Fully Packed Loop Configurations and Littlewood Richardson coefficients

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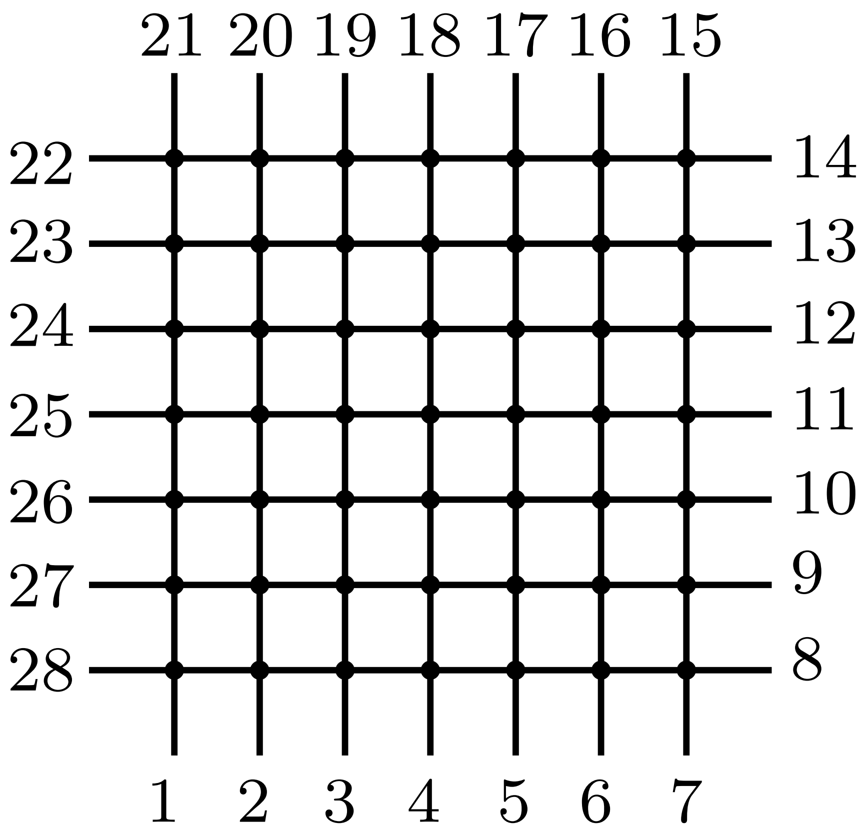
FPL configurations : Definition

Start with the square grid G_n with n^2 vertices and $4n$ external edges. In the example, we have $n = 7$.



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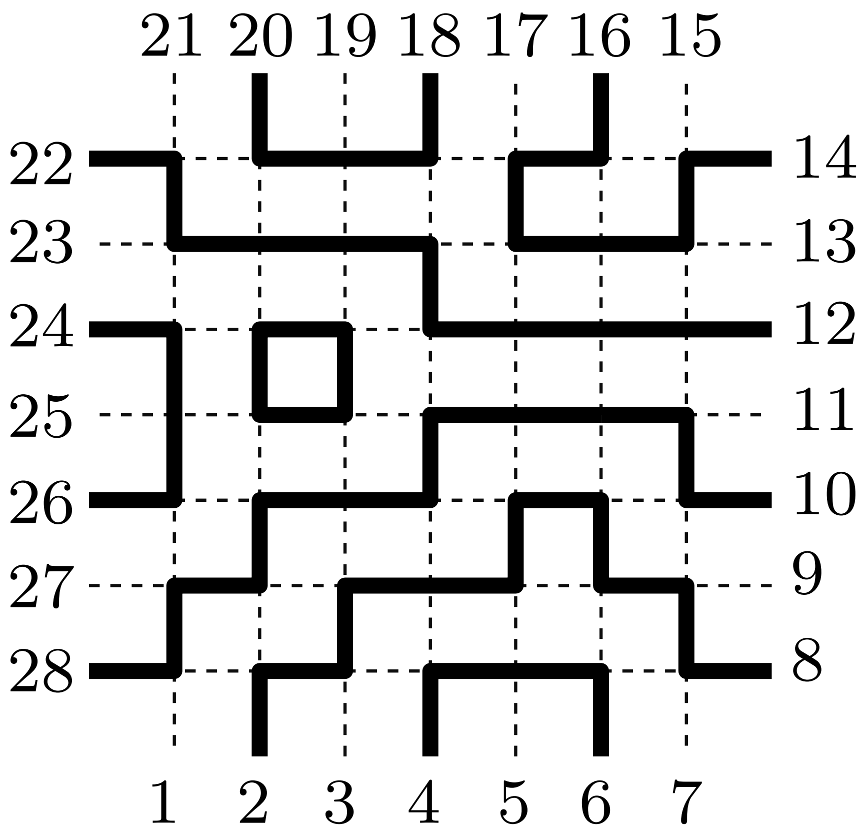


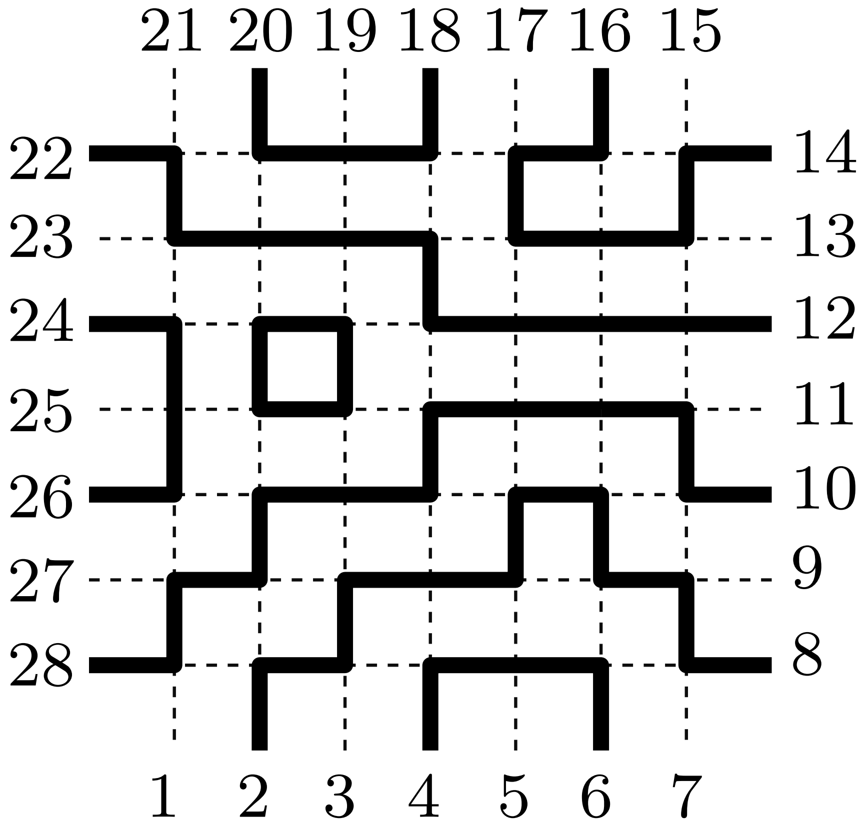
A FPL configuration of size n is a subgraph of the grid G_n

(1) such that around each vertex of G_n, 2 edges out of 4 are selected ; ("Fully Packed")

(2) containing every other external edge. ("Boundary condition")

FPL configurations : Enumeration

Such FPL configurations are in simple bijection with numerous objects : alternating sign matrices, height matrices, configurations of the six vertex model, Gog triangles,...



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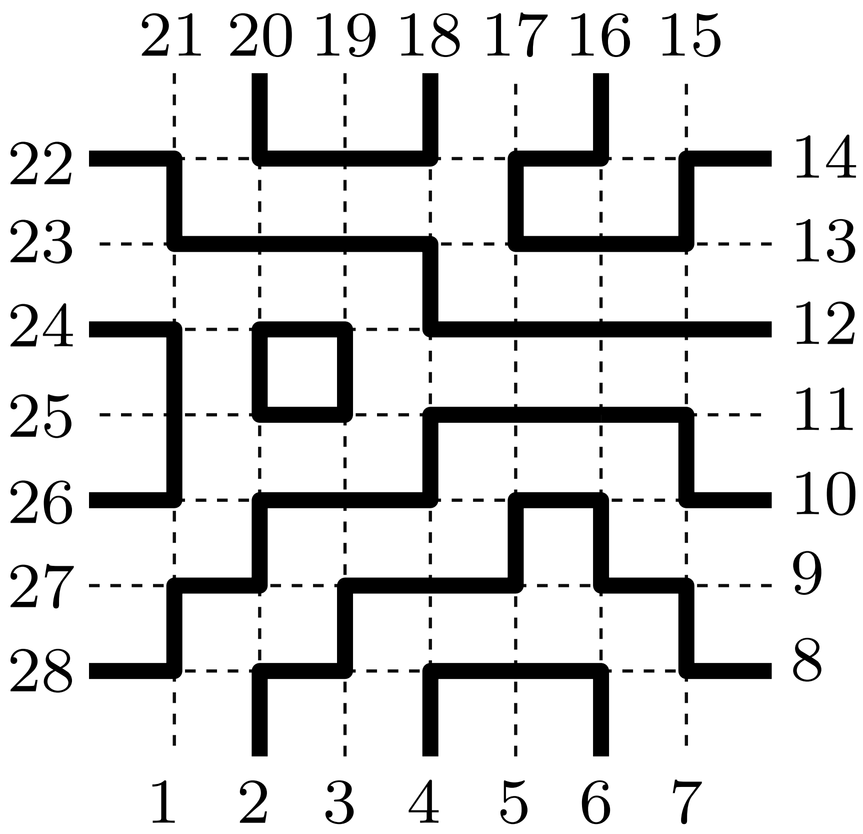

FPL of size n with even boundary

↕ bijection

Alternating sign matrices of size n

An ASM is a square matrix with coefficients in $\{1, 0, -1\}$ such that on each row or column 1 and -1 alternate, and the sum is 1.

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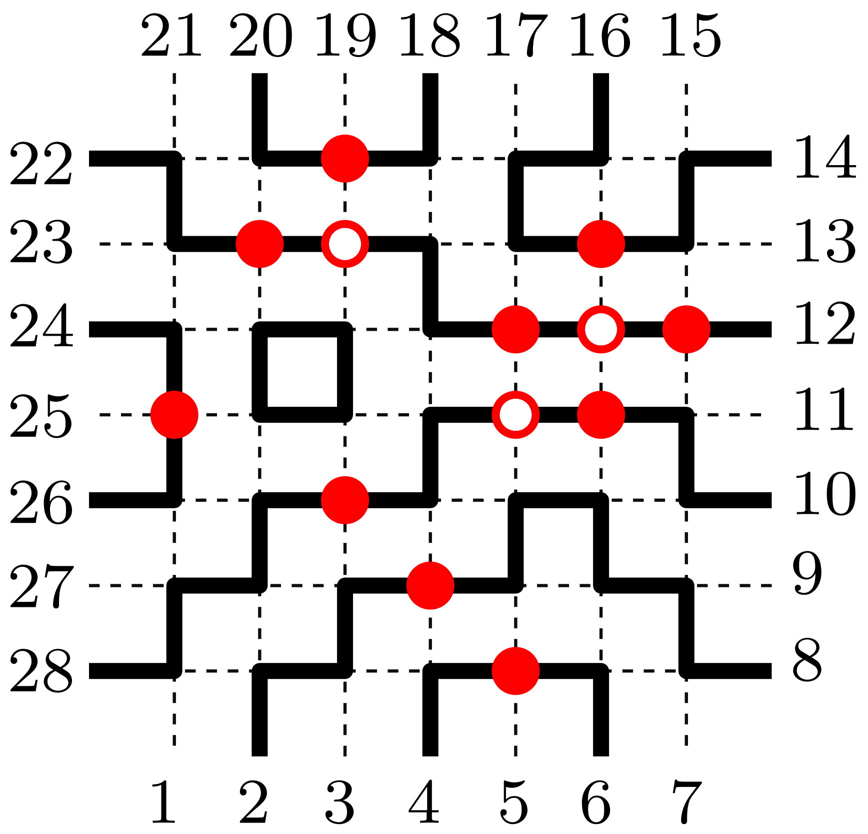

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Here $1 \longrightarrow$ ● and -1 $\longrightarrow$ ○

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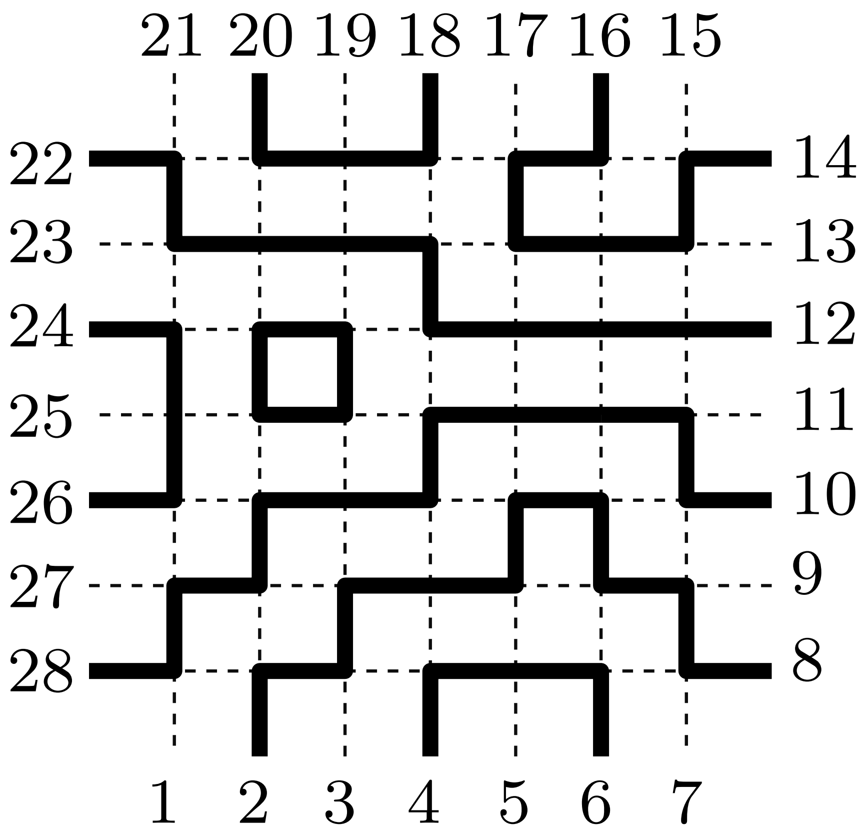


FPL of size n with even boundary

↕

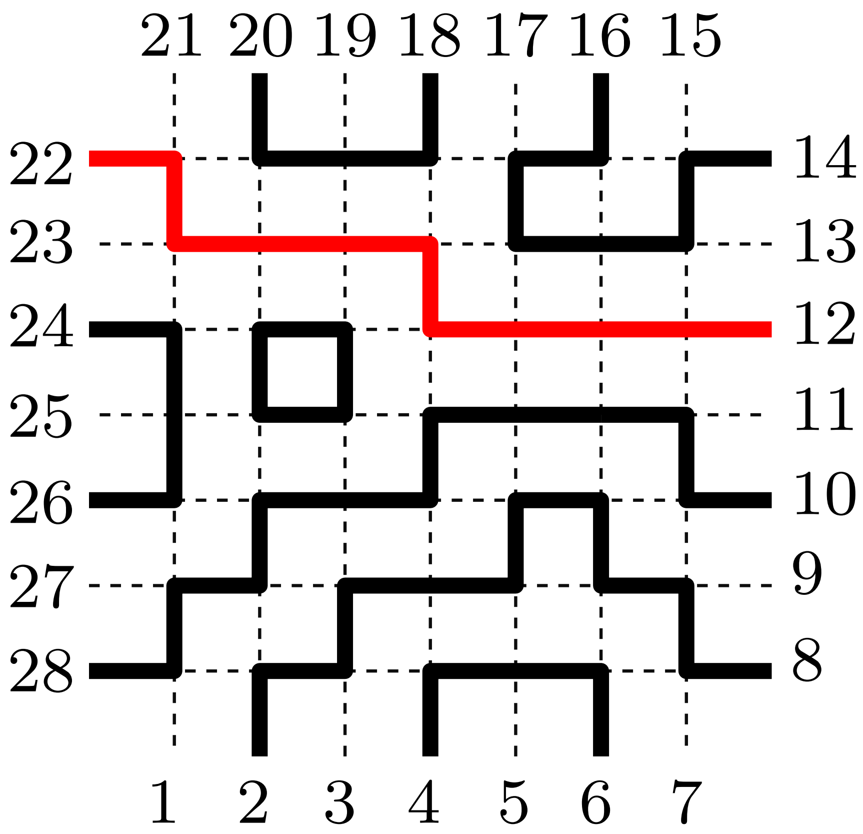
Alternating sign matrices of size n

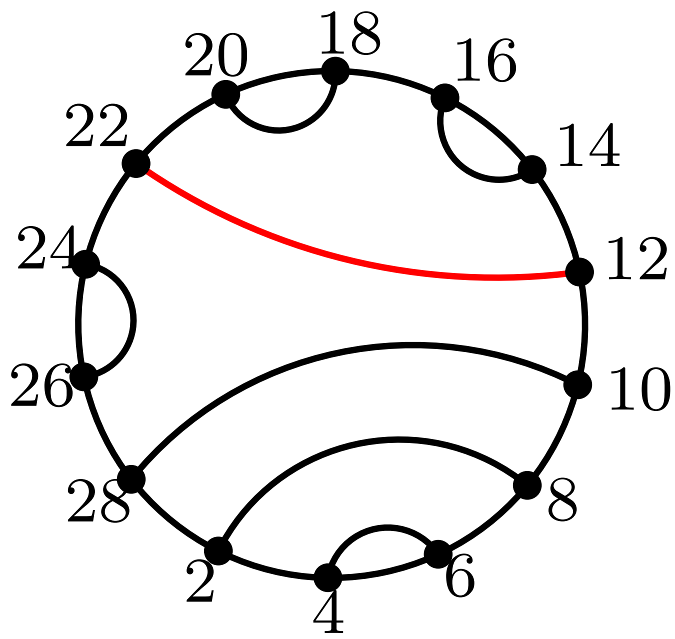
$$|FPL_n| = A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

[Zeilberger '96, Kuperberg '96]

FPL configurations : Refined enumeration

Every FPL configuration determines a link pattern on the odd or even external edges of the grid G_n.





