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Signed Enumeration of Ribbon Tableaux with Local Rules and Generalizations of the Schensted Correspondence.

Dominique Gouyou-Beauchamps and Philippe Nadeau

Abstract. Sergey Fomin defined the general framework of *dual graded graphs* that extends the classical Schensted correspondence to a much wider class of objects. His approach is both bijective and algebraic, the latter being inspired by the works of Stanley on differential posets.

In this work we present an extension of Fomin's work, through the signed enumeration of ribbon tableaux. We consider ribbons of different sizes, so we do not deal with graded graphs, yet algebraic and bijective techniques can still be developed. We will also give an application of our work to the computation of column sums in the character table of the symmetric group.

Résumé. Sergey Fomin a défini le cadre général de *graphes gradués en dualité* qui étend la correspondance de Schensted à une classe plus vaste d'objets ; son approche est bijective et possède également un pendant algébrique inspiré des travaux de Stanley sur les ensembles ordonnés différentiels.

Dans ce travail nous présentons une extension possible du cadre défini par Fomin, via l'énumération signée des tableaux de rubans. Nous considérons des rubans de tailles diverses, de sorte que nous n'avons plus affaire à des graphes gradués, et néanmoins des techniques algébriques et bijectives peuvent être développées ici aussi. Nous donnerons également une application de notre travail au calcul de la somme des entrées d'une colonne dans la table des caractères du groupe symétrique.

Introduction

In the articles [**Fom86, Fom94, Fom95**], Sergey Fomin founded a general theory of 'dual graded graphs' which can be seen as the theory of 'Schensted like' correspondences. One of the applications is the extension of the Schensted correspondence to the case of ribbon tableaux where all ribbons have the same size.

The article [**Whi83**] by White is a bijective proof of an orthogonality relation in the character theory of the symmetric group $\mathcal{S}_n$; it relies on the Murnaghan-Nakayama rule that interprets character values as sums over ribbon tableaux, where ribbons may have different sizes. White's proof is an insertion and deletion algorithm of which the classical Schensted correspondence is a special case.

We will use and extend Fomin's techniques to deal with the case of general ribbon tableaux. The benefits will go both ways. On the one hand we will be able to extend the results of White, in particular by being able to prove a symmetric version of his algorithm (Theorem 3.3); on the other hand the ideas developed can be applied to graphs more general than dual graded graphs.

Section 1 contains mainly standard definitions about ribbon tableaux. Section 2 introduces the concepts of hook permutations and involutions. Section 3 contains the main results of the paper concerning the enumeration of ribbon tableaux. Proofs of these results are sketched in sections 4 and 5, first in a bijective fashion, and then with an algebraic approach. This parallels the works of S. Fomin [**Fom95, Fom94**] where these two techniques are presented in the context of dual graded graphs. The bijective approach is a 'back and forth' algorithm whose proof relies on the involution principle of Garsia and Milne [**GM81b, GM81a**].

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In Section 6 we will use our results to compute the column sums of the character table of the symmetric group, and relate this enumeration to others. Finally in the conclusion we will explain how to extend the results in the context of the *ribbon graph*, and for more general *dual layered graphs.*

1. Definitions

1.1. Ribbons and Ribbon Tableaux. If $2k$ is a positive integer then we define its *double factorial* $(2k)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$. It is the number of partitions in 2-subsets of $[\![2k]\!]$, and also the number of fixed point free involutions on $[\![2k]\!]$; we use here the notation $[\![n]\!]$ to denote the integers between 1 and n. A permutation σ of $[\![\ell]\!]$ is a bijection from $[\![\ell]\!]$ to itself. Permutations will be in general written as words $\sigma_1 \dots \sigma_\ell$ where $\sigma_i = \sigma(i)$.

A composition is a finite sequence of positive integers, and a partition λ of a nonnegative integer n is a nonincreasing sequence $\lambda_1 \geqslant \lambda_2 \geqslant \dots \lambda_k > 0$ of integers such that $\sum_i \lambda_i = n$; k is the number of parts of λ, and $|\lambda| = n$ is its size. Similar definitions hold for compositions as well. A partition $\widetilde{\mu}$ can be associated to each composition μ by rearranging it in nonincreasing order; it has therefore the same number of parts counted with multiplicity. We will note $\mathbf{Y}$ the set of all partitions, and $\mathbf{Y}_n$ the set of the partitions of size n. We will identify a partition λ with its *Ferrers diagram*: it is a left justified set of unit cells in $\mathbb{Z}^2$, where the ith row has λ_i cells; see Figure 1.

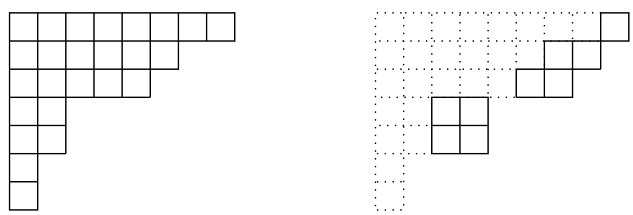


FIGURE 1. Ferrers diagrams representing the partition $(8,6,5,2,2,1,1)$ of size 25 (left) and the skew shape $(9,8,7,4,4,1,1)/(8,6,5,2,2,1,1)$ (right).

Two partitions $\lambda \subseteq \mu$ (inclusion of Ferrers diagrams) define a *skew shape* μ/λ. We will often think of a skew shape as the collection of cells $\mu\backslash\lambda$; this will cause no confusion, even though different skew shapes may define the same collection of cells. The size of μ/λ is its number of cells and is denoted $|\mu/\lambda|$. A set S of cells in $\mathbb{Z}^2$ is *connected* if given any 2 cells c and c' of S there exist cells $c = c_0, c_1, \dots, c_t = c'$ in S such that any 2 consecutive cells in the sequence share a common side.

A *ribbon* is then a connected skew shape containing no 2×2 square of cells ; they are also called rim hooks in the literature. We need some specific definitions concerning ribbons; let $r = \mu/\lambda$ be a non empty ribbon. It is said to be μ-addable and λ-removable. The height $\mathrm{ht}(r)$ is the number of rows of r minus one, and the sign $\varepsilon(r)$ is defined as $(-1)^{\mathrm{ht}(r)}$. By convention, the empty ribbons will have positive sign $\varepsilon(\lambda/\lambda) = 1$. The Southwestmost cell of r is its *tail*, and its Northeastmost cell is its *head*.

For i positive, we will note Rib_i the set of ribbons of size i, and Rib the set of all non empty ribbons. We shall often think of $\mathbf{Y}$ as the vertices of a nonoriented graph GR with edges given by Rib; each of these edges carries a sign given by the corresponding ribbon. This graph is partitioned in levels corresponding to partitions of a fixed size, and we will think of bigger partitions to be at a higher level: to add a ribbon is then to go up one step, and to remove a ribbon is to go down one step. See Figure 3 for the first five levels of the graph GR.

DEFINITION 1.1. A *ribbon tableau* of shape $\lambda \in \mathbf{Y}$ and length ℓ is a numbering of the cells of λ by numbers from 1 to ℓ such that :

- The numbers are nondecreasing in each row and each column.
- for each $i \in [\![\ell]\!]$ the cells numbered i form a non empty ribbon shape $r^{(i)}$.

