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A general bijection for a class of walks on the slit plane

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Abstract. We study walks in the plane $\mathbb{Z}^2$, with steps in a given finite set $\mathfrak{S}$, which start from the origin but otherwise never hit the half-line $\mathcal{H} = \{(k,0), k \leqslant 0\}$. These ***walks on the slit plane*** have received some attention these last few years, since in particular their enumeration leads to simple closed formulas; but only one bijection has been found so far, in the case of the square lattice, that explains such formulas.

Let $p = p(\mathfrak{S})$ be the smallest possible abscissa x such that there is a walk on the slit plane ending at $(x,0)$. Suppose that $|j| \leqslant 1$ for each step $(i,j) \in \mathfrak{S}$. The main result of this paper is the construction of a length preserving bijection between $\mathfrak{S}$-walks on the slit plane with a marked step ending at $(p,0)$, and a certain class of walks on the plane whose enumeration is much simpler. This allows us to interpret combinatorially previously known enumerations, and to give many new ones.

Résumé. Nous étudions des chemins dans le plan $\mathbb{Z}^2$, dont les pas appartiennent à un ensemble $\mathfrak{S}$ donné, qui partent de l'origine mais qui sinon évitent la demi-droite $\mathcal{H} = \{(k,0), k \leqslant 0\}$. Ces ***chemins sur le plan incisé*** ont éveillé un certain intérêt ces dernières années, notamment en raison d'énumérations menant à des formules closes simples ; cependant une seule bijection a jusqu'ici été trouvée, dans le cas du réseau carré, pour expliquer de telles formules.

Soit $p = p(\mathfrak{S})$ la plus petite abscisse x telle qu'il existe un chemin sur le plan incisé terminant en $(p,0)$. On suppose que $|j| \leqslant 1$ pour tout pas $(i,j) \in \mathfrak{S}$. Le résultat principal de cet article est la construction d'une bijection, préservant la longueur, entre les $\mathfrak{S}$-chemins dans le plan incisé avec un pas marqué terminant en $(p,0)$, et une certaine classe de chemins du plan dont l'énumération est plus aisée. Cela nous permet de donner des interprétations combinatoires de résultats d'énumération déjà connus, et d'en donner de nombreux autres.

1. Introduction

Walks on the slit plane were introduced in [**2**]. Given a finite set of steps $\mathfrak{S} \subset \mathbb{Z}^2$, they are defined as walks on the plane $\mathbb{Z}^2$ with steps in $\mathfrak{S}$ that start at the origin O, and otherwise avoid the half-line $\mathcal{H} = \{(k,0), k \leqslant 0\}$. In the paper [**2**] and the following paper [**3**], the goal was to give closed forms for various generating functions related to these walks, and study when such generating functions were algebraic.

One of the main results of [**3**] is that, in the case where all elements (i,j) of $\mathfrak{S}$ verify $|j| \leqslant 1$ ($\mathfrak{S}$ is said to have the *small height variation property*), then the generating functions $S(x,y,t), S_j(x,t), S_{i,j}(t)$ are algebraic, these series enumerating respectively walks according to length and endpoint, walks ending at height j according to length and final abscissa, and walks ending at (i,j) according to length.

For specific steps $\mathfrak{S}$, the closed form of the generating functions allows in fact to obtain expressions for the number of walks of length n ending at certain points (i,j). For instance, for $\mathfrak{S} = \{(\pm 1,0),(0,\pm 1)\}$ (the square lattice) or $\mathfrak{S} = \{(\pm 1, \pm 1)\}$ (the diagonal lattice), closed formulas are proved in [**2**] and [**6**] for various endpoints. There is a specific endpoint for which closed formulas are often obtained : it is $(p,0)$, where $p = p(\mathfrak{S})$ is the smallest possible abscissa x such that there is a walk on the slit plane ending at $(x,0)$; for instance, p equals 1 for the square lattice and 2 for the diagonal lattice.

In this paper, we construct bijections that will explain these closed formulas for the endpoint $(p,0)$. First we will define a bijection which is valid for sets $\mathfrak{S}$ with small variations and that are symmetric with respect

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to the x-axis; the main idea for the construction comes from the paper [**1**], where it was used to enumerate these walks on the square lattice. Then we will modify this bijection in the case of sets $\mathfrak{S}$ that are not necessarily symmetric; we will thus be able to bijectively prove, for instance, that if $\mathfrak{S} = \{(\pm 1, 1), (0, -1)\}$ then there are $4^{2n+1}\binom{2n+1}{n}/(4n+2)$ walks of length $4n+2$ on the slit plane that end at $(1, 0)$.

2. Preliminaries

In this section we will define standard notions concerning walks on the lattice $\mathbb{Z}^2$, and give specific definitions for the case of walks on the slit plane.

For the rest of this section, we let $\mathfrak{S}$ be a finite subset of $\mathbb{Z}^2$.

2.1. Walks. A *walk* with steps in $\mathfrak{S}$ is a finite sequence $w = (w_0, w_1, \dots, w_n)$ of points of $\mathbb{Z}^2$ such that $w_0 = (0, 0)$ and $w_i - w_{i-1} \in \mathfrak{S}$ for $1 \leqslant i \leqslant n$. We shall also say that w is a $\mathfrak{S}$-walk. Note that our walks are always assumed to start at the origin. The number of steps n is the *length* of w. The *endpoint* of w is w_n, and it is denoted $end(w)$. We also denote the (final) height and abscissa of w by $y(w)$ and $x(w)$, that is $end(w) = (x(w), y(w))$. A walk with a *marked step* is the data of a walk $w = (w_0, w_1, \dots, w_n)$ together with an integer $i \in [\![0, n-1]\!]$, so a step $w_{i+1} - w_i$ is distinguished (or more precisely, an occurrence of this step in w). For the figures of this paper, marked steps will be distinguished by a thicker line.