Link pattern= set of n noncrossing chords between $2n$ points on a disk

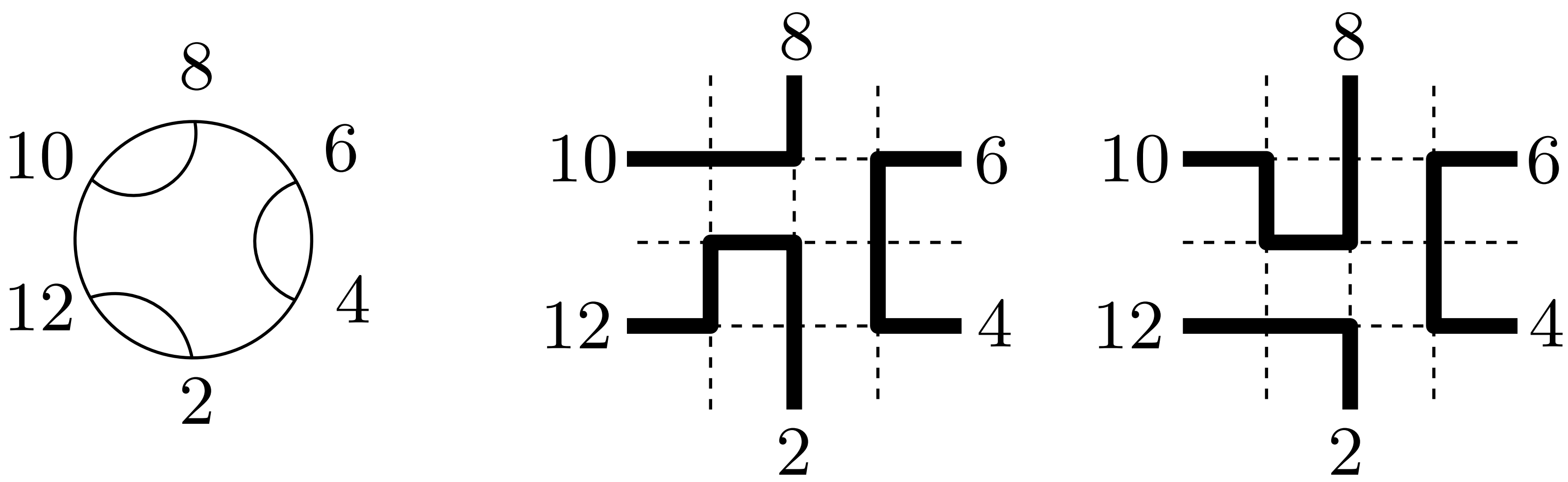
$$|LP_n| = C_n := \frac{1}{n+1}\binom{2n}{n}$$

FPL configurations : Refined enumeration

Now if we are given a pairing X of external edges, our main question will be : how many FPL configurations induce the link pattern X ?

Definition We note A_X this number.

If $X =$



For this link pattern we have $A_X = 2$.

FPL configurations : Refined enumeration

Now given a link pattern X, let X' be defined by

$$(i,j) \in X' \Leftrightarrow (i-1, j-1) \in X$$

Theorem [Wieland '00]

$$A_X = A_{X'}$$

This means that "rotating the link pattern" does not change the number of FPL configurations attached to it.

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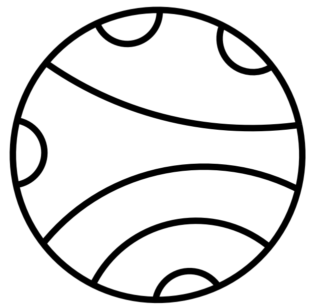
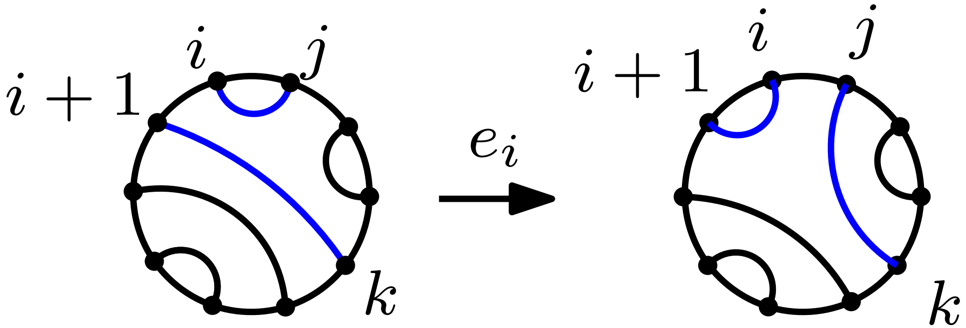
For enumeration purposes, we can then use unlabeled link patterns :



Motivation : the Razumov-Stroganov conjecture.

Definition : We define operators e_i on link patterns for $i = 1 \ldots 2n$ by $\{i, j\}, \{i+1, k\} \in X \to \{i, i+1\}, \{j, k\} \in e_i(X)$.



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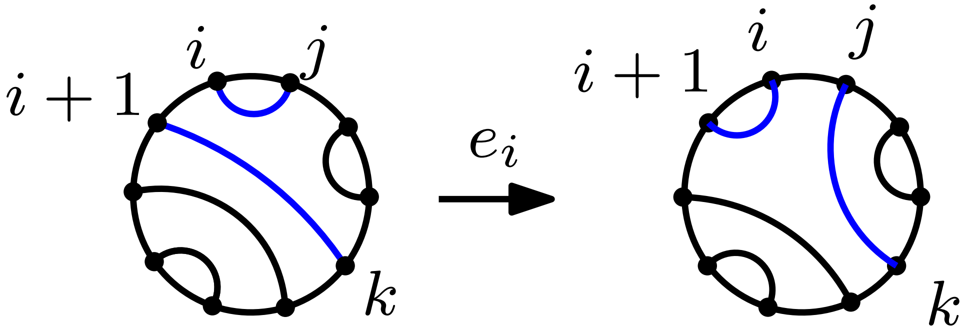

Markov chain $\mathcal{M}$

- States $= LP_n$;
- Transition probabilities : $P(X \to Y) = \frac{k}{2n}$ where k is the number of $i \in \{1, \ldots, 2n\}$ such that $e_i(X) = Y$.

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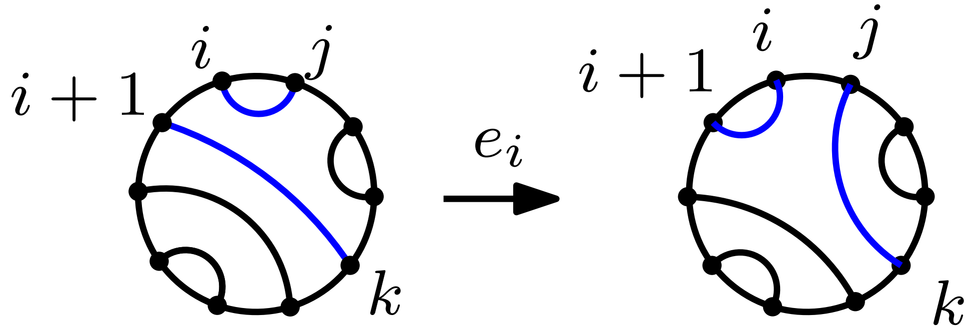

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Stationary distribution (ψ_X) of $\mathcal{M}$

Let P be the matrix defined by $P_{XY} = P(X \to Y)$ where $X, Y \in LP_n$. Then there is a unique probability distribution $(\psi)_X$ on LP_n such that $P\psi = \psi$.

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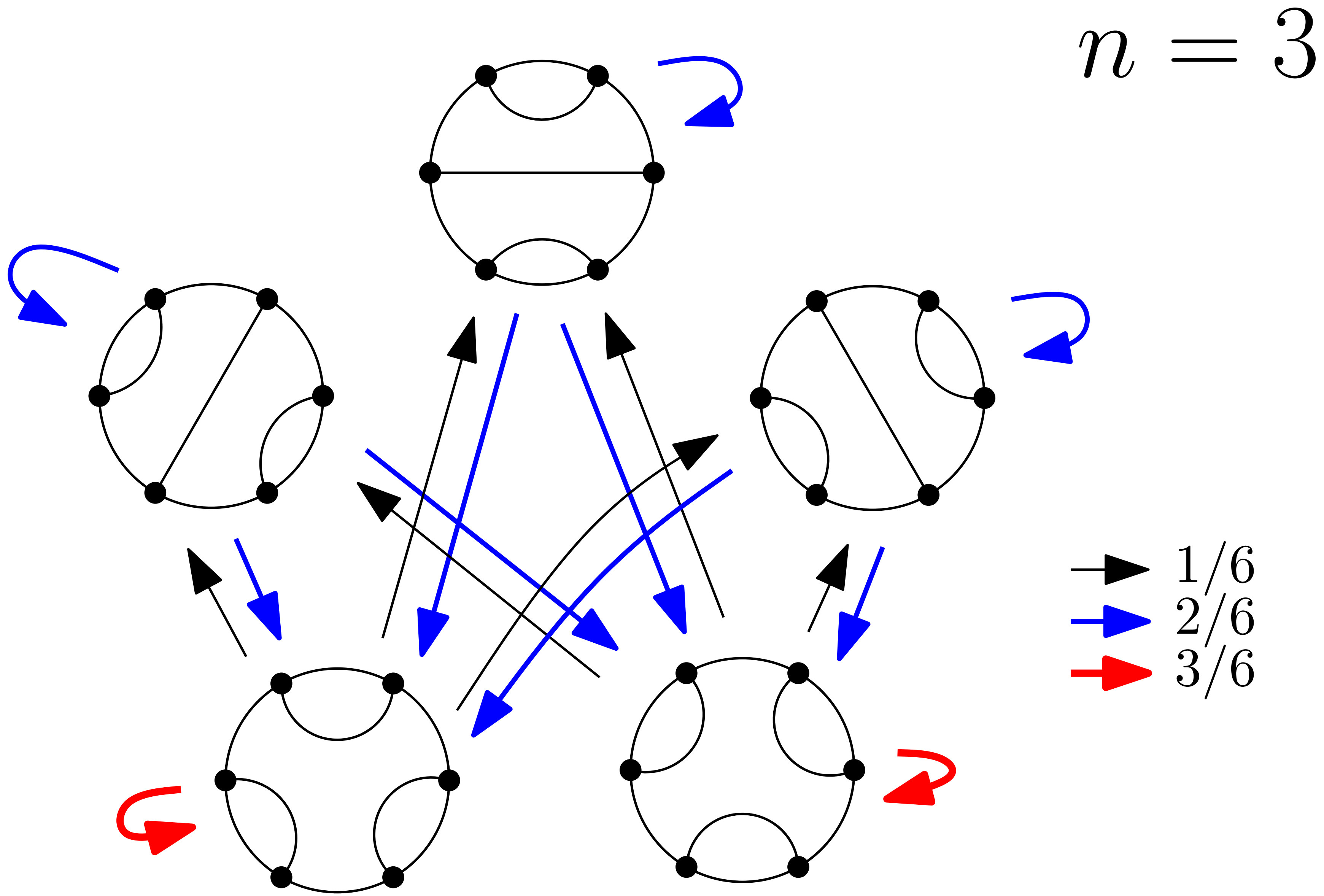

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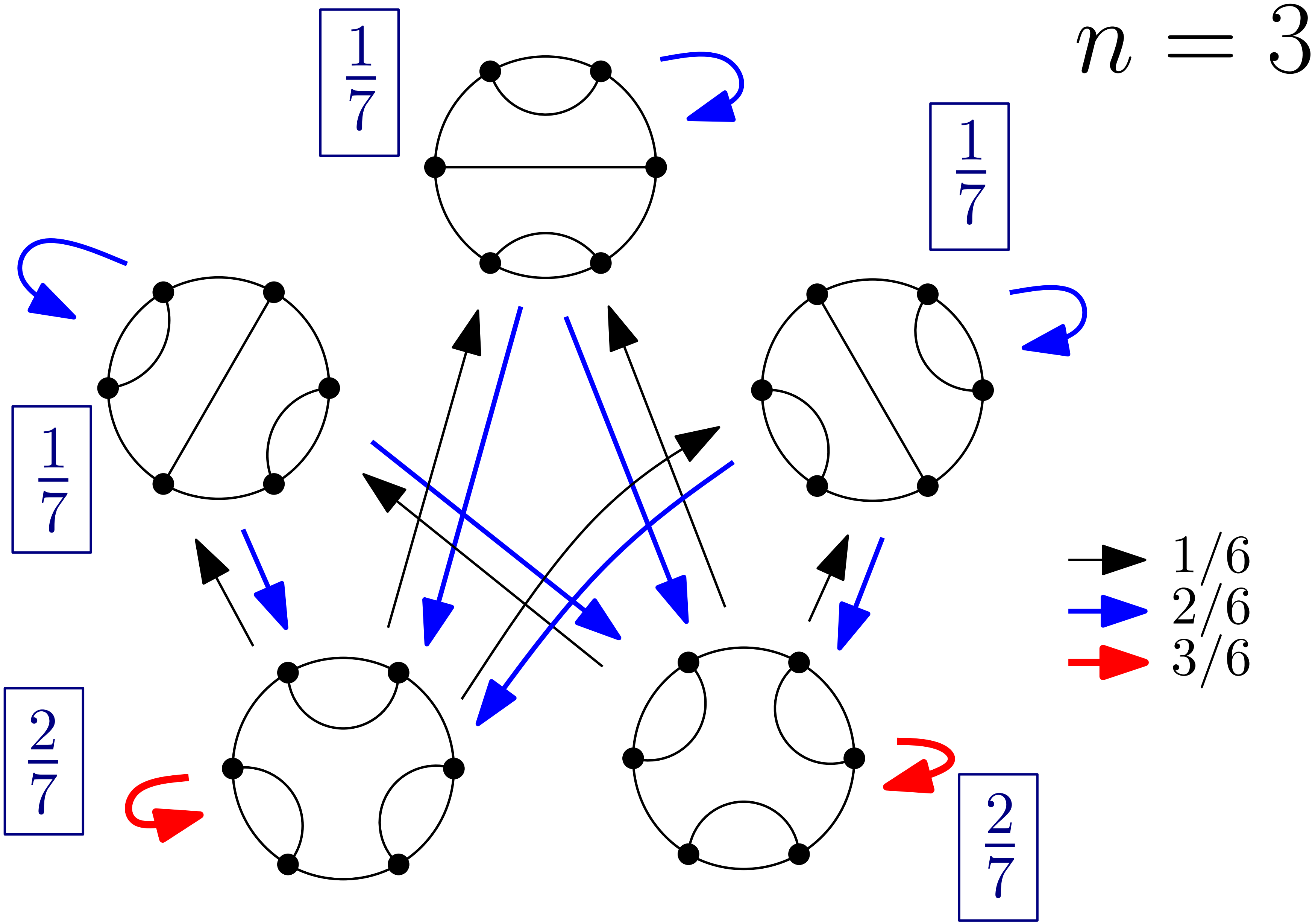

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RS conjecture : The stationary distribution $(\psi_X)_{X \in LP_n}$ is given by

$$\psi_X = \frac{A_X}{A_n}$$

Another formulation is : $\forall X, \quad 2nA_X = \sum_{(i,Y), e_i(Y)=X} A_Y$

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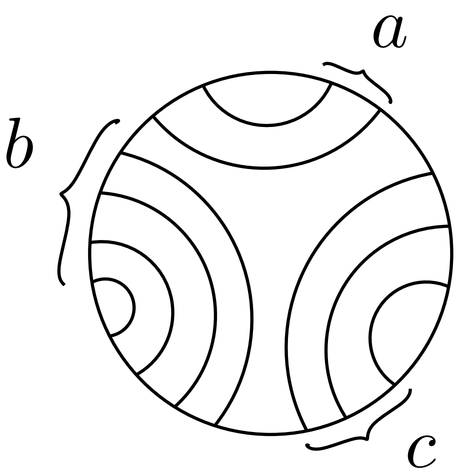
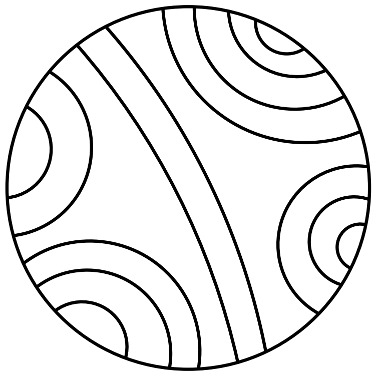
The numbers ψ_X were studied in detail by Di Francesco and Zinn-Justin
$\rightarrow$ integral expressions (up to a change of basis), multivariate versions, computation in special cases.

For the numbers A_X, very little is known in contrast.

Special cases for A_X



$$= \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}$$



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[Zinn-Justin, Zuber, Di Francesco, Caselli, Krattenthaler]

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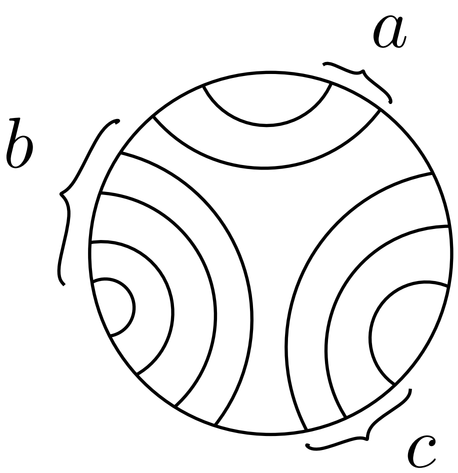

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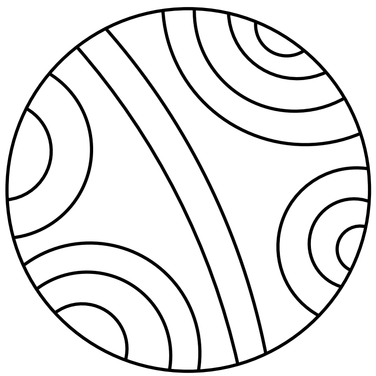


= Complicated determinant formulas

[Zinn-Justin, Zuber, Di Francesco, Caselli, Krattenthaler]

In this talk we will describe a possible approach for the computation of A_X.

Outline of the talk

(0) Long introduction

Why do we want to count FPLs with a given link pattern ?

(1) From the square to the triangle

We will explain a formula expressing numbers A_X in terms of FPL configurations in a certain triangle (TFPL), which uses *link patterns with nested arches*.

(2) FPL configurations in a triangle

We will collect various formulas and relations for FPL configurations in the triangle.

