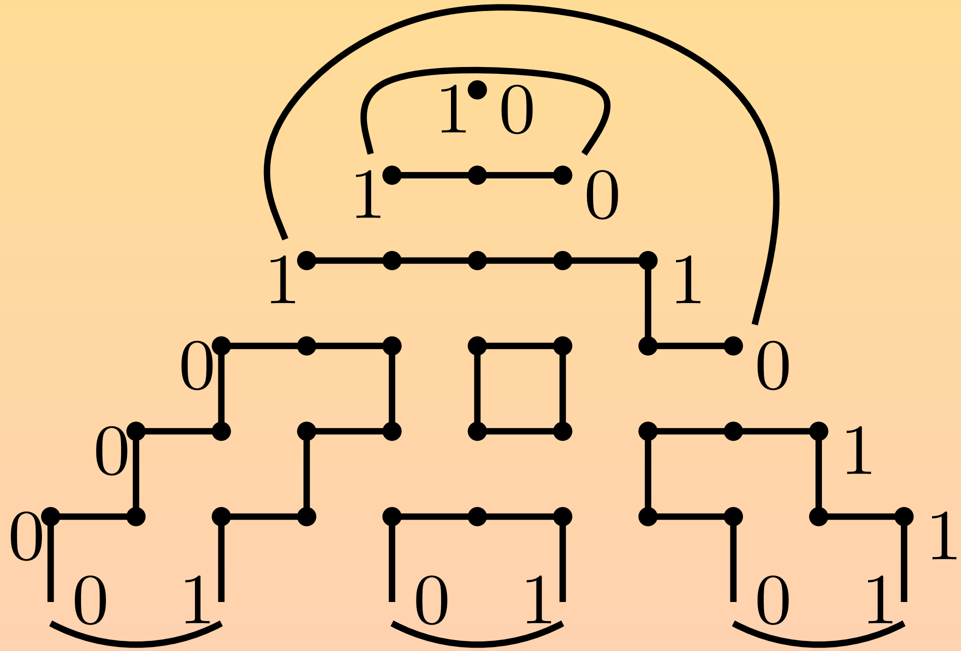
- verify $c_{\alpha\pi} = c_{\alpha^*\pi^*}$ and $c_{0\alpha 1, 0\pi 1} = c_{\alpha\pi}$.

Challenge : conjecture a direct combinatorial description of these coefficients.

The triangle $\mathcal{T}_n$

We now study the FPL configurations in the triangle, in short TFPL configurations. Given the boundary data σ, π, τ, we want to compute $t^{\pi}_{\sigma,\tau}$ which is the number of TFPL configurations with these boundaries.





The numbers $t^{\pi}_{\sigma,\tau}$

Proposition

$$t^{\pi}_{\sigma,\tau} = t^{\pi^*}_{\tau^*,\sigma^*}.$$

Proposition

$$\sum_{\substack{\sigma_1 \in \mathcal{D}_n \\ \sigma \to \sigma_1}} t^{\pi}_{\sigma_1,\tau} = \sum_{\substack{\tau_1 \in \mathcal{D}_n \\ \tau^* \to \tau_1^*}} t^{\pi}_{\sigma,\tau_1}.$$

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Theorem [CKLN '04]

$t^{\pi}_{\sigma,\tau} = 0$ unless $\sigma \leq \pi$.

Sketch of the proof : the idea is to show that there exist integers $N_i(f) \geq 0$ (with $N_0(f) = 0$) attached to a TFPL configuration f, such that if f has boundary data σ, π, τ, then $\sigma_i - \pi_i = N_i(f) - N_{i-1}(f)$ for all i.

Common prefixes and suffixes

Now we study the case where σ and π have common prefixes and suffixes.

Proposition

Let $\pi, \sigma, \tau \in \mathcal{D}_n$, and suppose that there exist words u, σ', π', v such that $\sigma = u\sigma' v$ and $\pi = u\pi' v$. Define a, b by $n - a = |u|_0 + |v|_0$ and $n - b = |u|_1 + |v|_1$.
Then $t^{\pi}_{\sigma,\tau} = 0$ unless $\tau = 0^{n-a}\tau' 1^{n-b}$ for a certain τ'.

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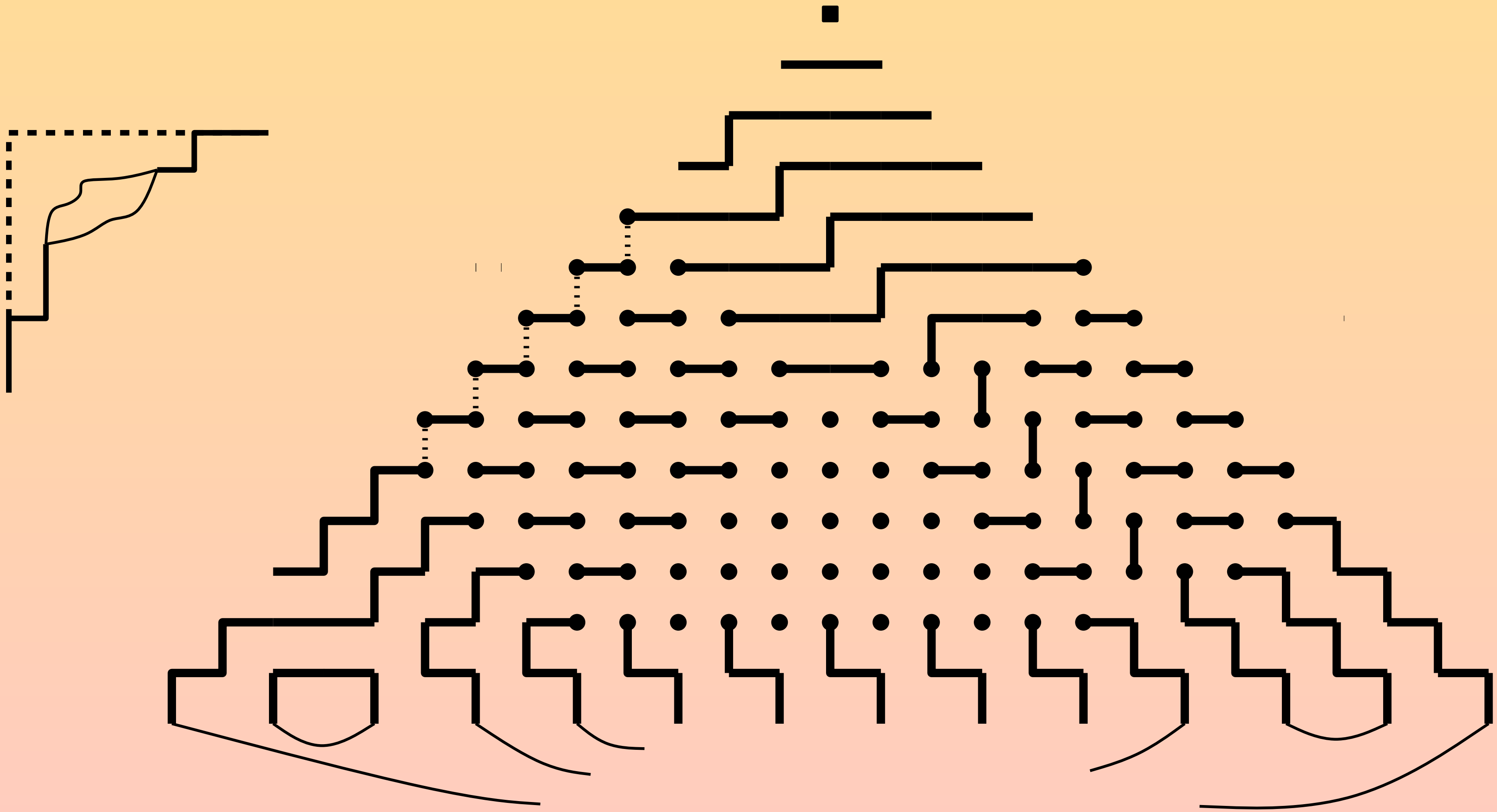
Proposition

If in addition $\pi' = 1^b 0^a$, then $t^{\pi}_{\sigma,\tau}$ is given by the determinant of a matrix of size a (or b) with entries given by certain binomial coefficients.