2.2. Walks and words. A walk is characterized by a finite sequence of steps of $\mathfrak{S}$. Hence, it will be convenient to consider walks as words on the alphabet $\mathfrak{S}$. The set of words $\mathfrak{S}^*$ is equipped with the usual concatenation product; as usual, ε denotes the empty word.

Any word w of $\mathfrak{S}^*$ will thus be thought of as a walk starting from $(0, 0)$, and we will in fact make no distinction between the walk and the word: for instance, if w_1 and w_2 are two walks, then $w_1 w_2$ is the walk w_1 followed by the walk w_2 (which is attached at the endpoint of w_1). Note that end is then a morphism from the monoid $\mathfrak{S}^*$ to $\mathbb{Z}^2$, where elements of $\mathbb{Z}^2$ are added componentwise. That is, when w_1 and w_2 are two walks, then we have $end(w_1 w_2) = end(w_1) + end(w_2)$.

2.3. Walks on the Slit Plane. We will use the terminology introduced in [**2, 3**]. We say that the walk w avoids the half-line $\mathcal{H} = \{(k, 0), k \leqslant 0\}$ if none of the vertices $w_1, \dots, w_n$ belong to $\mathcal{H}$. We then call w a ***walk on the slit plane***. For $(i, j) \neq (0, 0)$, we denote by $\mathcal{S}_{i,j}(n)$ the set of walks w on the slit plane of length n and such that $end(w) = (i, j)$. We also denote the cardinality of this set by $S_{i,j}(n)$. In this paper, we will consider $\mathcal{S}_{p,0}(n)$, where $p = p(\mathfrak{S})$ is the minimum positive integer x such that there is a $\mathfrak{S}$-walk ending at $(x, 0)$; we shall always assume that we deal with sets $\mathfrak{S}$ such that p is well defined.

If $s = (x, y) \in \mathbb{Z}^2$, we note $\tilde{s} = (x, -y)$ its symmetric with respect to the x-axis. We extend this definition to walks: if $w = s_1 s_2 \cdots s_n \in \mathfrak{S}^*$, then $\widetilde{w} = \tilde{s_1}\tilde{s_2}\cdots\tilde{s_n}$. Geometrically, $\widetilde{w}$ is the walk symmetric of w with respect to the x-axis.

We now define two properties of sets of steps, illustrated on Figure 1 :

DEFINITION 2.1. Let $\mathfrak{S}$ be a set of steps.

- $\mathfrak{S}$ is *symmetric* (with respect to the x-axis) if for all $s \in \mathfrak{S}$, then $\tilde{s} \in \mathfrak{S}$.
- The set $\mathfrak{S}$ is said to have *small height variations* if, for all $(i, j) \in \mathfrak{S}$, $|j| \leqslant 1$.

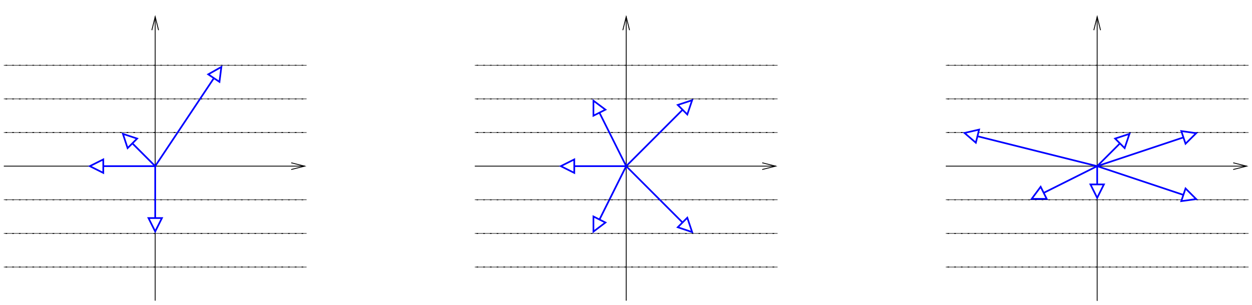

FIGURE 1. From left to right: a set $\mathfrak{S}$ which is neither symmetric nor with small variations, a symmetric set, and a set with small variations.

3. The main theorems

We consider in this section (and in the rest of the paper) a finite set $\mathfrak{S} \subset \mathbb{Z}^2$ of *steps* which has the *small height variations* property. Let $w = (w_0, \ldots, w_n)$ be an element of $\mathcal{S}_{p,0}(n)$, i.e. a walk on the slit plane of length n that ends at $(p, 0)$. Let also $i \in [\![0, n-1]\!]$.

First suppose that, in addition, $\mathfrak{S}$ is *symmetric*. Then define $\Psi(w, i) = (W_0, W_1, \ldots, W_n)$ as the walk

- whose beginning $(W_0, W_1, \ldots, W_{n-i})$ is obtained by reflecting $(w_i, \ldots, w_n)$ off the x-axis and translating it so that $W_0 = O$;
- and whose ending is obtained by appending $w = (w_0, \ldots, w_i)$ to $(W_0, W_1, \ldots, W_{n-i})$ through the translation of vector $W_{n-i} - w_0$

See Figure 2 for an illustration of this construction. If w is considered as a word in $\mathfrak{S}^*$ (see Section 2), the choice of i corresponds to a factorization $w = uv$ in $\mathfrak{S}^*$ with $v \neq \varepsilon$. Then one has simply $\Psi(w, i) = \tilde{v}u$.

We can now state our first theorem:

THEOREM 3.1. *The construction Ψ is a bijection between the following two sets:*
(1) $\mathfrak{S}$-walks on the slit plane of length n ending at $(p, 0)$ with a marked step.
(2) $\mathfrak{S}$-walks on the plane of length n ending at $(p, 2k)$ for a certain $k \in \mathbb{Z}$.