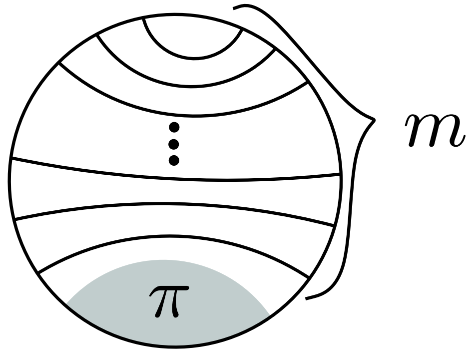
(3) Extremal TFPL configurations

In a special case, we will show that TFPL configurations are enumerated by the famous Littlewood-Richardson coefficients.

(1) From the square to the triangle

Link patterns with nested arches

We consider now integers $n, m \geq 0$, and link patterns with m nested arches, and π is a noncrossing matching with n arches.



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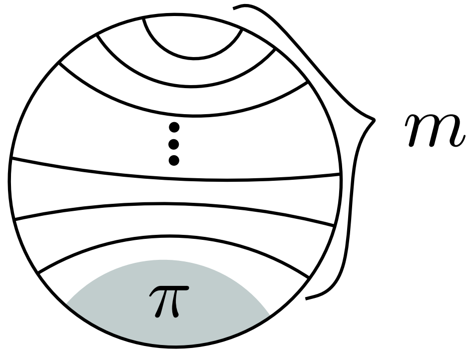

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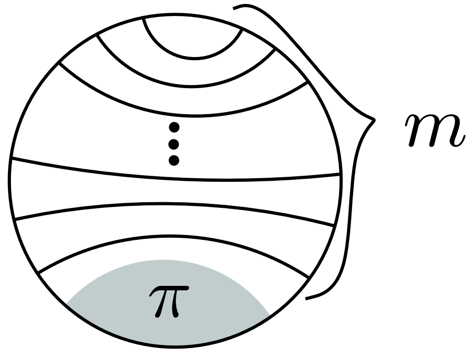

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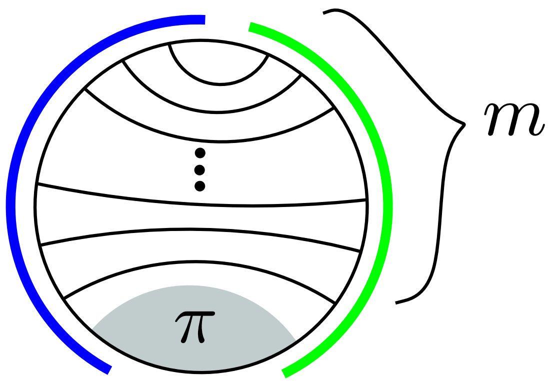
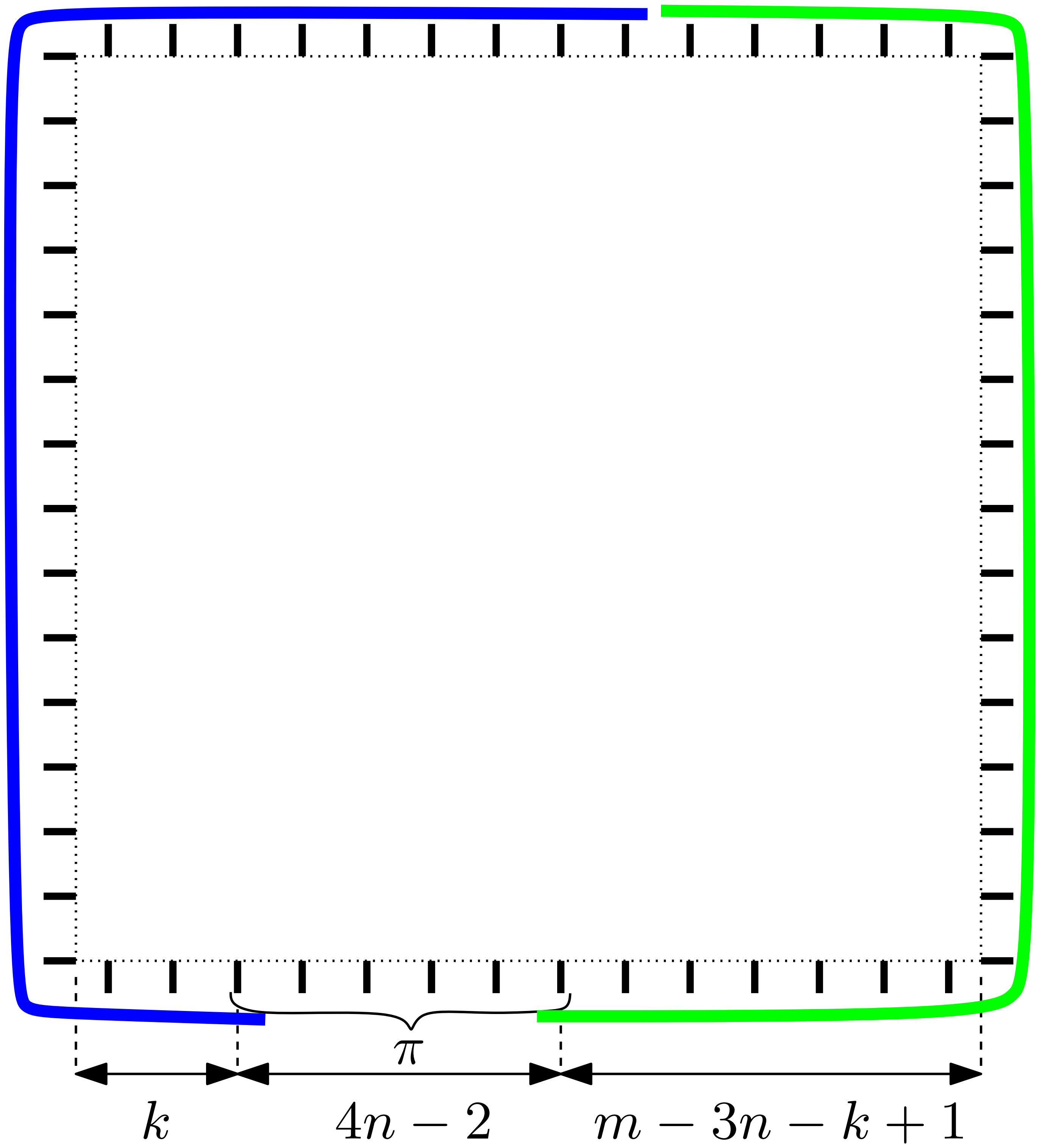
Idea : for m large enough, we derive an expression for $A_\pi(m)$ based on a certain combinatorial decomposition. It turns out that the expression is actually valid for all $m \geq 0$.

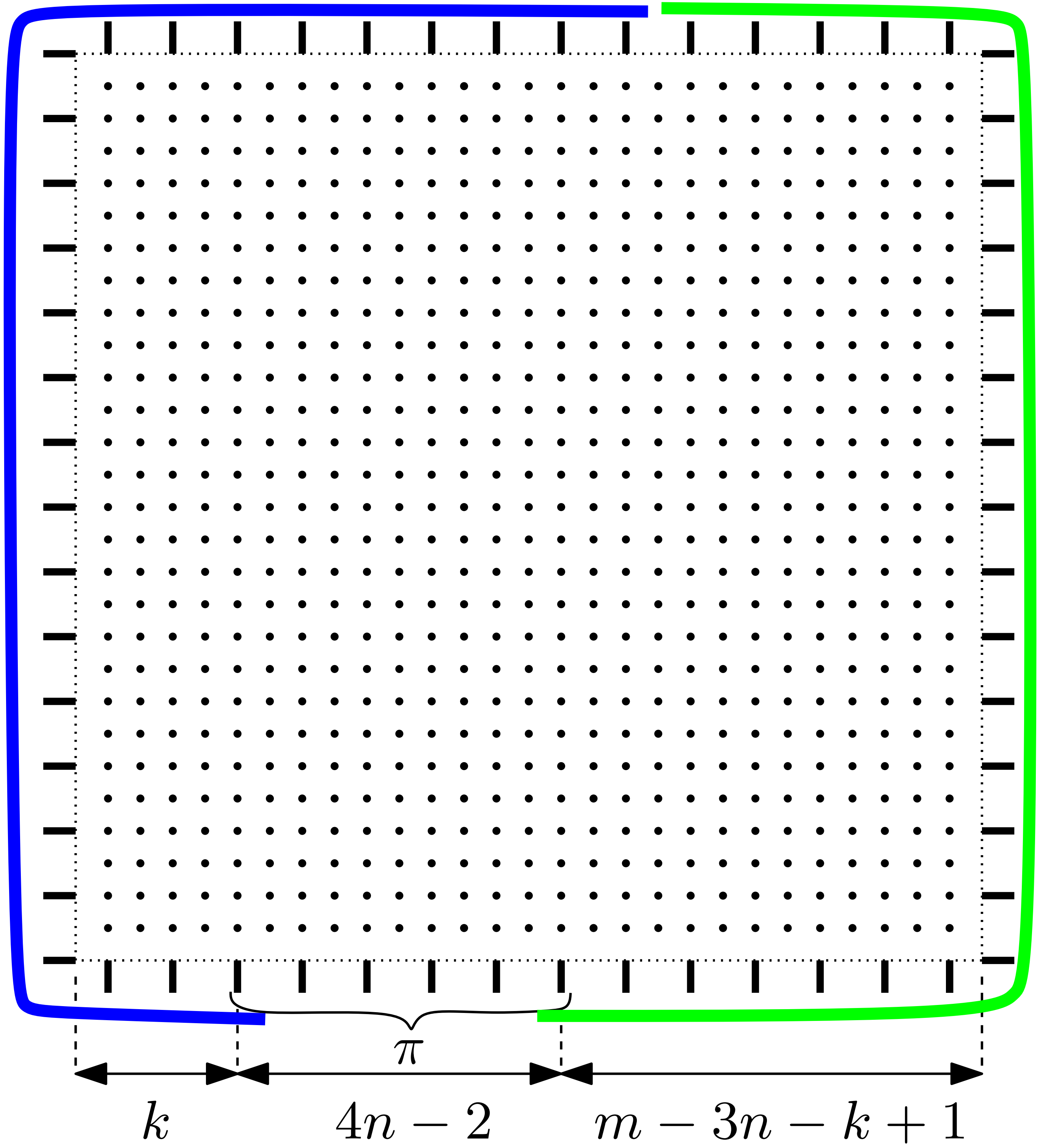
We suppose $m \geq 3n-1$, and choose k such that $0 \leq k \leq m-(3n-1)$.





Many edges of the grid belong to every FPL configuration respecting the link pattern.

⇒"Fixed edges"



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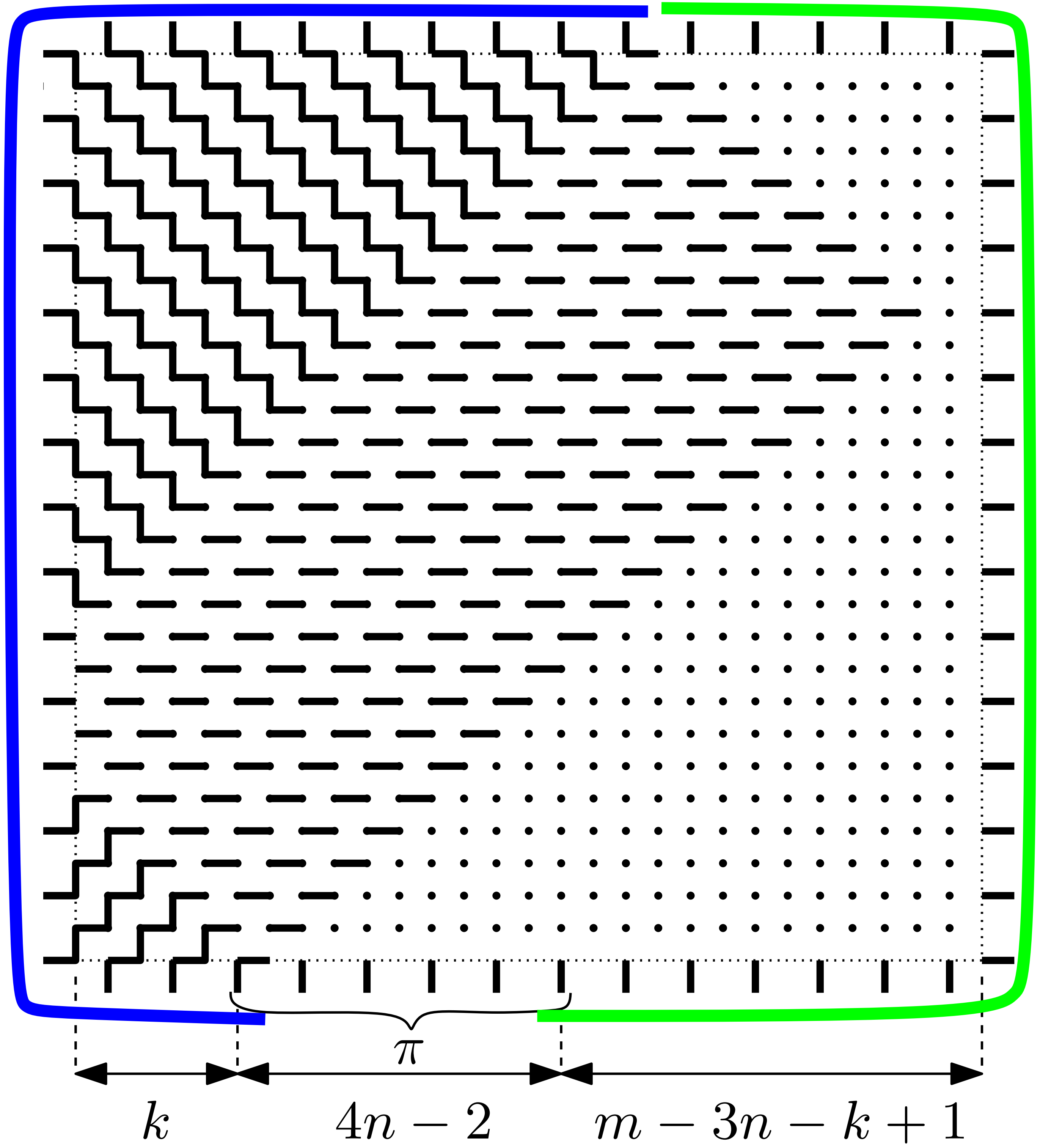

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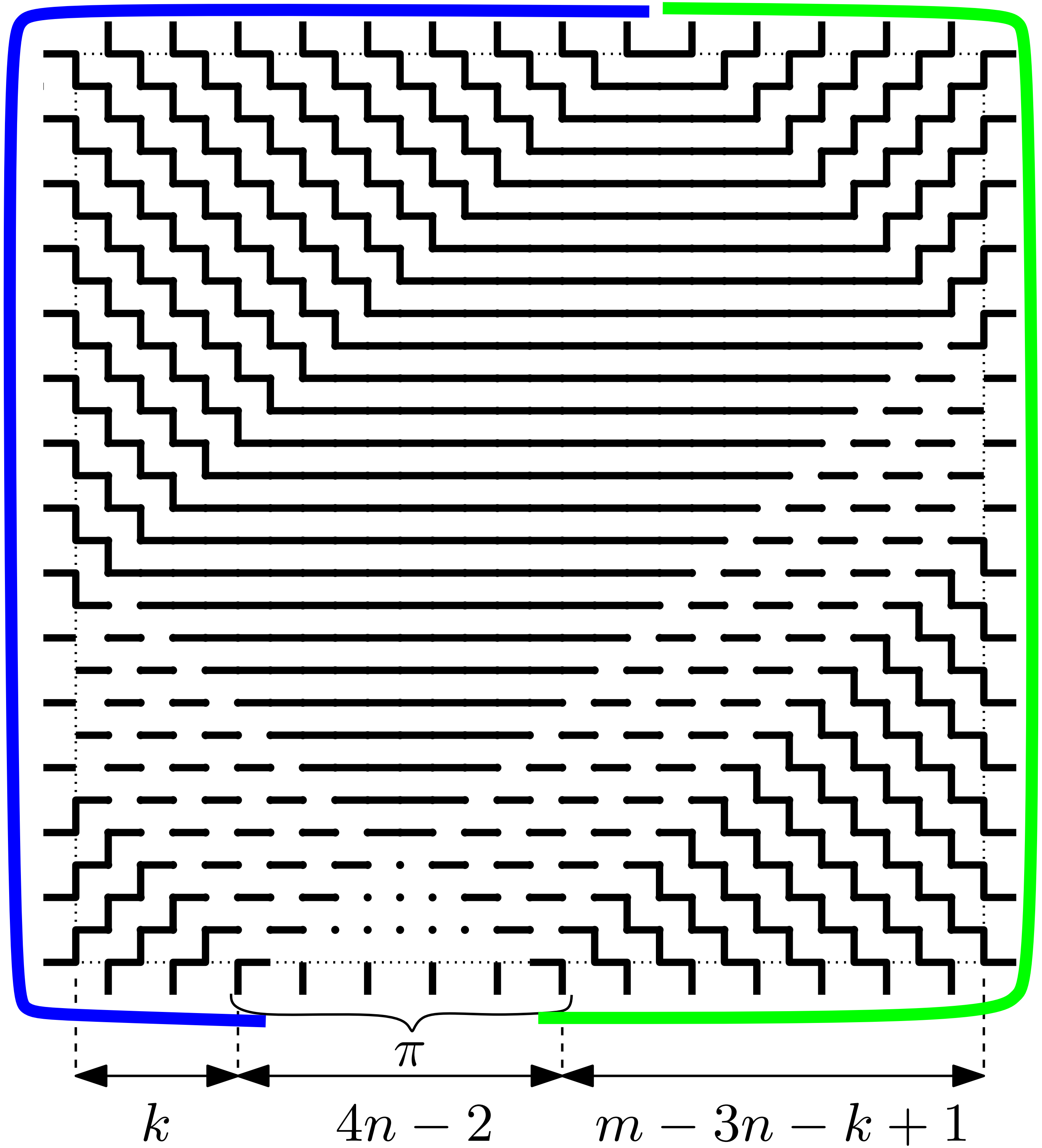

