BIJECTIONS FOR PERMUTATION TABLEAUX

SYLVIE CORTEEL AND PHILIPPE NADEAU

Authors' affiliations: LRI, CNRS et Université Paris-Sud, 91405 Orsay, France

Corresponding author: Sylvie Corteel
Sylvie.Corteel@lri.fr
LRI Université Paris-Sud 91405 Orsay Cedex
Phone: +33169156608 Fax:+3316915658

Abstract. In this paper we propose two bijections between permutation tableaux and permutations. These bijections show how natural statistics on the tableaux are equidistributed to classical statistics on permutations: descents, RL-minima and pattern enumerations. We then use those bijections to define subclasses of permutation tableaux that are in bijection with set partitions.

Keywords: enumeration, bijections, permutations, tableaux, permutation patterns.

1. Introduction

Permutation tableaux are fairly new objects that come from the enumeration of the totally positive Grassmannian cells [12, 15]. Surprisingly they are also connected to a statistical physics model called the Partially ASymmetric Exclusion Process [5, 7, 8]. As in [13], a *permutation tableau* T is a shape (the Ferrers diagram of a partition into non negative parts) together with a filling of the cells with 0's and 1's such that the following properties hold:

(1) Each column contains at least one 1.
(2) There is no 0 which has a 1 above it in the same column *and* a 1 to its left in the same row.

An example of a permutation tableau is given in Figure 1. Different statistics on permutation tableaux were defined in [8, 13]. We list a few here. The *length* of a tableau is the number of rows plus the number of columns of the tableau. A zero in a permutation tableau is *restricted* if there is a one above it in the same column. A row is *unrestricted* if it does not contain a restricted entry. A one is *superfluous* if it contains a one above itself in the same column.

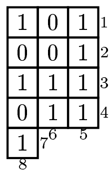

Figure 1: Example of a permutation tableau

We label the South-East border of the shape of the tableau from 1 to its length, going from top-right to bottom-left. On Figure 1, a permutation tableau of shape $(3, 3, 3, 3, 1)$ and length 8 is given. The rows 1, 3 and 7 are unrestricted and the rows 2 and 4 are restricted.

Our main interest here is that there exist $n!$ permutation tableaux of length n. To our knowledge two bijections between permutations and permutation tableaux are known and appeared in [2, 13]. The bijection given in [13] is quite complicated; but a lot of statistics of the permutation (weak excedances, crossings [5], alignments [15]...) can be read from the tableau. In particular the set of weak excedances of the permutation corresponds to the set of rows of the tableau. See [13] for many more details. The bijection in [2] is the same as the one in [13], except that before applying the map some of the entries equal to one are changed into zero.

In this paper, we focus on descent statistics and generalized pattern enumeration and give two bijections between permutation tableaux and permutations.

Let us consider a permutation $\sigma = (\sigma_1, \dots, \sigma_n)$ of $[n] = \{1, 2, \dots, n\}$. For $i < n$, we say that σ_i is a descent if $\sigma_i > \sigma_{i+1}$, otherwise we call it an ascent. The *shape* of a permutation of n is a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_k \geq 0$ such that the i^{th} step of the boundary of λ is West (resp. South) if i is a descent (resp. ascent) of σ. For example, if $\sigma = (7, 1, 2, 6, 4, 3, 5)$, then the descents are $7, 6$ and 4 and the shape of σ is $(3, 3, 3, 2)$. As in [1], the generalized pattern (31-2) occurs in σ if there exist $i < j$ such that $\sigma_{i-1} > \sigma_j > \sigma_i$. The number of occurrences of (31-2) in σ is the cardinality of the set $\{1 < i < j \mid \sigma_{i-1} > \sigma_j > \sigma_i\}$, and will be written 31-2(σ). In the previous example, σ has six occurrences of the pattern (31-2). An entry σ_i is a *RL-minimum* of a permutation σ if and only if $\sigma_i < \sigma_\ell$ for any $\ell > i$.

Our main result is the following:

Theorem 1. *There exists a bijection ξ between permutations of $[n]$ and permutation tableaux of length n. This bijection is such that if $T = \xi(\sigma)$ then*

(1) *the shape of T is the same as the shape of σ.*
(2) *i is an unrestricted row of T if and only if i is a RL-minimum of σ.*
(3) *T has s superfluous ones if and only if there are s occurrences of the pattern (31-2) in σ.*

Remark. Theorem 1 without item (2) is implied by the composition of the two bijections presented in [13]. Our map is different from this composition or any variation of it and gives the full Theorem 1.

In Section 2, we give a very simple proof that there are $n!$ permutation tableaux of length n. We present in Section 3 a first bijection between permutation tableau and permutations which gives Theorem 1 without item (3). To prove Theorem 1 we give the other bijection in Section 4. We give some applications to pattern enumeration in Section 5, define some families of tableaux counted by Bell numbers in 6 and we conclude in Section 7.

2. How many tableaux?

Let $t(n, k, \ell)$ be the number of tableaux of length n with $k+1$ unrestricted rows and ℓ ones in the first row, and let $T_n(x, y) = \sum_{k,\ell} t(n, k, \ell) x^k y^\ell$.

Proposition 1. *If $n > 1$,*

$$T_n(x, y) = \prod_{i=0}^{n-1} (x + y + i)$$

and $T_1(x, y) = 1$. In particular $T_n(1, 1)$, the number of tableaux of length n, is equal to $n!$.

Proof. The proof uses an argument close to the one used in [15] to enumerate permutation tableaux with at most two rows. Given a tableau of length $n-1$ with $j+1$ unrestricted rows and ℓ ones in the first row, one can

- add an empty row and create a tableau of length n with $j+2$ unrestricted rows and ℓ ones in the first row or,

- one can add a column to this tableau. In this case, the cells of the new column need to be filled with zeros and ones. The cells of the restricted rows must get a zero to make sure the tableau is still a permutation tableau. The cells of the unrestricted rows can get a zero or a one. Number the unrestricted rows from 0 to j starting from the top. If $k-i+1$ cells get a one and the topmost one is put in the i^{th} unrestricted row, then the tableau has $k+1$ unrestricted rows and there are $\binom{j-i}{k-i}$ ways to do this. Therefore there are $\binom{j}{k}$ ways to create a tableau of length n with $k+1$ unrestricted rows and $\ell+1$ ones in the first row and $\sum_{i\geq 1}\binom{j-i}{k-i} = \binom{j}{k-1}$ ways to create a tableau of length n with $k+1$ unrestricted rows and ℓ ones in the first row.

