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On the Maximum Order of Torsion Elements in $GL(n, \mathbf{Z})$ and $\mathrm{Aut}(F_n)$

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We study the maximum order of torsion elements in $GL(n, \mathbf{Z})$ and $\mathrm{Aut}(F_n)$, denoted $G(n)$ and $H(n)$, respectively. We prove a Landau-type estimate $\log G(n) \sim \sqrt{n \log n}$, and we show that $H(n) = G(n)$ if and only if $n \neq 2, 6, 12$. © 1998 Academic Press

0. INTRODUCTION

The maximum order $g(n)$ of an element of the symmetric group Σ_n was first studied by Landau [7] (see also [10]). Denoting log the natural logarithm, he proved

$$\log g(n) \sim \sqrt{n \log n}$$

as $n \to +\infty$ (as usual, $\varphi(n) \sim \psi(n)$ means $\lim_{n \to +\infty}(\varphi(n)/\psi(n)) = 1$; we say that φ is asymptotic to ψ). More precise results were obtained later (see [9]).

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Here we study $G(n)$, the maximum order of torsion elements in the general linear group $GL(n, \mathbf{Z})$, and $H(n)$, the maximum order in the automorphism group $\mathrm{Aut}(F_n)$ of a free group of rank n.

Note that Σ_n naturally embeds into $GL(n, \mathbf{Z})$, yielding the inequality $g(n) \le G(n)$. The inequality $H(n) \le G(n)$ also holds, as the natural epimorphism from $\mathrm{Aut}(F_n)$ to $GL(n, \mathbf{Z})$ has torsion-free kernel [1].

Our main results are the following.

THEOREM 1. *Let $G(n)$ be the maximum order of torsion elements in $GL(n, \mathbf{Z})$.*

(1) *Landau's estimate*

$$\log G(n) \sim \sqrt{n \log n}$$

holds.

(2) *As $n \to +\infty$, the number of distinct primes dividing $G(n)$ is asymptotic to $2\sqrt{(n/\log n)}$. The largest prime number dividing $G(n)$ is asymptotic to $\sqrt{n \log n}$. For a fixed prime λ, let λ^{a_λ} be the largest power of λ dividing $G(n)$. Then $a_\lambda \log \lambda \sim \frac{1}{2} \log n$ as $n \to +\infty$.*

THEOREM 2. *Let $H(n)$ be the maximum order of torsion elements in $\mathrm{Aut}(F_n)$. Then $H(n) = G(n)$ if and only if $n \ne 2, 6, 12$, and $H(2k+1) = H(2k)$ if and only if $2k \ne 2, 6, 12$.*

We make a few comments about these results.

(1) Statements similar to Theorem 1 were proved in [11] for $g(n)$. Other results about $G(n)$ will be given in [13]. In particular, it will be shown that

$$\lim_{n \to +\infty} \frac{G(n)}{g(n)} = +\infty.$$

(2) The exceptional cases in Theorem 2 are

$$H(2) = 4, \qquad G(2) = 6$$
$$H(6) = 24, \qquad G(6) = 30$$
$$H(12) = 180, \quad G(12) = 210.$$

(3) It follows from [5] or [6] that $H(n)$ is also the maximum order of torsion elements in the outer automorphism group $\mathrm{Out}(F_n)$ for $n \ge 3$. There is an element of order 6 in $GL(2, \mathbf{Z})$ and $\mathrm{Out}(F_2)$, but not in $\mathrm{Aut}(F_2)$.

(4) Automorphism groups of free groups are often compared to mapping class groups of closed surfaces. Note however that torsion in mapping class groups is bounded by a *linear* function of the genus.

(5) The maximum order of a finite subgroup of $\mathrm{Aut}(F_n)$ or $\mathrm{Out}(F_n)$ is known to be $2^n n!$ [14]. Though $GL(n, \mathbf{Z})$ contains bigger finite subgroups for certain values of n, it is conjectured that the maximum order $K(n)$ of a finite subgroup of $GL(n, \mathbf{Z})$ satisfies

$$\lim_{n \to +\infty} \left(\frac{K(n)}{n!} \right)^{1/n} = 2$$

(see [3]).