Common prefixes and suffixes

Proof : lots of fixed edges.

Case where σ and π have common prefix and suffix.



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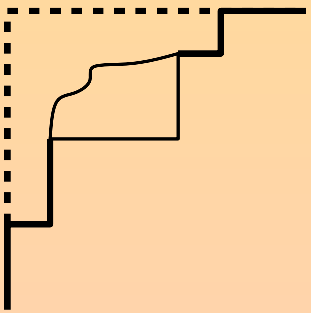

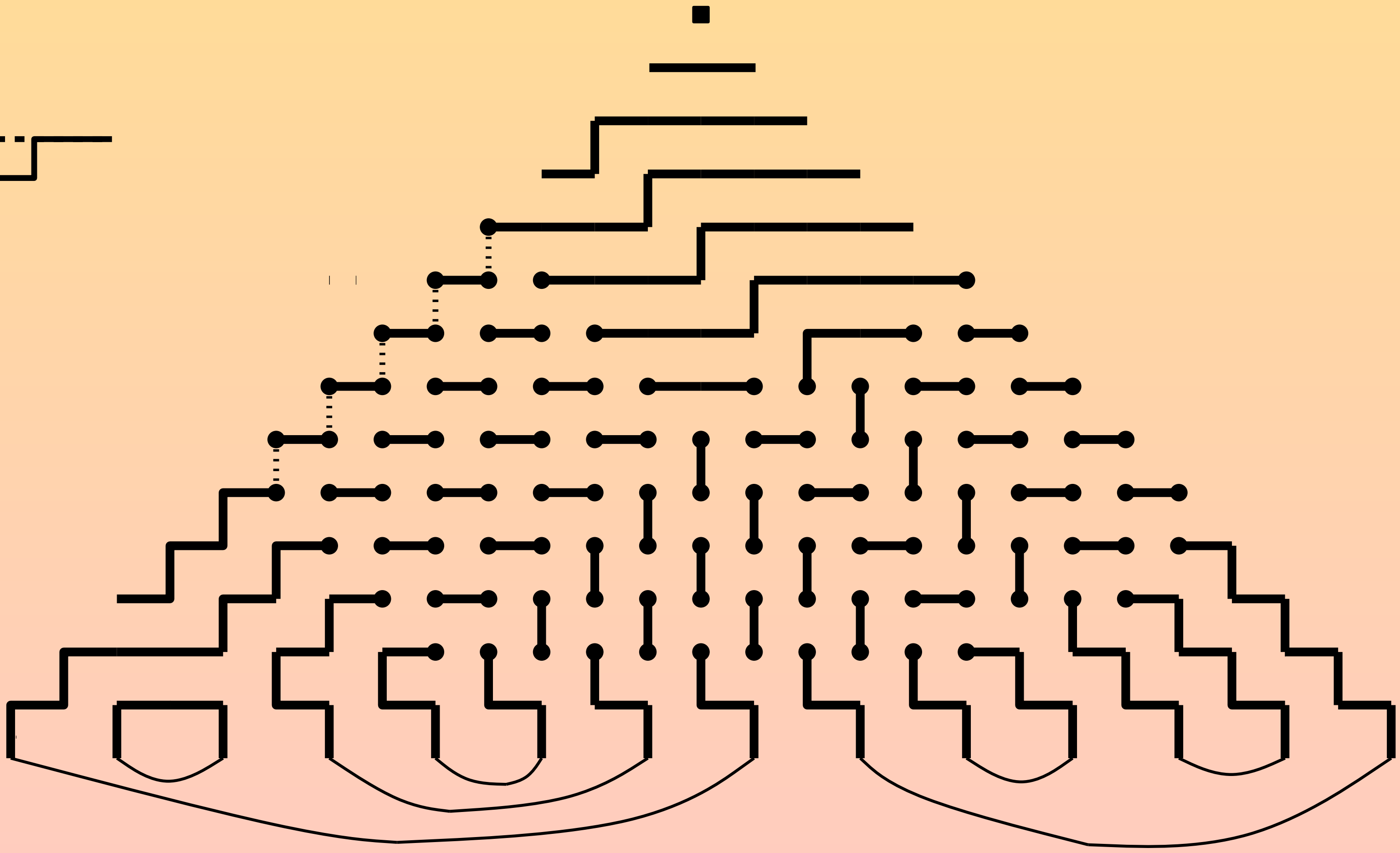

Triangles and Littlewood-Richardson coefficients

Thapper proved the following :

$$t^{\pi}_{\sigma,\tau} = 0 \text{ unless } \ell(\sigma) + \ell(\tau) \leq \ell(\pi).$$

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Following his idea, we can say something about the case of equality :

Proposition For every $\pi \in \mathcal{D}_n$

$$\frac{1}{h(\pi)} = \sum_{\substack{\sigma,\tau\in\mathcal{D}_n \\ \ell(\sigma)+\ell(\tau)=\ell(\pi)}} t^{\pi}_{\sigma,\tau} \cdot \frac{1}{2^{\ell(\sigma)}h(\sigma)} \cdot \frac{1}{2^{\ell(\tau)}h(\tau)}$$

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Sketch of the proof : remember that $A_{\pi}(m)$ is a polynomial of degree $\ell(\pi)$ and leading coefficient $1/h(\pi)$. It can be written as

$$\sum_{\sigma,\tau} t^{\pi}_{\sigma,\tau} \cdot SSYT(\sigma, n+k) \cdot SSYT(\tau^*, m+1-k-2n).$$

Choose $k = m/2$, and compare the coefficients in degrees $\ell(\pi)$ and higher to get the formula.