Therefore if $0 \leq l < n$ and $0 \leq k < n$ we have

$$t(n,k,\ell) = \sum_{j=k}^{n-1} \binom{j}{k} t(n-1,j,\ell-1) + \binom{j}{k-1} t(n-1,j,\ell), \tag{1}$$

while $t(1,0,0)=1$, and $t(n,k,\ell)=0$ otherwise. Using this recurrence, we directly get that $T_n(x,y) = (x+y)T_{n-1}(x+1,y)$ if $n>1$ and $T_1(x,y)=1$. This completes the proof. □

In particular, $T_n(x,y) = T_n(y,x)$ and we get a symmetry result which was proved combinatorially in [8].

Corollary 1. *The number of permutation tableaux of length n with $k+1$ unrestricted rows and ℓ ones in the first row is equal to the number of tableaux of length n with $\ell+1$ unrestricted rows and k ones in the first row.*

The proposition also implies a result proved in [6] thanks to the bijection of [13] :

Corollary 2. *The number of permutation tableaux of length n with $k+1$ unrestricted rows (or k ones in the first row) is equal to the first Stirling number $s(n,k)$ which enumerates the number of permutations of $[n]$ with k cycles.*

3. Bijection I

In this section we exhibit a bijection between permutation tableaux of length n and permutations of $[n]$. This bijection is such that if σ is the image of T then

(1) the shape of T and the shape of σ are the same and
(2) the list of the RL-minima of σ is the same as the list of the labels of the unrestricted rows of T.

Therefore this proves the first two items of Theorem 1.

A zero in a permutation tableau is a rightmost restricted zero if it is a restricted zero and there is no restricted zero to its right in the same row. The bijection relies on the following claim. A permutation tableau is uniquely determined by its topmost ones and rightmost restricted zeros. Indeed if one knows the positions of the topmost ones (resp. rightmost restricted zeros), then all the cells above (resp. to their left) them are filled with zeros. The rest of the cells are filled with superfluous ones.

From tableaux to permutations. We start with the tableau T of shape λ. Then we initialize the permutation σ to the list of the labels of the unrestricted rows in increasing order. Now for each column, starting from the left proceeding to the right, if the column is labeled by j and if (i, j) is the topmost one of the column then we add j to the left of i in the permutation σ. Moreover if column j contains rightmost restricted zeros in rows $i_1, \ldots, i_k$ then we add $i_1, \ldots, i_k$ in increasing order to the left of j in the permutation σ.

It is easy to see that that the result is a permutation of shape λ. We now prove that the unrestricted rows correspond to the RL-minima of the permutation. This is true when we initialize the permutation to the list of the labels of the unrestricted rows. When we add a descent this does not change the RL-minima, as a descent can not be a RL-minima. When we add the label of a restricted row, it is always inserted to the left of the label of an unrestricted row that has a smaller label. Therefore the RL-minima do not change.

Example 1. We start with the tableau in Figure 1. The unrestricted rows are rows 1,3 and 7. The rightmost restricted zeros are in cells $(2, 8)$ and $(4, 8)$. We start with the permutation $(1, 3, 7)$, We add 8 to the left of 1 and add 2 and 4 to the left of 8. We get $(2, 4, 8, 1, 3, 7)$. We add 6 to the left of 3 and get $(2, 4, 8, 1, 6, 3, 7)$. Finally we add 5 to the left of 1. The permutation is $(2, 4, 8, 5, 1, 6, 3, 7)$.
Example 2. We start with the tableau in Figure 2. The unrestricted rows are rows 1 and 6. The rightmost restricted zeros are in cells $(4, 7)$ and $(2, 3)$. We start with the permutation $(1, 6)$. We add 8 to the left of 1 and get $(8, 1, 6)$. We add 7 to the left of 1 and 4 to the left of 7 and get $(8, 4, 7, 1, 6)$. We then add 5 to the left of 4 and get $(8, 5, 4, 7, 1, 6)$. Finally we add 3 to the left of 1 and 2 to the left of 3. The result is $(8, 5, 4, 7, 2, 3, 1, 6)$.

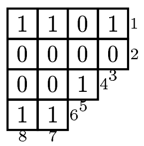

Figure 2: Image of the permutation $(8, 5, 4, 7, 2, 3, 1, 6)$.

The reverse is as easy to define.

From permutations to tableaux. Let σ be a permutation of $\{1, \ldots, n\}$ and let T be its image. We first draw the shape of the tableau T which is the same as the shape of σ. For i from 1 to n, we draw a West step if i is a descent and a South step otherwise and we label those steps from 1 to n. An example for $\sigma = (2, 4, 8, 5, 1, 6, 3, 7)$ is given in Figure 1, as 5,6 and 8 are the descents of σ. Now let us fill the cells of the tableau T with topmost ones and rightmost restricted zeros. As remarked at the beginning of the section, the rest of the entries can be filled in a unique way when the topmost ones and the rightmost restricted zeros are known. Let (i, j) be the eastmost and southmost cell that is not yet visited and is such that i and j are adjacent in the permutation.

- if i is before j, fill cell (i,j) with a rightmost restricted zero and delete i from the permutation σ.
- otherwise fill cell (i,j) with a topmost one and delete j from the permutation σ.

At the end of this process, σ is the list of the labels of the unrestricted rows of T in increasing order. Then fill the rest of the cells of T. One can see that this is the reverse mapping.

Example 1. We start with $\sigma = (2,4,8,5,1,6,3,7)$, and we draw the shape of T (see Figure 1). We first fill (1,5) with a topmost one and delete 5 from the permutation. The permutation is now $\sigma = (2,4,8,1,6,3,7)$. We then fill cell $(3,6)$ with a topmost one and delete 6 from the permutation. The permutation is now $\sigma = (2,4,8,1,3,7)$. Then cell $(4,8)$ gets a rightmost restricted zero and 4 is deleted. The permutation is now $\sigma = (2,8,1,3,7)$. Finally cell $(2,8)$ gets a rightmost restricted zero and 2 is deleted. The permutation is now $(8,1,3,7)$. Cell $(1,8)$ gets a topmost one and 8 is deleted. The permutation is finally $(1,3,7)$. The result is given in Figure 1.