As an immediate corollary, we have:

COROLLARY 3.1. *Let $S_{p,0}(n)$ be the number of walks on the slit plane that end at $(p, 0)$. Let also $W_{p,even}(n)$ be the number of walks on the plane that end at $(p, 2k)$ for a certain integer k.*

Then we have the identity

$$n \cdot S_{p,0}(n) = W_{p,even}(n)$$

We now generalize this bijection in the case where $\mathfrak{S}$ is not assumed to be symmetric (but still has the small height variation property). Note that the construction of the previous theorem cannot function as such, because, after reflection, some of the steps may not be elements of $\mathfrak{S}$.

Let us write $\mathfrak{S}_{sym}$ for the set of steps whose elements are the symmetric of those of $\mathfrak{S}$ off the x-axis; in other words $\mathfrak{S}_{sym}$ equals $\widetilde{\mathfrak{S}}$. Then we define $\overline{\mathfrak{S}} = \mathfrak{S} \cup \mathfrak{S}_{sym}$. We also need to define $\mathfrak{S}^\delta = \{s \in \mathfrak{S} \mid y(s) = \delta\}$ for $\delta \in \{-1, 0, 1\}$, and similarly $\overline{\mathfrak{S}^1}$ and $\overline{\mathfrak{S}^{-1}}$. We can now state our second theorem:

THEOREM 3.2. *Let $\mathfrak{S}$ be a set of steps with small variations, and n be a positive integer. Assume that there is a $\mathfrak{S}$-walk ending on the positive x-axis, and let $p = p(\mathfrak{S})$ be the smallest positive abscissa that can be reached. Then we have a bijection between the following sets:*
(1) Walks on the slit plane of length n with steps in $\mathfrak{S}$ that end at $(p, 0)$.
(2) Walks of length n with steps in $\overline{\mathfrak{S}}$, ending at abscissa p, with an even number $2m$ of steps in $\overline{\mathfrak{S}^1} \cup \overline{\mathfrak{S}^{-1}}$, such that, among these steps, the first m ones are in $\overline{\mathfrak{S}^1}$ and the last m ones are in $\overline{\mathfrak{S}^{-1}}$

Notice that if $\mathfrak{S}$ is symmetric then the walks in 3.1(2) and 3.2(2) coincide; in fact the bijections will be identical in this case.

4. Proof of Theorem 3.1

We consider here a set $\mathfrak{S}$ of steps that is symmetric and has the small height variation property. We keep the notations of Section 3 concerning Theorem 3.1; in particular (u, v) is the factorization of w afforded by the marked step. Note first that the steps of $\Psi(w, i)$ are in $\mathfrak{S}$, since we assumed that $\mathfrak{S}$ is symmetric. An example of $\Psi(w, i)$ is shown on Figure 2.

The proof will proceed as follows : first we show that Ψ is well defined. Then we construct a function Γ from $\mathcal{W}_{p,even}(n)$ to $\mathcal{S}_{p,0}(n) \times [\![0, n-1]\!]$. Finally, we prove that Ψ and Γ are actually inverse to one another.

Ψ is well-defined. We have to show that $\Psi(w, i)$ is a walk with endpoint at abscissa p and even ordinate. This is obvious from the geometric construction, and follows from a simple computation. Define (h, k) by $end(u) = (h, k)$. Since $end(w) = (p, 0)$, it follows that $end(v) = end(w) - end(u) = (p - h, -k)$, and consequently $end(\tilde{v}) = (p - h, k)$. Finally, we have $\Psi(w, i) = \tilde{v}u$, so $end(\Psi(w, i)) = end(\tilde{v}) + end(u) = (p, 2k)$, which shows that indeed $\Psi(w, i)$ is an element of $\mathcal{W}_{p,even}(n)$.

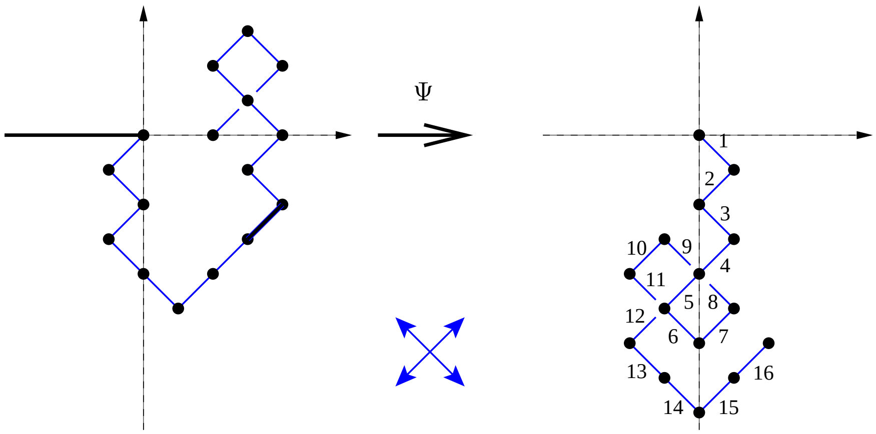

FIGURE 2. Example of the bijection Ψ in the case of the diagonal lattice. Here $n = 16$, $i = 7$, and steps are numbered in the second walk for easier understanding.

Definition of the inverse. Let us now define a function Γ from $\mathcal{W}_{p,even}(n)$ to $\mathcal{S}_{p,0}(n) \times [\![0, n-1]\!]$. Let $W = s_1 s_2 \cdots s_n$ be a $\mathfrak{S}$-walk that ends at $(p, 2l)$ with $l \in \mathbb{Z}$. Since $\mathfrak{S}$ has the small height variation property, there are points in W with ordinate l. Among such points, let (m, l) be the one with minimal abscissa. Let finally $i \in [\![1, n]\!]$ be maximal such that s_i is a step starting from (m, l). Note that such a step always exists because (m, l) cannot be the endpoint of W. Indeed, this is clear if $l \neq 0$; and in the case $l = 0$, we have $m \leqslant 0$ because W starts from O, whereas the walk W ends in $(p, 0)$ with $p > 0$.