(6) The number $H(n)$ is related to dynamics of automorphisms of F_n (see [8]). This was the first motivation for the present paper.

1. TORSION IN $GL(n, \mathbf{Z})$

Landau's work about the symmetric group (see [7, 10]) is based on the formula

$$g(n) = \max_{n_1 + \cdots + n_k = n} \mathrm{lcm}(n_1, \ldots, n_k)$$

with $n_1, \ldots, n_k \in \mathbf{N}^* = \mathbf{N} \setminus \{0\}$.

Using the inequality $a + b \leq ab$, valid for any integers $a, b \geq 2$, he obtains

$$g(n) = \max_{\sum p_i^{\alpha_i} \leq n} \prod p_i^{\alpha_i},$$

where the p_i's are distinct primes and $\alpha_i \geq 1$. (Here, and in the whole paper, the symbol p, with or without a subscript, will always denote a prime number. We write $\prod p_i^{\alpha_i}$ for the decomposition into prime factors.)

This may be rephrased as

$$g(n) = \max_{l(k) \leq n} k,$$

where l: $\mathbf{N}^* \to \mathbf{N}$ is the additive function characterized by $l(1) = 0$ and $l(p^\alpha) = p^\alpha$. Recall that a function l is *additive* if $(m, n) = 1 \Rightarrow l(mn) = l(m) + l(n)$.

We derive a similar formula to study torsion in $GL(n, \mathbf{Z})$. Let L: $\mathbf{N}^* \to \mathbf{N}$ be the additive function defined by

$$\begin{cases} L(1) = L(2) = 0 \\ L(p^\alpha) = \varphi(p^\alpha) = p^\alpha - p^{\alpha - 1} \qquad \text{if } p^\alpha \geq 3. \end{cases}$$

Thus, if $k = \prod p_i^{\alpha_i}$, then $L(k) = \sum \varphi(p_i^{\alpha_i})$ if $k \not\equiv 2 \mod 4$, whereas $L(k) = \sum \varphi(p_i^{\alpha_i}) - 1$ if $k \equiv 2 \mod 4$. Note that $L(k)$ is always even, and $L(2p^\alpha) = L(p^\alpha)$ if p is an odd prime.

PROPOSITION 1.1. *An integer k is the order of an element of $GL(n, \mathbf{Z})$ if and only if $L(k) \leq n$.*

Before proving this proposition (which is implicit in [6]), we state two obvious corollaries.

COROLLARY 1.2. *Orders of torsion elements are the same in $GL(2p, \mathbf{Z})$ and $GL(2p+1, \mathbf{Z})$. In particular, $G(2p+1) = G(2p)$.*

COROLLARY 1.3. $G(n) = \max_{L(k) \leq n} k$.

The method of dynamical programming used in [12] to compute $g(n)$ has been easily adapted to get the table of $G(n)$ given at the end of the paper.

Proof of Proposition 1.1. We assume $k > 2$. Let A have order k in $GL(n, \mathbf{Z})$. Factoring the minimal polynomial of A into a product of cyclotomic polynomials, one shows that there exist integers $\delta_1, \dots, \delta_s$ with

$$\begin{cases} \varphi(\delta_1) + \cdots \varphi(\delta_s) \leq n \\ k = \operatorname{lcm}(\delta_1, \dots, \delta_s). \end{cases}$$

We would like to argue, as in Landau's situation, that we may require the δ_i's to be powers of distinct primes. But, as $\varphi(a) + \varphi(b) \leq \varphi(a)\varphi(b)$ holds only for $a, b \geq 3$, this is true only if k is odd or divisible by 4. If $k \equiv 2 \mod 4$, we conclude that some set $\{\delta_1, \dots, \delta_s\}$ as above has the form $\{2p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_s^{\alpha_s}\}$, with $p_1^{\alpha_1}, \dots, p_s^{\alpha_s}$ distinct odd primes and $\alpha_i \geq 1$.
Since $L(2p^\alpha) = \varphi(2p^\alpha)$ if p is an odd prime, we have in all cases