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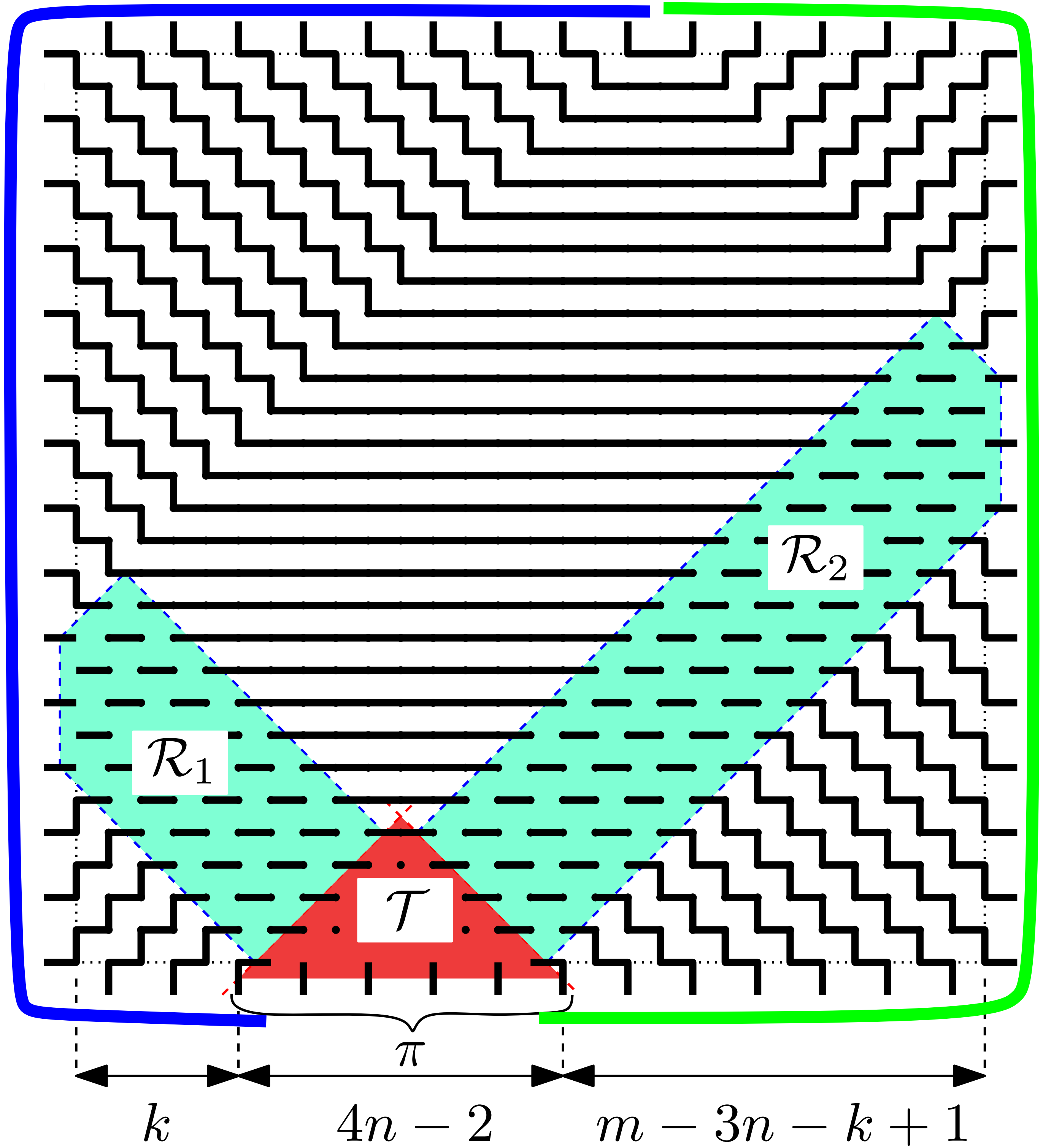
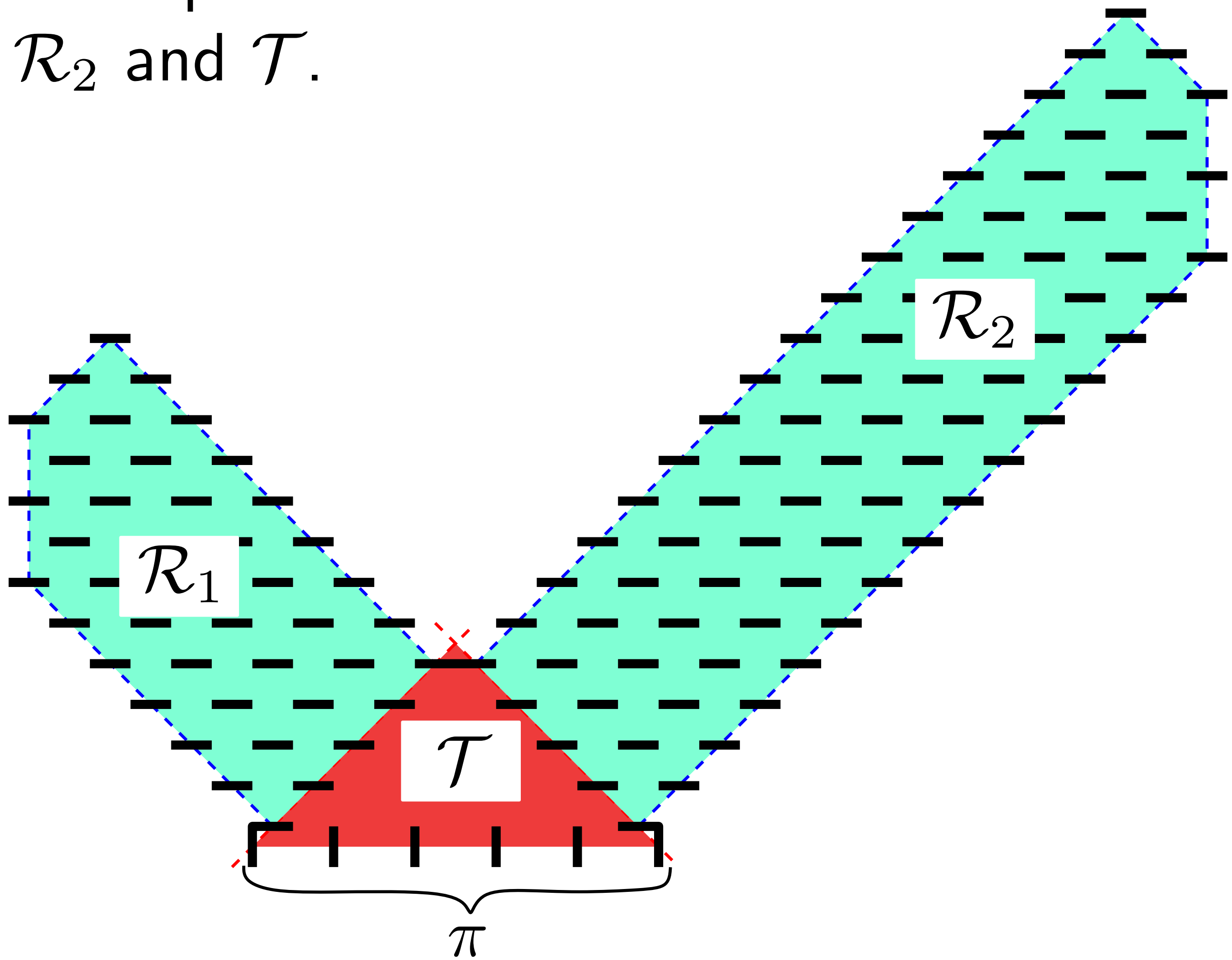


To compute the numbers $A_\pi(m)$, we will count FPL configurations separately in $\mathcal{R}_1, \mathcal{R}_2, \mathcal{T}$.

For this, we need to encode the possible boundaries between $\mathcal{R}_1$ and $\mathcal{T}$, and between $\mathcal{R}_2$ and $\mathcal{T}$.



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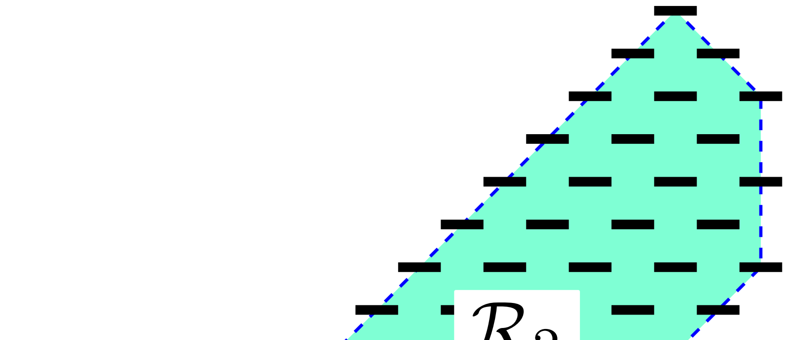
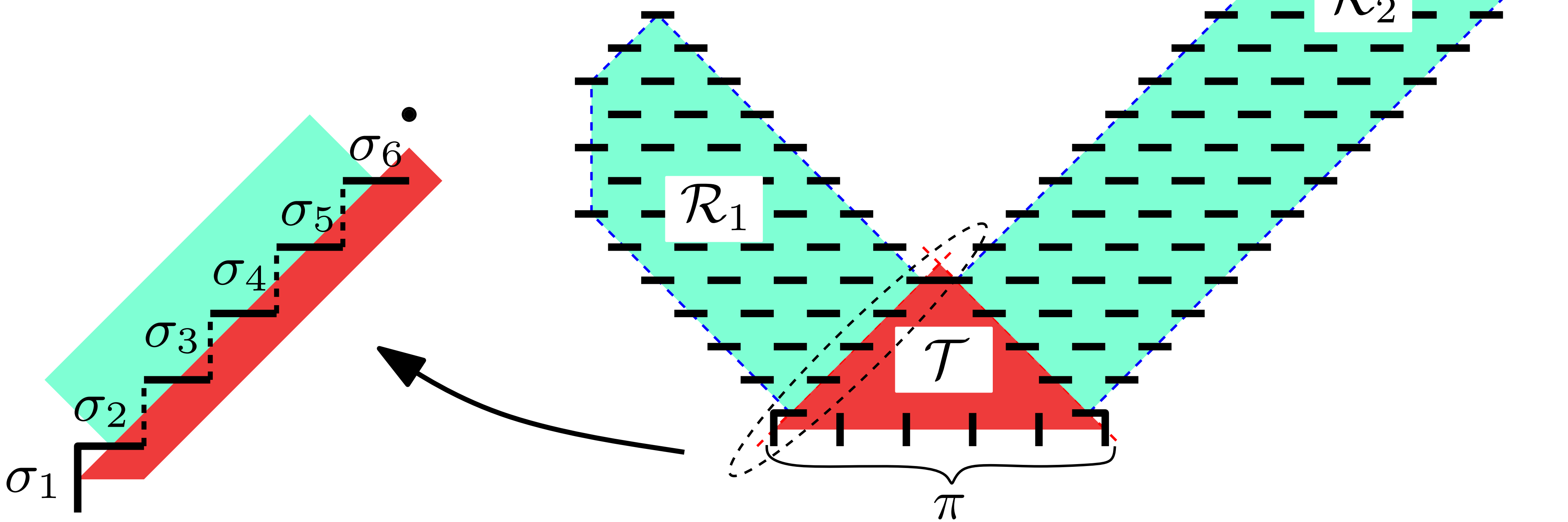
Word $\sigma = \sigma_1 \ldots \sigma_{2n}$ in $\{0,1\}^{2n}$, where

$\sigma_i = 0 \Leftrightarrow$ a vertical edge is present





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Word $\tau = \tau_1 \dots \tau_{2n}$ in $\{0,1\}^{2n}$, where

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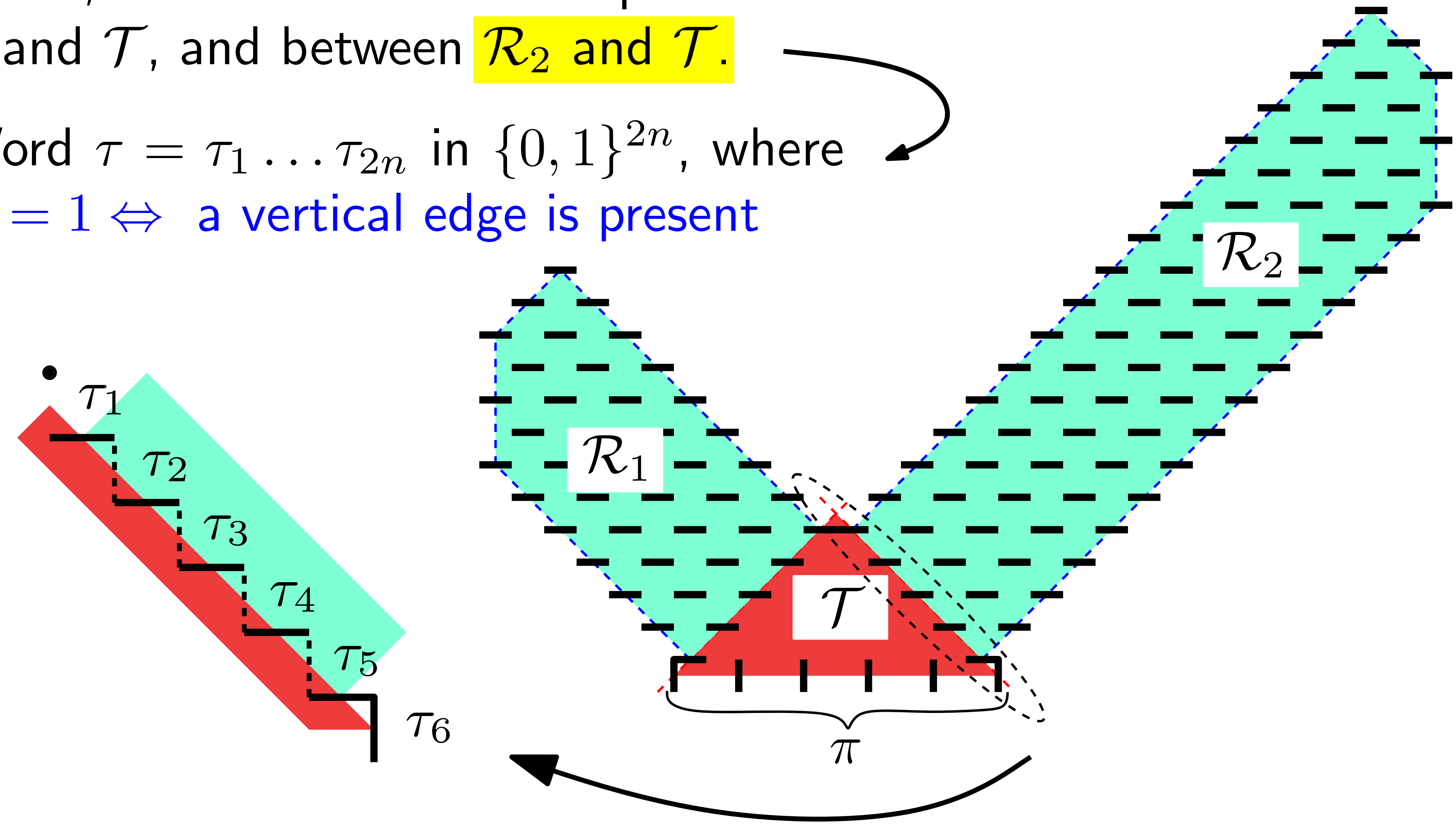


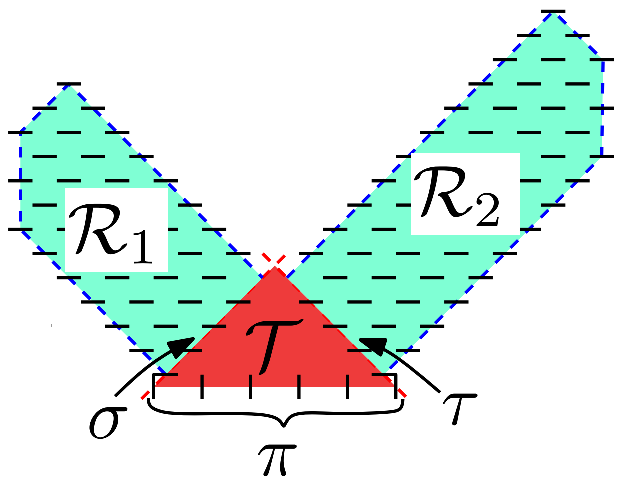
Putting things together

We can then write, for $m \geq 3n - 1$ and $0 \leq k \leq m - (3n - 1)$

$$A_\pi(m) = \sum_{\sigma,\tau} |\mathcal{R}_1(\sigma, k)| \times t^\pi_{\sigma,\tau} \times |\mathcal{R}_2(\tau, m - 3n - k + 1)|$$

where

- σ, τ are words of length $2n$ on $\{0, 1\}$;



- $\mathcal{R}_1(\sigma, .), \mathcal{R}_2(\tau, .)$ are the sets of FPL configurations in the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ with boundaries σ, τ respectively ;

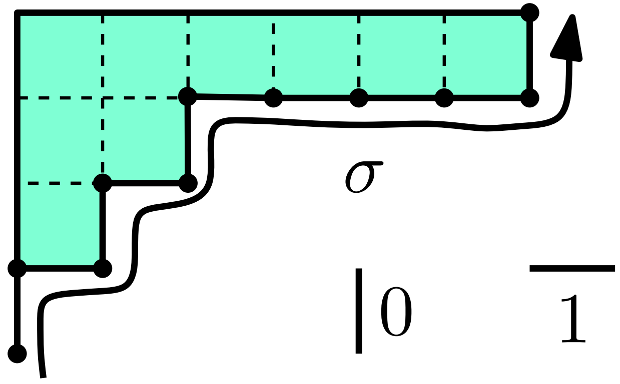
- $t^\pi_{\sigma,\tau}$ is the number of FPL configurations in the triangle $\mathcal{T}$ with boundary data $\{\sigma, \pi, \tau\}$.

Words and Shapes

Let $\sigma = \sigma_1 \ldots \sigma_p$ be a word in $\{0,1\}^p$; we write $|\sigma| := p$.