Let us define $W_1 = s_1 \cdots s_{i-1}$ and $W_2 = s_i \cdots s_n$; we then set

$$\begin{aligned} \Gamma(W) &= (W_2\widetilde{W_1}, n-i+1) \text{ if } i > 1 \\ &= (W_2, 0) \text{ if } i = 1 \end{aligned}$$

See Figure 3 for an example.

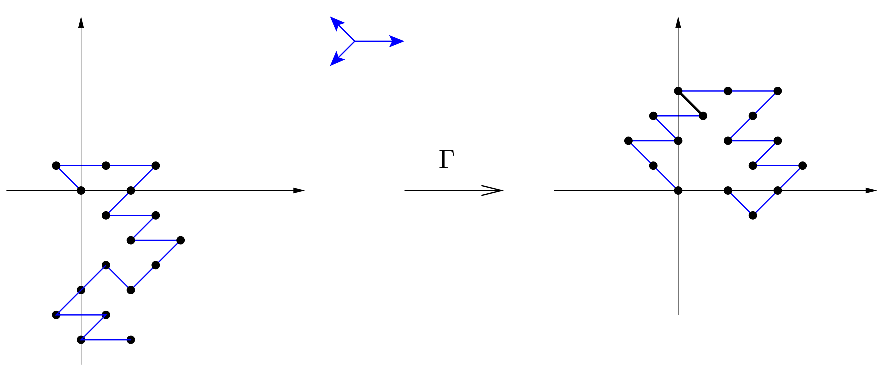

FIGURE 3. Example of the inverse bijection Γ when $\mathfrak{S} = \{(2,0), (-1,1), (-1,-1)\}$. Here we have $n = 16$, $(m, l) = (1, -3)$ and $i = 12$.

Let us rephrase the properties of W_1 in the language of words; we will state this as a lemma for easier reference:

LEMMA 4.1. *Let U be a prefix of W such that $end(U) = (k, l)$. Then we have $k \geqslant m$, and the inequality is strict if U is a prefix strictly longer than W_1.*

We claim that the walk thus obtained is an element of $\mathcal{S}_{p,0}(n)$; by abuse of notation, we will write $\Gamma(W)$ for $W_2\widetilde{W_1}$. Firstly, we have $end(W_1) = (m, l)$, so that $end(\widetilde{W_1}) = (m, -l)$, and $end(W_2) = (p - m, l)$. So we have $end(\Gamma(W)) = (p, 0)$ which shows that $\Gamma(W)$ has the good endpoint.

Now we have to prove that for every nonempty prefix w_1 of $\Gamma(W)$ such that $end(w_1) = (x, 0)$, then x is a positive integer. Let w_1 be such a prefix: there are two cases to consider, depending on whether w_1 is a shorter prefix than W_2 or not. If $w_1 \cdot u = W_2$, then $W_1 w_1$ is a prefix of W strictly longer than W_1 and its endpoint is $(m + x, l)$. By Lemma 4.1, we have indeed $x > 0$. If $W_2 \cdot u = w_1$, then u is a prefix of $\widetilde{W_1}$, so that $\tilde{u}$ is a prefix of W_1. But $end(\widetilde{u}) = end(\widetilde{w_1}) - end(\widetilde{W_2}) = (m - p + x, l)$. By Lemma 4.1, we have $x - p \geqslant 0$, which implies again $x > 0$ because $p > 0$.

This completes the proof that Γ is a well-defined function from $\mathcal{W}_{p,even}(n)$ to $\mathcal{S}_{p,0}(n)$.

End of the proof of Theorem 3.1. We finally have to show that Γ is the inverse of Ψ. It is clear that $\Psi(\Gamma(W)) = W$, so that we need to prove that $\Gamma(\Psi(w, i)) = (w, i)$. This is clearly equivalent to showing that $\widetilde{v}$ is equal to the prefix W_1 of $\Psi(w, i)$ defined in the construction of Γ. If $end(u) = (h, k)$, we have already computed that $end(\widetilde{v}) = (p - h, k)$ and $end(\Psi(w, i)) = end(\widetilde{v}u) = (p, 2k)$. So what we have to show is that (1) $p - h = m$, where m is defined as in the construction of Γ, and that (2) if U is a prefix of $\Psi(w, i) = \widetilde{v}u$, longer than $\tilde{v}$, and whose endpoint equals (x, k) for a certain x, then $x > m$.

Suppose that $p - h > m$. Assume first that $\widetilde{v}$ is a strict prefix of W_1, so that there exists $u_0 \neq \varepsilon$ such that $W_1 = \widetilde{v}u_0$, and $end(u_0) = (x, 0)$ where $x = m - (p - h) < 0$. Since $W_1 W_2 = \widetilde{v}u$, this implies $u = u_0 W_2$, which is absurd since u_0 is a prefix of w hitting the forbidden half-line. We have a contradiction, so $\widetilde{v}$ is a suffix of W_1, or, equivalently, $\widetilde{W_1}$ is a prefix of v: there follows that $u\widetilde{W_1}$ is a prefix of w with endpoint $(m + h, 0)$. But we supposed that $m + h < p$, so this contradicts the definition of p. (Note that this is the only place where the definition of p is used in the proof). Finally $p - h \leqslant m$, and the definition of m forces $p - h \geqslant m$, so (1) is proved.