$$L(k) = \sum_{i=1}^{s} L(\delta_i) = \sum_{i=1}^{s} \varphi(\delta_i) \leq n.$$

Conversely, suppose $L(k) \leq n$. We construct a block diagonal matrix of order k in $GL(n, \mathbf{Z})$. If $k \not\equiv 2 \mod 4$, we write $k = \prod p_i^{\alpha_i}$ and we use as building blocks elements of order $p_i^{\alpha_i}$ in $GL(\varphi(p_i^{\alpha_i}), \mathbf{Z})$. If $k \equiv 2 \mod 4$, we have to use one block of size $\varphi(p^\alpha)$ and order $2p^\alpha$ (with p an odd prime). ∎

We will often use the following consequence of Corollary 1.3:

$$M > G(n) \Rightarrow L(M) > L(G(n)).$$

Corollary 1.3 makes it possible to extend to G most of the results proved about g in [11] or [9], see also [10]. We content ourselves with proving results relevant to Theorem 1. The proofs given here are fairly different from those in [11].

PROPOSITION 1.4. *Let λ be a fixed prime number, and λ^{a_λ} the largest power of λ dividing $G(n)$. Then*

$$\left(1 - \frac{1}{\lambda}\right)\lambda^{a_\lambda} \lesssim \sqrt{n \log n}$$

(*by* $\varphi(n) \lesssim \psi(n)$, *we mean* $\limsup_{n \to +\infty}(\varphi(n)/\psi(n)) \leq 1$).

Proof. Let $G(n) = \prod p_i^{\alpha_i}$ be the decomposition of $G(n)$ into prime factors. Then

$$\sum_i (p_i - 1) \leq \sum_i \varphi(p_i^{\alpha_i}) \leq L(G(n)) + 1 \leq n + 1,$$

which we rewrite as $\sum_{p|G(n)} p \lesssim n$.

Fix $\varepsilon > 0$. It is known (cf. [7]) that $\sum_{p \leq x} p \sim x^2/(2 \log x)$, and therefore $\sum_{p \leq u\sqrt{n \log n}} p \sim u^2 n$ for $u > 0$. In particular, the sum of all primes between $\sqrt{2n}$ and $(1 + \varepsilon)\sqrt{n \log n}$ is asymptotic to $(1 + \varepsilon)^2 n$ as $n \to +\infty$. Thus, for n large, we can find a prime p, not dividing $G(n)$, with

$$\sqrt{2n} < p < (1 + \varepsilon)\sqrt{n \log n}\,.$$

Now consider the number

$$G^+(n) = \frac{p}{\lambda^{[a_\lambda/2]}} G(n),$$

where $[x]$ denotes integral part. Since $p > \sqrt{2n}$ and

$$\frac{1}{2}\lambda^{a_\lambda} \leq \lambda^{a_\lambda} - \lambda^{a_\lambda - 1} \leq L(G(n)) \leq n,$$

we have $G^+(n) > G(n)$.

Corollary 1.3 then forces $L(G(n)) < L(G^+(n))$. This gives

$$\lambda^{a_\lambda} - \lambda^{a_\lambda - 1} < p - 1 + \lambda^{a_\lambda - [a_\lambda/2]} - \lambda^{a_\lambda - [a_\lambda/2] - 1}.$$

Proposition 1.4 follows easily, using the inequality $p < (1 + \varepsilon)\sqrt{n \log n}$. ∎

Remark 1.5. Note that the estimates just given are uniform, in the following sense: given $\varepsilon > 0$, there exists n_0 such that

$$n \geq n_0 \Rightarrow \left(1 - \frac{1}{\lambda}\right)\lambda^{a_\lambda} \leq (1+\varepsilon)\sqrt{n \log n}$$

for all primes λ such that $a_\lambda \geq 2$. In particular, the largest prime p such that p^2 divides $G(n)$ satisfies $p \lesssim \sqrt[4]{n \log n}$.

COROLLARY 1.6. $\log G(n) \sim \sqrt{n \log n}$.