Littlewood Richardson coefficients

Let λ, μ, ν be partitions, and $\Lambda(x)$ be the ring of symmetric functions of the variables $x_1, x_2, \ldots$. The Schur functions $s_\lambda(x)$ can be defined as

$$s_\lambda(x) = \sum_T x_i^{T_i},$$

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Schur functions form a basis of $\Lambda(x)$. We can expand $s_\mu(x)s_\nu(x)$ on this basis, where the coefficients $c^\lambda_{\mu,\nu}$ are often called the Littlewood-Richardson (LR) coefficients.

$$s_\mu(x)s_\nu(x) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda(x)$$

Since all terms in s_λ have degree $\ell(\lambda)$, we get

$$c^\lambda_{\mu,\nu} = 0 \text{ unless } \ell(\lambda) = \ell(\mu) + \ell(\nu).$$

Littlewood Richardson coefficients

These coefficients appear in other places in the theory of symmetric functions ; we have for instance :

$$s_{\lambda/\mu}(x) = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_{\nu}(x)$$

We have also, if $s_\lambda(x,y)$ is the symmetric function s_λ in the variables $x_1, x_2, \ldots, y_1, y_2, \ldots$

$$s_\lambda(x,y) = \sum_{\mu,\nu} c^{\lambda}_{\mu,\nu} s_\mu(x) s_\nu(y)$$

If we evaluate this at $x_i = y_i = 1$ for $i = 1, \ldots, m/2$, $x_i = y_i = 0$ for $i > m/2$, we obtain polynomials in m which give the following identity in top degree $\ell(\lambda)$:

$$\frac{1}{h(\lambda)} = \sum_{\mu,\nu} c^{\lambda}_{\mu,\nu} \cdot \frac{1}{2^{\ell(\mu)} h(\mu)} \cdot \frac{1}{2^{\ell(\nu)} h(\nu)}$$

Littlewood Richardson coefficients

As a consequence, there exist $a_{\sigma\tau} > 0$ such that, for any $\pi \in \mathcal{D}_n$,

$$\sum_{\sigma,\tau} a_{\sigma\tau} c^{\pi}_{\sigma,\tau} = \sum_{\sigma,\tau} a_{\sigma\tau} t^{\pi}_{\sigma,\tau} \qquad (E)$$

in which σ, τ go through all words such that $\ell(\sigma) + \ell(\tau) = \ell(\pi)$

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Theorem [N. '09]

For all words $\pi, \sigma, \tau \in \mathcal{D}_n$ verifying $\ell(\sigma) + \ell(\tau) = \ell(\pi)$,
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Theorem [N. '09]

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Thanks to equation (E), we need only prove that $c^{\pi}_{\sigma,\tau} \leq t^{\pi}_{\sigma,\tau}$ for all valid σ, τ, π.

Computing LR coefficients

There are many objects that are counted by LR-coefficients. We use here Knutson-Tao puzzles.

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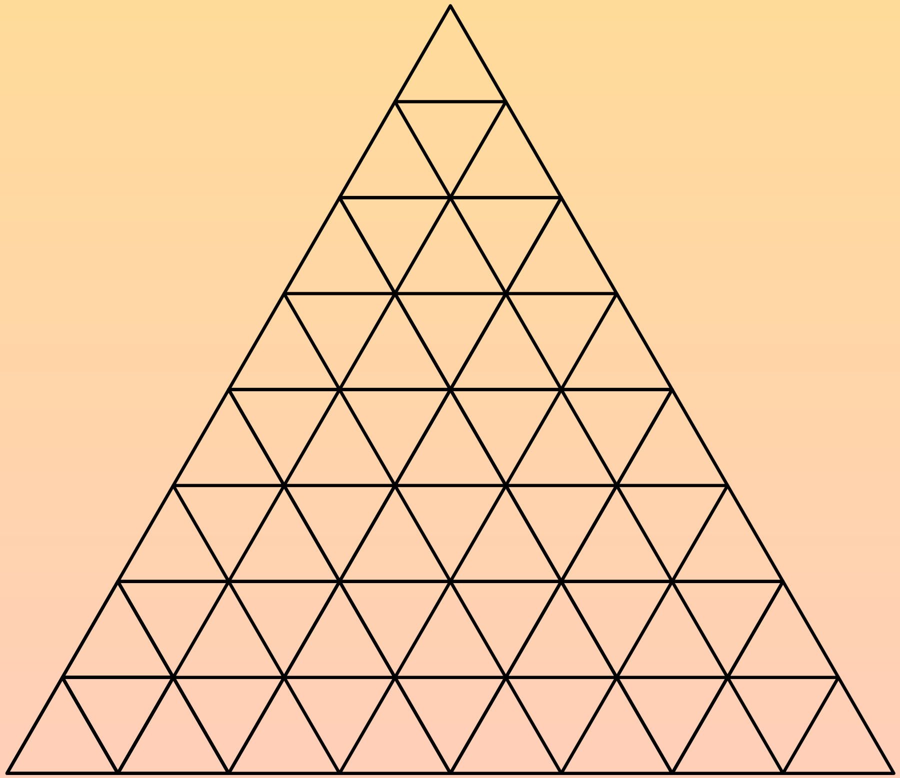


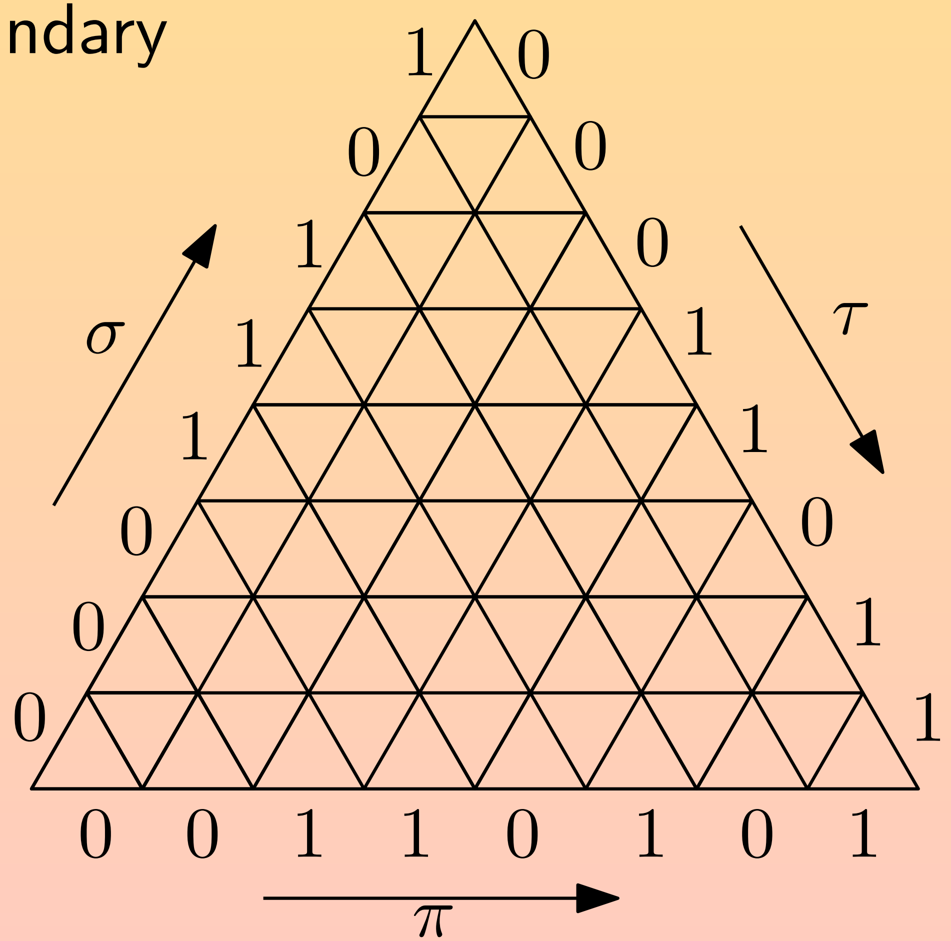
Computing LR coefficients

There are many objects that are counted by LR-coefficients. We use here Knutson-Tao puzzles.