We have thus defined in this Section a simple bijection that possesses the first two properties of Theorem 1; to get all three properties, we will define another bijection in a quite different way.

4. Bijection II

4.1. **Reduction of the tableaux.** We give in this Section a recursive decomposition of the tableaux that was used in [15] to enumerate permutation tableaux with two rows. This decomposition will be essential to define our second bijection.

Let T be a tableau of length $n > 0$ and of shape $(\lambda_1, \lambda_2, \ldots, \lambda_m)$. We suppose that the last row of T is labeled by k and that the length of this row is t. Then three cases are possible:

- Type 1 : The last row does not contain any ones.
- Type 2 : The rightmost entry of the last row contains a topmost one.
- Type 3 : The rightmost entry of the last row contains a superfluous one.

From the definition of the permutation tableaux we know that these are the only three possible cases. Indeed if the rightmost entry of the last row is a zero then all the entries of the row are zeros.

We can then reduce a tableau T according to its type:

- If the tableau T is of type 1, then we can delete the last row and get a tableau of length $n-1$ and shape $(\lambda_1, \ldots, \lambda_{m-1})$.
- If the tableau is of type 2, then we can delete the column $k+1$ and get a tableau of length $n-1$ and shape $(\lambda_1 - 1, \ldots, \lambda_m - 1)$.
- If the tableau is of type 3, we can delete the rightmost entry of the last row and get a tableau of length n and shape $(\lambda_1, \ldots, \lambda_{m-1}, \lambda_m - 1)$.

The resulting tableau is denoted $red(T)$; note that when applying this reduction, the sum of the length of the tableau plus its number of superfluous ones decreases by one. Therefore, given a tableau of length n with j superfluous ones, exactly $n+j$ reductions will give the empty tableau. If each time we reduce the tableau, we keep in mind the type $1(t)$ (t is the length of the last row), 2 or 3, this gives an encoding of the tableau, since it allows us to inverse the specific reduction that took place.

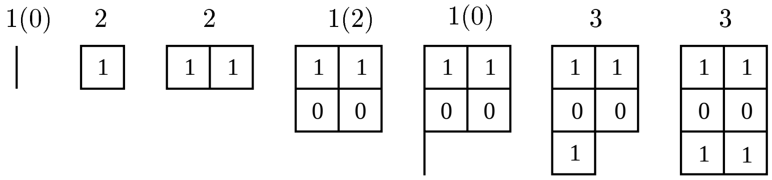

Figure 3: Successive reductions of a tableau (from right to left).

Let us give a simple example in Figure 3. The tableau of shape $(2,2,2)$ at the extreme right is reduced successively, and $1(0),2,2,1(2),1(0),3,3$ is the code obtained in the process.

4.2. **Reduction of a permutation.** Given a permutation $\sigma=(\sigma_1,\dots,\sigma_n)$ with $\sigma_j=k$, we denote by (31-2)(k) the cardinality of the set $\{1<i<j \mid \sigma_{i-1}>k>\sigma_i\}$. This corresponds to the number of occurrences of the pattern 31-2 where k is the "2" of the pattern. For example, if $\sigma=(5,2,1,6,3,4)$ then (31-2)$(4)=2$. Let σ be a permutation of shape $\lambda=(\lambda_1,\dots,\lambda_m)$ such that k is the largest ascent. We suppose that $\sigma_0=0$ and $\sigma_{n+1}=n+1$. We say that σ_i is a peak (resp. double descent, resp. valley, resp. double ascent) if $\sigma_{i-1}<\sigma_i>\sigma_{i+1}$ (resp. $\sigma_{i-1}>\sigma_i>\sigma_{i+1}$, resp. $\sigma_{i-1}>\sigma_i<\sigma_{i+1}$, resp. $\sigma_{i-1}<\sigma_i<\sigma_{i+1}$).

Three types of permutations exist :

- Type 1 : k is a double ascent in σ and (31-2)$(k)=0$. (This implies that k is to the left of $k+1$.)
- Type 2 : $k+1$ is to the left of k and (31-2)$(k+1)=0$ and one of the following holds
  - $k+1$ is a double descent
  - k and $k+1$ are adjacent
- Type 3 : None of the previous configurations appears. That is
  (1) k is to the left of $k+1$ and adjacent to $k+1$ and is a valley; or
  (2) $k+1$ is to the left of k and adjacent to k and $k+1$ is a peak and (31-2)$(k+1)>0$; or
  (3) k is to the left of $k+1$ and k is a double ascent and (31-2)$(k)>0$; or
  (4) $k+1$ is to the left of k and $k+1$ is a double descent and (31-2)$(k+1)>0$; or
  (5) k is to the left of $k+1$ and not adjacent to $k+1$ and k is a valley; or
  (6) $k+1$ is to the left of k and not adjacent to k and $k+1$ is a peak.

This takes care of all the possible cases.

We define a reduction RED of the permutation σ whose largest ascent is k :

- If σ is of type 1 : Delete k and decrease by one all the entries greater than k. The result is a permutation of $[n-1]$ and shape $(\lambda_1,\dots,\lambda_{m-1})$.
- If σ is of type 2 : delete $k+1$ and decrease by one all the entries greater than k and get a permutation of $[n-1]$ and shape $(\lambda_1-1,\dots,\lambda_m-1)$.
- If σ is of type 3 : apply bijection Φ defined below and get a permutation of $[n]$ and shape $(\lambda_1,\dots,\lambda_{m-1},\lambda_m-1)$ with one less occurrence of (31-2).