Words = Ferrers shapes in a box.

$$\sigma = 0101011110 \qquad |\sigma| = 10, |\sigma|_0 = 4, |\sigma|_1 = 6$$



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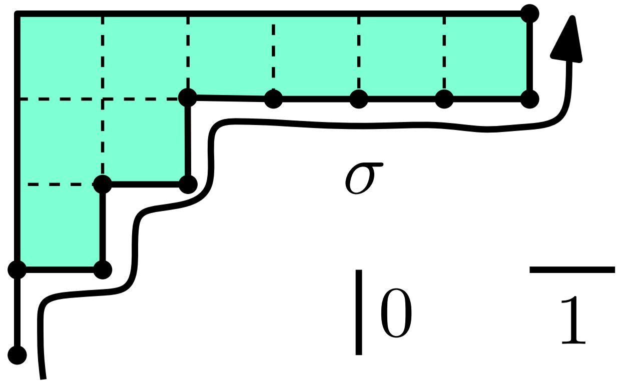
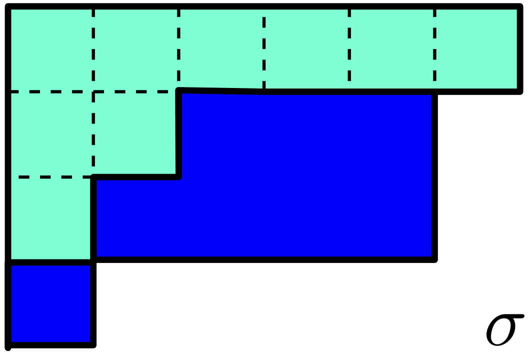


$$d(\sigma) = 9$$

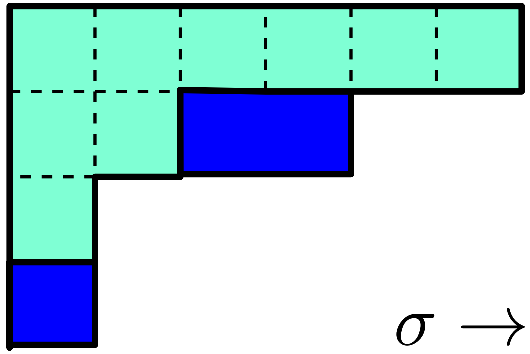
$$\sigma^* = 1000010101$$

Length $d(\sigma)$:= the number of boxes in the diagram σ.

Transpose $\sigma^* := (1-\sigma_p)\cdots(1-\sigma_2)(1-\sigma_1)$



$\sigma \leq \sigma'$



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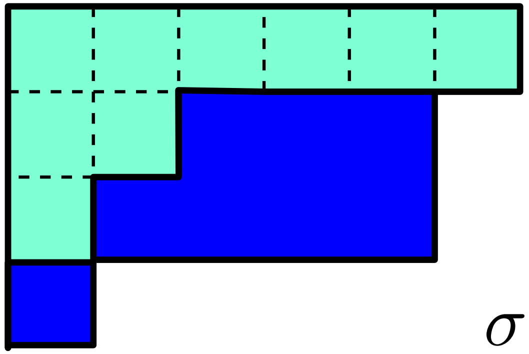
At most one more box per column



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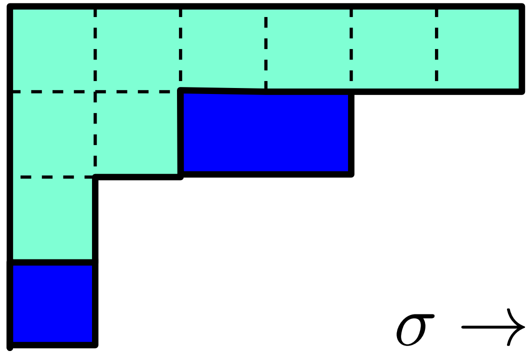

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Definition

A semi standard Young tableau of shape σ and entries bounded by N is a filling of the shape σ by integers in $\{1, \ldots, N\}$ such that entries are strictly increasing in columns and weakly increasing in rows.

The number of such tableaux is given by $SSYT(\sigma, N)$, a polynomial in N with leading term $\frac{1}{H(\sigma)} N^{d(\sigma)}$.

(Here $H(\sigma)$ is the product of *hook lengths* of the shape σ.)

Regions $\mathcal{R}_1$ and $\mathcal{R}_2$

Proposition [Caselli,Krattenthaler,Lass,N. '05]

Let σ be a word of length $2n$, and $k \in \mathbb{N}$. There is a bijection between FPLs in $\mathcal{R}_1(\sigma, k)$ and semistandard Young tableaux of shape σ and length $n + k$.

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So for $m \geq 3n-1$ (and $k=0$) we obtain :

$$A_\pi(m) = \sum_{\sigma,\tau} |\mathcal{R}_1(\sigma, 0)| \;\cdot\; t^{\pi}_{\sigma,\tau} \;\cdot\; |\mathcal{R}_2(\tau, m-3n+1)|$$

$$= \sum_{\sigma,\tau} SSYT(\sigma, n) \cdot t^{\pi}_{\sigma,\tau} \cdot SSYT(\tau^*, m-2n+1)$$

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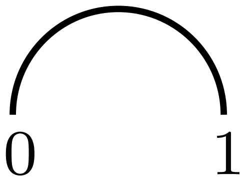
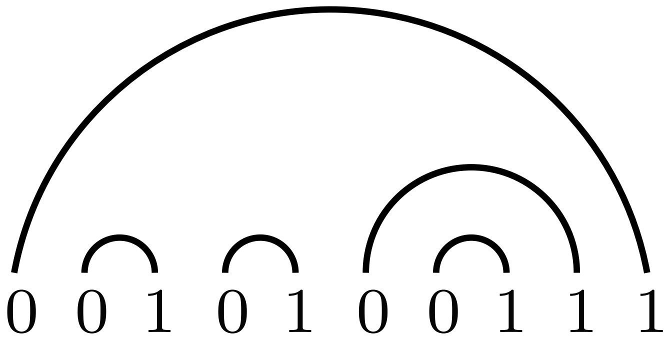
Theorem [CKLN '05]

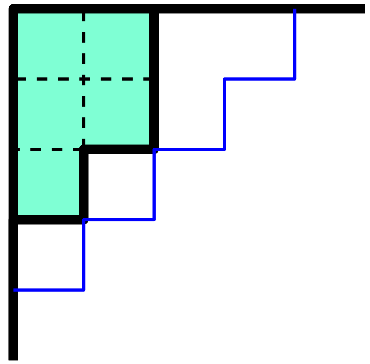
> $A_\pi(m)$ is a polynomial function of m for $m \geq 0$

Some more definitions

Given a noncrossing matching π of size n, we can associate to it a word, and thus a Ferrers shape :







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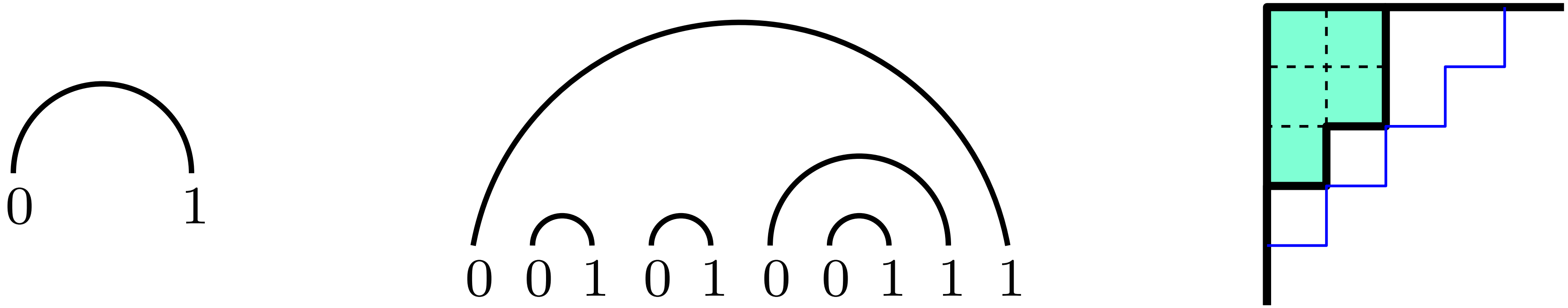

The words obtained from matchings are the famous Dyck words :

Definition We note $\mathcal{D}_n$ the words w such that $|w|_0 = |w|_1 = n$ and which are smaller than $(01)^n$.

We write $\mathbf{0}_n := 0^n 1^n$, and $\mathbf{1}_n := (01)^n$.Then $(\mathcal{D}_n, \leq)$ forms a poset with minimum $\mathbf{0}_n$ and maximum $\mathbf{1}_n$.

The final expression for $A_\pi(m)$

Theorem [CKLN '04]

> For all σ, τ, π, we have $t^{\pi}_{\sigma,\tau} \neq 0$ implies $\sigma \leq \pi$.
> Moreover, $t^{\pi}_{\pi,\mathbf{0}_n} = 1$ and $t^{\pi}_{\pi\tau} = 0$ if $\tau \neq \mathbf{0}_n$.

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As a consequence, the expression for $A_\pi(m)$ can be restricted to words $\sigma, \tau \in \mathcal{D}_n$: for any $m \geq 0$

$$A_\pi(m) = \sum_{\sigma,\tau \in \mathcal{D}_n} SSYT(\sigma, n) \cdot t^\pi_{\sigma,\tau} \cdot SSYT(\tau^*, m - 2n + 1)$$

One can show then that $A_\pi(m)$ has leading term $\frac{1}{H(\pi)} m^{d(\pi)}$.

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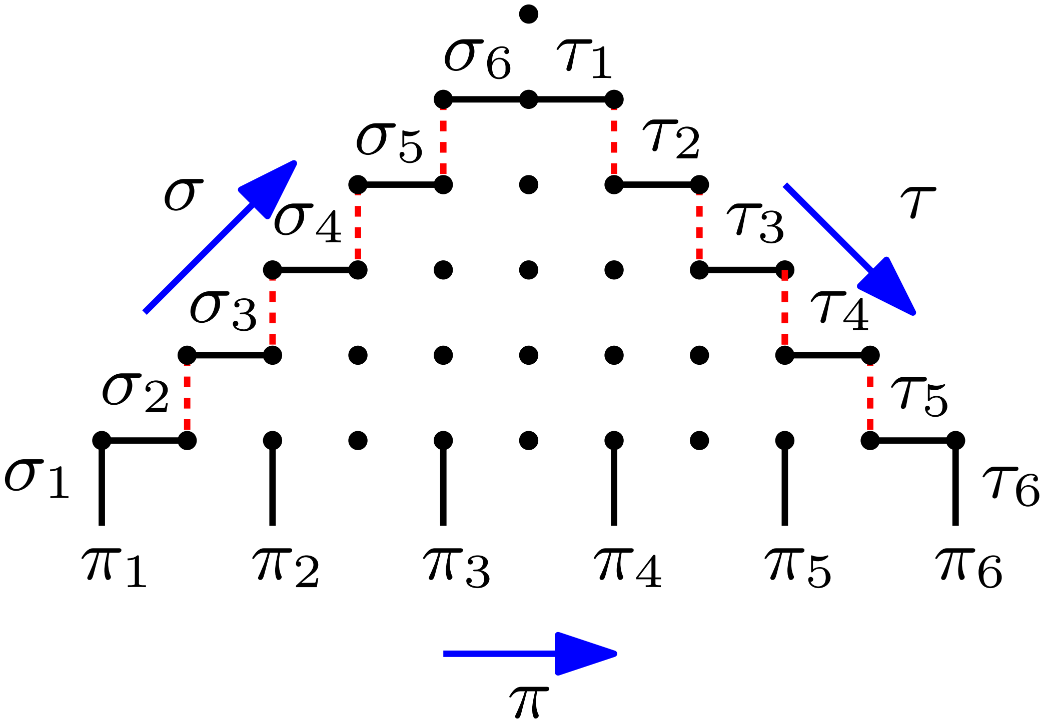
Our goal is to obtain a formula for $A_\pi(m)$, so the problem is now to evaluate the numbers $t^\pi_{\sigma,\tau}$, i.e. the number of FPLs in a triangle.

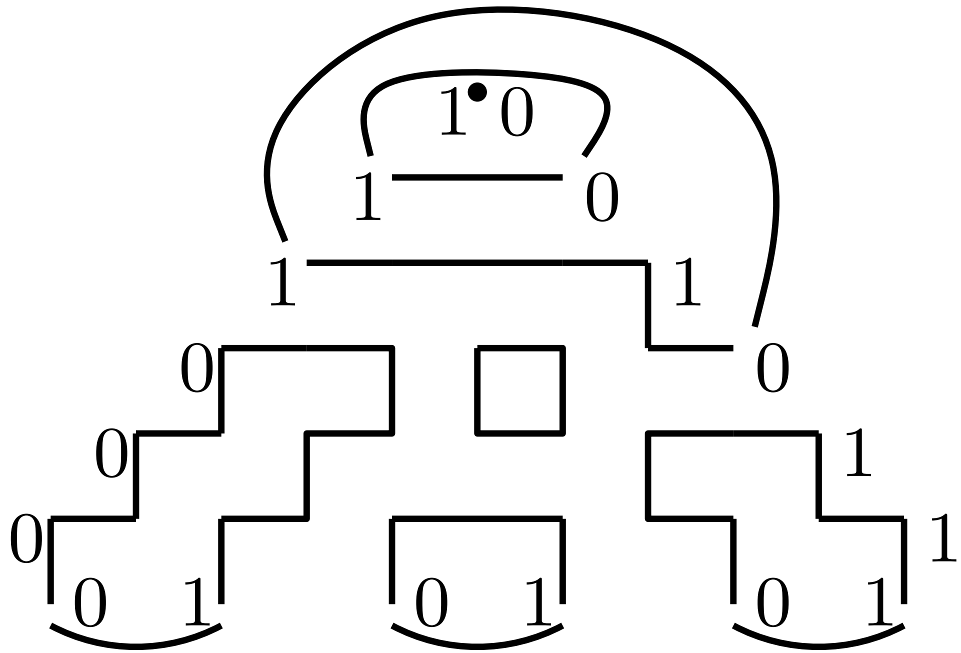
(2) FPL configurations in a triangle

The triangle $\mathcal{T}_n$

We now study the FPL configurations in the triangle, in short TFPL configurations.

Goal : understand the structure of TFPL configurations with given boundaries, and deduce enumerative results.





First properties

A vertical symmetry gives immediately

$$t^{\pi}_{\sigma,\tau} = t^{\pi^*}_{\tau^*,\sigma^*}.$$

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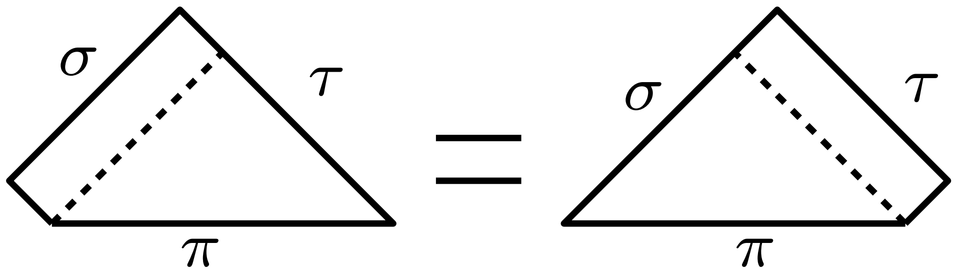
There holds also the following identity, the proof of which is based on Wieland's rotation :

Théorème [N '09]

$$\sum_{\substack{\sigma_1 \in \mathcal{D}_n \\ \sigma \to \sigma_1}} t^{\pi}_{\sigma_1,\tau} = \sum_{\substack{\tau_1 \in \mathcal{D}_n \\ \tau^* \to \tau_1^*}} t^{\pi}_{\sigma,\tau_1}.$$



Theorem [CKLN '04, N]

> $t^{\pi}_{\sigma,\tau} \neq 0$ implies $\sigma \leq \pi$.

Proof (sketch) the idea is to attach to any TFPL f certain integers $N_i(f) \geq 0$ such that if f has boundaries σ, π, τ, then

$$\pi_i - \sigma_i = N_i(f) - N_{i-1}(f)$$

for all $i \geq 1$, and $N_0(f) = 0$. These integers $N_i(f)$ actually *count* certain edges in the configuration f.
One obtains then :

$$\forall j, \sum_{i \leq j} (\pi_i - \sigma_i) = N_j(f) \geq 0,$$

which is equivalent to $\sigma \leq \pi$.

Common prefixes and suffixes

For $\sigma = \pi$,there is just one possible TFPL, which verifies $\tau = \mathbf{0}_n$.
What happens when σ is “close” to π ?
A partial answer : σ and π share a common prefix and/or suffix.

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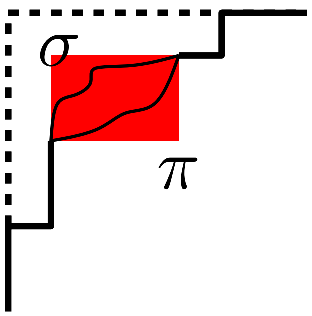
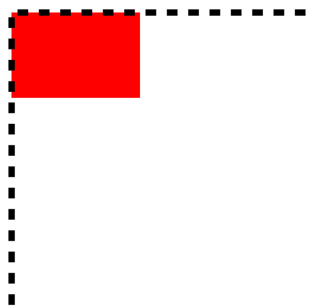
Proposition [N]

Let $\pi, \sigma, \tau \in \mathcal{D}_n$. Let also u, v, σ', π', v be such that

$$\sigma = u\sigma'v \quad \text{and} \quad \pi = u\pi'v.$$

Write $a := |u|_0 + |v|_0$ and $b := |u|_1 + |v|_1$.
Then $t^{\pi}_{\sigma,\tau} \neq 0$ implies $\tau = 0^a\tau'1^b$ for a certain τ'.

i.e. 

implies $\tau \subseteq$ 

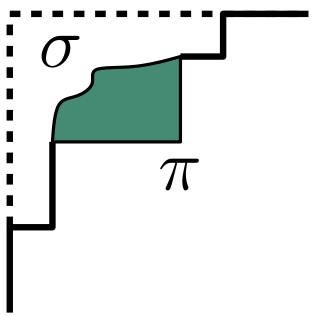
Common prefixes and suffixes

In a special case we can actually evaluate the coefficient $t^{\pi}_{\sigma,\tau}$.

Proposition

If $\pi' = 1^{n-b}0^{n-a}$, then $t^{\pi}_{\sigma,\tau}$ can be written as a determinant of size $\min(n-a, n-b)$, the entries of which are certain binomial coefficients.

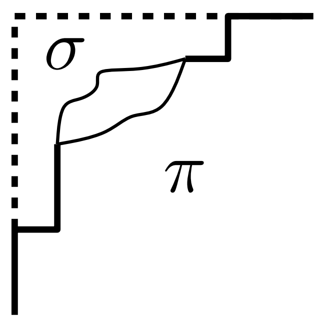
This corresponds to the case where the skew shape π/σ is a "rotated diagram".