Equivalently, a ribbon tableau is a chain of partitions of length ℓ from $\emptyset$ to λ such that successive partitions form ribbons, i.e. a path from $\emptyset$ to λ going up in the graph GR. We will note the set of ribbon tableaux of shape λ and length ℓ by $RT_{\lambda,\ell}$. The sign $\varepsilon(P)$ of a ribbon tableau P is the product of the signs

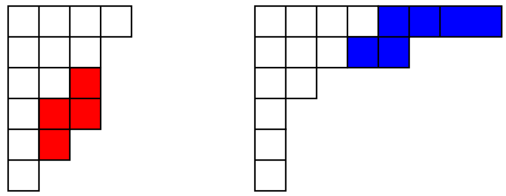


FIGURE 2. Examples of ribbons. The left one has size 4, height 2 and sign +1, while the one on the right has size 6, height 1 and sign −1.

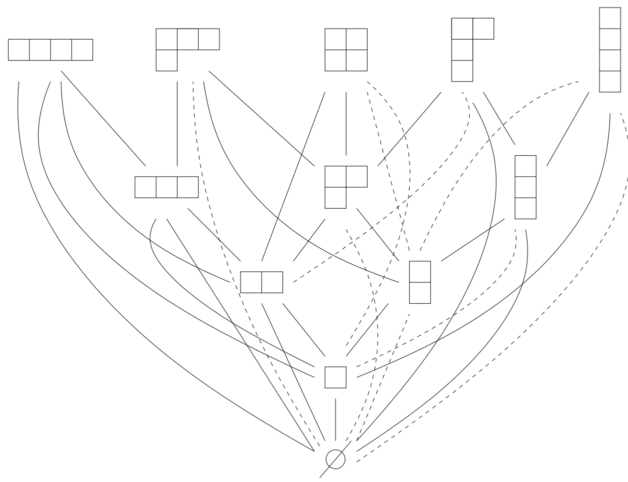


FIGURE 3. First levels of the graph of ribbons $GR = (\mathbf{Y}, Rib)$; negative edges are represented by dashed lines.

of its defining ribbons $r^{(i)}$. The content $c(P)$ of a ribbon tableau P is the composition consisting of the successive sizes of the ribbons forming P, and we will let $RT_{\lambda,\mu}$ be the set of ribbon tableaux of shape λ and content μ.

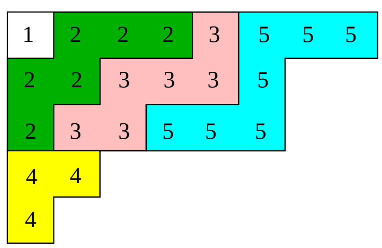


FIGURE 4. A ribbon tableau of shape $(8, 6, 6, 2, 1)$, of content $(1, 6, 6, 3, 7)$, and of sign $(-1)^0(-1)^2(-1)^2(-1)^1(-1)^2 = -1$

1.2. Signed sets and signed bijections. In this work we have to deal with signed enumerations, so we need some definitions and notations to explain what we mean by a bijection in this context. All sets are assumed to be finite.

DEFINITION 1.2 (Signed Sets). A signed set is a set A together with a partition $A = A^+ \sqcup A^-$ (with either of these sets possibly empty). The members of A^+ are *positive* elements, those of A^- are *negative*.

A function f between two signed sets is sign preserving (resp. sign reversing) if a and $f(a)$ have the same sign (resp. different signs) for all a. Fixed points of a function i form the set $Fix(i)$.

DEFINITION 1.3 (Signed bijections). A signed bijection between the signed sets A and B is the data of 3 functions i_A, i_B and φ such that i_A (resp. i_B) is an involution on A (resp. B) which is sign reversing outside its fixed points , and φ is a sign preserving bijection between $Fix(i_A)$ and $Fix(i_B)$.

The *signed cardinal* of a signed set A is $|A|_\pm = |A^+| - |A^-|$. A *signed bijection* between A and B proves that $|A|_\pm = |B|_\pm$, and this is equivalent to $|A^+| + |B^-| = |B^+| + |A^-|$. A bijection proving this last equality , i.e. a bijection between the unsigned sets $A^+ \sqcup B^-$ and $B^+ \sqcup A^-$, is clearly equivalent to a signed bijection between A and B. This explains why signed bijections are the correct generalizations of bijections, in that they give a combinatorial explanation of the equality of signed cardinals.

Our objects of study here are the sets $RT_{\lambda,\mu}$, the sign being given by the function ε; the signed cardinal of a set $R = RT_{\lambda,\mu}$ can thus be written by $\sum_{P\in R} \varepsilon(P)$. Note also that unless explicitly stated, usual sets are naturally considered as positive sets.

2. Hook Permutations and Hook Involutions

2.1. Hook Permutations. A *hook* is a non empty ribbon of shape $\lambda/\emptyset$, or equivalently a partition of the type $(k, 1, 1, \cdots, 1)$. Note that a hook is characterized by the data of its size s and height $h \in [\![0, s-1]\!]$.

DEFINITION 2.1. A hook permutation (H, σ) is an ordered sequence $H = (H_1, \ldots, H_\ell)$ of ℓ hooks, together with a permutation σ of $[\![\ell]\!]$. The length of a hook permutation is ℓ, its size is $\sum_i |H_i|$, and its content is the composition $(|H_1|, \ldots, |H_\ell|)$.

We will write $\mathcal{HP}$ for the set of hook permutations, its elements of content μ forming $\mathcal{HP}(\mu)$ where μ is any composition. Hook permutations can be represented by the list H where the cells of hook H_i are numbered by $\sigma(i)$, or by square matrices of size ℓ such that entry (i, j) is empty unless $j = \sigma(i)$ in which case it is occupied by the ith hook H_i. Illustrations are given Figure 5.

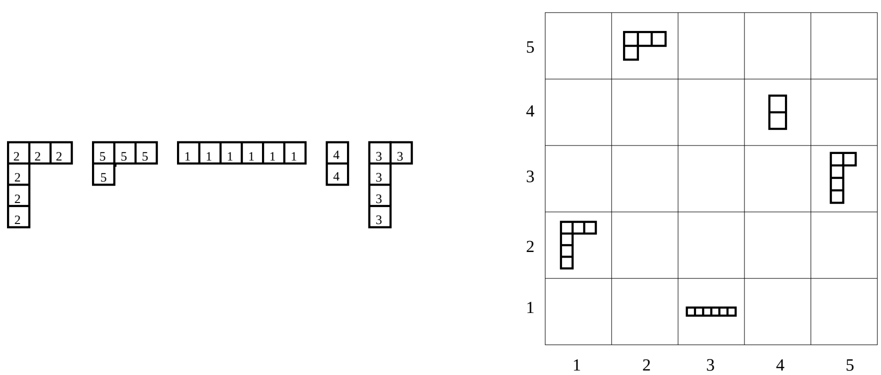


FIGURE 5. Two representations of the same hook permutation of length 5, size 23 and content $(6, 4, 6, 2, 5)$.

2.2. Hook Involutions.

DEFINITION 2.2. Hook involutions are hook permutations with a matrix representation that is symmetric with respect to the diagonal $i = j$.

In other words, they are hook permutations (H, σ) such that σ is an involution, and $H_i = H_j$ if $j = \sigma(i)$. For a hook involution $I = (H, \sigma)$, we define its *sign* as $\varepsilon(I) = \prod_{i/\sigma(i)=i} \varepsilon(H_i)$. It is the product of the signs of the hooks associated to fixed points (considering all hooks would not change the value of the product, by definition of a hook involution).

We define $\mathcal{HI}$ as the signed set of hook involutions, $\mathcal{HI}(\mu)$ as the signed subset of those hook involutions with content μ, and finally $\mathcal{HI}_{spec}(\mu)$ as the elements of $\mathcal{HI}(\mu)$ all of whose fixed points are hooks of odd size and of height 0. $\mathcal{HI}_{spec}(\mu)$ is clearly a positive set.