We now give a bijection Φ between permutations of $[n]$ of type 3 of shape λ with j occurrences of (31-2) and permutations of shape $(\lambda_1,\dots,\lambda_{m-1},\lambda_m-1)$ with

$j-1$ occurrences of (31-2). The basic idea is to exchange k and $k+1$ in σ in order to transform k into a descent, $k+1$ into an ascent. This will work unless k and $k+1$ are adjacent. Moreover we will decrease by one the number of occurrences of (31-2), unless k is to the left and not adjacent to $k+1$ or k is adjacent to $k+1$ and to its right. In those cases, we will have to do a bit more.

We give the details in the following paragraph and illustrate in parallel the bijection on Figure 4. We write the permutation $\sigma = (\sigma_1, \dots, \sigma_n)$ as the word $0\sigma_1 \dots \sigma_n(n+1)$. We suppose that $p_1, p_2, \dots$ are words with elements smaller than k; $G_1, G_2, \dots$ are words with elements larger than k; and that X, Y, Z are words. The words denoted X, Y, Z may be empty, while the p_i and G_i are nonempty unless explicitly stated otherwise:

(1) If k is a valley and is just to the left of $k+1$, then σ can be written as $Xp_1G_1k(k+1)p_2Y$. We set $\Phi(\sigma) = Xp_1(k+1)G_1kp_2Y$.
(2) If $k+1$ is a peak, just to the left of k, and $(31\text{-}2)(k+1) > 0$, then σ can be written as $XG_1p_1(k+1)kG_2Y$. We set $\Phi(\sigma) = XG_1kp_1(k+1)G_2Y$.
(3) If k is to the left of $k+1$, k is a double ascent and $(31\text{-}2)(k) > 0$, then σ can be written as $Xp_1G_1p_2kG_2Y(k+1)p_3Z$. We set $\Phi(\sigma) = Xp_1(k+1)G_1p_2G_2Ykp_3Z$. (Here G_2Y may be empty.)
(4) If $k+1$ is on the left of k, $k+1$ is a double descent and $(31\text{-}2)(k+1) > 0$, then $\sigma = XG_1p_1G_2(k+1)p_2YkZ$ and $\Phi(\sigma) = XG_1kp_1G_2p_2Y(k+1)Z$. (Here p_2Y may be empty.)
(5) If k is a valley, on the left of $k+1$ but not adjacent to it, then σ can be written as $Xp_1G_1kG_2Y(k+1)Z$. We set $\Phi(\sigma) = Xp_1G_1(k+1)G_2YkZ$.
(6) If $k+1$ is a peak, on the left of k but not adjacent to it, then σ can be written as $Xp_1(k+1)p_2YkZ$. We set $\Phi(\sigma) = Xp_1kp_2Y(k+1)Z$.

The six cases are pictured on Figure 4. The dots represent k and $k+1$, and possible prefixes and suffixes are not pictured since they are not modified by Φ. One sees, in the first four cases, how the number of occurrences of (31-2) is decremented by suitably moving one of the entries among $k, k+1$ to the left; this is not required in the last two cases, where the mere exchange of k and $k+1$ suffices to decrement 31-2(σ).

To show that this is indeed a bijection, we give the inverse algorithm. Start with a permutation π where $k+1$ is the largest ascent and k is a descent. Note that k and $k+1$ can not be adjacent in the permutation.

(1) If $k+1$ is to the left of k and $k+1$ is a double ascent:
    (a) If all entries between $k+1$ and k are greater than $k+1$, then $\pi = Xp_1(k+1)G_1kp_2Y$ and $\Phi^{-1}(\pi) = Xp_1G_1k(k+1)p_2Y$.
    (b) Otherwise $\pi = Xp_1(k+1)G_1p_2G_2Ykp_3Z$ and $\Phi^{-1}(\pi) = Xp_1G_1p_2kG_2Y(k+1)p_3Z$ (G_2Y may be empty here) .
(2) If k is to the left of $k+1$ and k is a double descent:
    (a) If all entries between k and $k+1$ are smaller than k, then $\pi = XG_1kp_1(k+1)G_2Y$ and $\Phi^{-1}(\pi) = XG_1p_1(k+1)kG_2Y$.
    (b) Otherwise, $\pi = XG_1kp_1G_2p_2Y(k+1)Z$ and $\Phi^{-1}(\pi) = XG_1p_1G_2(k+1)p_2YkZ$ (p_2Y may be empty here).
(3) If $k+1$ is to the left of k and $k+1$ is a valley, then $\pi = Xp_1G_1(k+1)G_2YkZ$ and $\Phi^{-1}(\pi) = Xp_1G_1kG_2Y(k+1)Z$.

