ABELIAN REPETITIONS IN STURMIAN WORDS

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The Parikh vector of a finite word w enumerates the occurrences of each letter of the alphabet in w. Therefore, two words have the same Parikh vector if and only if one can be obtained from the other by permuting letters. A finite word w is called an abelian repetition of (abelian) period m and exponent $|w|/m$ if w can be written as $w = u_0u_1 \cdots u_{j-1}u_j$, for an integer $j \geq 3$, where for $0 < i < j$ all the u_i's have the same Parikh vector $\mathcal{P}$ whose sum of components is m and the Parikh vectors of u_0 and u_j are contained in $\mathcal{P}$. When u_0 and u_j are both empty, w is called an abelian power.

We study the maximal exponent of abelian powers and abelian repetitions occurring in infinite Sturmian words. More precisely, we give a formula for computing the maximal exponent of an abelian power of period m occurring in any Sturmian word s_α of slope α, and a formula for computing the maximal exponent of an abelian power of period m starting at a given position n in the Sturmian word $s_{\alpha,\rho}$ with slope α and intercept ρ.

Fixed a Sturmian word s_α of slope α, we prove that if k_m (resp. k'_m) denotes the maximal exponent of an abelian power (resp. an abelian repetition) of period m, then $\limsup k_m/m = \limsup k'_m/m \geq \sqrt{5}$, and the equality holds for the infinite Fibonacci word. We further prove that the above superior limits are finite if and only if the development in continued fraction of α has bounded partial quotients, equivalently if and only if s_α is β-power free for some real number β.

Concerning the infinite Fibonacci word, we prove that the longest prefix that is an abelian repetition of period F_j, $j > 1$, has length $F_j(F_{j+1} + F_{j-1} + 1) - 2$ if j is even or $F_j(F_{j+1} + F_{j-1}) - 2$ if j is odd, where F_n is the nth Fibonacci number.We also prove that the smallest abelian period of any factor of the Fibonacci word is a Fibonacci number.

From the previous results, we derive the exact formula for the smallest abelian periods of the Fibonacci finite words. More precisely, we prove that for $j \geq 3$, the Fibonacci word f_j, of length F_j, has smallest abelian period equal to $F_{\lfloor j/2 \rfloor}$ if $j = 0, 1, 2 \mod 4$, or to $F_{1+\lfloor j/2 \rfloor}$ if $j = 3 \mod 4$.