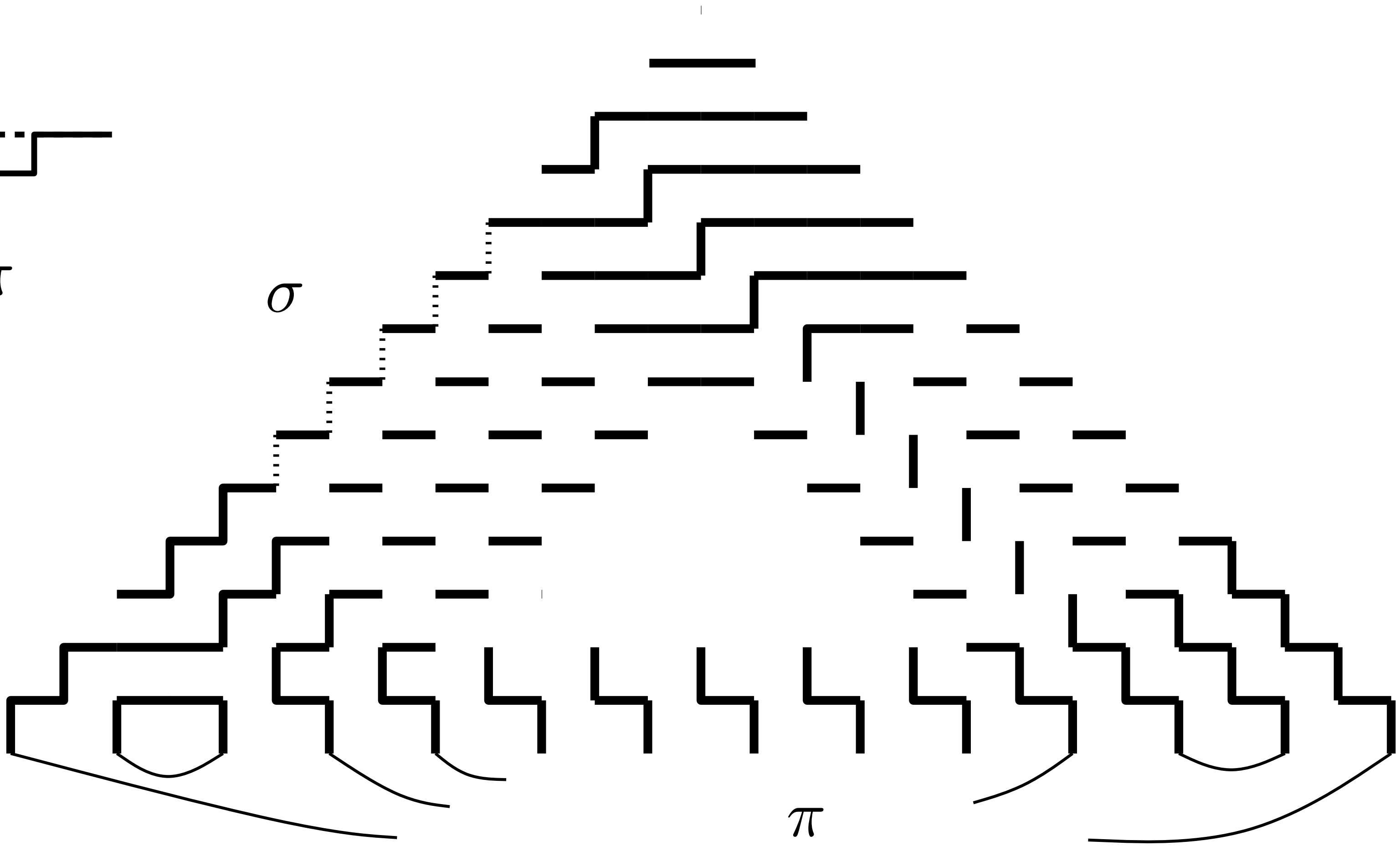


Common prefixes and suffixes

Idea of Proof : there are many fixed edges.

$$\sigma = 00100\sigma' 1011$$
$$\pi = 00100\pi' 1011$$





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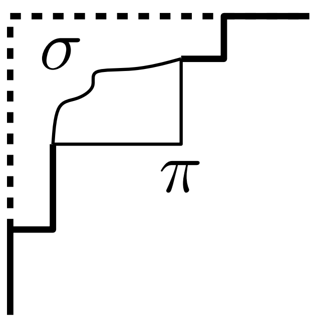

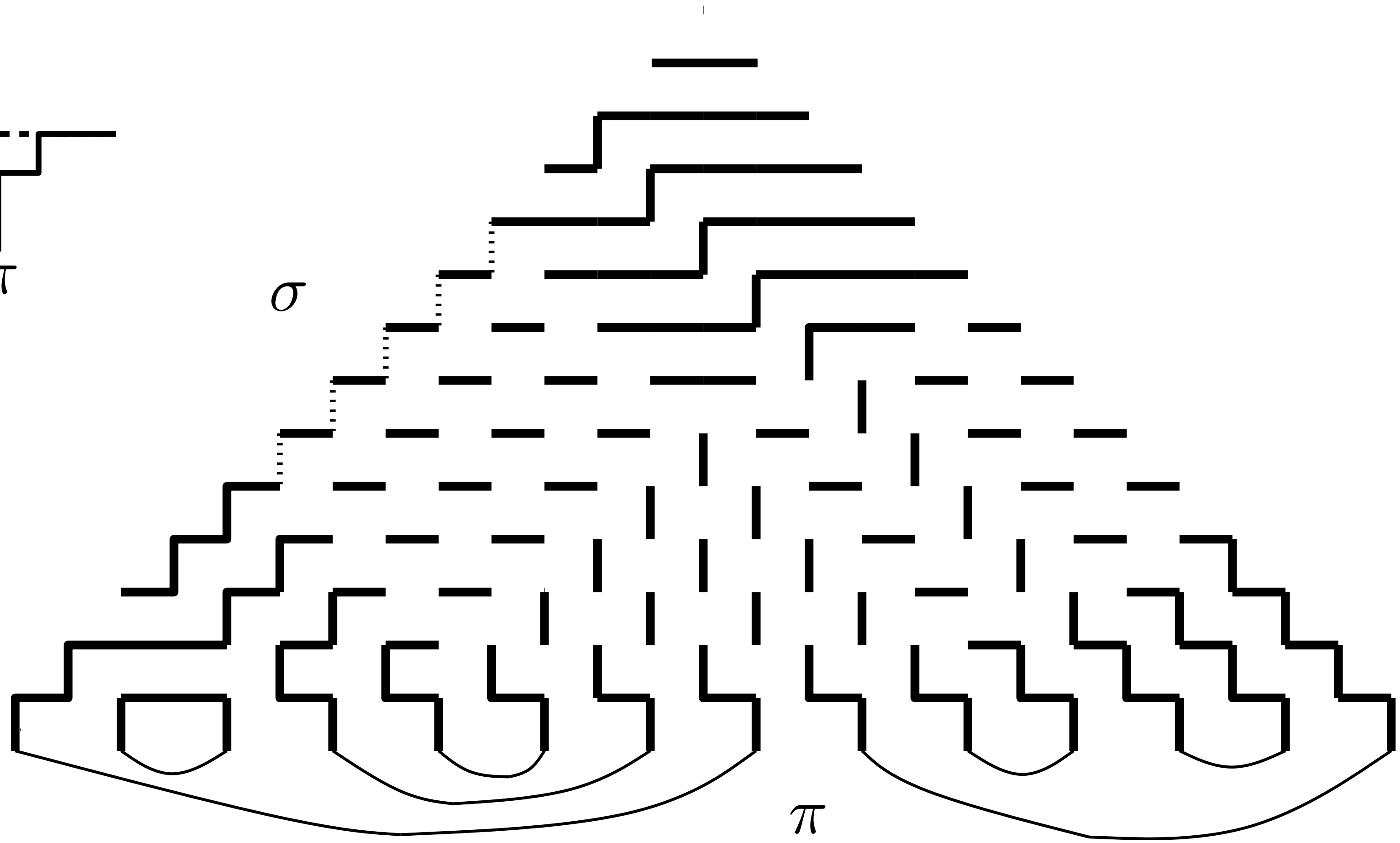

(3) Extremal TFPL
and
Littlewood-Richardson coefficients.

Extremal configurations

We previously saw the “non vanishing” constraint $\sigma \leq \pi$. Thapper proved another important such constraint :

> $t^{\pi}_{\sigma,\tau} \neq 0$ implies $d(\sigma) + d(\tau) \leq d(\pi)$.

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> $t^{\pi}_{\sigma,\tau} \neq 0$ implies $d(\sigma) + d(\tau) \leq d(\pi)$.

Following his idea, one obtains a certain identity in the case $d(\sigma) + d(\tau) = d(\pi)$:

Proposition For any$\pi \in \mathcal{D}_n$,

$$\frac{1}{H(\pi)} = \sum_{\substack{\sigma,\tau \in \mathcal{D}_n \\ d(\sigma)+d(\tau)=d(\pi)}} t^{\pi}_{\sigma,\tau} \cdot \frac{1}{2^{d(\sigma)} H(\sigma)} \cdot \frac{1}{2^{d(\tau)} H(\tau)}$$

Definition : We name extremal the TFPL with boundaries $\{\sigma, \pi, \tau\}$ verifying $d(\sigma) + d(\tau) = d(\pi)$.

Sketch of proof

(a) $t^{\pi}_{\sigma,\tau} \neq 0$ implies $d(\sigma) + d(\tau) \leq d(\pi)$.

(b) $$\frac{1}{H(\pi)} = \sum_{\substack{\sigma,\tau \in \mathcal{D}_n \\ d(\sigma)+d(\tau)=d(\pi)}} t^{\pi}_{\sigma,\tau} \cdot \frac{1}{2^{d(\sigma)} H(\sigma)} \cdot \frac{1}{2^{d(\tau)} H(\tau)}$$

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(a) $t^{\pi}_{\sigma,\tau} \neq 0$ implies $d(\sigma) + d(\tau) \leq d(\pi)$.

(b) $$\frac{1}{H(\pi)} = \sum_{\substack{\sigma,\tau \in \mathcal{D}_n \\ d(\sigma)+d(\tau)=d(\pi)}} t^{\pi}_{\sigma,\tau} \cdot \frac{1}{2^{d(\sigma)} H(\sigma)} \cdot \frac{1}{2^{d(\tau)} H(\tau)}$$

Let us recall that $A_\pi(m)$ is a polynomial of degree $d(\pi)$ whose leading coefficient is $1/H(\pi)$, and that

$$A_\pi(m) = \sum_{\sigma,\tau} t^{\pi}_{\sigma,\tau} \cdot SSYT(\sigma, n+k) \cdot SSYT(\tau^*, m+1-k-2n).$$

for k between 0 and $m-(3n-1)$. We choose then $k = m/2$ for m even and large enough. Then we obtain
(a) by comparing coefficients in degree $> d(\pi)$ and
(b) by comparing them in degree $= d(\pi)$.

Littlewood Richardson coefficients

Let λ, μ, ν be partitions, and $\Lambda(x)$ be the ring of symmetric functions of the variables $x_1, x_2, \ldots$. The Schur functions $s_\lambda(x)$ can be defined as

$$s_\lambda(x) = \sum_T \prod_i x_i^{T_i},$$

where T goes through all semistandard Young tableaux of shape λ, and T_i is the number of cells labeled i.

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Schur functions form a basis of $\Lambda(x)$. We can expand $s_\mu(x)s_\nu(x)$ on this basis, where the coefficients $c^\lambda_{\mu,\nu}$ are often called the Littlewood-Richardson (LR) coefficients.

$$s_\mu(x)s_\nu(x) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda(x)$$

Littlewood Richardson coefficients

By homogeneity of Schur functions, we have

$$c^{\lambda}_{\mu,\nu} \neq 0 \text{ implies } d(\lambda) = d(\mu) + d(\nu).$$

We have also, if $s_\lambda(x, y)$ is the symmetric function s_λ in the variables $x_1, x_2, \dots, y_1, y_2, \dots$

$$s_\lambda(x, y) = \sum_{\mu,\nu} c^{\lambda}_{\mu,\nu} s_\mu(x) s_\nu(y)$$

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If we evaluate this at $x_i = y_i = 1$ for $i = 1, \ldots, m/2$, $x_i = y_i = 0$ for $i > m/2$, we obtain polynomials in m which give the following identity in top degree $d(\lambda)$:

$$\frac{1}{H(\lambda)} = \sum_{\mu,\nu} c^{\lambda}_{\mu,\nu} \cdot \frac{1}{2^{d(\mu)} H(\mu)} \cdot \frac{1}{2^{d(\nu)} H(\nu)}$$

Littlewood Richardson coefficients

As a consequence, there exist $a_{\sigma\tau} > 0$ such that, for any $\pi \in \mathcal{D}_n$,

$$\sum_{\sigma,\tau} a_{\sigma\tau} c^{\pi}_{\sigma,\tau} = \sum_{\sigma,\tau} a_{\sigma\tau} t^{\pi}_{\sigma,\tau} \qquad (E)$$

in which σ, τ go through all words such that $d(\sigma) + d(\tau) = d(\pi)$

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Theorem [N. '09]

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Thanks to equation (E), we need only prove that $c^{\pi}_{\sigma,\tau} \leq t^{\pi}_{\sigma,\tau}$ for all σ, τ, π such that $d(\sigma) + d(\tau) = d(\pi)$.

Computing LR coefficients

There are many objects that are counted by LR-coefficients. We use here Knutson-Tao puzzles.

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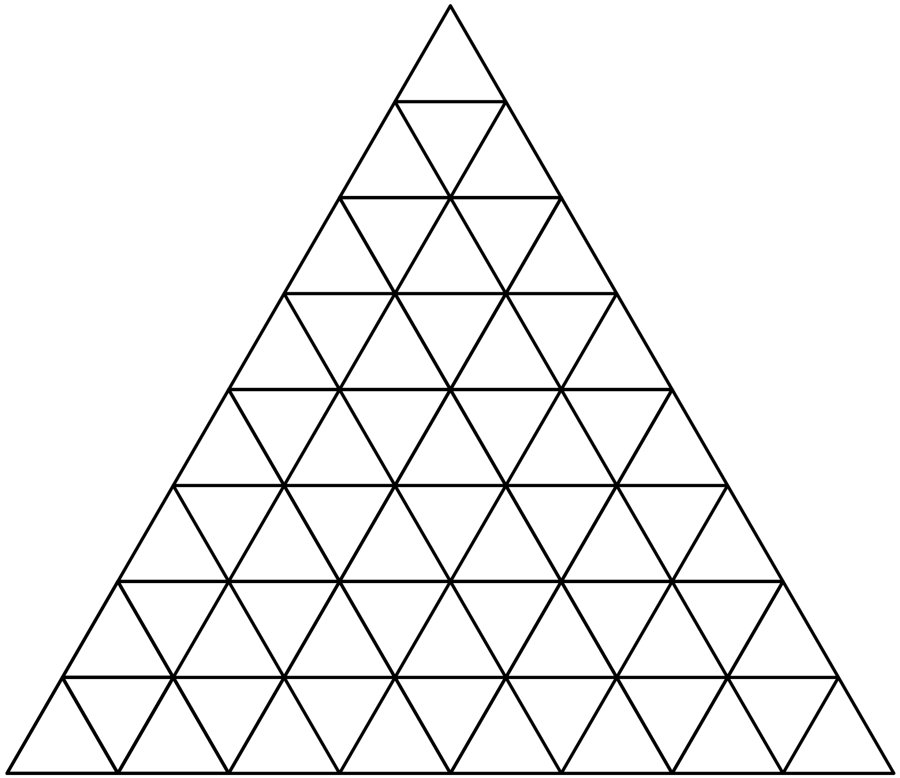
Consider a triangle of size $2n$ on the triangular lattice.



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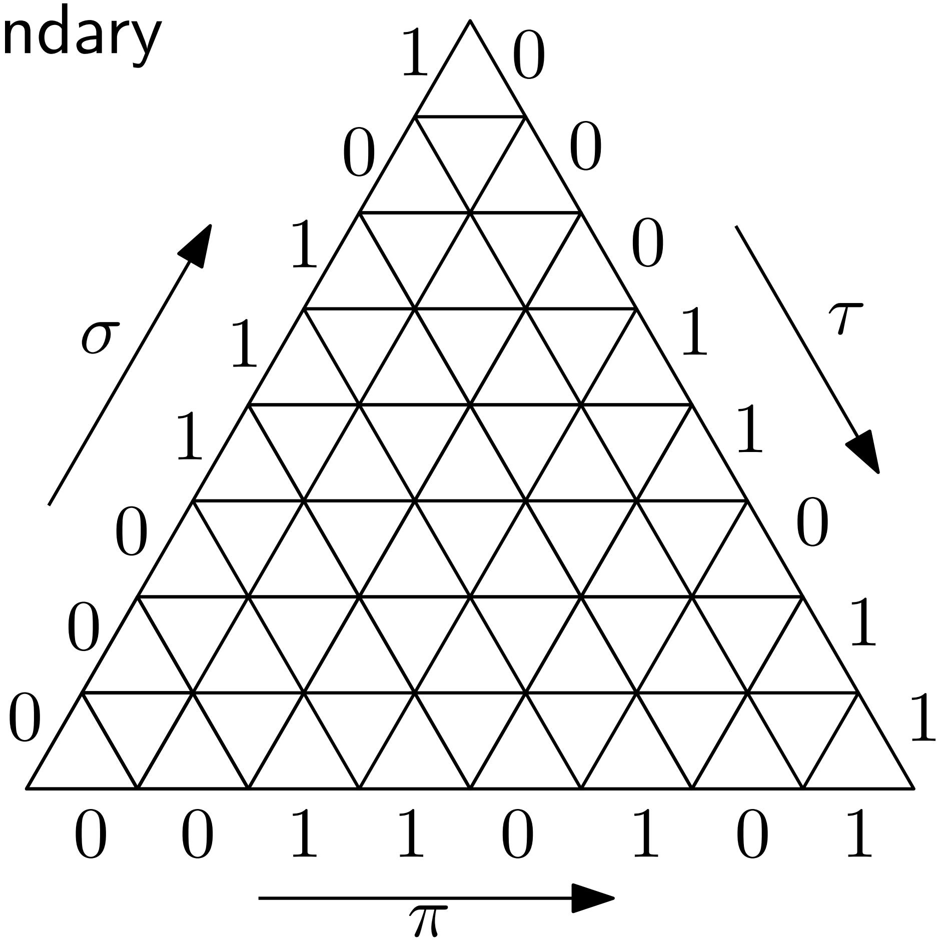
There are many objects that are counted by LR-coefficients. We use here Knutson-Tao puzzles.

Consider a triangle of size $2n$ on the triangular lattice.

Fix $\sigma, \pi, \tau \in \mathcal{D}_n$, and label the boundary edges of the triangle.