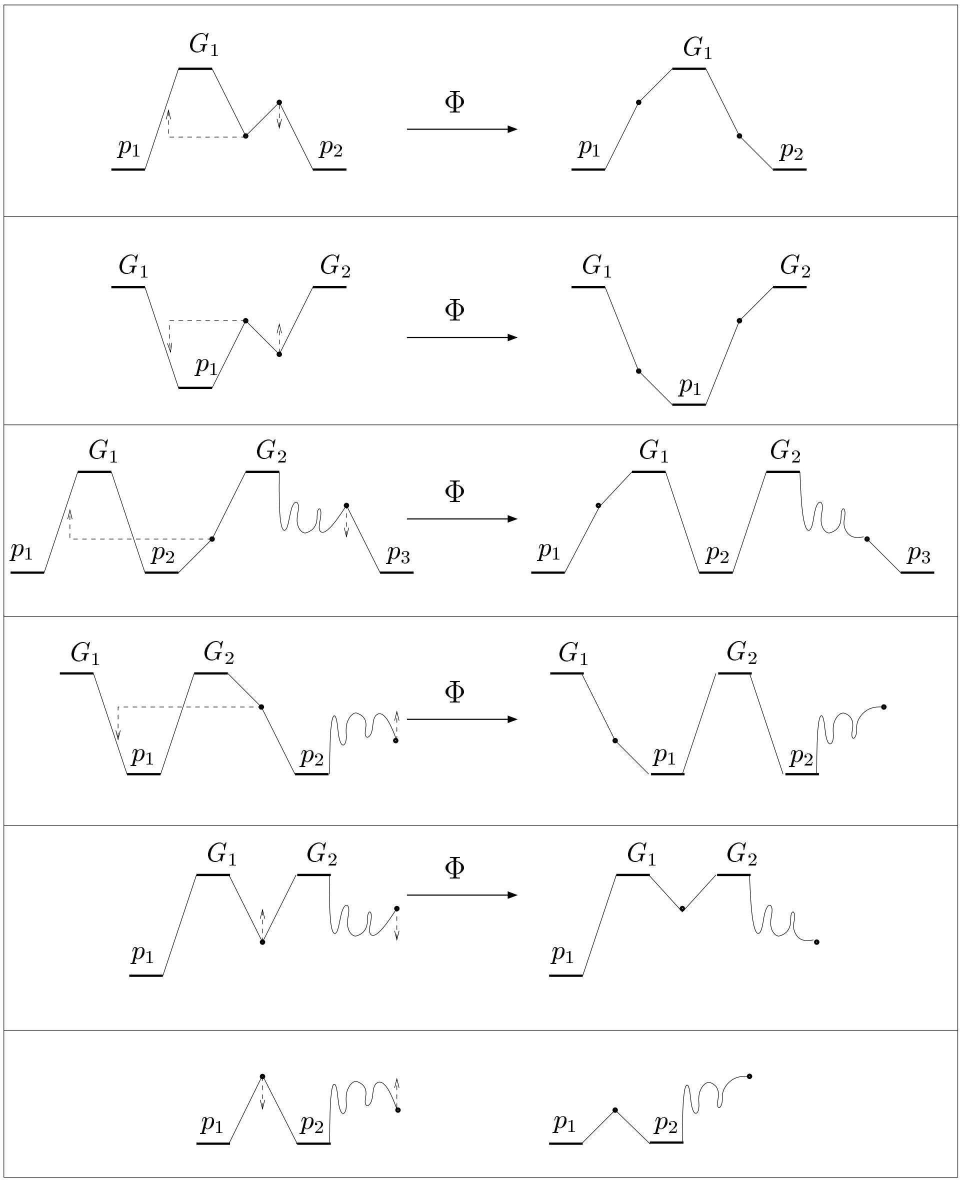

Figure 4: The six cases in the definition of Φ.

(4) Otherwise k is to the left of $k+1$ and k is a peak, then $\pi = Xp_1kp_2Y(k+1)Z$ and $\Phi^{-1}(\pi) = Xp_1(k+1)p_2YkZ$.

Proposition 2. *Φ is a bijection between permutations of $[n]$ of type 3 of shape λ with j occurrences of (31-2) and permutations of shape $(\lambda_1, \ldots, \lambda_{m-1}, \lambda_m - 1)$ with $j-1$ occurrences of (31-2)*

Proof. We gave details of the construction of Φ, as well as its tentative inverse : it is then an easy task (albeit a bit tedious) to check that there is in fact a bijective correspondence between cases 1 to 6 in the definition of Φ and, respectively, cases $1(a), 2(a), 1(b), 2(b), 3$ and 4 of the definition of Φ^{-1}. Details can be found in [10]. □

From the previous result, we can derive easily an algorithmic bijection between permutation tableaux and permutations. This is what we explain in the following section.

4.3. The bijection ξ.

From permutations to tableaux. Let σ be a permutation of $[n]$ and k its largest ascent. If σ is the empty permutation then $\xi(\sigma)$ is the empty tableau.

Otherwise we define $\xi(\sigma)$ by induction. Let T' be the tableau $\xi(RED(\sigma))$.

- If σ is of type 1 : $\xi(\sigma)$ is the tableau T' with one extra row of length $n-k$ filled with zeros.
- If σ is of type 2 : $\xi(\sigma)$ is the tableau T' with one extra column made of as many rows as T' with its lower cell at the end of the last row of T'. This lower cell is filled with a one and all the cells above it with zeros.
- If σ is of type 3 : $\xi(\sigma)$ is the tableau T' with one extra cell added to the last row and filled with a superfluous one.

This can be expressed by the encoding described at the end of paragraph 4.1: if c is the encoding of the tableau T', then $T = c, 1(n-k)$ (resp. $T = c, 2$, resp. $T = c, 3$) if σ is of type 1 (resp. of type 2, resp. of type 3).

An example is given on Figure 5. When $\sigma = (2,4,8,5,1,6,3,7)$. We first compute $RED(\sigma) = (2,4,7,5,1,6,3,8)$, and get $\xi(RED(\sigma))$ (which is supposed known by induction) on the left of Figure 5. As σ is of type 3, we add a cell with a one at the end of the last row and get $\xi(\sigma)$ on the right of Figure 5.

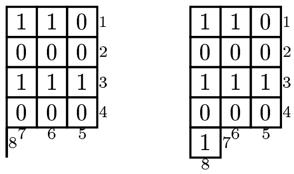

Figure 5: Images of (2,4,7,5,1,6,3,8) and $(2,4,8,5,1,6,3,7)$

To give a full example, let us consider the permutation $(2,5,1,4,3)$. It is of type 3, and its successive reductions are $(2,5,3,1,4)$ of type 3, $(2,4,3,1,5)$ of type 1, $(2,4,3,1)$ of type 1, $(3,2,1)$ of type 2, $(2,1)$ of type 2, (1) of type 1, and the empty permutation. From this one can build the corresponding tableau inductively, and the result is none other than Figure 3, and $\xi((2,5,1,4,3))$ is thus the tableau on the far right of this figure.

Proof of Theorem 1. We now prove by induction that

1. (1) the shape of $T = \xi(\sigma)$ is the same as the shape of σ.
2. (2) i is an unrestricted row of T if and only if i is a RL-minimum of σ.
3. (3) T has s superfluous ones if and only if there are s occurrences of the pattern (31-2) in σ.

If σ is the empty partition, the claims are true. Now we suppose by induction that everything holds for $T' = \xi(RED(\sigma))$. As explained in the subsection on the reduction of the permutation, the shape of $RED(\sigma)$ is the same as the shape of σ with one deleted row, column or cell depending on the type on σ. Since we add back the same row, column or cell to the shape of T' to create T, the shape of T is the same as the shape of σ, so (1) is proved.