LEMMA 2.3. *There is an involution on $\mathcal{HI}(\mu)$ which is sign reversing outside the set $\mathcal{HI}_{spec}(\mu)$.*

Proof: If $I = (H, \sigma) \notin \mathcal{HI}_{spec}(\mu)$ then let $i = \sigma(i)$ be its smallest fixed point contradicting the definition of $\mathcal{HI}_{spec}(\mu)$. Let h be the height of the hook H_i. If H_i is of even size , then we let H'_i be the hook of the same size and of height $h+1$ (resp. $h-1$) if h is even (resp. odd). If H_i is of odd size, so that necessarily $h \neq 0$ by the definition of H_i, then we let H'_i be the hook of the same size and of height $h+1$ (resp. $h-1$) if h is odd (resp.even).

Let H' be the hook list equal to H except in position i where H'_i replaces H_i. If we define $f(I) = (H', \sigma)$, then we have the desired sign reversing involution on $\mathcal{HI}(\mu) \setminus \mathcal{HI}_{spec}(\mu)$. □

Corollary 2.4. $|\mathcal{HI}(\mu)|_\pm = |\mathcal{HI}_{spec}(\mu)|$.

We will give some consequences of this corollary in Section 6.

3. Main Results

When the contents of tableaux are partitions , such that the first one is smaller than the second for the reverse lexicographical order, the following theorem is the work of White [**Whi83**]. We'll prove it in the following section by formulating his 'insertion and deletion' algorithm in a version with 'local rules' in the spirit of Fomin [**Fom95**]; the advantages, as we will see, are the wider fields of applications of such techniques. One could also argue that it simplifies somewhat the proof of White.

Theorem 3.1. *There is a signed bijection $(\emptyset, i, \varphi)$ between hook permutations of size n and length ℓ, and pairs of ribbon tableaux of size n and length ℓ. This bijection* preserves contents *in the following sense: if $i(P,Q) = (P_1, Q_1)$, then $c(P) = c(P_1)$ and $c(Q) = c(Q_1)$; and if $\varphi(H, \sigma) = (P, Q)$, then $c(H) = c(Q)$ and $c(\sigma(H)) = c(P)$.*

This theorem has the following enumerative consequences :

Corollary 3.2. *Let μ, ν be two compositions of n with ℓ parts.*

$$\sum_{\substack{\lambda \in \mathbf{Y}_n \\ P \in RT_{\lambda,\mu}, Q \in RT_{\lambda,\nu}}} \varepsilon(P)\varepsilon(Q) = \delta_{\widetilde{\mu}\widetilde{\nu}} \cdot 1^{j_1}(j_1!)2^{j_2}(j_2!)\cdots , \tag{3.1}$$

where μ has j_1 parts of size 1, j_2 parts of size 2,...

$$\sum_{\substack{\lambda \in \mathbf{Y}_n \\ P,Q \in RT_{\lambda,\ell}}} \varepsilon(P)\varepsilon(Q) = \binom{n+\ell-1}{2\ell-1} \cdot \ell! \tag{3.2}$$

The Schensted correspondence sends involutions to identical standard tableaux. We were also able to prove a result in this symmetric case for general ribbon tableaux :

Theorem 3.3. *There is a signed bijection between ribbon tableaux of size n and length ℓ, and hook involutions of size n and length ℓ; this bijection preserves contents.*

We will outline bijective proofs of all these results in Section 4; section 5 contains an algebraic proof of Corollary 3.2. In Section 6 we will apply Theorem 3.1 to the computation of the column sums of the character table of the symmetric group.

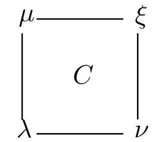
4. Bijective approach

We will prove Theorem 3.1 in the part 4.2, but we have to first define a *local* correspondence concerning ribbons. From there we will be able to deduce the global correspondence of the theorem. In a third part we will prove Theorem 3.3, and finally we will deduce Corollary 3.2 in the last part.

4.1. Local Rules. In this subsection we fix μ, ν be two partitions of size m and n, and we let i, j be nonnegative integers.

We define $\mathcal{U}_i(\mu, \nu)$ to be the set of partitions of size $\max(m, n) + i$ such that ξ/μ and ξ/ν are 2 ribbons. $\mathcal{U}_i(\mu, \nu)$ is a signed set by $sgn(\xi) := \varepsilon(\xi/\mu) \cdot \varepsilon(\xi/\nu)$.Similarly, $\mathcal{D}_j(\mu, \nu)$ is the set of partitions λ of size $\min(m, n) - j$ such that λ/μ and λ/ν are 2 ribbons. $\mathcal{D}_j(\mu, \nu)$ is a signed set by $sgn(\lambda) := \varepsilon(\mu/\lambda) \cdot \varepsilon(\nu/\lambda)$. Notice that $\mathcal{U}_i(\mu, \mu)$ and $\mathcal{D}_i(\mu, \mu)$ are positive sets.

We draw a square with North West corner labeled by μ and South East corner labeled by ν. Elements of $\mathcal{U}_i(\mu,\nu)$ will appear in the North East corner while elements of $\mathcal{D}_i(\mu,\nu)$ will appear in the South West corner . In the case $\lambda=\mu=\nu$, the interior C can be either empty or filled by a hook; in all other cases it is empty.



To define local rules, we need to define many operations on ribbons and partitions, already used by other authors; these definitions are given in Appendix A.

Now, given $((\lambda,C),\mu,\nu)$ written on a square as above, to apply a direct local rule is to find out which operation has to be performed according to the list below, then*erase* λ and C from the square and finally write the outcome of the direct rule on the square: in the Northeast corner for the rules D1 to D6, and in the Soutwest corner for rule S. We 'erase and write' in a similar fashion when applying inverse rules.

Direct rules: In what follows we let λ be an element of $D_i(\mu,\nu)$, and C is an empty hook except possibly when $\lambda=\mu=\nu$ (in which case it may be any hook). We let r, r' be the (possibly empty) ribbons μ/λ and ν/λ.

- If $\lambda=\mu=\nu$ and C is empty, then $\xi=\lambda$. (D1)
- If $\lambda=\mu=\nu$ and C is a hook h, then $\xi=\lambda\cup first(\lambda,h)$. (D2)
- If $\lambda\neq\mu=\nu$, then $\xi=\mu\cup next(\mu,\mu/\lambda)$. (D3)
- If $\lambda=\mu\neq\nu$ (*resp.* $\lambda=\nu\neq\mu$), then $\xi=\nu$ (*resp.* $\xi=\mu$). (D4)
- If $\lambda\neq\mu\neq\nu$, then:
  - if r and r' have different heads and tails, then $\xi=\lambda\cup bumpout(r,r')$. (D5)
  - if r and r' have the same head but different tails, or the same tail but different heads, then:
    * if $slideout(\lambda,r,r')$ is defined, then $\xi=slideout(\lambda,r,r')$. (D6)
    * otherwise, define $\lambda'=switchout(\lambda,r,r')\in D_i(\mu,\nu)$. (S)

Inverse Rules: In what follows ξ is an element of $U_i(\mu,\nu)$. Define r, r' be the (possibly empty) ribbons ξ/μ and ξ/ν. Unless explicitly stated, the hook C is left empty.