Now let U be a prefix of $\widetilde{v}u$, with $end(U) = (x, k)$, such that $U = \widetilde{v}u_0$ with $u_0 \neq \varepsilon$. Then u_0 is a nonempty prefix of u with $end(u) = (x + h - p, 0) = (x - m, 0)$. Since $w = uv$ is a walk on the slit plane, we must have $x - m > 0$, which proves (2), and completes the proof of Theorem 3.1.

REMARK 4.2. This construction is a generalization of the one in **[1]**. Indeed, in the special case of the square lattice, what they did is marking a specific step in each walk, namely the last one with origin at the smallest possible abscissa. If this is the marked step in our bijection, then it reduces to theirs.

Marking a unique step in this fashion is not always feasible, but it is possible for a certain category of sets $\mathfrak{S}$. We state this a corollary, and it is a direct generalization of the construction of **[1]** :

COROLLARY 4.1. *Let n be a positive integer, $\mathfrak{S}$ be a symmetric set of steps with small variations, which contains only one step with positive abscissa, namely $(1, 0)$. Then there is a bijection between $\mathcal{S}_{1,0}(n)$ and walks of length $n - 1$ that end at $(0, 2k)$ for a certain $k \in \mathbb{Z}$ and stay in the right half-plane $x \geqslant 0$.*

REMARK 4.3. The construction Ψ is easily seen to remain injective when $\mathfrak{S}$ is not symmetric (and even when $\mathfrak{S}$ has "large variations"). Let us describe the image of Ψ in this case, as we will use it in the next Section:

PROPOSITION 4.1. *Let $\mathfrak{S}$ have the small heigth variation property, and n be a positive integer. Then Ψ is a bijection between:*

(1) $\mathfrak{S}$-walks on the slit plane of length n ending at $(p, 0)$ with a marked step.
(2) $\overline{\mathfrak{S}}$-walks on the plane of length n ending at $(p, 2k)$ for a certain $k \in \mathbb{Z}$, such that if W_1 and W_2 are such as defined in the construction of Γ, then W_1 has its steps in $\mathfrak{S}^{sym}$ and W_2 has its steps in $\mathfrak{S}$.

Note that this proposition is not true in general if $\mathfrak{S}$ does not have small variations, because in this case there may be walks with steps in $\overline{\mathfrak{S}}$ that end at $(p, 2k)$ that do not have any point at height k, so that W_1 and W_2 are not well-defined.

5. Proof of Theorem 3.2

We will deal in all this section with a set of steps $\mathfrak{S}$ with the small height variation property. We also still assume that $p(\mathfrak{S})$ is well defined, that is, there exists a $\mathfrak{S}$-walk that ends on the positive axis.

The bijection announced in Theorem 3.2 will be defined by first introducing some intermediate objects. The reader is advised to have a look at Figure 4 for an illustration of the different constructions of this section.