The RL-minima of $RED(\sigma)$ are the same as the RL-minima of σ (up to an obvious renumbering in types 1 and 2 for σ), *unless* $\sigma_n = n$ and σ has one extra RL-minimum. Equivalently T and T' have the same unrestricted rows unless we add a row of length zero to T' which is indeed unrestricted. This happens only if $\sigma_n = n$, which proves (2).

Finally the number of occurrences of (31-2) of $RED(\sigma)$ is the same as for σ unless σ is of type 3, in which case σ has one extra occurrence of (31-2). The number of superfluous ones of T and T' differ at most by one. They differ by one exactly when a cell is appended to the last row, which is exactly done when σ is of type 3 and proves (3).

To finish the proof, we need to prove that ξ is a bijection and we give the reverse mapping, where we will use the notations p_i, G_i, X, Y introduced in the definition of the function Φ.

From tableaux to permutations. If T is the empty tableau then $\xi^{-1}(T)$ is the empty permutation. Otherwise we will define $\xi^{-1}(T)$ by induction; let σ be the permutation $\xi^{-1}(red(T))$:

- If T is of type 1 and its last row is of length $n-k$: increase all the entries of σ greater than or equal to k by one. Insert k to the left of the leftmost entry greater than k, so that we transform $p_1 G_1 X$ in $p_1 k G_1 X$.
- If T is of type 2, then let k be the largest ascent of the permutation σ. Increase by one all the entries greater then k.
  (1) If there is no entry larger than k to its left, then insert $k+1$ to the left of k; that is, we transform $p_1 k X$ in $p_1 (k+1) k X$.
  (2) Otherwise let i be the leftmost element greater than k such that i is to the left of k and the element after i is smaller than $k+1$. Insert $k+1$ to the right of i in σ: thus we transform $p_1 G_1 X k Y$ in $p_1 G_1 (k+1) X k Y$.
- If T is of type 3 then σ becomes $\Phi^{-1}(\sigma)$.

In each case the permutation $\xi^{-1}(T)$ is defined to be the permutation τ obtained; it is respectively of type $1, 2$ and 3, and $RED(\tau)$ is exactly the permutation σ. This proves Theorem 1.

5. Permutation patterns

5.1. **Bijection between permutation tableaux and PT-words.** We will show that the reduction defined in Section 4 directly defines a bijection Υ between permutation tableaux and certain words on the alphabet $\{D, U, V\}$. We define the height h of the letters $h(D) = -1$, $h(U) = h(V) = 1$. The height of a word is the sum of the heights of its letters. To define Υ, it is easier to define first a function Υ_0 as follows: if T is the empty tableau then $\Upsilon_0(T)$ is the empty word. Otherwise, let t be the length of the last row of T :

- If T is of type 1, then $\Upsilon_0(T) = \Upsilon_0(red(T))D^i U$, where i is such that $h(\Upsilon_0(T)) = t+1$.

- If T is of type 2, then $\Upsilon_0(T) = \Upsilon_0(red(T))U$.
- If T is of type 3, then $\Upsilon_0(T) = \Upsilon_0(red(T))V$,

where $red(T)$ is the reduction defined in Section 4.1.

We add $t+1$ letters D at the end of $\Upsilon_0(T)$ if the last row of T has length t, and this gives us finally the word $\Upsilon(T)$.

There is an easily equivalent non-recursive description of $\Upsilon(T)$ as follows: consider the rows of T from top to bottom, read from left to right. For each row, first write down a U, and then a U (respectively a V) every time you encounter a topmost one (resp. a superfluous one) in the row. When you reach the end of the row, consider the word formed up until then (i.e. with the possible previous rows), and add as many Ds as necessary so that its height is equal to the number of restricted zeros of the following row. The resulting word is then $\Upsilon(T)$.

Example 3. Consider the tableau T_0 on the extreme right of Figure 3, the word $\Upsilon_0(T_0)$ is $U \cdot U \cdot U \cdot DU \cdot DDDU \cdot V \cdot V$, and one adds DDD at the end to obtain the final word $\Upsilon(T) = UUUDUDDDUVVDDD$. To take a bigger example, consider the tableau T_1 of Figure 1. We have $\Upsilon(T_1) = \Upsilon_0(T_1)DD$ because the last row of T_1 has length 1. Then one checks that

$$\Upsilon_0(T_1) = UUUDDUVDDDUVUVDDDUVVDDDDUV.$$

We explicit the family of words given by this construction:

Definition 1. *A PT-word is a word w on the alphabet $\{D, U, V\}$ such that for each prefix X of w, $h(X) \geq 0$ and $h(w) = 0$; a letter D can not be followed by a letter V; and w can be decomposed into $w_1 D^{d+1} UMD w_2$ with M a word on the alphabet $\{U, V\}$ and d maximal if and only if M contains at most d letters V. Finally, only letters U can precede the first letter D.*

Proposition 3. *Υ is a bijection between permutation tableaux of length n, k superfluous ones and j unrestricted rows and PT-words of length $2n+2k$, with k letters V and j prefixes of height 0.*

Proof. We will describe the inverse bijection by induction on the length of the PT-words. Let a nonempty PT-word w be given, and consider its factorization $w = w_0 D^t UMD^u$, where M is a word on the alphabet $\{U, V\}$, and t is chosen maximal:

(1) if M is empty, then define $w' = w_0 D^{t+u-1}$;
(2) if M ends with a U, i.e $M = M'U$, then $w' = w_0 D^t UM' D^{u-1}$.
(3) if M ends with a V, i.e $M = M'V$, then $w' = w_0 D^t UM' D^{u-1}$.

It is immediate to check that w' is a PT-word; so, by induction, there exists a unique tableau T' such that $w' = \Upsilon(T')$. We then define T as the tableau T' to which a certain operation is applied according to the three cases above:

(1) T is obtained by adding a row of zeros of length $u-1$ under T'.
(2) T is obtained by inserting a column in T', with as many rows as T', and with its lower cell at the end of the last row of T'. This cell contains a one, and all other cells above it are filled with zeros.
(3) T is obtained by adding a cell containing a one at the end of the last row of T'.