Consider a triangle of size $2n$ on the triangular lattice.

Fix $\sigma, \pi, \tau \in \mathcal{D}_n$, and label the boundary edges of the triangle.

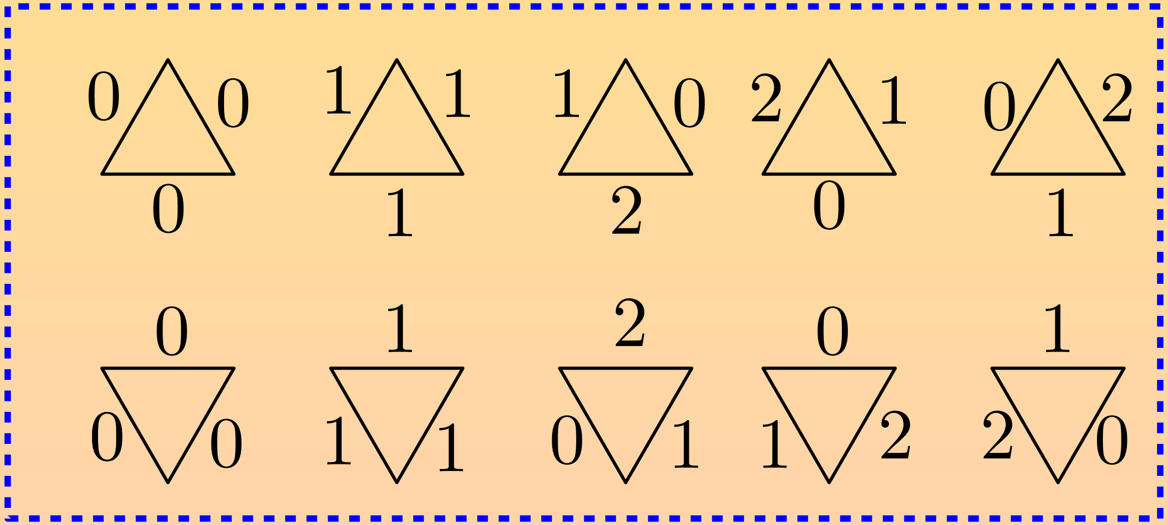
$$\pi = 00110101$$
$$\sigma = 00011011$$
$$\tau = 00011011$$

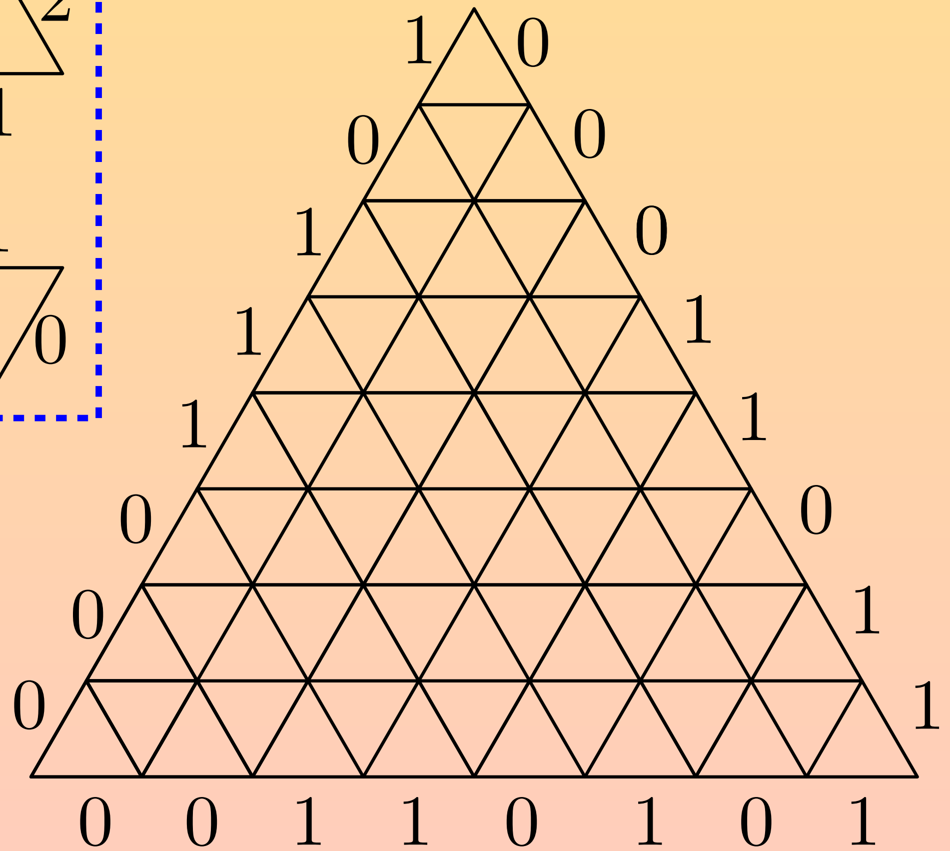


Definition

A Knutson-Tao puzzle with boundary data σ, π, τ is a labeling of each edge of the triangle by $0, 1$ or 2, such that :

- the labels on the boundary is given by σ, π, τ ;
- on each triangle, the induced labeling must be among :