Proof. Since $g(n) \leq G(n)$ and $\log g(n) \sim \sqrt{n \log n}$, it suffices to fix $\varepsilon > 0$ and to show $G(n) \leq g((1+\varepsilon)n)$ for n large. Fix a prime p with $1 - 1/p > (1+\varepsilon/2)/(1+\varepsilon)$.

Write $G(n) = \prod p_i^{\alpha_i}$. We have $\sum_i (1 - 1/p_i)p_i^{\alpha_i} \leq n + 1$, and therefore

$$\left(1 - \frac{1}{p}\right)\sum_i p_i^{\alpha_i} \leq n + 1 + \sum_{p_i \leq p}\left(\frac{1}{p_i} - \frac{1}{p}\right)p_i^{\alpha_i} \leq n + 1 + \frac{1}{2}\sum_{p_i \leq p} p_i^{\alpha_i}.$$

The term $\frac{1}{2}\sum_{p_i \leq p} p_i^{\alpha_i}$ is less than $(\varepsilon/4)n$ for n large, since $\lambda^{a_\lambda} = o(n)$ for any fixed prime λ by Proposition 1.4. We get for n large

$$\sum_i p_i^{\alpha_i} \leq \frac{n + 1 + (\varepsilon/4)n}{1 - 1/p} \leq \frac{(1+\varepsilon/2)n}{1 - 1/p} \leq (1+\varepsilon)n,$$

showing $G(n) \leq g((1+\varepsilon)n)$. ∎

PROPOSITION 1.7. *Let f be the number of distinct primes dividing $G(n)$. Then $f \sim 2\sqrt{n/\log n}$.*

Proof. For $u > 0$, the sum of the first $2u\sqrt{n/\log n}$ prime numbers is asymptotic to $u^2 n$. Thus the inequality $\sum_{p|G(n)} p \lesssim n$ implies $f \lesssim 2\sqrt{n/\log n}$.

To get a lower bound for f, first observe that, if $G(n) = \prod p_i^{\alpha_i}$, then

$$\frac{1}{2}\sum p_i^{\alpha_i} \leq L(G(n)) + 1 \leq n + 1.$$

Further, by the classical inequality between the arithmetic and geometric mean, we have

$$G(n) = \prod_{i=1}^{f} p_i^{\alpha_i} \leq \left(\frac{1}{f}\sum_{i=1}^{f} p_i^{\alpha_i}\right)^f \leq \left(\frac{2n+2}{f}\right)^f.$$

From Corollary 1.6 we have $\log G(n) \sim \sqrt{n \log n}$. This implies $f \gtrsim 2\sqrt{n/\log n}$. ∎

PROPOSITION 1.8. *Let q be the largest prime dividing $G(n)$. Then* $q \sim \sqrt{n \log n}$.

Proof. Denoting r_f as the fth prime number, we have $q \geq r_f \sim f \log f \sim \sqrt{n \log n}$ by the prime number theorem and Proposition 1.7. We shall now prove the opposite inequality.

First we claim that the number of primes $p \leq \sqrt{n \log n}$ dividing $G(n)$ is asymptotic to $2\sqrt{n/\log n}$. Indeed, if more than $2\varepsilon\sqrt{n/\log n}$ primes greater than $\sqrt{n \log n}$ divide $G(n)$ for a fixed $\varepsilon > 0$ and infinitely many n, we get a contradiction writing

$$n \gtrsim \sum_{p|G(n)} p \gtrsim (1-\varepsilon)^2 n + 2\varepsilon\sqrt{n/\log n}\sqrt{n \log n} = (1+\varepsilon^2)n.$$

In particular, given $\varepsilon > 0$, the interval $((1-\varepsilon)\sqrt{n \log n}, \sqrt{n \log n})$ contains a prime p which divides $G(n)$, for n large enough. Now assume that there exists $c > 1$ such that $q \geq c\sqrt{n \log n}$ for infinitely many values of n. We consider only these values. Choose ε with $1 - \varepsilon > 1/c$, and let p be as above.