- If $\xi=\mu=\nu$, then $\lambda=\xi$. (I1)
- If $\xi\neq\mu=\nu$, then
  - if $prev(\xi,r)=\emptyset$ set $\lambda=\mu$ and C is the hook with the same size and height as r; (I2)
  - otherwise $\lambda=\mu\backslash prev(\xi,r)$. (I3)
- If $\xi=\mu\neq\nu$ (*resp.* $\xi=\nu\neq\mu$), then $\lambda=\nu$ (*resp.*$\lambda=\mu$). (I4)
- If $\xi\neq\mu\neq\nu$, then:
  - if r and r' have different heads and tails, then $\lambda=\xi\backslash bumpin(r,r')$. (I5)
  - if r and r' have the same head but different tails, or same tail but different heads, then:
    * if $slidein(\xi,r,r')$ is defined, then $\lambda=slidein(\xi,r,r')$; (I6)
    * otherwise, define $\xi'=switchin(\xi,r,r')\in U_i(\mu,\nu)$. (T)

PROPOSITION 4.1. *D1-D6 are the respective inverses of rules I1-I6, whereas S and T are both involutions. Moreover, D1-D6 and I1-I6 are sign preserving between $\mathcal{D}_i(\mu,\nu)$ and $\mathcal{U}_i(\mu,\nu)$, while S and T are sign reversing, respectively on $\mathcal{D}_i(\mu,\nu)$ and $\mathcal{U}_i(\mu,\nu)$.*

This was proved in [**SW02, Whi83**] though not exactly in the form stated here. We will summarize the local signed bijections afforded by these rules in the following theorem:

THEOREM 4.1. (a) *If $\mu=\nu$, there is a bijection φ_1 between $\mathcal{U}_i(\mu,\mu)$ and $\mathcal{D}_i(\mu,\mu)\sqcup [\![0,i-1]\!]$.*
(b) *If $\mu\neq\nu$, there is a* signed *bijection (i_D,i_U,φ_2) between $\mathcal{D}_i(\mu,\nu)$ and $\mathcal{U}_i(\mu,\nu)$.*

The first bijection is also signed, but since all elements are positive in the two sets a signed bijection is nothing else than a bijection.

The proofs can be found in the aforementioned papers; let us notice say here that the proofs in [**Whi83**] can be simplified by using the encoding of partitions as *edge sequences*: see [**vL99**] and the appendix to [**SW02**] for information about this encoding.

4.2. Proof of Theorem 3.1. Let ℓ be fixed, and consider a square grid $G = G_\ell$ of size $\ell \times \ell$, made up of ℓ^2 squares; the square numbered (i, j) is the one in the ith row from the bottom and the jth row from the left.The goal is to apply the local rules repeatedly in the squares of G, and obtain a global correspondence as a result.

The ℓ^2 squares of the grid are ordered by $(i, j) \preceq (i', j')$ iff $i \leqslant i'$ and $j \leqslant j'$. We now fix a total order $\leqslant$ extending this partial order.Each square sq except $(1, 1)$ has then a predecessor $pred(sq)$, and each square sq except (ℓ, ℓ) has a successor $succ(sq)$; we set $next(sq, dir)$ to be $pred(sq)$ when $dir = -1$, and $succ(sq)$ when $dir = 1$.

Given a square sq and a direction $dir \in \{+1, -1\}$ then:

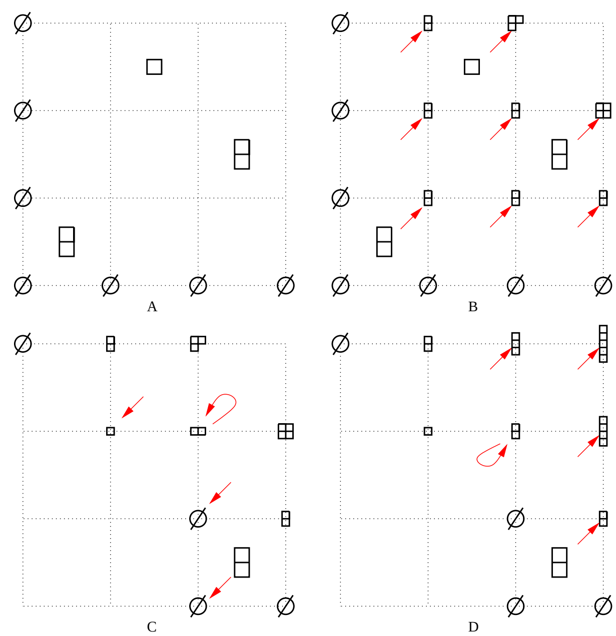
- if $dir = 1$, and μ, ν, λ, C are as in the definition of direct rules, apply the adequate direct local rule;
- if $dir = -1$, and μ, ν, ξ are defined, apply the adequate inverse local rule.

We call this procedure Apply_local_rule and we write loc:=Apply_local_rule(dir,sq) where loc is the label of the local rule that has been applied.

We are now in a position to describe the sign-preserving bijection φ of Theorem 3.1, in an algorithmic fashion :

Algorithm φ :
Input: hook permutation (H, σ).
0utput: pair of ribbon tableaux of the same shape (P, Q).
begin
$sq := (1, 1)$; $dir := 1$;
repeat
loc:=Apply_local_rule(dir,sq);
if ((loc=S) or (loc=T)) then $dir := -dir$; end if;
sq:=next(sq,dir);
until ((sq:=(ℓ, ℓ) and $dir = 1$) or ($sq = (1, 1)$ and $dir = -1$));
end

We will explain the algorithm on a small example. Here the total order chosen is $(i, j) < (i', j')$ if $j < j'$, or if $j = j'$ and $i < i'$. So a square is greater than another if it is in a column strictly to the right, or if it is in the same column but in a strictly higher row.
First start with a hook permutation drawn on the grid (see A).
Then we apply local rules a total of eight times to get to the configuration B: the rules are successively D2,D4,D4,D4,D1,D2,D4,D2.
Then we have to apply rule S, so that direction changes and we have to apply inverse rules starting in square (2,3); rules are I5,I2 and I5, and we attain configuration C.
We apply T, change direction once again, and apply rules D3,D2,D3 et D6 to reach the final configuration D.



We define the involution i algorithmically in the same way as φ, only the input is a pair of tableaux (P, Q), and initially $sq = (\ell, \ell)$ and $dir = -1$. The **repeat . . . until** loop is then exactly the same as φ; if the algorithm ends in (ℓ, ℓ), this defines $i(P, Q)$, whereas if it ends in $(1, 1)$ we retrieve $\sigma = \varphi^{-1}(P, Q)$.

We will not give here a complete proof that these algorithms work correctly and that they prove Theorem 3.1, but we give the main ingredients: the proof uses mainly the original algorithm of White, the works of Fomin and also the *Involution Principle* of Garsia and Milne [**GM81b, GM81a**] (see Appendix B).

Note that each change of direction in algorithm φ corresponds to a change of sign, because S and T are sign reversing; this is why φ cannot end in $(1,1)$ since hook permutations are always positive.

Finally, the fact that the signed bijection preserves contents is a simple property of the local rules themselves: the size of the ribbons corresponding to parallel edges of the square is unchanged unless rule D2 or I2 is applied, in which case the size of top and right ribbons match the size of the hook C.

4.3. Proof of Theorem 3.3. We will now consider just one half of the grid G_ℓ, namely the squares (i,j) with $i \geqslant j$; let us write H_ℓ this set of squares. Ribbon tableaux will be represented by chains of partitions on the top vertices, and we will represent hook involutions by their restriction to the squares of H_ℓ. We also fix a total order $s < s'$ on H_ℓ that extends the restriction of the order $\preceq$. We define $succ(sq)$,$pred(sq)$ and $next(dir, sq)$ as before.