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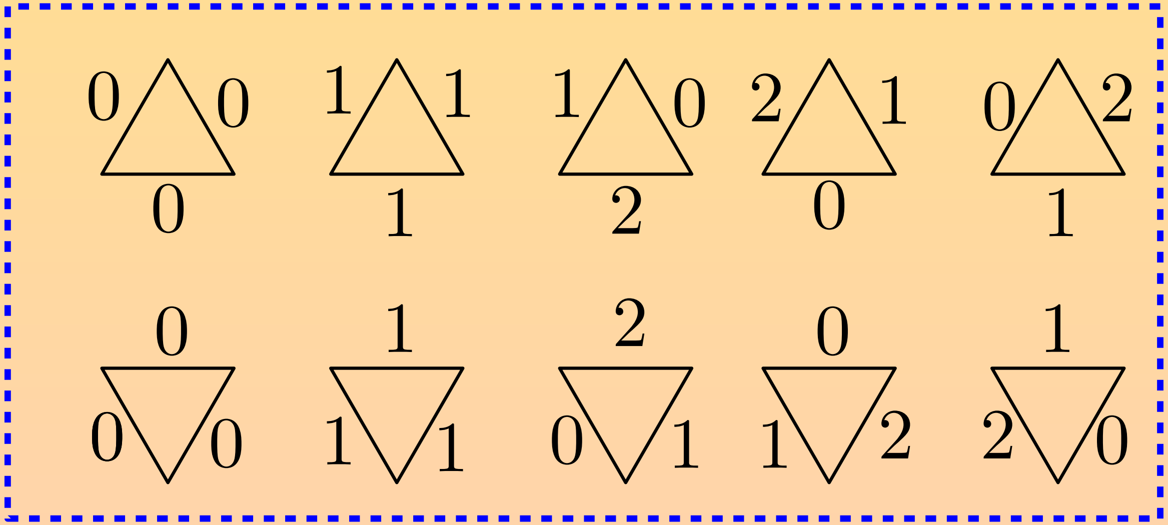
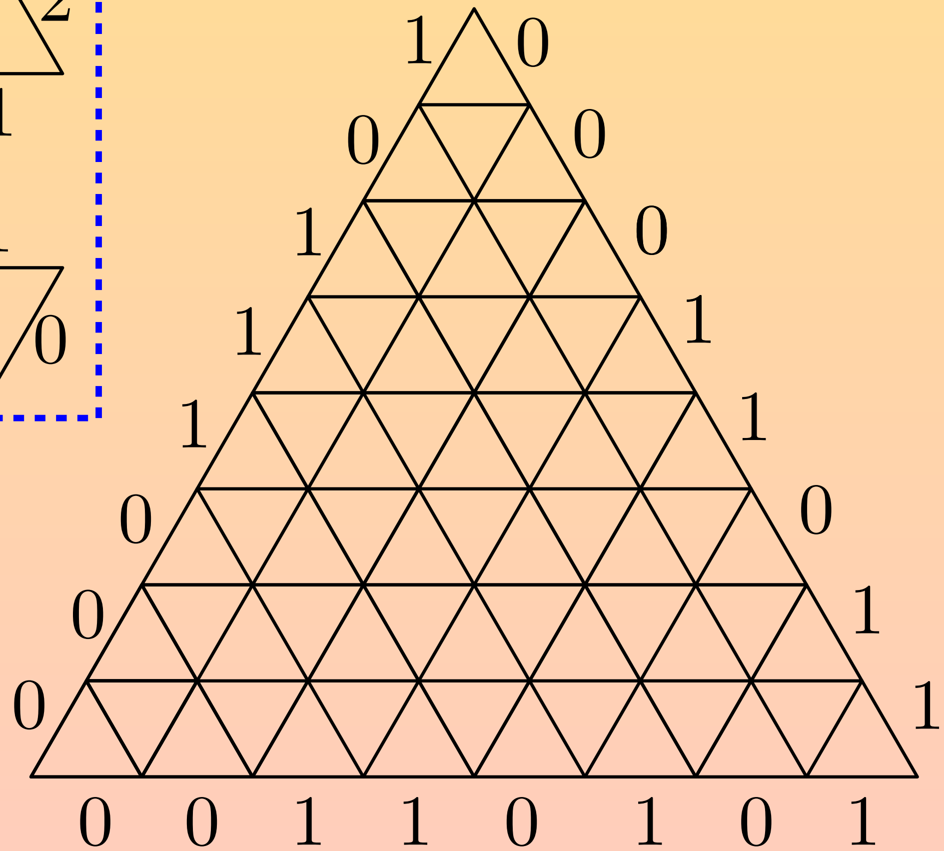




We will picture the labeling of edges as follows :

— label 0

▬ label 1

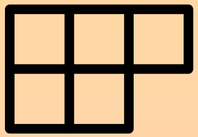
- - - - label 2

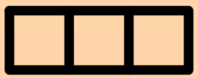
Theorem [Knutson, Tao '03][K., T. and Woodward '03]

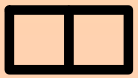
Let $\sigma, \tau, \pi \in \mathcal{D}_n$. Then the number of KT-puzzles with boundary data σ, π, τ is equal to the LR coefficient $c^{\pi}_{\sigma,\tau}$.

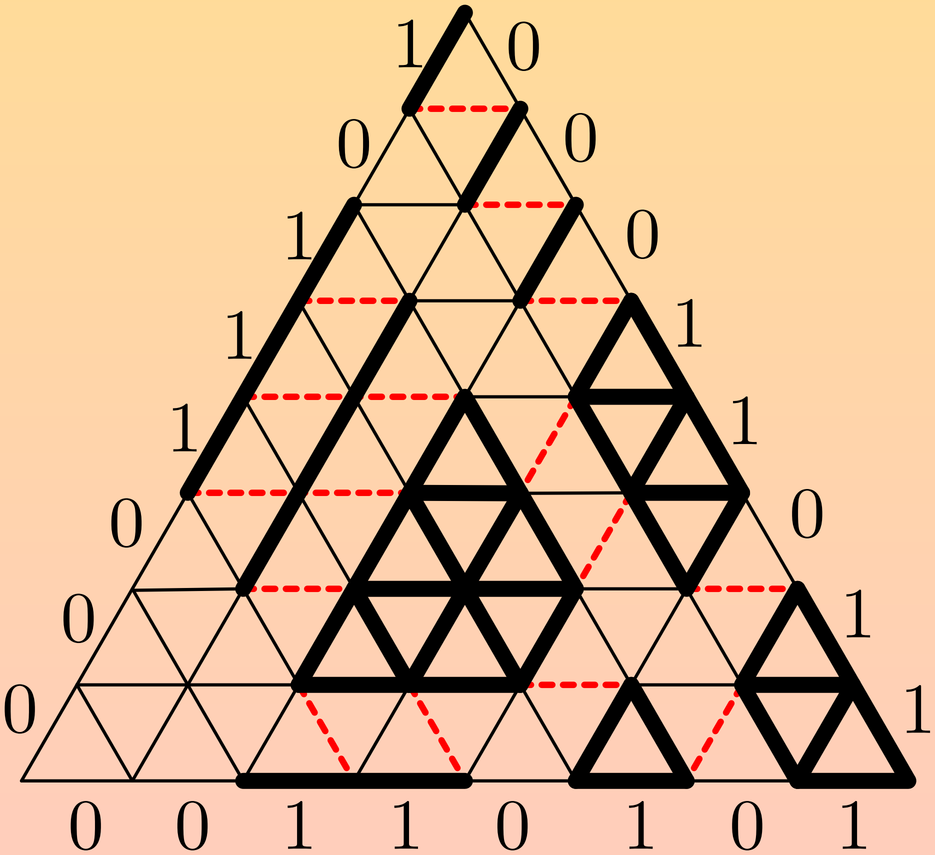
For example, it is easy to see that there is only one puzzle with the boundary data of the example.

so $c^{\lambda}_{\mu,\nu} = 1$ where

$\lambda =$ 

$\mu =$ 

$\nu =$ 



From KT puzzles to TFPL configurations.