Arguing as in [2, Lemma 2], we see that all primes in the interval $I = (\sqrt{pq}, \frac{p+q}{2})$, except at most one, divide $G(n)$: if $p_1, p_2 \in I$ do not divide $G(n)$, then $G^+(n) = \frac{p_1 p_2}{pq} G(n) > G(n)$ satisfies $L(G^+(n)) \leq L(G(n))$, a contradiction.

But the number ρ_I of primes contained in I satisfies

$$\rho_I \gtrsim \frac{(p+q)/2 - \sqrt{pq}}{\log((p+q)/2)} \gtrsim \frac{\left(1 - 1/\sqrt{c}\right)^2}{2} \frac{q}{\log q} \gtrsim \left(\sqrt{c} - 1\right)^2 \sqrt{\frac{n}{\log n}}.$$

Since $\sqrt{pq} \geq \sqrt{n \log n}$, this is a contradiction, because the number of primes $\geq \sqrt{n \log n}$ dividing $G(n)$ is $o(2\sqrt{n/\log n})$. ∎

The proof of Theorem 1 will be complete when we show $a_\lambda \log \lambda \gtrsim \frac{1}{2}\log n$. The following lemma says that we may assume $\lambda = 2$.

LEMMA 1.9. *Let λ, μ be distinct primes, appearing in $G(n)$ with exponents a_λ, a_μ, respectively. Then $\mu^{a_\mu - 1} < 4\lambda^{a_\lambda + 1}$.*

Proof. The proof is similar to that of the similar statement about $g(n)$ [11, Propriété 4]. Since the result is obvious for $a_\mu \leq 1$, we may assume $a_\mu \geq 2$. We define the integer b by $\mu < \lambda^b < \lambda\mu$, and we consider $G^+(n) = (\lambda^b/\mu)G(n)$. From $L(G^+(n)) > L(G(n))$, we get

$$\lambda^{a_\lambda + b} + \mu^{a_\mu - 1} - \mu^{a_\mu - 2} > \mu^{a_\mu} - \mu^{a_\mu - 1}.$$

We then write

$$2\lambda^{a_\lambda+1}(\mu-1) \geq \lambda^{\alpha_\lambda+1}\mu > \lambda^{a_\lambda+b} > \mu^{a_\mu-1}\left(1-\frac{1}{\mu}\right)(\mu-1)$$
$$\geq \frac{1}{2}\mu^{a_\mu-1}(\mu-1),$$

and the result follows. ∎

We need to show $a_2 \log 2 \gtrsim \frac{1}{2}\log n$. By Proposition 1.8, it suffices to prove that, for any integer k, there exists a number C such that

$$2^{a_2} \geq Cq^{1-1/k}$$

for all n (as before, q denotes the largest prime dividing $G(n)$).

Denote r_j the jth prime. We define integers $b_1, \dots, b_k$ (depending on n) by $r_j q^{1/k} > r_j^{b_j} > q^{1/k}$, and we consider

$$G^+(n) = \frac{G(n)}{q}\prod_{j=1}^{k} r_j^{b_j}.$$

We have $G^+(n) > G(n)$, and therefore $L(G^+(n)) > L(G(n))$ by Corollary 1.3. Denoting θ_j as the exponent of r_j in $G(n)$, we easily obtain

$$\sum_{j=1}^{k} r_j^{\theta_j+b_j} > q-1.$$

Since $r_j^{b_j} < r_k q^{1/k}$, and $r_j^{\theta_j} < 8r_k 2^{a_2}$ by Lemma 1.9, we get the required estimate $2^{a_2} > Cq^{1-1/k}$ with C depending only on k. This completes the proof of Theorem 1.

2. TORSION IN Aut(F_n)

There is a natural epimorphism from Aut(F_n) to $GL(n, \mathbf{Z})$, defined by considering the action of an automorphism on the abelianization of the free group F_n. This epimorphism has torsion-free kernel [1]. Thus $H(n) = G(n)$ if and only if $GL(n, \mathbf{Z})$ contains an element of order $G(n)$ that lifts to a torsion element of Aut(F_n).

PROPOSITION 2.1. *Let $n \geq 2$. The following conditions are equivalent*:

(1) $H(n) = G(n)$.