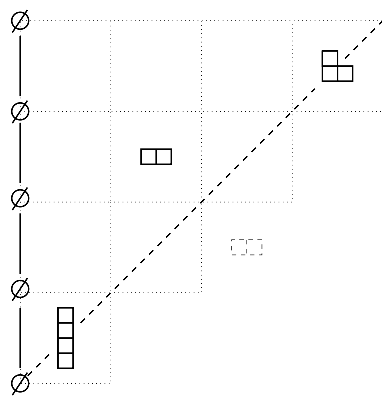


FIGURE 6. The reduced grid for the symmetric case, and a hook involution.

Now we will run the same algorithms as the ones defined in the previous part, but on this reduced grid H_ℓ; to run the algorithm on the diagonal squares, one has first to write a 'ghost copy' of the northwest partition in the southeast corner, before applying the rule. Note that only rules D1-D3 and I1-I3 may then be applied.

This defines the signed bijection of Theorem3.3.

Signs are defined differently than in Theorem 3.1, in which :

- pairs of tableaux (P, P) have all positive sign $1 = \varepsilon(P)^2$;
- hook permutations are always positive, whereas hook involutions are signed in Theorem 3.3.

The proof is thus not a simple consequence of Theorem 3.1, but uses another ingredient: define $\mathcal{U}_i(\mu)$ to be a signed set identical to $\mathcal{U}_i(\mu,\mu)$ as a set, but with sign now given for $\lambda \in \mathcal{U}_i(\mu)$ by $\varepsilon(\lambda/\mu)$; similarly,$\mathcal{D}_i(\mu)$ is $\mathcal{D}_i(\mu,\mu)$ but with sign $\varepsilon(\mu/\lambda)$. Then local rules D2, D3 and I2,I3 prove the following:

PROPOSITION 4.2 ([**SW02**]). *If $k \in [\![0, i-1]\!]$ is a signed set by $sgn(k) = (-1)^k$, then there is a sign preserving bijection between $\mathcal{U}_i(\mu)$ and $\mathcal{D}_i(\mu) \sqcup [\![0, i-1]\!]$*

This is the key result that insures that sign reversal happens exactly when direction reversal does. This is the reason why we chose 'Shimozono and White' rules (D2-D3,I2-I3) instead of 'Stanton and White' rules (used in [**SW85, Whi83**]):the latter are not a signed bijection in the sense of the above proposition.

In particular, White's original bijection does not have Theorem 3.3 as a special case.

4.4. Proof of corollary 3.2. Let us start with the first identity. The signed bijection of Theorem preserves contents, so if a hook permutation corresponds to a pair of tableaux, then the contents of this tableaux must rearrange to the same partition, since this is true for the contents of H and of $\sigma(H)$. Therefore if $\widetilde{\mu} \neq \widetilde{\nu}$, the terms on the left cancel out by the involution i. In the case $\widetilde{\mu} = \widetilde{\nu}$, it is easily seen that $1^{j_1}(j_1!)2^{j_2}(j_2!)\cdots$ is the number of hook permutations of with $c(H) = \nu$ and $c(\sigma(H)) = \mu$, and so the proof is a consequence of Theorem 3.1.

To prove the second identity, it is enough to prove that there are $\binom{n+\ell-1}{\ell-1}$ hook lists of length ℓ and size n. This is bijectively explained on Figure 7.

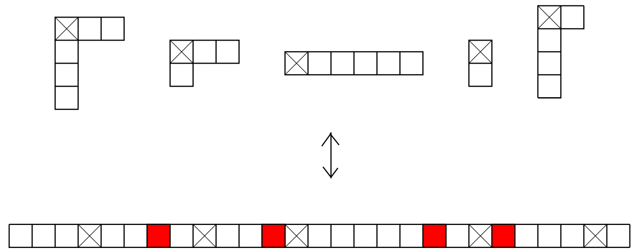


FIGURE 7. Here we have $n = 23$ and $\ell = 5$, so that $n + \ell - 1 = 27$ and $2\ell - 1 = 9$. The 9-subset of $[\![27]\!]$ in bijection with the given hook list is then $\{4, 7, 9, 12, 13, 19, 21, 22, 26\}$.

5. Algebraic approach

The goal here is to prove Corollary 3.2 without using Theorem 3.1

5.1. Elementary linear operators. Let $\mathbb{K}$ be a field of characteristic zero, and consider $\mathbb{K}\mathbf{Y} = \oplus_n \mathbb{K}\mathbf{Y}_n$, the graded vector space of formal linear combinations of partitions with coefficients in $\mathbb{K}$. For i a positive integer we first define two linear operators U_i and D_i :

DEFINITION 5.1.

$$U_i\lambda = \sum_{r=\mu/\lambda\in Rib_i} \varepsilon(r)\mu \quad ; \quad D_i\lambda = \sum_{r=\lambda/\mu\in Rib_i} \varepsilon(r)\mu.$$

U_i and D_i extend indeed to endomorphisms of $\mathbb{K}\mathbf{Y}$, which send $\mathbb{K}\mathbf{Y}_n$ to $\mathbb{K}\mathbf{Y}_{n+i}$ and $\mathbb{K}\mathbf{Y}_{n-i}$ respectively. They were actually already defined by Stanley in [**Sta88**] but for different purposes. The fundamental property of these operators is the following ($AB = A \circ B$ here):

PROPOSITION 5.1.

$$D_iU_i = U_iD_i + i \cdot \mathrm{Id} \tag{5.1}$$

$$D_iU_j = U_jD_i \quad \textit{if } i \neq j \tag{5.2}$$

Let us define $< \lambda, \mu >= \delta_{\lambda\mu}$, and extend it to $\mathbb{K}\mathbf{Y} \times \mathbb{K}\mathbf{Y}$ by linearity. Notice that U_i and D_i are dual for this bilinear form : $< U_i\lambda, \mu >=< \lambda, D_i\mu >$

Proof: Take μ and ν two partitions. To prove the first equality we have to check that $< D_iU_i(\mu), \nu >=< U_iD_i + i \cdot \mathrm{Id}(\mu), \nu >$, i.e. that $< D_iU_i(\mu), \nu >=< U_iD_i(\mu), \nu > +\delta_{\mu,\nu}$; the second equality means $< D_iU_j(\mu), \nu >=< U_jD_i(\mu), \nu >$. These are in turn equivalent to $|\mathcal{U}_i(\mu, \mu)| = |\mathcal{D}_i(\mu, \mu)| + i$ and $|\mathcal{U}_i(\mu, \nu)|_\pm = |\mathcal{D}_i(\mu, \nu)|\pm$ for $\mu \neq \nu$. This is exactly what the bijections of Theorem 4.1 prove. □

5.2. Algebraic proof of Corollary 3.2. Take now $\mathbb{K} = \mathbb{Q}((q))$, the field of formal Laurent series in q . Let us consider $\mathbf{Y}^{\mathbb{K}} = \prod_n \mathbb{K}\mathbf{Y}_n$, the vector space of functions from $\mathbf{Y}$ to $\mathbb{K}$. We shall write such functions as infinite linear combinations of partitions with coefficients in $\mathbb{K}$. We may then extend the bilinear form $< ., . >$ to $\mathbb{K}\mathbf{Y} \times \mathbf{Y}^{\mathbb{K}}$.

With these conventions we define the operators $\mathbf{U}$ and $\mathbf{D}$:

DEFINITION 5.2.

$$\mathbf{U} = \sum_i q^i U_i \quad ; \quad \mathbf{D} = \sum_j q^j D_j$$

The endomorphisms U_i and D_j for any i, j extend to endomorphisms of $\mathbf{Y}^{\mathbb{K}}$, and $\mathbf{U}$ and $\mathbf{D}$ are then themselves endomorphisms of $\mathbf{Y}^{\mathbb{K}}$ (though they are not endomorphisms of $\mathbb{K}\mathbf{Y}$).