This construction from w to T is the inverse of the bijection Υ; details can be found in [10]. The preservation of the different statistics is immediate. ☐

5.2. **Shape of a tableau T given $\Upsilon(T)$.** We can easily describe the shape of a tableau T given its associated PT-word $\Upsilon(T)$: if $\Upsilon(T)$ is empty then T is the empty tableau. Otherwise, decompose $\Upsilon(T)$ in the form

$$\Upsilon(T) = U^{k_0} D^{l_1} M_{k_1} \cdots D^{l_t} M_{k_t} D^{l_{t+1}},$$

where all k_i and l_i are positive, and M_{k_i} is a word on the alphabet $\{U, V\}$ for each i. Define v_i as the number of letters V in the word M_{k_i}; by definition of a PT-word we have $v_i \leqslant l_i - 1$. Then the South East border of the tableau T is given by

$$SW^{l_1-1-v_1} SW^{l_2-1-v_2} S \cdots W^{l_t-1-v_t} SW^{l_{t+1}-1}.$$

This is easily proved by induction.

For the word $\Upsilon(T_1)$ of Example 3, we have $l_1 = 2, v_1 = 1; l_2 = 3, v_2 = 2; l_3 = 3, v_3 = 2; l_4 = 4, v_4 = 1$ and finally $l_5 = 2$. This gives a South East border encoded by $SSSSWWSW$, in concordance with the tableau of Figure 1.

5.3. **One occurrence of (31-2).** It is well known that the number of permutations of $[n]$ with no occurrence of the pattern (31-2) is equal to the n^{th} Catalan number [3]. The bijection between permutation tableaux and PT-words given in Section 5.1 gives another proof of this fact. Indeed if the permutation tableau has no superfluous ones, the corresponding word is a Dyck word. Thanks to this approach, we can also give the first bijective proof of the following fact :

Proposition 4. [4] *The number of permutations of $[n]$ with one occurrence of the pattern (31-2) is equal to*

$$\binom{2n}{n-3}.$$

Proof. There exist simple bijections between

(1) PT-words of length $2n+2$ with one letter V
(2) Words on $\{D, U\}$ of length $2n$ which end at height -2 such that the height after the last D step is strictly larger than the minimal height of the path.
(3) Words on $\{D, U\}$ of length $2n$ that end at height -6.

These bijections imply the result as the number of words on $\{D, U\}$ of length $2n$ that end at height -6 is $\binom{2n}{n-3}$. The reader is advised to follow the constructions on Figure 6.

(1) $\leftrightarrow$ (2). Let w be a PT-word of length $2n+2$ with one letter V. Then w can be decomposed uniquely into $w_0 D^2 U^t V w_1$ with $t > 0$. Then the image of w is $w' = w_1 D w_0 D U^{t-1}$, which belongs to the family (2). It is easy to see that this is a bijection, where the inverse construction goes like this. A path P from (2) can be decomposed a $p_0 D p_1 D U^u$ with $u \geq 0$, where $p_0 D$ is the prefix after which P reaches its minimal height for the first time. Then the word of (1) corresponding to P is $p_1 D^2 U^{u+1} V p_0$.

(2) $\leftrightarrow$ (3). Let w' be a path of the family (2). Then it can be uniquely decomposed into $w_2 D w_3 D U^i$ such that: $h(w_2) = -i-2$, the height of every prefix of w_2 is greater than or equal to $-i-2$, and w_2 is maximal for these properties. The image of w' is $w_2 D \tilde{w}_3 D U^i$ where $\tilde{w}_3$ is the word w_3 where every U is changed into D and every D into U. It is easy to check that this word ends at height -6, and that this is indeed a bijection. □

Another bijective proof of this result can be done by using the Françon-Viennot correspondence [9] and similar arguments (see [10]). Actually, Parviainen [11] proved Proposition 4 using a variation of the Françon-Viennot correspondence, and in fact gave a general procedure to extract formulas for the number of permutations of $[n]$ with k occurrences of the pattern (31-2). Nevertheless, though more combinatorial than in [4], his results are not fully bijective, and thus do not explain the simplicity of $\binom{2n}{n-3}$ in a completely satisfying way.

$(3, 2, 4, 1, 9, 7, 6, 10, 11, 8, 5)$

0	0	0	0	1	0
0	0	0	0	0	1
1	0	1	0		
0	1	1	1		
0					

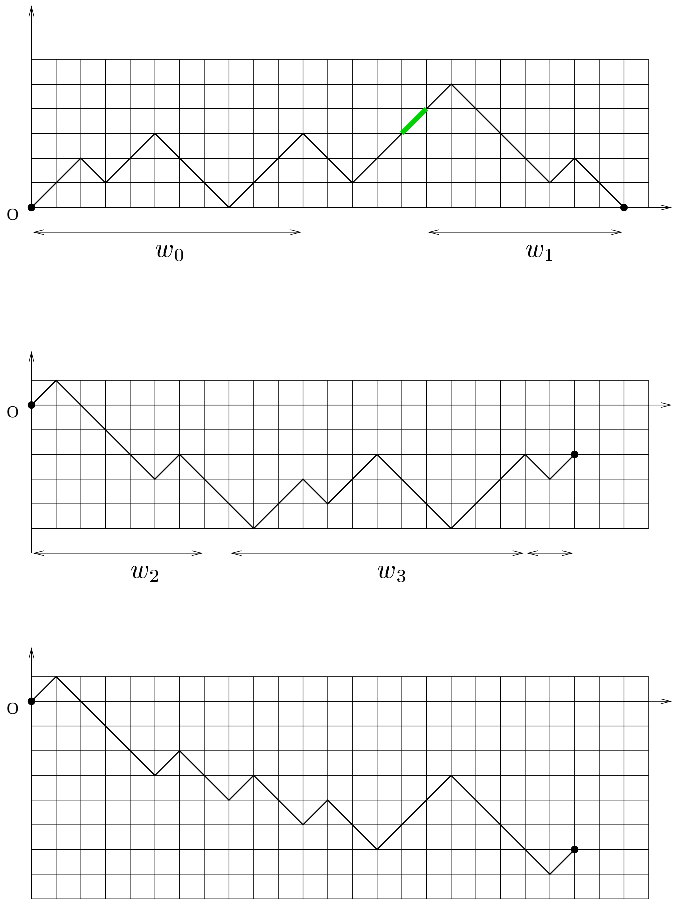

Figure 6: A permutation with $31\text{-}2(\sigma) = 1$, its tableau $\xi^{-1}(\sigma)$, and the three types of paths used in the proof of Proposition 4