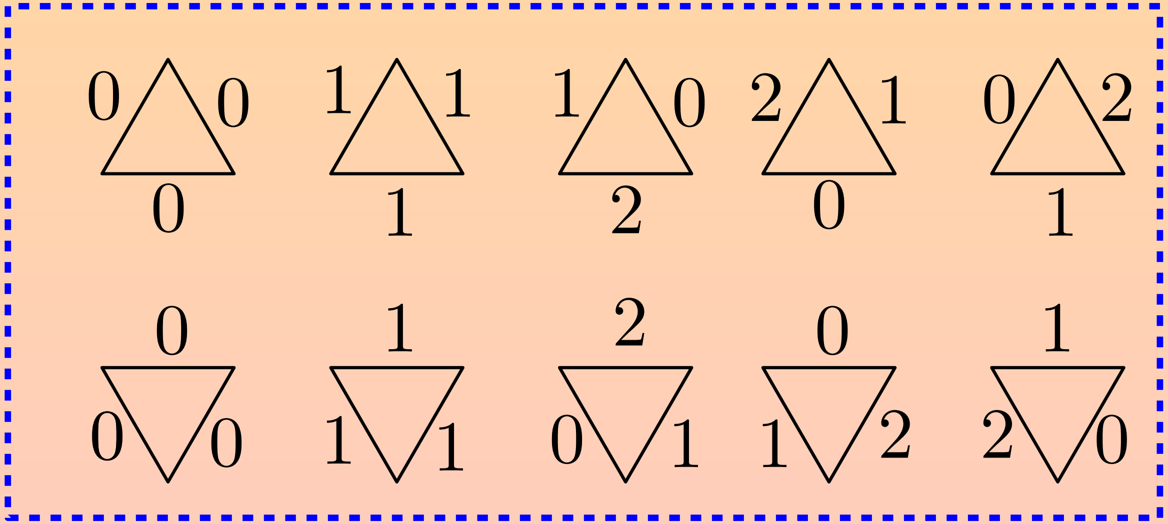
We fix $\sigma, \pi, \tau \in \mathcal{D}_n$, such that $\ell(\sigma) + \ell(\tau) = \ell(\pi)$. We will define a map Φ.

KT puzzles with boundary data σ, π, τ

↓

TFPL configurations with boundaries σ, τ, π

The map is local : it changes every small labeled triangle of the puzzle to a piece of a path of a TFPL configuration.



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$\downarrow \Phi$

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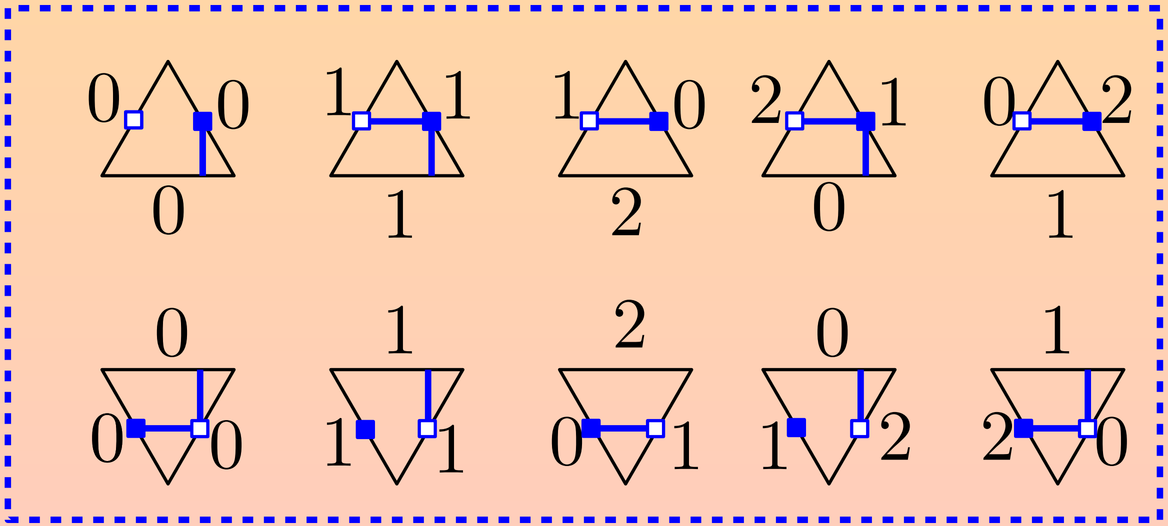

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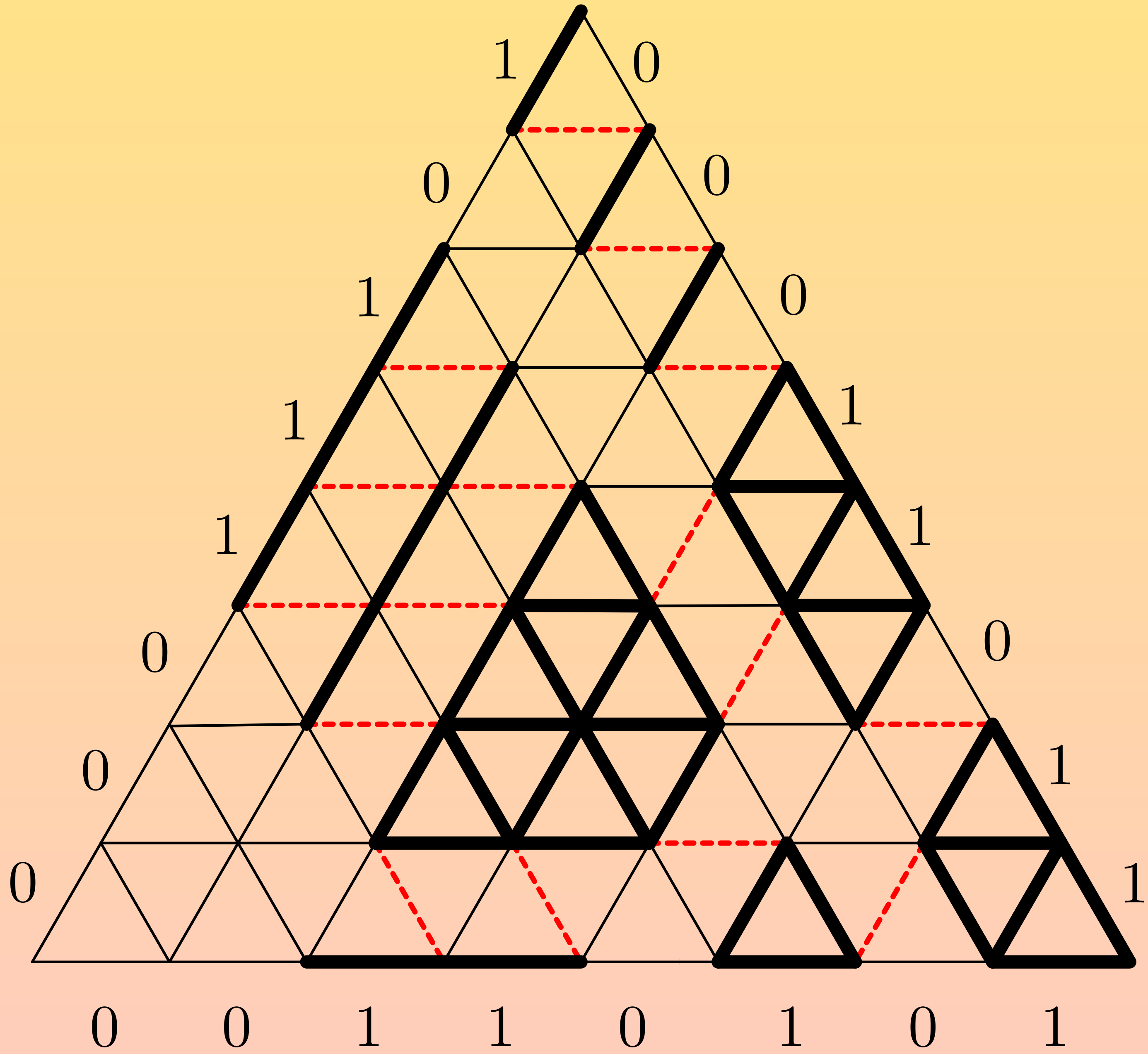