Then we have as a direct consequence of the definitions of the operators $\mathbf{U}$ and $\mathbf{D}$:

$$\sum_{\substack{\lambda \in \mathbf{Y}_n \\ P,Q \in RT_{\lambda,\ell}}} \varepsilon(P)\varepsilon(Q) = [q^{2n}] < \emptyset, \mathbf{D}^\ell \mathbf{U}^\ell \emptyset > \tag{5.3}$$

Proposition 5.1 can be restated in the single following commutation relation:

$$\mathbf{DU} = \mathbf{UD} + \frac{q^2}{(1-q^2)^2}\,\mathrm{Id} \tag{5.4}$$

We are now in a position to use the following result of Stanley:

THEOREM 5.3 (Stanley [**Sta88**]). *If two operators satisfy $DU = UD + rI$, then we have*

$$D^\ell U^\ell = (UD + rI)(UD + 2rI) \cdots (UD + \ell r I) \tag{5.5}$$

Therefore, if there is an element $\widehat{O}$ such that $D\widehat{O} = 0$, then $< \widehat{O}, D^\ell U^\ell \widehat{O} >= r^\ell \ell!$

We have such a relation with $\widehat{O} = \emptyset$ and $r = q^2/(1-q^2)^2$, so the second identity of Corollary 3.2 follows from 5.4, 5.5 and the following computation

$$r^\ell \ell! = \ell! \cdot q^{2\ell} \cdot \frac{1}{(1-q^2)^{2\ell}} = \sum_{n \geqslant 2\ell} \left[\binom{n+\ell-1}{2\ell-1} \cdot \ell!\right] q^{2n}.$$

Let us turn to the proof of the first equality of 3.2. The left hand side of the formula in the corollary is equal to $< \widehat{O}, D_{\nu_1} \cdots D_{\nu_\ell} U_{\mu_\ell} \cdots U_{\mu_1} \widehat{O} >$. Then we have the following lemma:

LEMMA 5.4. $< \widehat{O}, D_{\nu_1} \cdots D_{\nu_\ell} U_{\mu_1} \cdots U_{\mu_\ell} \widehat{O} >= \nu_\ell \times \sum_\rho < \widehat{O}, D_{\nu_1} \cdots D_{\nu_{\ell-1}} U_\rho \widehat{O} >$, *where ρ goes through the multiset of the compositions of length $\ell - 1$ deduced from μ by deleting a part equal to ν_ℓ.*

The proof uses repeatedly relations of Proposition 5.1, and the simple fact that $< U_i\lambda, \widehat{O} >= 0$. By induction on ℓ, the sum on the right can be computed and a proof of Corollary 3.2 easily follows.

6. Column Sums of the character table of S_n

The link between hook involutions and ribbon tableaux is suited to the study of column sums of the character table of S_n. For background concerning the representation theory of the symmetric group good references are [**Ful97, Sag01**].

6.1. A formula for $\sum_\lambda \chi^\lambda_\mu$. Let λ, μ be partitions of n, and let χ^λ_μ b e the irreducible character of S_n indexed by λ evaluated at a permutation with cycle type μ. The Murnaghan-Nakayama rule states that χ^λ_μ is equal to the signed sum of ribbon tableaux of content μ and shape λ. So $\sum_\lambda \chi^\lambda_\mu$ is equal to the signed sum of ribbon tableaux of content μ. Then, by Theorem 3.3, this is equal to the signed sum of hook involutions of content μ.

Defining $C(\mu) = \sum_\lambda \chi^\lambda_\mu$ as the sum of the entries of column μ of the Character table of S_n, what precedes amounts to $C(\mu) = |RT_{\lambda,\mu}|_\pm = |\mathcal{HI}|_\pm$. By the corollary 2.4, we finally have

$$C(\mu) = |\mathcal{HI}_{spec}(\mu)| \tag{6.1}$$

THEOREM 6.1. *Let $\mu = (1^{m_1} 2^{m_2} \cdots)$ be a partition.Then $C(\mu) = \prod_{i>0} c_{i,m_i}$ with*

$$c_{i,m_i} = \begin{cases} 0 & \text{if } i \text{ is even and } m_i \text{ is odd;} \\ m_i!! \cdot i^{(m_i/2)} & \text{if } i \text{ is even and } m_i \text{ is even;} \\ \sum_{k=0}^{\lfloor \frac{m_i}{2} \rfloor} \binom{m_i}{m_i - 2k} \cdot (2k)!! \cdot (i)^k & \text{if } i \text{ is odd.} \end{cases}$$

We will give two proofs, one bijective and the other algebraic.

First Proof: The computation of $|\mathcal{HI}_{spec}(\mu)|$ for general μ reduces to the case $\mu = k^{a_k}$ where μ has only one part (counted with multiplicity). In this case an element of $\mathcal{HI}_{spec}(\mu)$ is an involution on $[\![1, a_k]\!]$ with a choice of a hook of size a_k for each cycle of length 2. Remembering that elements of $\mathcal{HI}_{spec}(\mu)$ have no fixed points corresponding to even parts, the proof is complete.

□

Second Proof: We will now sketch a proof that does not use Theorem 3.3. For this we need an algebraic formulation of Corollary 4.2 using the operators D_i and U_i:

PROPOSITION 6.1. $D_i\mathbf{Y} = U_i\mathbf{Y} + [\![i \text{ is odd}]\!] \cdot \mathbf{Y}$

Here $\mathbf{Y}$ stands for the characteristic vector $\sum_{\lambda\in\mathbf{Y}} \lambda$, and $[\![P]\!]$ equals 1 if property P is true and 0 otherwise. We may then deduce easily the following lemmas:

LEMMA 6.2. $D_i^m\mathbf{Y} = [\![i \text{ is odd}]\!] \cdot D_i^{m-1}\mathbf{Y} + (m-1)i \cdot D_i^{m-2}\mathbf{Y} + U_iD_i^{m-1}\mathbf{Y}$.

LEMMA 6.3. *$D_i^m\mathbf{Y} = c_{i,m}\mathbf{Y} + U_iA_{i,m}\mathbf{Y}$, where $c_{i,m}$ is given by Theorem 6.1 and $A_{i,m}$ is a certain endomorphism of $\mathbf{Y}^{\mathbb{K}}$.*

LEMMA 6.4. *$< D_\mu D_i^m\mathbf{Y}, \widehat{\emptyset} >= c_{i,m} < D_\mu\mathbf{Y}, \widehat{\emptyset} >$ if all parts of μ are greater than k.*

Proofs only use the relations of propositions 6.1 and 5.1. $C(\mu)$ is equal by the Murnaghan-Nakayama rule to $< D_\mu\mathbf{Y}, \widehat{\emptyset} >$; applying the previous lemma by induction, one obtains eventually Theorem 6.1. □

6.2. Link with other works. The formula given in 6.1 is certainly not new, but it is to our knowledge the first *(signed) bijective proof* of the result, the signed bijection being the result of going from ribbon tableaux to hook involutions, and then from hook involutions to the enumeration.