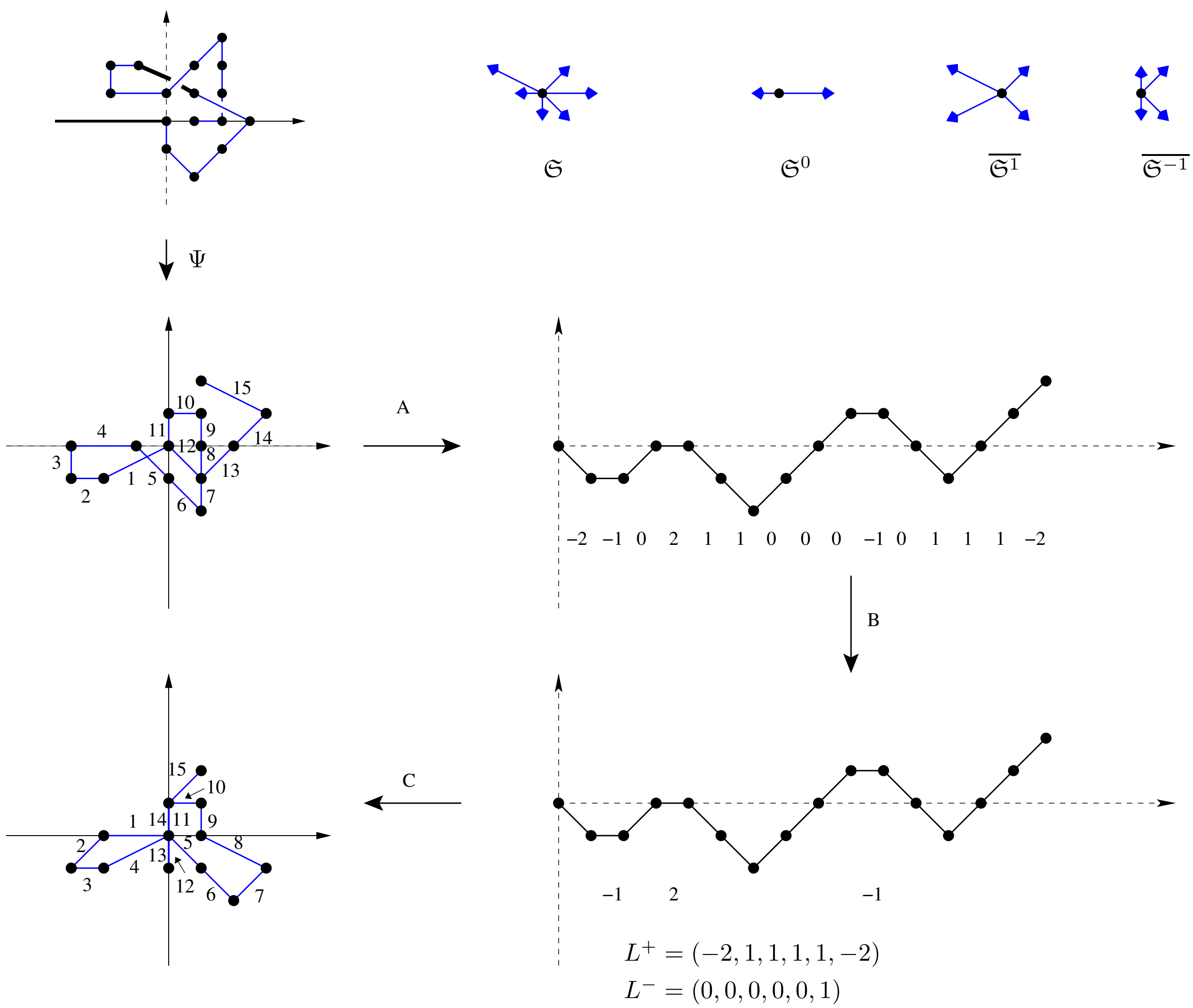

FIGURE 4. Illustration of the different steps of the bijection of Theorem 3.2. Construction A is involved in Lemma 5.3, construction B is the bijection of Theorem 5.4, and construction C is explained at the end of Section 5.

Let $\mathfrak{M}$ be the set of steps $\{(1,1),(1,0),(1,-1)\}$; the letter $\mathfrak{M}$ stands for Motzkin, since $\mathfrak{M}$-walks that end on the x-axis and remain in the upper half plane are the famous *Motzkin paths*, see for instance [**5**] for more information.

DEFINITION 5.1. Let n be a positive integer, and $M = m_1 \cdots m_n$ be a $\mathfrak{M}$-walk of length n. A *labeling* of M is a function l from $[\![1,n]\!]$ to $\mathbb{Z}$. We also associate to a labeling l the function $\hat{l}$ defined by $\hat{l}(i) = l(1) + \cdots + l(i)$ for $i \in [\![0,n]\!]$.

One should think of l as being associated to the occurrences of the steps of M, and $\hat{l}$ as associated to the point of abscissa i in M. Now for $\delta \in \{-1,0,1\}$, we define $S^\delta \subset \mathbb{Z}$ as the abscissas of the walks in $\mathfrak{S}^\delta$.

DEFINITION 5.2. Let $M = m_1 \cdots m_n$ be an $\mathfrak{M}$-walk of length n ending at height $2k$ ($k \in \mathbb{Z}$), together with a labeling l. Consider all integers i such that (i,k) is a point of M, and consider among them those with $\hat{l}(i)$ minimal; denote $i_{max} = i_{max}(M,l)$ the maximal integer with this property.

Then M is *well* $\mathfrak{S}$*-labeled* by l (or l is a *good* $\mathfrak{S}$*-labeling* of M) if the following conditions are verified:
– for all $i \leqslant i_{max}$ and $\delta \in \{-1,0,1\}$, $l(i) \in S^\delta$ iff $m_i = (1,-\delta)$;
– for all $i > i_{max}$ and $\delta \in \{-1,0,1\}$, $l(i) \in S^\delta$ iff $m_i = (1,\delta)$;

This seemingly artificial definition is explained by the following lemma:

LEMMA 5.3. *There is a bijection between*

(1) $\mathfrak{S}$-walks on the slit plane of length n ending at $(p,0)$
(2) couples (M,l) where M is an $\mathfrak{M}$-walk of length n ending at even height, well $\mathfrak{S}$-labeled by l, and verifying $\hat{l}(n)=p$.

PROOF. First apply Proposition 4.1. Then transform a walk $w=s_1\cdots s_n$ thus obtained in the following way: for every $i\in[\![1,n]\!]$, define $m_i=(1,y(s_i))$ and $l(i)=x(s_i)$. The resulting path $M=m_1\cdots m_n$ with the function l give then the desired bijection, as is easily seen: notice that the length of W_1 in the intermediate walk is equal to the integer i_{max} in the definition of a good labeling. □

We will now state the main step of the bijection:

THEOREM 5.4. *Let $M=m_1\cdots m_n$ be an $\mathfrak{M}$-path of length n ending at even height. There is a bijection between:*

(1) good $\mathfrak{S}$-labelings l of M with $\hat{l}(n)=p$.
(2) 3-uples (L^+,L^-,l_0) such that l_0 is a function from $\{i\in[\![1,n]\!]\mid m_i=(1,0)\}$ to S^0, and L^+ (respectively L^-) is a sequence of m elements of S^+ (resp. S^-) where $2m$ is defined as the number of i such that $m_i\in\{(1,1),(1,-1)\}$.

SKETCH OF THE PROOF. Let us define this bijection. First, $l_0(i)$ is simply defined as $l(i)$: the labels for horizontal steps remain unchanged. To define $L^+(i)$ and $L^-(i)$, recall the definition of i_{max} given in 5.2. For i increasing, L^+ consists first of all labels $l(i)$ of steps $m_i=(1,-1)$ for $i\leqslant i_{max}$, followed by the labels of steps $m_i=(1,1)$ for $i>i_{max}$. Similarly, L^- consists of all labels $l(i)$ of steps $m_i=(1,1)$ for $i\leqslant i_{max}$ followed by the labels of steps $m_i=(1,-1)$ for $i>i_{max}$. One checks easily that this is well defined.

The main problem is to inverse this construction, and for this we must determine the abscissa i_{max}, so that we can be assured that we obtain a *good labeling*. Indeed, let us try to define the function l given by L^+,L^- and l_0. Of course we have to set $l(i)=l_0(i)$ for all i such that m_i is horizontal. There is clearly only one way to define labels for the first abscissas i, up until we hit an abscissa i_0 whose corresponding point m_{i_0} is at height k (where k is defined by $y(M)=2k$): we have to use (in their original order) the elements of L^+ to label steps of the form $(1,-1)$, and elements of L^- to label those of the form $(1,1)$. Now we have to know whether i_{max} is equal to i_0 or not, in order to know if we have to switch the roles of L^+ and L^-. Clearly, if there is only one way to define i_{max}, then we have found the only possible inverse construction.