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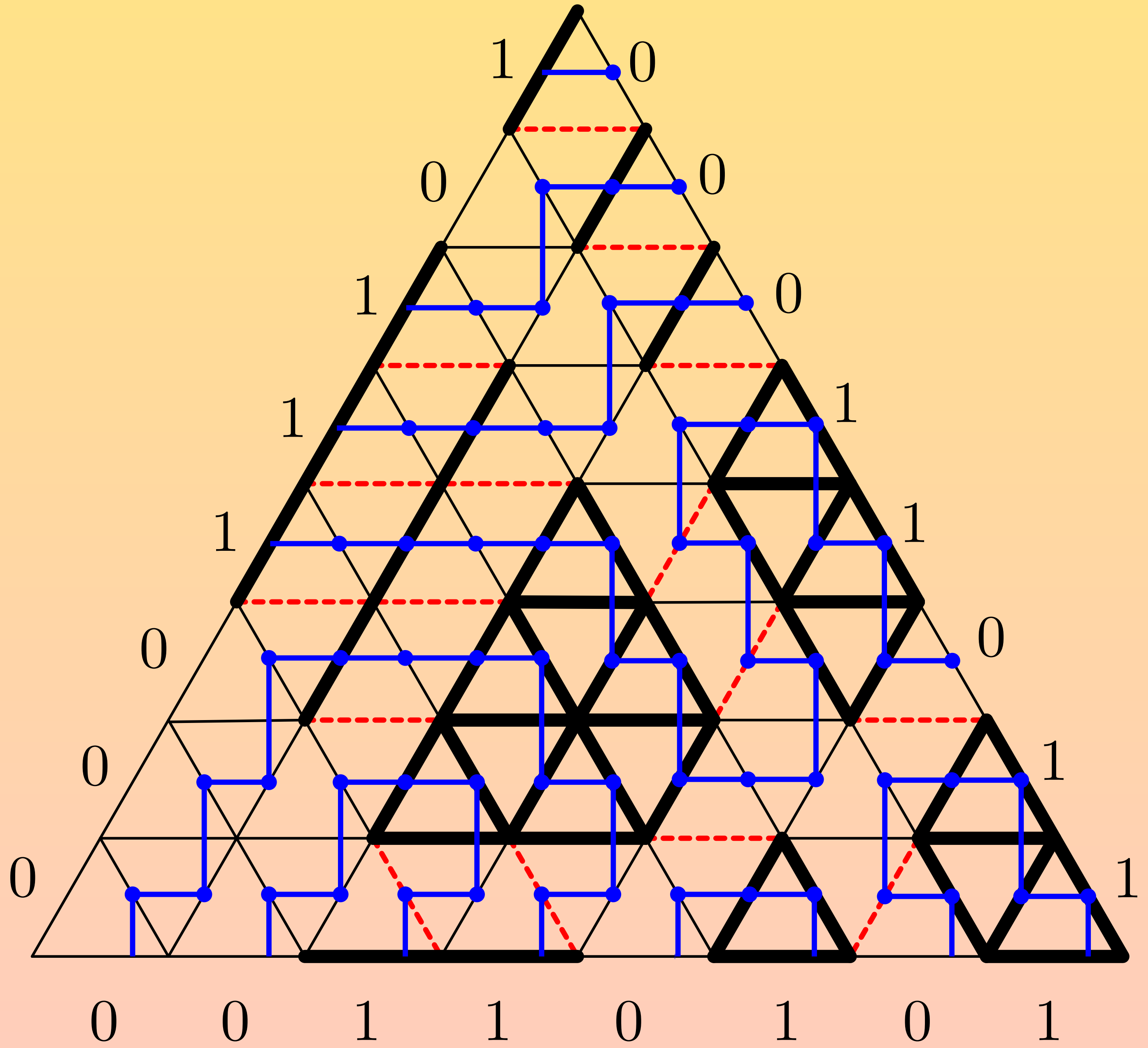

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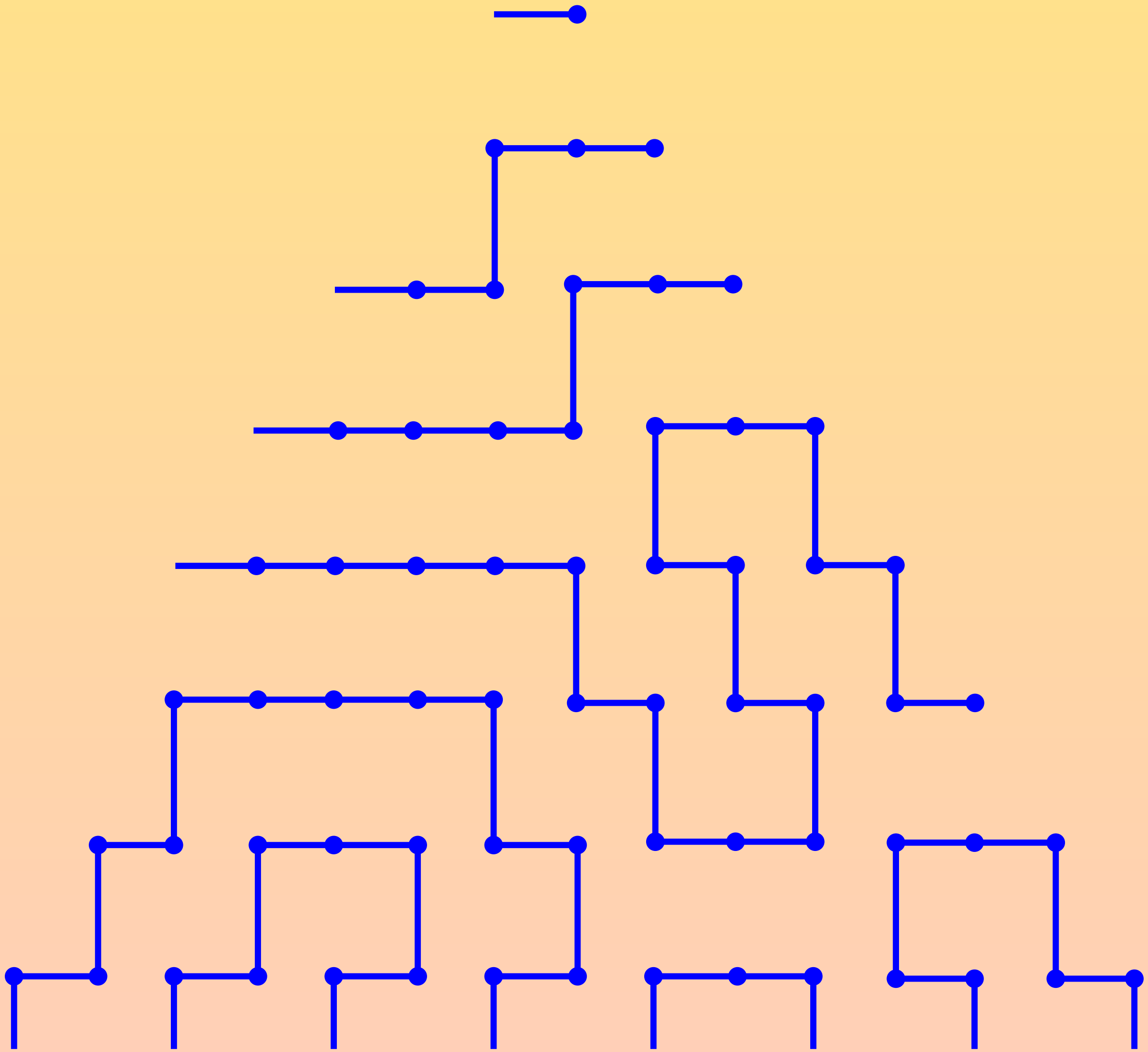