The computation of $C(\mu)$ is given in Macdonald [**Mac98**], p.122 ex.11[1], and proved by symmetric function techniques. It states that $C(\mu)$ is equal to $\prod_{i\geqslant 1} a_i^{(m_i)}$, where $a_i^{(m)}$ is the coefficient of $t^m/(m!)$ in $\exp(t + \frac{1}{2}it^2)$ (*resp.* $\exp(\frac{1}{2}it^2)$) if i is odd (*resp.* even). It is a simple exercise to expand these generating functions and then deduce Theorem 6.1 . Another enumeration appears in Exercise 7.69 of [**Sta99**], which follows actually from a general result of Frobenius and Schur in the character theory of finite groups (see the classic reference [**Isa94**] for instance):

THEOREM 6.5 ([**Sta99, Isa94**]). *Let σ be a permutation of $[\![1, n]\!]$ with cycle type μ. Then $C(\mu)$ is equal to the number of square roots of σ, i.e. the number of permutations τ such that $\tau^2 = \sigma$.*

It is possible to prove this theorem by constructing a bijection $HiToRoot$ between $\mathcal{HI}_{spec}(\mu)$ and the square roots of σ.

6.3. Equations on partitions. We can also use the formula to answer the question : for a given integer k, what are the partitions μ such that the column sum $C(\mu)$ is equal to k ? Let $\mathcal{OD}$ be the set of partitions with odd distinct parts. The answers for the first integers are:

- $C(\mu) = 0$ iff μ has at least an even part with odd multiplicity.
- $C(\mu) = 1$ iff $\mu \in \mathcal{OD}$.
- $C(\mu) = 2$ iff 1 has multiplicity 2 and $\mu - 1^2 \in \mathcal{OD}$, or 2 has multiplicity 2 and $\mu - 2^2 \in \mathcal{OD}$
- $C(\mu) = 3$ has no solution.
- $C(\mu) = 4$ iff 3 has multiplicity 2 and $\mu - 3^2 \in \mathcal{OD}$, or 4 has multiplicity 1 and $\mu - 4^1 \in \mathcal{OD}$, or 2 and 1 have multiplicity 2 and $\mu - 1^2 2^2 \in \mathcal{OD}$.

The number of solutions to $C(\mu) = 0$ is sequence A085642 in Sloane's Online Encyclopedia [**Slo**]. The article [**BO04**] proves bijectively that another family of partitions is in bijection with $\mathcal{OD}$, namely the partitions with at least one part congruent to 2 modulo 4.

7. Conclusion

The results described here can easily be extended. As has been noticed, ribbon tableaux can be fruitfully seen as paths going up in the graph $GR = (\mathbf{Y}, R)$. And a pair of ribbon tableaux of the same shape and the same number of ribbons is a walk on GR from $\emptyset$ to $\emptyset$ with ℓ up steps followed by ℓ down steps . But other walks from $\emptyset$ to $\emptyset$ with prescribed conditions on up or down steps can be studied, and so can *oscillating* ribbon tableaux, i.e. walks on GR of length 2ℓ from $\emptyset$ to $\emptyset$.

For standard tableaux, there exist the procedures of *jeu de taquin* and *evacuation* due to Schutzenberger. Stanton and White explained how this could be generalized to ribbon tableaux, to show combinatorially that $|RT_{\lambda,\mu}|_\pm = |RT_{\lambda,\nu}|_\pm$ as soon as μ and ν are 2 compositions verifying $\widetilde{\mu} = \widetilde{\nu}$.It is possible to give local rules

[1]actually the result in [**Mac98**] is stated with t^m instead of $t^m/(m!)$ but we corrected it here

realizing this, building on Fomin's version of jeu de taquin in his appendix to Stanley's book [**Sta99**]. Let us justpoint out that the algebraic way to express the is simply $D_iD_j = D_jD_i$ for all i, j.

As a matter of fact, algebraic and bijective techniques can be generalized to other graphs, and extend parts of Fomin's seminal work [**Fom94, Fom95**]. Consider a simple nonoriented graph $G = (V, E)$ with a sign function $\varepsilon : E- > \{+1, -1\}$. Suppose that V is a disjoint union of finite sets V_i, $i \in \mathbb{N}$ where V_0 is a singleton $\{O\}$. We say that G is a *layered graph with zero*. Introduce operators U_i, D_j on $\mathbb{Z}V$ defined for $v \in V_k$ by $U_i(v) = \sum_e \varepsilon(e)v'$ where e runs through edges from v to the vertices $v' \in V_{k+i}$; a similar definition holds for operator D_j that sends $\mathbb{Z}V_k$ to $\mathbb{Z}V_{k-i}$.

We say that G is *self dual* if it satisfies, for certain integers α_i:

$$D_iU_i = U_iD_i + \alpha_i \cdot \mathrm{Id} \tag{7.1}$$

$$D_iU_j = U_jD_i \quad \text{if } i \neq j \tag{7.2}$$

We only defined the self dual case, but layered graphs in duality can be defined likewise following [**Fom94**]. It is then easy to see that algebraic techniques developed for the ribbon graph can be applied to such graphs ; signed bijections can even be constructed if one defines local rules that prove combinatorially the previous relations. Note that this concerns only generalizations of Theorem 3.1 and its Corollary: to have a Theorem similar to Theorem 3.3 one needs additional properties.

One should try to generalize other properties of the Schensted correspondence to ribbon tableaux: for instance, what relation is there between a hook permutation and the common shape of the ribbon tableaux associated to it ? This might help in finding a combinatorial explanation of the positivity of the row sums of the character table $\left(\chi^\lambda_\mu\right)_{\lambda,\mu}$ (see Exercise 7.71 in [**Sta99**]).

References

[BO04] Christine Bessenrodt and Jørn Olsson, *On the sequence A085642*, The On-line Encyclopedia of Integer Sequences (2004).

[Fom86] S. V. Fomin, *The generalized Robinson-Schensted-Knuth correspondence*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **155** (1986), no. Differentsialnaya Geometriya, Gruppy Li i Mekh. VIII, 156–175, 195. MR MR869582 (88b:06003)

[Fom94] Sergey Fomin, *Duality of graded graphs*, J. Algebraic Combin. **3** (1994), no. 4, 357–404. MR MR1293822 (95i:05088)

[Fom95] ———, *Schensted algorithms for dual graded graphs*, J. Algebraic Combin. **4** (1995), no. 1, 5–45. MR MR1314558 (95m:05246)

[Ful97] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997, With applications to representation theory and geometry. MR MR1464693 (99f:05119)

[GM81a] A. M. Garsia and S. C. Milne, *Method for constructing bijections for classical partition identities*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981), no. 4, part 1, 2026–2028. MR MR611112 (82d:05020)

[GM81b] ———, *A Rogers-Ramanujan bijection*, J. Combin. Theory Ser. A **31** (1981), no. 3, 289–339. MR MR635372 (84g:05018)

[Isa94] I. Martin Isaacs, *Character theory of finite groups*, Dover Publications Inc., New York, 1994, Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)]. MR MR1280461

[Ker99] Adalbert Kerber, *Applied finite group actions*, second ed., Algorithms and Combinatorics, vol. 19, Springer-Verlag, Berlin, 1999. MR MR1716962 (2000j:05129)

[Mac98] I. G. Macdonald, *Symmetric functions and orthogonal polynomials*, University Lecture Series, vol. 12, American Mathematical Society, Providence, RI, 1998, Dean Jacqueline B. Lewis Memorial Lectures presented at Rutgers University, New Brunswick, NJ. MR MR1488699 (99f:05116)

[Sag01] Bruce E. Sagan, *The symmetric group*, second ed., Graduate Texts in Mathematics, vol. 203, Springer-Verlag, New York, 2001, Representations, combinatorial algorithms, and symmetric functions. MR MR1824028 (2001m:05261)

[Slo] N. Sloane, *On-line encyclopedia of integer sequences*, `http://www.research.att.com/~njas/sequences/`.