(2) $G(n)$ *is divisible by* 4, *or n is odd*.

(3) $GL(n, \mathbf{Z})$ *contains a block diagonal matrix of order $G(n)$, each block being a square matrix of size $p^\alpha - p^{\alpha-1}$ and order p^α for some prime p and $\alpha \geq 1$.*

Proof. (2) ⇒ (3). This follows from the proof of Proposition 1.1 when $G(n)$ is divisible by 4. If n is odd, we construct a block diagonal matrix of order $G(n)/2$ in $GL(n-1, \mathbf{Z})$, and we complete it by placing -1 in the lower right corner.

It is easy to lift a block diagonal matrix as in (3) to a finite order automorphism ψ of F_n, for instance by representing ψ as an automorphism of a finite graph (see [4] or [6]). This proves (3) ⇒ (1).

The hard implication is (1) ⇒ (2). It follows from [6], or from the result of [4] asserting that

$$H(n) = \max_{L'(k) \le n} k,$$

where L' is the additive function characterized by $L'(p^\alpha) = \varphi(p^\alpha)$ for *all* $p \ge 2$ and $\alpha \ge 1$. ∎

Theorem 2 now follows from:

PROPOSITION 2.2. *Let n be an even integer. Then $G(n)$ is divisible by* 4 *if and only if $n \ne 2, 6, 12$.*

Note that $G(n)$ is divisible by arbitrarily large powers of 2 for n large (see Theorem 1).

Proof of Proposition 2.2. Direct computation proves the result for $n \le 12$. We then note that $G(n)$ is always even, and we show

$$G(n) \equiv 2 \mod 4 \Rightarrow n \le 12.$$

In the arguments below, we rule out various possibilities by constructing $M > G(n)$ with $L(M) \le L(G(n))$. The primes that we do not mention are always assumed to appear with the same exponent in M as in $G(n)$.

- $G(n)$ is not divisible by p^2, for p an odd prime ($G(n)$ is quadratfrei).

Indeed, suppose that p^α is a prime factor of $G(n)$, with $\alpha \ge 2$. We define b by $p < 2^b < 2p$, and we construct M by replacing $2.p^\alpha$ by $2^{b+1}.p^{\alpha-1}$. The value of $L(M) - L(G(n))$ is

$$\begin{aligned} 2^b + p^{\alpha-1} - p^{\alpha-2} - \left(p^\alpha - p^{\alpha-1}\right) &= 2^b - p^{\alpha-2}(p-1)^2 \\ &< 2p - p^{\alpha-2}(p-1)^2. \end{aligned}$$

It is negative if $\alpha \ge 3$, or if $\alpha = 2$ and $p > 3$. If $p^\alpha = 3^2$, we have $L(M) = L(G(n))$.

- $G(n)$ is not divisible by 11.

Assume it is. If $G(n)$ is divisible by 3, we replace 2.3.11 by $2^3.3^2$. If not, we replace 2.11 by $2^3.3$.

- $G(n)$ is not divisible by 13.

If it is, we replace 2.3.13 by $2^3.11$, or 2.13 by $2^2.3^2$.

- $G(n)$ is not divisible by $p > 13$.

If it is, we define b by $p/11 < 2^b < 2p/11$, and we replace $2.p$ by $2^{b+1}.11$. We have

$$L(M) - L(G(n)) = 2^b + 10 - (p-1) < \frac{2p}{11} + 10 - (p-1) < 0$$

for $p > 121/9 = 13.44\ldots$.

It follows from the items above that $G(n)$ divides $2.3.5.7 = 210$, whence $n \le 12$ since $G(14) = 420$. ∎

3. VALUES OF $G(n)$

Table I contains the values of $G(n)$ for $n \le 300$. Since $G(2p+1) = G(2p)$, we assume n to be even. We omit n if $G(n) = G(n-2)$.