Here is the way to do it: if there is no other point of height k, then clearly $i_{max}=i_0$. If not, let $i_1,\ldots,i_t$ be all other abscissas whose corresponding points m_i are at height k. The key point is that, for every j, $\hat{l}(i_j)$ does not depend on whether we have already decided (haphazardly) to switch the roles of L^+ and L^- after a certain i_l or not. Indeed, one verifies that we have to use the same elements of both lists L^+ and L^- whatever our choice: this is a direct consequence of the fact that between two points m_{i_l}, there are as many steps up and down since all these points are at the same height k. So, now that we know the values of $\hat{l}$ for the abscissas i_l, there is only one way to define i_{max}, and we can construct the labeling l.

It is clear that these constructions are inverse to one another; then it remains to check that these constructions are well defined to complete the proof. This is easy and will be omitted in this abstract.

□

Finally, let us show that this theorem implies Theorem 3.2. Let M and (L^+,L^-,l_0) be as in the above theorem, and we will bijectively associate to such data a $\overline{\mathfrak{S}}$-walk such as described in Theorem 3.2(2). The construction is simple: let $i_1<\ldots<i_m<j_1<\ldots<j_{2m}$ be the indices i such that $m_i\in\{(1,1),(1,-1)\}$. Then define a $\overline{\mathfrak{S}}$-walk $w=s_1\cdots s_n$ by $s_i=(l_0(i),0)$ when $m_i=(1,0)$, $s_{i_t}=(L^+(t),y(m_{i_t}))$ and $s_{j_t}=(L^-(t),y(m_{j_t}))$ for $t\in[\![1,m]\!]$ (see Figure 4). It is straightforward to show that this construction is well defined and bijective. By Lemma 5.3, this completes the proof of Theorem 3.2.

REMARK 5.5. We will quickly explain explain where the idea for the walks described in Theorem 3.2(2) comes from. Let $A_\delta(x)=\sum_{i\in S^\delta}x^i$ for $\delta\in\{-1,0,1\}$. Mireille Bousquet-Mélou [**3**] proves the following:

THEOREM 5.6 ([**3**]). *Let $\Delta(x;t)$ be the following polynomial in x, x^{-1} and t:*

$$\Delta(x;t) = (1 - tA_0(x))^2 - 4t^2 A_1(x)A_{-1}(x)$$

Then the generating function for walks on the slit plane ending at $(p,0)$ is

$$\sum_{n=1}^{\infty} S_{p,0}(n)t^n = [x^p] \log\left(\frac{1}{\sqrt{\Delta(x;t)}}\right)$$

For $i \in \mathbb{Z}$, define $a_i = |\{(j,k)\ /\ (j,1),(k,-1) \in \mathfrak{S} \text{ and } i = j+k\}|$. That is, a_i is equal to the number of couples $(s^+, s^-) \in \mathfrak{S}^+ \times \mathfrak{S}^-$ such that $end(s^+s^-) = (i,0)$. Then we have easily $A_1(x)A_{-1}(x) = \sum_i a_i x^i$, and

$$(A_1A_{-1})^m = (\sum_i a_i x^i)^m = \sum_{\substack{m_1,\ldots,m_k \\ \sum_i m_i = m}} \binom{m}{m_1,\ldots,m_k} \left(\prod_i a_i^{m_i}\right) x^{\sum_i i m_i}$$

Then some standard calculations using Theorem 5.6 lead to an expression of $S_{p,0}(n) = [x^p t^n] \log\left(\frac{1}{\sqrt{\Delta(x;t)}}\right)$, that can naturally be interpreted as in Theorem 3.2(2).

6. Applications

6.1. Examples. We will apply both theorems to particular sets of steps for which closed formulas exist.

First, let us deal with the diagonal lattice (for which $p = 2$). Let $C_n = \binom{2n}{n}/(n+1)$ be the nth Catalan number.

PROPOSITION 6.1 ([**3**]). *Let n be a positive integer. There are $4^n C_n/2$ walks on the slit plane with steps in $\{(\pm 1, \pm 1)\}$ of length $2n$ that end in $(2,0)$.*

PROOF. Let D_n be this number. By Theorem 3.1, we have to enumerate walks of length $2n$ that end at $(2, 2l)$ where $l \in \mathbb{Z}$. In fact the condition that walks end at an even ordinate is superfluous because the walks are of even length, so we have to enumerate walks that end at abscissa 2.

Let us first choose the occurrences of steps with positive abscissa $(1,1)$ and $(1,-1)$; since the walks end at abscissa $(2,0)$, there are $n+1$ occurrences, so that there are $\binom{2n}{n+1}$ choices. To define a walk completely, it remains to choose if the steps go up or down, and there are clearly $2^{2n} = 4^n$ ways to do that.

Finally, by Theorem 3.1, we have

$$2nD_n = 4^n \binom{2n}{n+1},$$

which is equivalent to the desired formula. □

PROPOSITION 6.2 ([**4**]). *Let n be a positive integer. There are $4^n\binom{3n}{n}/(n+1)$ walks on the slit plane with steps in $\{(2,0),(-1,1),(-1,-1)\}$ of length $3n+1$ that end in $(2,0)$.*

PROOF. Let K_n be this number. We have to enumerate walks of length $3n+1$ with endpoint at abscissa 2 and at an even ordinate. Let a be the number of steps $(2,0)$ in such a walk. By focusing at abscissas we have $2a - (3n+1-a) = 2$, so that $a = n+1$. Choosing the occurrences of these steps is then counted by $\binom{3n+1}{n+1}$, and then it remains to choose for the remaining $2n$ steps between $(-1,1)$ and $(-1,-1)$. Note that here again the condition that the endpoint has an even ordinate is superfluous. We finally obtain

$$(3n+1)K_n = 4^n \binom{3n+1}{n+1},$$

which gives us the desired enumeration. □

Let us give an example of application of Theorem 3.2.