[Sta88] Richard P. Stanley, *Differential posets*, J. Amer. Math. Soc. **1** (1988), no. 4, 919–961. MR MR941434 (89h:06005)

[Sta99] ———, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR MR1676282 (2000k:05026)

[SW85] Dennis W. Stanton and Dennis E. White, *A Schensted algorithm for rim hook tableaux*, J. Combin. Theory Ser. A **40** (1985), no. 2, 211–247. MR MR814412 (87c:05014)

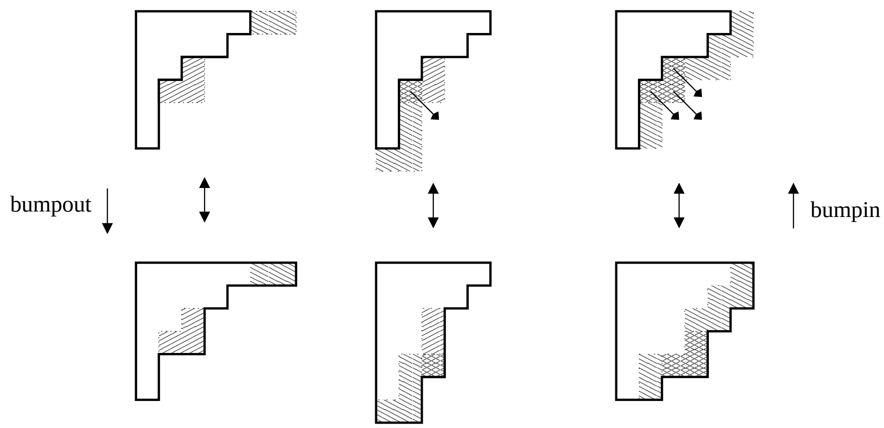
[SW02] Mark Shimozono and Dennis E. White, *Color-to-spin ribbon Schensted algorithms*, Discrete Math. **246** (2002), no. 1-3, 295–316, Formal power series and algebraic combinatorics (Barcelona, 1999). MR MR1887492 (2003e:05147)

[vL99] Marc A. A. van Leeuwen, *Edge sequences, ribbon tableaux, and an action of affine permutations*, European J. Combin. **20** (1999), no. 2, 179–195. MR MR1676191 (2000f:05087)

[Whi83] Dennis E. White, *A bijection proving orthogonality of the characters of S_n*, Adv. in Math. **50** (1983), no. 2, 160–186. MR MR723200 (85b:20019)

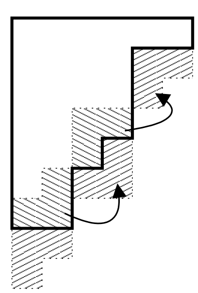
Appendix A. Operations on Ribbons

- If r and r' are the set of cells corresponding to 2 ribbons, then *bumpout*(r, r') is the set of cells $(r \backslash r') \cup (r' \backslash r) \cup (r' \cap r)_{\searrow}$ where $A_{\searrow}$ is the translate of the set A by the vector $Southeast = \{1, -1\}$. *bumpin*(r, r') is defined in a similar fashion by pushing common cells in the northwest direction. These definitions differ slightly from the standard ones.



- Let λ be a partition, k a positive integer, and h a nonnegative integer. By a result of Shimozono and White [**SW02**], λ-addable ribbons of size k and height h and λ-removable ribbons of size k and height h are $r_0 < r'_1 < r_1 < \ldots < r'_t < r_t$ where:
  - $rib_1 < rib_2$ if the head of rib_1 is weakly southwest of the head of rib_2;
  - $(r_i)_{i=0\ldots t}$ are the λ-addable ribbons of size k and height h and $(r'_i)_{i=1\ldots t}$ are the λ-removable ribbons of size k and height h.

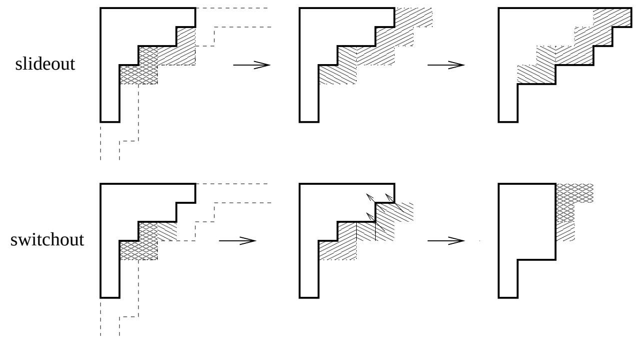
  We can now define $first(\lambda, h)$ to be the ribbon r_0 defined above, with $k = |r_0| = |h|$ and $h = ht(r_0) = ht(r_1)$; and we define $next(\lambda, r'_i) = r_i$ for $i \in [\![1, t]\!]$. Conversely, we define $prev(\lambda, r_i) = r'_i$ for $i \in [\![1, t]\!]$, and $prev(\lambda, r_0) = \emptyset$. The arrow is 'next' in the Figure below.



- Let λ be a partition, and r, r' be λ-addable ribbons having the same tail but different heads, and without loss of generality we assume that $|r| > |r'|$. The outside rim of λ consists of cells immediately to the right and below λ, or in the first column and below λ, or in the first row and to the right of λ.Then consider the set τ of $|r'|$ contiguous cells of the outside rim of λ that lie Northwest of r and are adjacent to it.
  - if $\tau \cup r$ forms a λ-addable ribbon, then define *slideout*$(\lambda, r, r') = \lambda \cup r \cup \tau$.
  - otherwise define *switchout*$(\lambda, r, r') = (\lambda \cup r') \backslash \tau_{\nwarrow}$, where $A_{\nwarrow}$ is the translate of the set A by the vector $Northwest = \{-1, 1\}$.

  If r and r' have the same head but different tails, we operate the same procedures on the transpose of λ, r, r' and transpose again at the end.

  switchin and *slidein* operations are defined similarly for λ-removable ribbons (see [**Whi83**]).



Appendix B. The Garsia and Milne Principle

Garsia and Milne in [**GM81b, GM81a**] gave the first bijective proof of one of the famous Rogers-Ramanujan identities. For this they created and used the involution principle, of which we give a possible version in the following:

Let A, B be two signed sets, and i_A, i_B be two sign reversing involutions on A and B respectively. Let also φ be a sign preserving bijection between A and B. With these conditions, we have an equality $|Fix(i_A)|_\pm = |Fix(i_B)|_\pm$; yet suitable restrictions of i_A, i_B and φ do not necessarily induce a signed bijection between $Fix(i_A)$ and $Fix(i_A)$.

The principle of Garsia and Milne is the construction of such a signed bijection (ψ, j_A, j_B): let $a \in A$, and apply φ, then $\varphi^{-1} \circ i_B$, then $\varphi \circ i_A$, then $\varphi^{-1} \circ i_B$,then $\varphi \circ i_A$, etc ... until the image y is either :

- in $Fix(A)$, in which case one defines $j_A(a) := x$,
- or in $Fix(B)$,in which case one defines $\psi(a) := x$.

In order to define j_B (and ψ^{-1}), do a symmetric procedure starting from $b \in B$. These procedures always end and give the wanted signed bijection; for a proof of this fact see for instance [**Ker99**, p.76].

LABORATOIRE DE RECHERCHE EN INFORMATIQUE, UNIVERSITÉ PARIS 11 AND CNRS, ORSAY FRANCE
E-mail address: dgb@lri.fr

LABORATOIRE DE RECHERCHE EN INFORMATIQUE, UNIVERSITÉ PARIS 11 AND CNRS, ORSAY FRANCE
E-mail address: nadeau@lri.fr