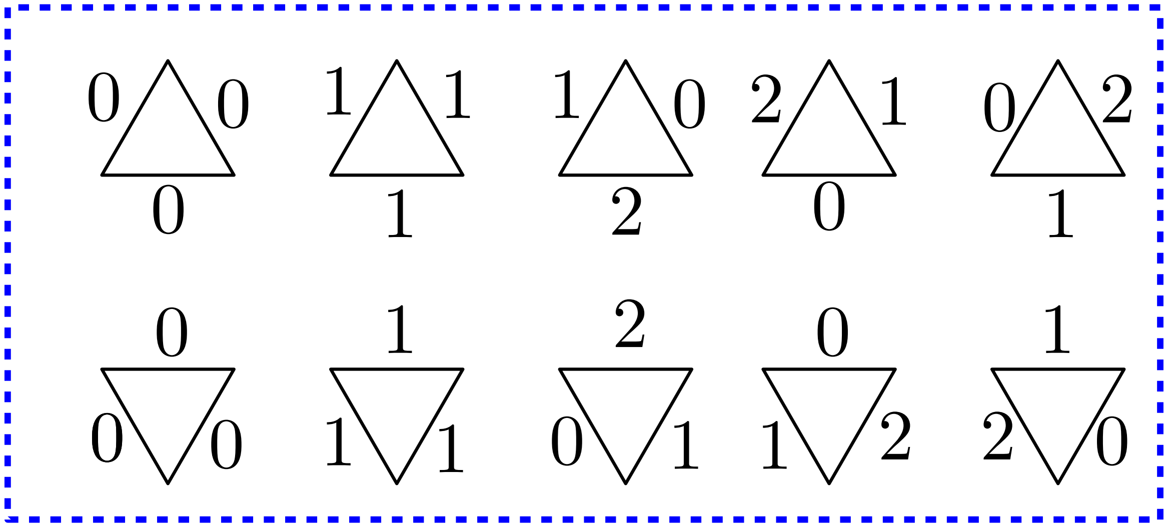
$$\pi = 00110101$$
$$\sigma = 00011011$$
$$\tau = 00011011$$



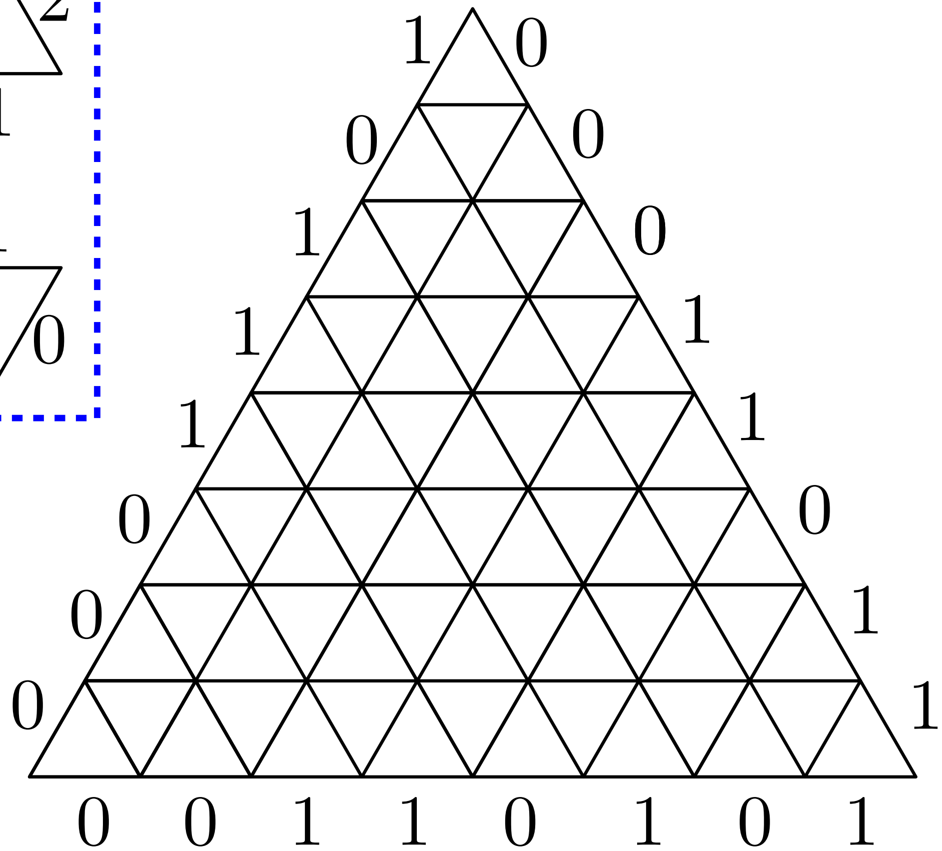
Definition

A Knutson-Tao puzzle with boundary data σ, π, τ is a labeling of each edge of the triangle by $0, 1$ or 2, such that :

- the labels on the boundary are given by σ, π, τ ;
- on each unit triangle, the induced labeling must be among :



"Only 0s, only 1s, or $0, 1, 2$ counterclockwise"



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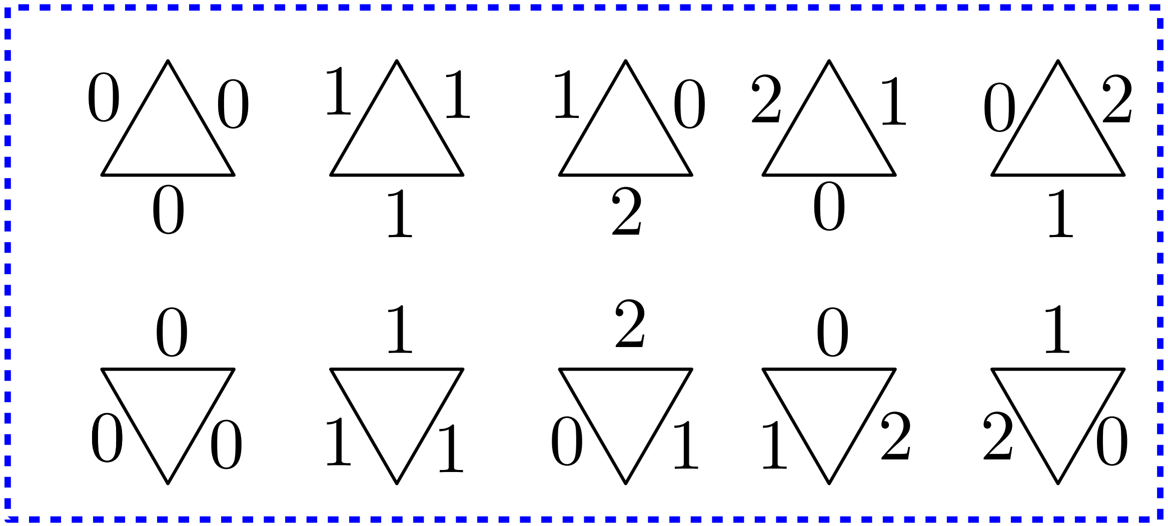
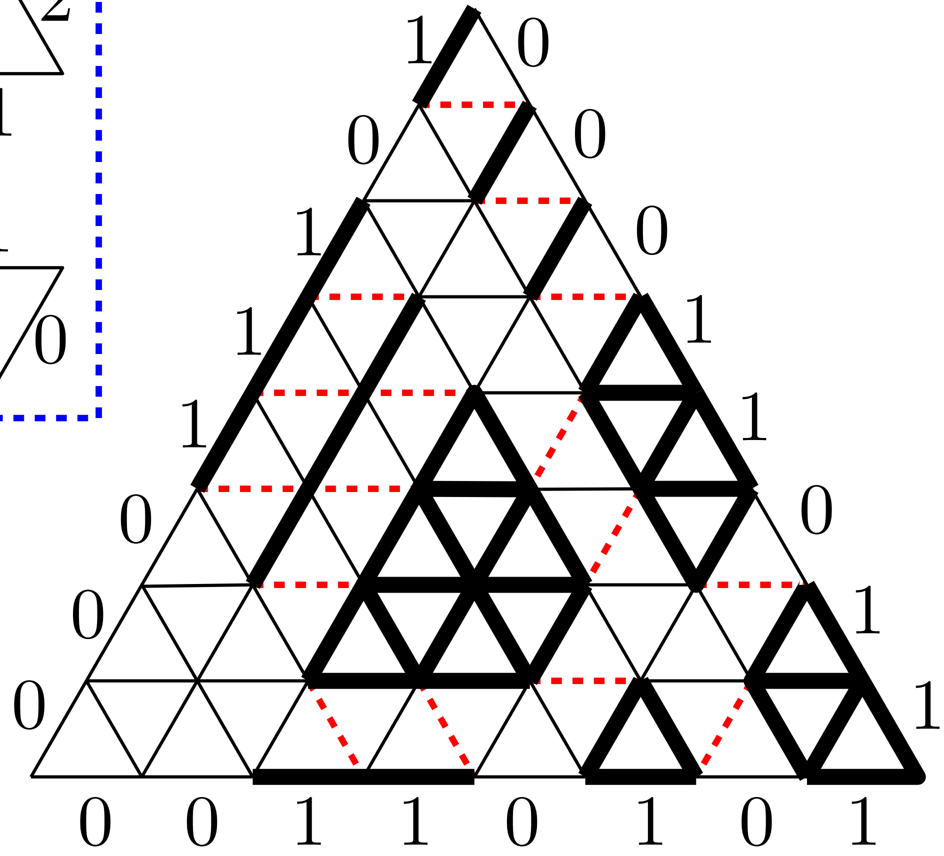


We will picture the labeling of edges as follows :

label 0

label 1

label 2



Theorem [Knutson, Tao '03][K., T. and Woodward '03]

> Let $\sigma, \tau, \pi \in \mathcal{D}_n$. Then the number of KT-puzzles with boundary data σ, π, τ is equal to the LR coefficient $c^{\pi}_{\sigma,\tau}$.

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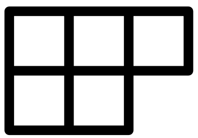
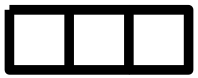
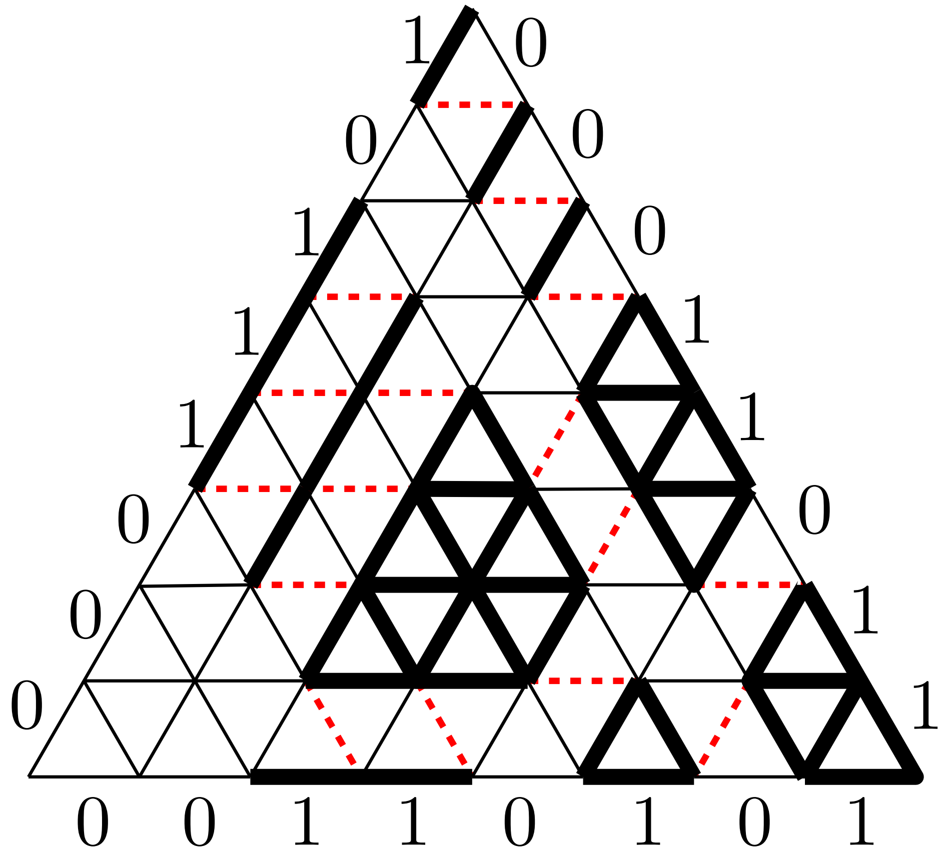
For example, it is easy to see that there is only one puzzle with the boundary data of the example.

so $c^{\lambda}_{\mu,\nu} = 1$ where

$\lambda =$ 

$\mu =$ 

$\nu =$ 



From KT puzzles to TFPL configurations.

We fix $\sigma, \pi, \tau \in \mathcal{D}_n$, such that $d(\sigma) + d(\tau) = d(\pi)$. We will define a map Φ.

$$\text{KT puzzles with boundary data } \sigma, \pi, \tau$$
$$\Big\downarrow \Phi$$
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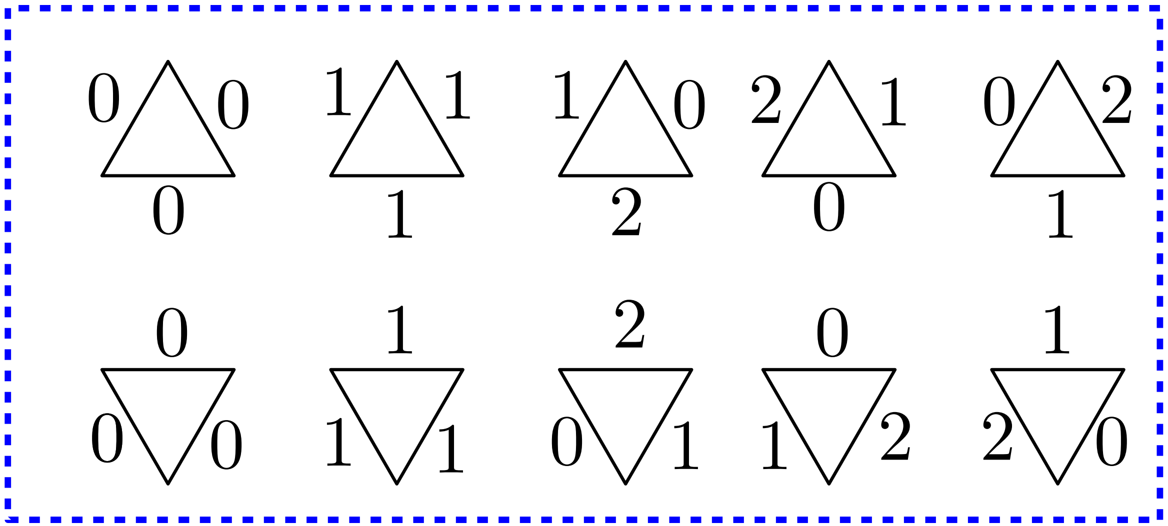
KT puzzles with boundary data σ, π, τ

$$\downarrow \Phi$$

TFPL configurations with boundaries σ, π, τ

The map is local : it changes every small labeled triangle of the puzzle to a piece of a path of a TFPL configuration.



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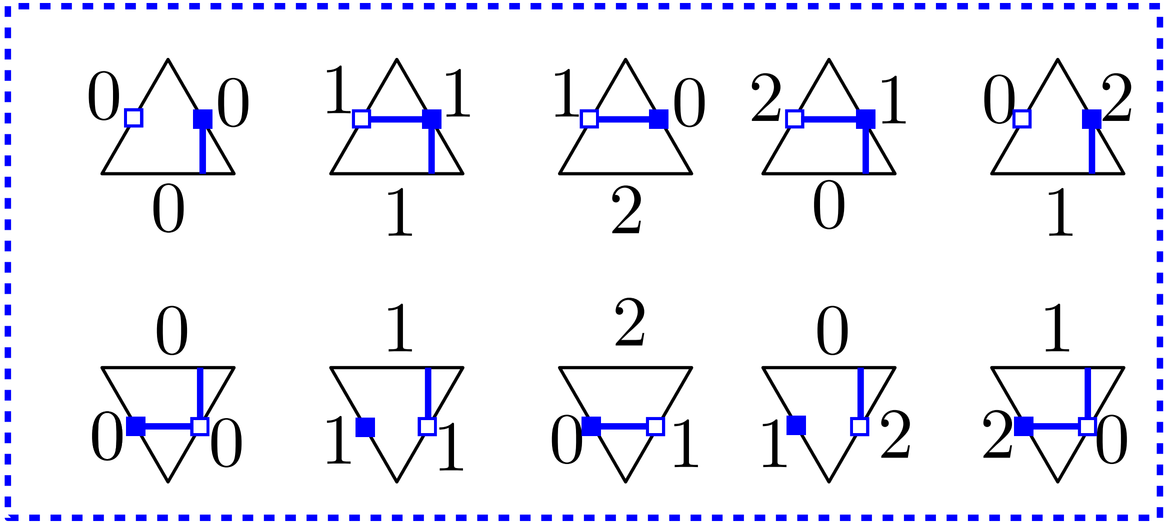