PROPOSITION 6.3. *Let n be a positive integer. There are $4^{2n+1}\binom{2n+1}{n}/(4n+2)$ walks on the slit plane with steps in $\{(0,-1),(-1,1),(1,1)\}$ of length $4n+2$ that end in $(1,0)$.*

PROOF. Let M_n be this number. By Theorem 3.2, after having marked a step, we have to enumerate walks of length $4n+2$ whose $2n+1$ first steps are elements of $\{(\pm 1, \pm 1)\}$ and $2n+1$ last steps are elements of $\{(0, \pm 1\}$, and ending at abscissa 1. To ensure the condition on the abscissa, there has to be $n+1$ steps in $\{(1, \pm 1)\}$. One then easily obtains the following identity

$$(4n+2)M_n = 2^{2n+1}\binom{2n+1}{n} \cdot 2^{2n+1},$$

which concludes the proof.

□

In fact, *for all possible sets $\mathfrak{S}$ of 3 steps, not all of them horizontal*, we can find and prove bijectively closed formulas for $S_{p,0}(n)$. Actually, for such sets the number of occurrences of each step for $w \in \mathcal{S}_{p,0}(n)$ is determined by the length n, and the enumeration of the corresponding walks through the bijection becomes easy. One also always obtains closed formulas for sets of cardinality 4 of the form $\{(a, \pm 1), (-b, \pm 1)\}$ where a and b are nonnegative integers.

6.2. Mean number of returns to a given ordinate. There is an obvious refinement of the bijections of Theorems 3.1 and 3.2: the walks on the slit plane marked at height k are sent to walks that end at height $2k$. We note that the case $k=0$ is a consequence of the "cyclic lemma" stated in [**2**].

This can be applied to answer the following question: given $\mathfrak{S}$ and n, assume uniform distribution on the walks of $\mathcal{S}_{p,0}(n)$, how many times on average do these walks hit the height j? Let H_j^n be the random variable defined on $\mathcal{S}_{p,0}(n)$ (with uniform distribution) by: $H_j^n(w) = |\{i > 0 \mid y(w_i) = j\}$. The following proposition, whose proof is immediate, gives a precise answer:

PROPOSITION 6.4. *Let $\mathfrak{S}$ be symmetric with small height variations, n be a positive integer, and j an integer. Then the expectation $\mathbb{E}(H_j^n)$ is equal to n times the quotient of the number of walks of length n ending at ordinate $2j$ by the number of walks of length n ending at even ordinate.*

A generalization can be stated for non symmetric $\mathfrak{S}$. From this proposition, one can obtain closed formulas for a great number of sets $\mathfrak{S}$. For instance, on the square lattice one can gets :

$$\mathbb{E}(H_j^{2n+1}) = \frac{(2n+2)\binom{2n+1}{n+j+1}\binom{2n+1}{n-j+1}}{\binom{4n+2}{2n+1}}$$

By Stirling's formula, this is asymptotically equivalent to $\frac{2\sqrt{2}}{\sqrt{n}}$ for fixed j and n going to infinity.

7. Concluding remarks

7.1. Cyclic lemma for walks in the upper plane. There is a variation on walks on the slit plane considered already in [**3**]. It deals with walks on the slit plane that in addition stay in the upper half plane $\{y \geqslant 0\}$. Let us consider, for a given set of steps $\mathfrak{S}$, the walks that end at $(p, 0)$ where $p = p(\mathfrak{S})$ is defined as before. Then we have the following theorem :

THEOREM 7.1. *Let n be a positive integer. Walks of length n in the region $(y \geqslant 0) \cap \mathcal{H}$ with a marked step ending at $(p, 0)$ are in bijection with walks in the plane of length n ending at $(p, 0)$.*

SKETCH OF THE PROOF. The bijection is defined in a manner similar Theorem 3.1, and is actually simpler because there is no reflection involved: if $w = uv$ is the factorization given by the marked step, then we associate to it the walk $w' = uv$.

For the inverse bijection, let W be a walk on the plane ending at $(p, 0)$. Then define W_1 as the longest prefix of W such that $y(W_1)$ is minimal, and $x(W_1)$ is minimal among all the prefixes U with $y(U)$ minimal. Then, if $W = W_1 W_2$, the inverse construction is defined by $W_2 W_1$ where the first step of W_1 is marked (or the first step if $W_1 = \varepsilon$).

The proof that these are well defined functions which are inverse to one another follows then the same lines as the proof of Theorem, and is actually simpler. 3.1. □

7.2. Other endpoints. In this paper we have dealt with walks on the slit plane ending at a specific endpoint, which is the most natural in that it respects the symmetry of the half line and is as close as possible to it. But, at least for certain sets $\mathfrak{S}$, there are other endpoints that lead to closed formulas for enumeration. Though most of them can be proved using generating functions, only for those ending at $(p,0)$ exists a bijective proof which was the object of this paper

So an obvious problem is to find bijective proofs for such formulas; our belief is that an approach similar to this paper is feasible (though the construction described here does not work as such), and that a construction for a given set of steps can be most certainly generalized to a whole class of steps.

Another question is to relate some of the problems just mentioned between them; for instance, it was noticed in **[2]** that among the walks of length $2n+1$ that go from $(0,0)$ to $(1,0)$ on the square square lattice, exactly as many avoid the horizontal half-line $\{(k,0) \mid k \leqslant 0\}$ as the diagonal half-line $\{(k,k) \mid k \leqslant 0\}$. There are many examples with various endpoints or steps of this phenomenon, which clearly needs a direct combinatorial explanation.

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