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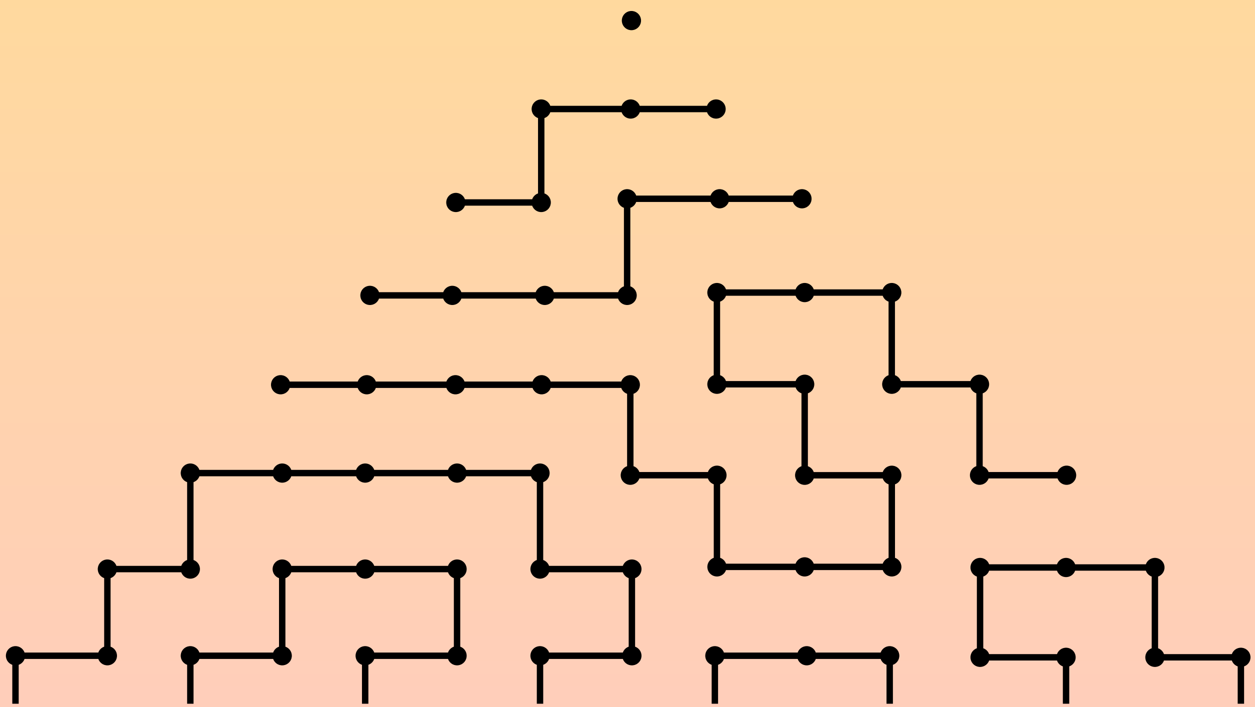

From KT puzzles to TFPL configurations.

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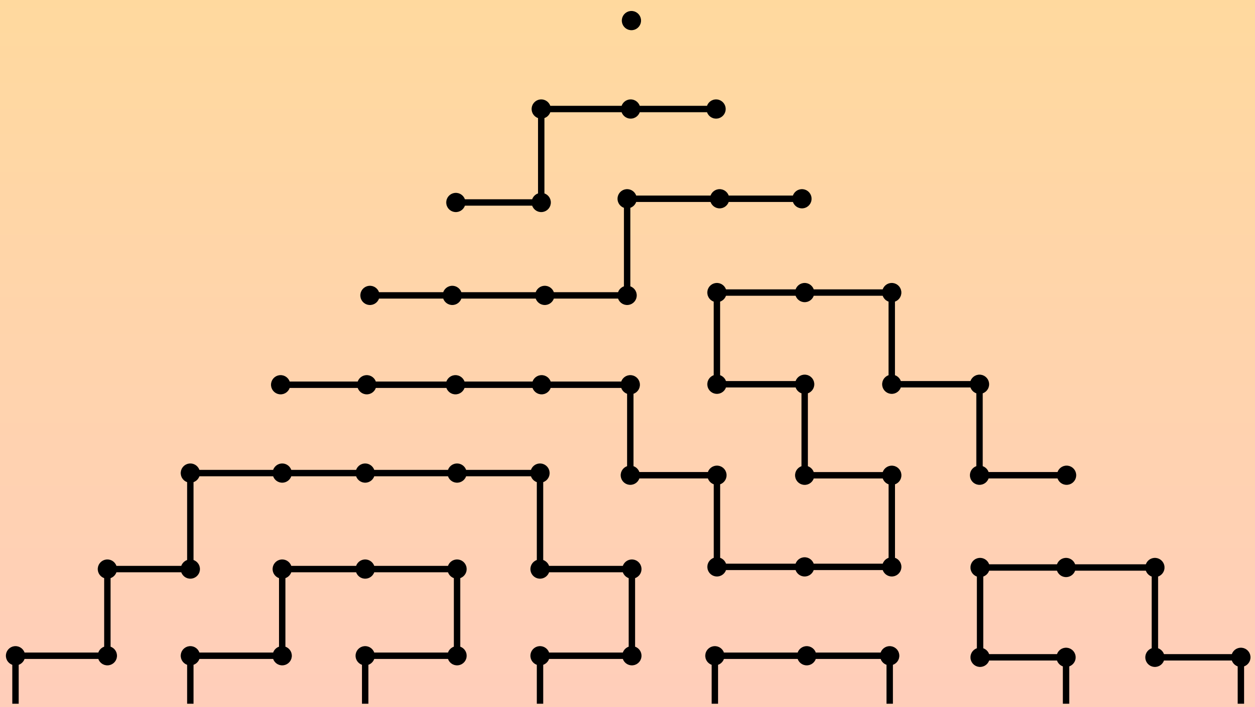
To finish the proof, one checks that this map Φ is :

– well defined, i.e. $\Phi(puzzle)$ is fully packed, and verifies the boundary data σ, π, τ

– injective ;



Conclusion

This diagram shows the possible indices for the numbers $t^{\pi}_{\sigma,\tau}$ when π is fixed ; in blue are coefficients we managed to compute, and in red are those involved in the definition of the $c_{\alpha\pi}$.

$t^{\pi}_{\sigma,\tau}$ for fixed π.