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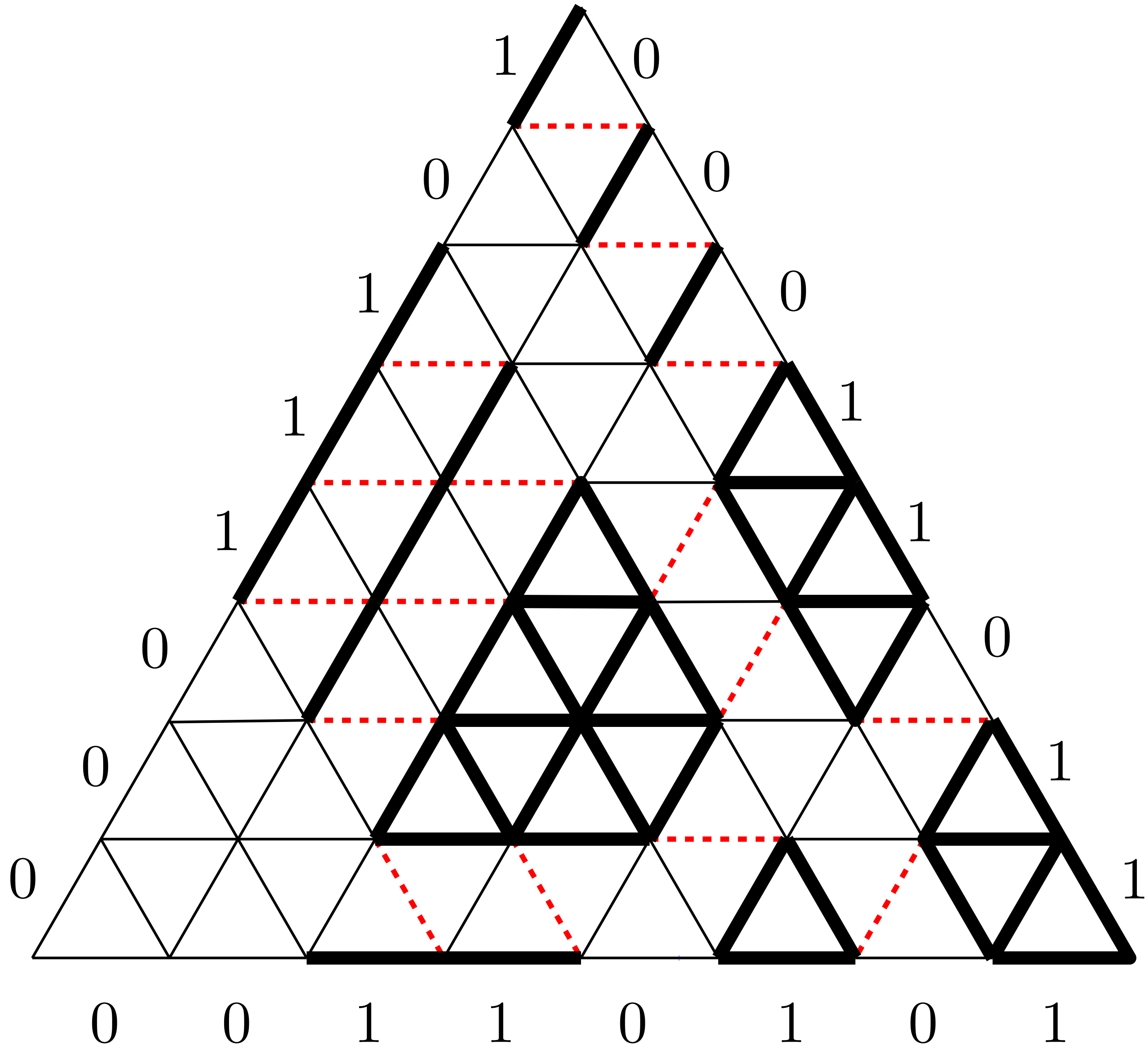

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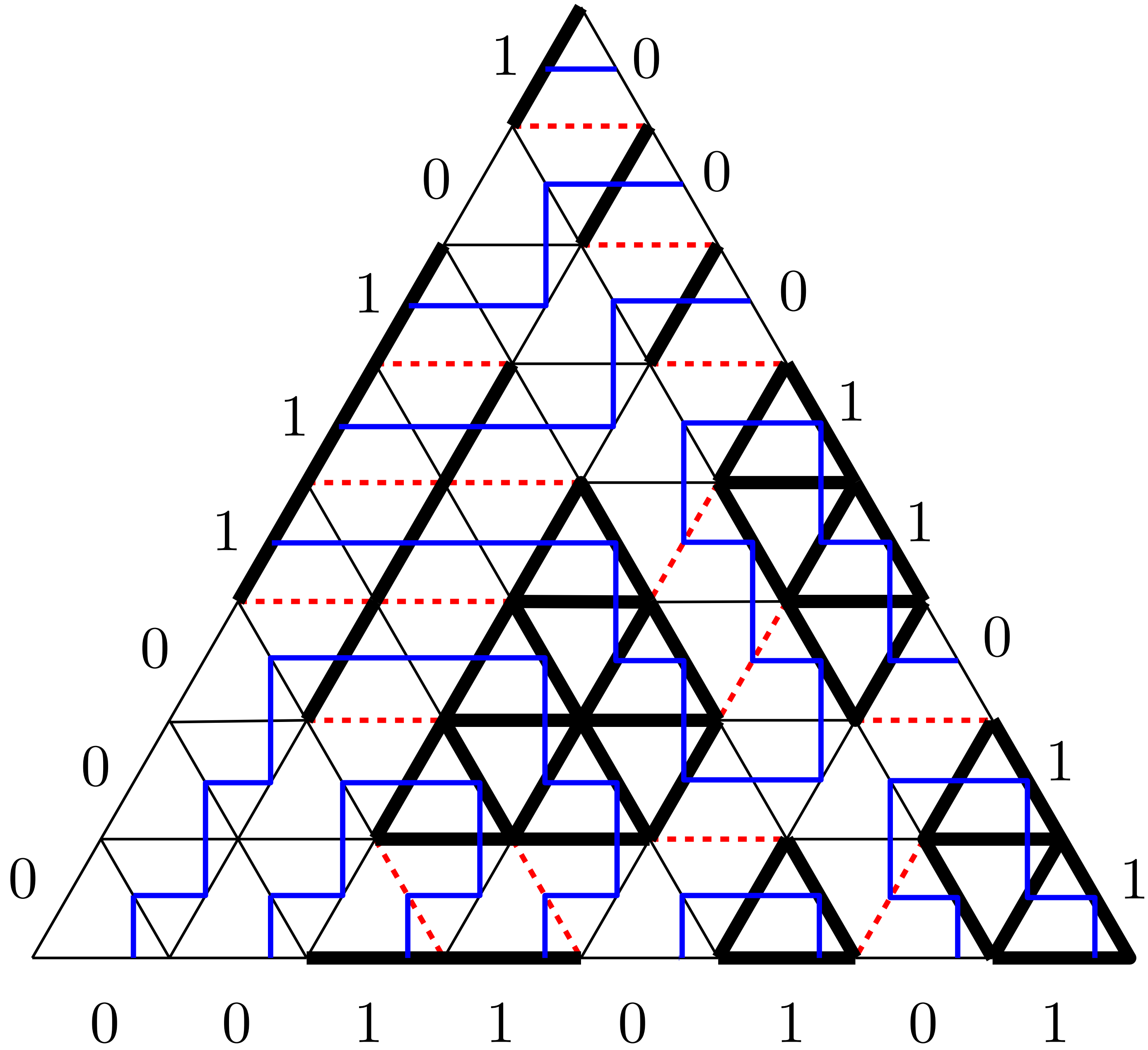

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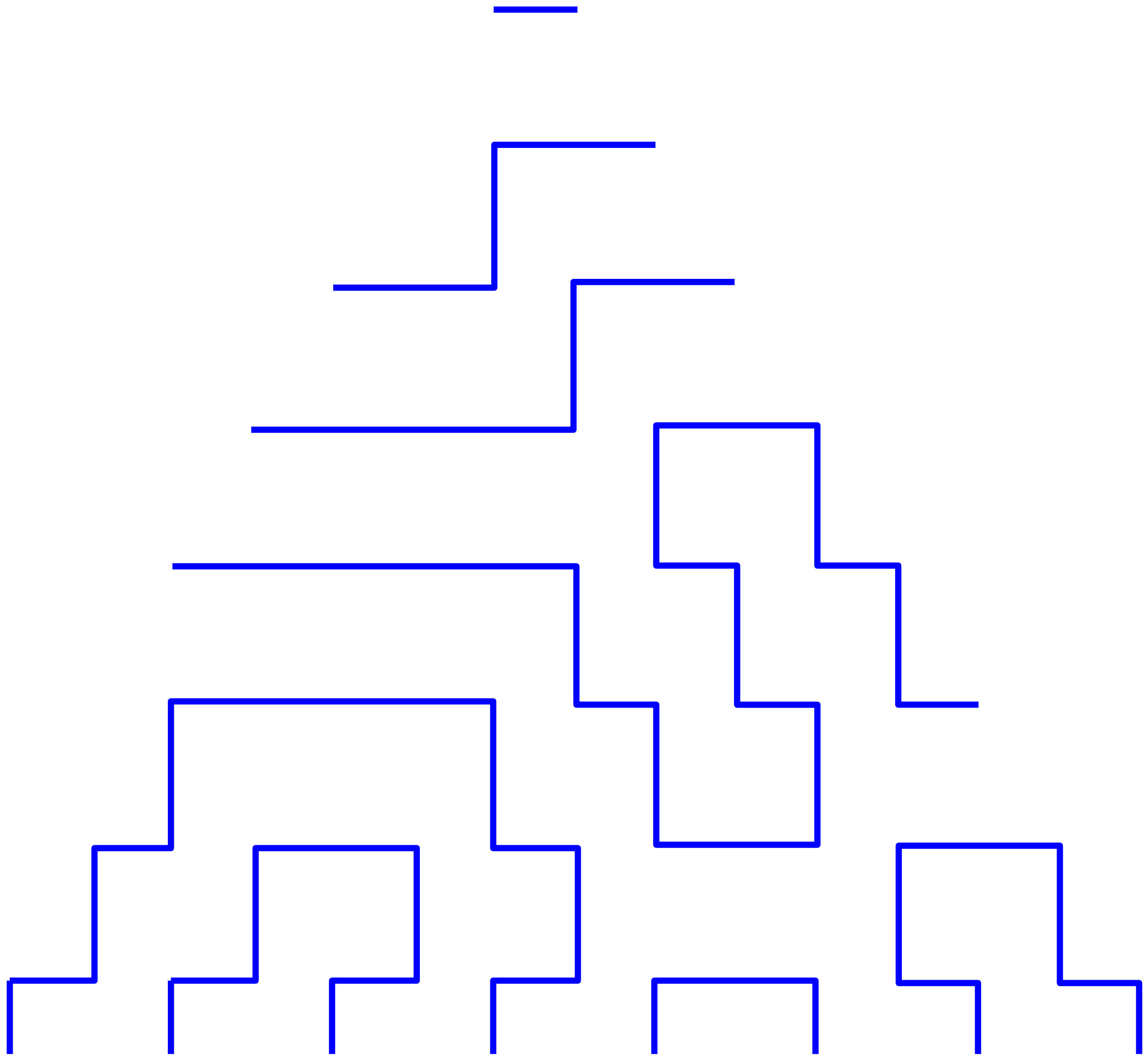
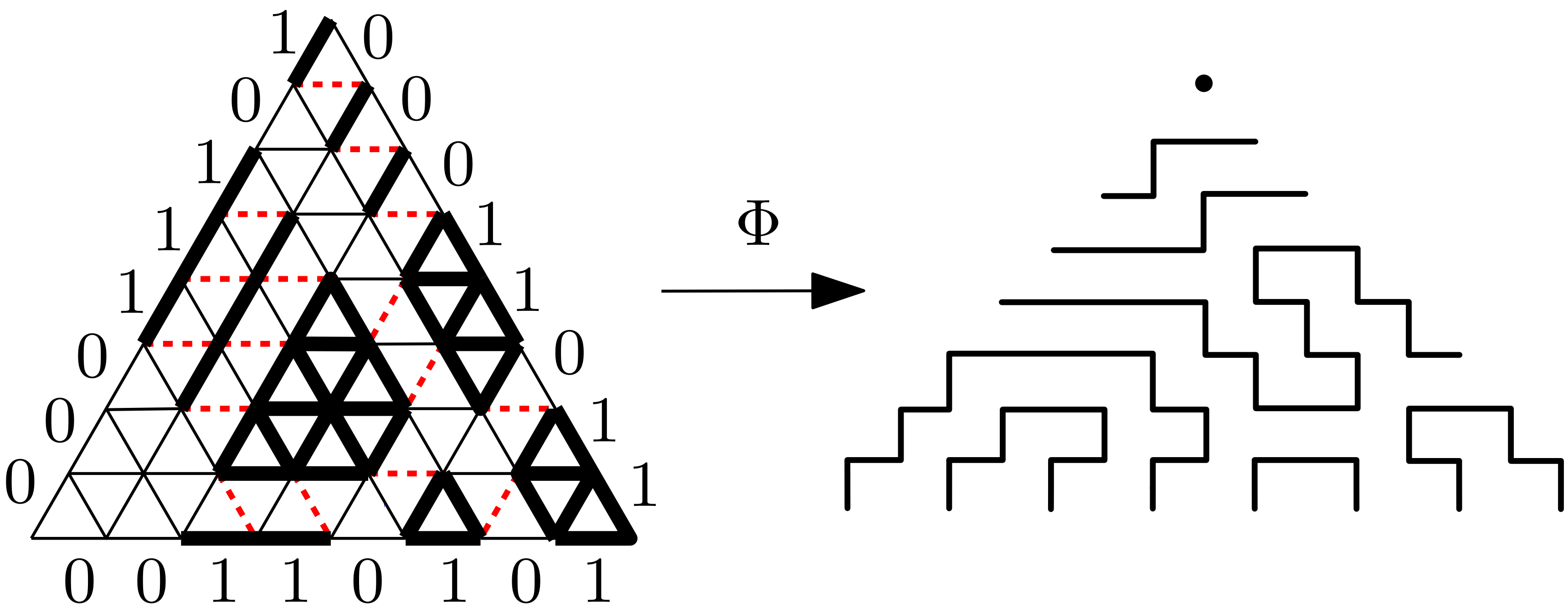


Φ is the wanted bijection

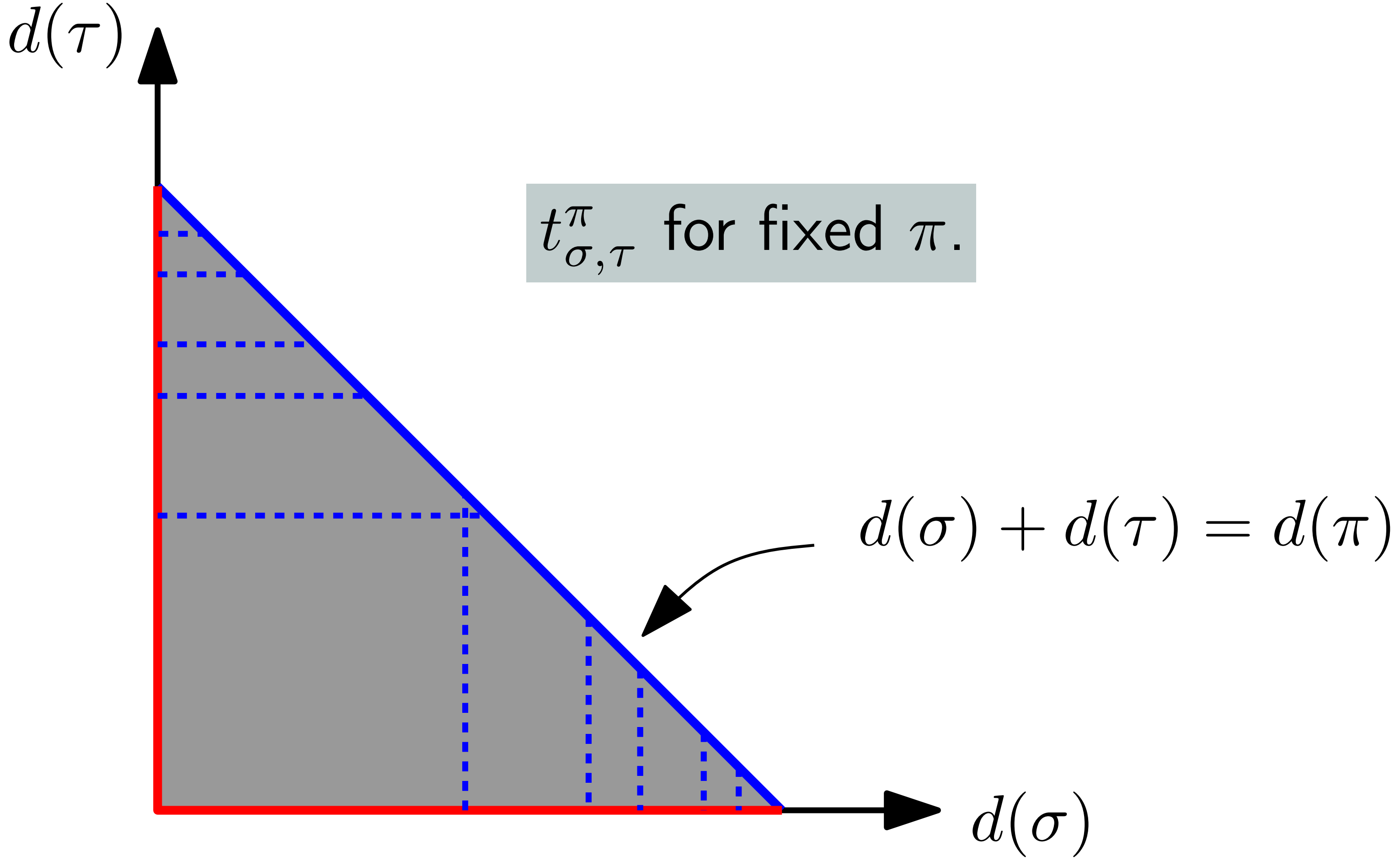
One has to prove that Φ is :

1. well defined :
   - the vertices of $\Phi(puzzle)$ are of degree 2 ,
   - $\Phi(puzzle)$ verifies the boundary conditions σ, τ.
   - the connectivity of external edges given by π is respected.
2. injective.



Conclusion

We have obtained enumerative results for certain numbers $t^{\pi}_{\sigma,\tau}$ (in blue). In red are the coefficients $t^{\pi}_{\sigma,\mathbf{0}_n}$.



Conclusion

– To compute A_X, one needs all coeffs $t^{\pi}_{\sigma,\tau}$, and not only the extremal ones. A natural parameter to partition these numbers is $exc(\pi, \sigma, \tau) := d(\pi) - d(\sigma) - d(\tau) \geq 0$.
The LR coefficients form the base case $exc(\pi, \sigma, \tau) = 0$; what are the general $t^{\pi}_{\sigma,\tau}$?

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The LR coefficients form the base case $exc(\pi,\sigma,\tau) = 0$; what are the general $t^{\pi}_{\sigma,\tau}$?

– Other direction (based on [Thapper '07]).The polynomials $A_\pi(m)$ verify linear recurrences

$$A_\pi(m) = \sum_{\alpha \leq \pi \in \mathcal{D}_n} c_{\alpha\pi} A_\alpha(m-1),$$

where $c_{\alpha\pi}$ are integers, defined in terms of the coefficients $t^{\pi}_{\sigma\mathbf{0}_n}$. What are these coefficients $c_{\alpha\pi}$?

Conclusion

– To compute A_X, one needs all coeffs $t^{\pi}_{\sigma,\tau}$, and not only the extremal ones. A natural parameter to partition these numbers is $exc(\pi,\sigma,\tau) := d(\pi) - d(\sigma) - d(\tau) \geq 0$.
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– Related work : joint with T. Fonseca, nice conjectures about the polynomials $A_\pi(m)$ pointing to combinatorial reciprocity for them ; cf arXiv.CO two days ago.

Vielen Dank für Ihre Aufmerksamkeit !