It would be interesting to pursue this approach to give bijective proofs of the following simple formulas, first proved analytically by Claesson and Mansour:

Proposition 5. [4] *The number of permutations of* $[n]$ *with two occurrences of (31-2) is*

$$\frac{n(n-3)}{2(n+4)}\binom{2n}{n-3}$$

and the number of permutations of $[n]$ *with three occurrences of (31-2) is*

$$\frac{1}{3}\binom{n+2}{2}\binom{2n}{n-5}.$$

6. Bell tableaux

In this Section we give two subfamilies of permutation tableaux that are in bijection with set partitions. A set partition of the set $[n]$ is a set of pairwise disjoint nonempty subsets of $[n]$ whose union is $[n]$. A set partition can also be seen as a permutation where all the cycles are increasing cycles. Recall that a one is topmost if it has no ones above itself in its column. A one is leftmost if it has no ones to its left in its row and rightmost if it has no ones to its right in its row.

6.1. **L-Bell tableaux.**

Definition 2. *An* L-Bell tableau *is a permutation tableau where all the topmost ones are also leftmost ones.*

Proposition 6. *There exists a bijection between L-Bell tableaux of length* n *such that the sum of the number of columns and the number of zero rows is* k *and set partitions of* $[n]$ *with* k *blocks.*

Proof. We start by giving the map from the tableaux to the set partitions. For every column of the tableau, construct a block of the set partition that is made of the label of the column and the labels of the rows that have a leftmost one in this column. The labels of zero rows form blocks of size 1.

The reverse is as easy. Given a set partition, the shape of the corresponding tableau is drawn such that the labels of the columns correspond to the largest element of each block of size at least 2. Then the tableau is filled from left to right and top to bottom :a cell is filled with a one if the label of its row is in the same block as the label of its column, or if it has a one above and to the left of itself; otherwise it is filled with a zero. □

For example, given the tableau on Figure 7, we get the set partition $\{1,7,8\},\{3,4,6\},\{2,5\}$.

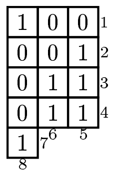

Figure 7: Example of a tableau where the topmost ones are also leftmost

6.2. **R-Bell tableaux.**

Definition 3. *An* R-Bell tableau *is a permutation tableau where all the topmost ones are also rightmost ones.*

Proposition 7. *There exists a bijection between R-Bell tableaux of length n such that the sum of the number of columns and the number of zero rows is k and set partitions of $[n]$ with k blocks.*

Proof. We propose a bijection based on the bijection of [13]. We apply this bijection to construct a permutation σ. This bijection is such that for each row with label i, if the row has no ones then $\sigma(i) = i$. Otherwise start with the leftmost one of row i and travel South and East changing direction each time a one is reached until the border is reached. Then $\sigma(i) = j$, where j is the label of the border. Apply the same process for the columns, starting at the topmost one and traveling East and South. It is easy to see that the tableau is an R-Bell tableau if and only if $\sigma(i) < i$ implies that $\sigma(\sigma(i)) \geq \sigma(i)$ and there does not exist $j < i$ such that $\sigma(j) < \sigma(i) < j < i$. Then we can transform σ in the set partition $\Pi = \{\Pi_1, \ldots, \Pi_k\}$ such that k is the number of non excedances plus the number of fixed points of σ and such that in each block $\{\pi_1, \pi_2, \ldots, \pi_\ell\}$ then ($\ell = 1$ and $\sigma(\pi_\ell) = \pi_\ell$) or $\pi_i = \sigma(\pi_{i-1})$ for all $1 < i \leq \ell$ and $\sigma(\pi_\ell) < \pi_\ell$. □

6.3. **Bijection between R-Bell and L-Bell tableaux.**

One might be surprised that R-Bell and L-Bell tableaux of length n are in bijection with set partitions of $[n]$, since there is no apparent left-right symmetry in the definition of permutation tableaux. Indeed we can show that

Proposition 8. *There is a bijection between R-Bell tableaux of shape λ and L-Bell tableaux of shape λ.*

Proof. This is direct using the bijection between permutation tableaux and PT-words defined in Section 5. Indeed a PT-word corresponds to a L-Bell tableau (resp. R-Bell) if and only if each subword on the alphabet $\{U, V\}$ is of the form U^tV^n where $t = 1$ or 2 and $n \geq 0$ (resp. UV^nU^t where $t = 0$ or 1 and $n \geq 0$). Given a word $A = a_1 \ldots a_n$, we define $\overline{A}$ to be the word $a_n \ldots a_1$. Then given a PT-word $w = UA_1D^{b_1}UA_2D^{b_2} \ldots$ we define $I(w) = U\overline{A_1}D^{b_1}U\overline{A_2}D^{b_2} \ldots$. The function I is an involution on the set of PT-words. The previous remarks imply that w is a PT-word that corresponds to a L-Bell tableau if and only of $I(w)$ is a PT-word that corresponds to a R-Bell tableau. The shapes of the tableaux are the same, as is immediately implied by the result of section 5.2. We could also define this involution directly on the tableaux, but it is less straightforward. □

7. Conclusion and open problems

In this paper we give two bijections between permutation tableaux and permutations that send the columns of the tableaux to the descent of the permutation. We also relate the superfluous ones of the tableaux to the number of occurrences of the pattern (31-2) of the permutation. We then use this approach to enumerate permutations with one occurrence of the pattern (31-2). We finally introduce Bell tableaux that are in bijection with set partitions. It is well known that set partitions are in one-to-one correspondence with permutations with no occurrences of the pattern 32-1 [3]. It would be interesting to find the statistic on permutation tableaux that has the same distribution as the number of occurrences of 32-1.

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LRI, CNRS et Université Paris-Sud, 91405 Orsay, France
E-mail address: `Sylvie.Corteel@lri.fr`

LRI, CNRS et Université Paris-Sud, 91405 Orsay, France and Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, 1090 Vienna, Austria
E-mail address: `nadeau@lri.fr`