TABLE I

n	$G(n)$	Prime factors of $G(n)$
1	2	2
2	6	2.3
4	12	$2^2.3$
6	30	$2.3.5$
8	60	$2^2.3.5$
10	120	$2^3.3.5$
12	210	$2.3.5.7$
14	420	$2^2.3.5.7$
16	840	$2^3.3.5.7$
18	1260	$2^2.3^2.5.7$
20	2520	$2^3.3^2.5.7$
24	5040	$2^4.3^2.5.7$
26	9240	$2^3.3.5.7.11$
28	13860	$2^2.3^2.5.7.11$
30	27720	$2^3.3^2.5.7.11$
32	32760	$2^3.3^2.5.7.13$
34	55440	$2^4.3^2.5.7.11$
36	65520	$2^4.3^2.5.7.13$
38	120120	$2^3.3.5.7.11.13$
40	180180	$2^2.3^2.5.7.11.13$
42	360360	$2^3.3^2.5.7.11.13$

TABLE I—Continued

n	$G(n)$	Prime factors of $G(n)$
46	720720	$2^4.3^2.5.7.11.13$
50	942480	$2^4.3^2.5.7.11.17$
52	1113840	$2^4.3^2.5.7.13.17$
54	2042040	$2^3.3.5.7.11.13.17$
56	3063060	$2^2.3^2.5.7.11.13.17$
58	6126120	$2^3.3^2.5.7.11.13.17$
60	6846840	$2^3.3^2.5.7.11.13.19$
62	12252240	$2^4.3^2.5.7.11.13.17$
64	13693680	$2^4.3^2.5.7.11.13.19$
68	17907120	$2^4.3^2.5.7.11.17.19$
70	24504480	$2^5.3^2.5.7.11.13.17$
72	38798760	$2^3.3.5.7.11.13.17.19$
74	58198140	$2^2.3^2.5.7.11.13.17.19$
76	116396280	$2^3.3^2.5.7.11.13.17.19$
80	232792560	$2^4.3^2.5.7.11.13.17.19$
84	281801520	$2^4.3^2.5.7.11.13.17.23$
86	314954640	$2^4.3^2.5.7.11.13.19.23$
88	465585120	$2^5.3^2.5.7.11.13.17.19$
92	698377680	$2^4.3^3.5.7.11.13.17.19$
94	892371480	$2^3.3.5.7.11.13.17.19.23$
96	1338557220	$2^2.3^2.5.7.11.13.17.19.23$
98	2677114440	$2^3.3^2.5.7.11.13.17.19.23$
102	5354228880	$2^4.3^2.5.7.11.13.17.19.23$
108	6750984240	$2^4.3^2.5.7.11.13.17.19.29$
110	10708457760	$2^5.3^2.5.7.11.13.17.19.23$
114	16062686640	$2^4.3^3.5.7.11.13.17.19.23$
118	26771144400	$2^4.3^2.5^2.7.11.13.17.19.23$
122	32125373280	$2^5.3^3.5.7.11.13.17.19.23$
124	38818159380	$2^2.3^2.5.7.11.13.17.19.23.29$
126	77636318760	$2^3.3^2.5.7.11.13.17.19.23.29$
128	82990547640	$2^3.3^2.5.7.11.13.17.19.23.31$
130	155272637520	$2^4.3^2.5.7.11.13.17.19.23.29$
132	165981095280	$2^4.3^2.5.7.11.13.17.19.23.31$
138	310545275040	$2^5.3^2.5.7.11.13.17.19.23.29$
140	331962190560	$2^5.3^2.5.7.11.13.17.19.23.31$
142	465817912560	$2^4.3^3.5.7.11.13.17.19.23.29$
144	497943285840	$2^4.3^3.5.7.11.13.17.19.23.31$
146	776363187600	$2^4.3^2.5^2.7.11.13.17.19.23.29$
148	829905476400	$2^4.3^2.5^2.7.11.13.17.19.23.31$
150	931635825120	$2^5.3^3.5.7.11.13.17.19.23.29$
152	995886571680	$2^5.3^3.5.7.11.13.17.19.23.31$
154	1552726375200	$2^5.3^2.5^2.7.11.13.17.19.23.29$
156	2406725881560	$2^3.3^2.5.7.11.13.17.19.23.29.31$
160	4813451763120	$2^4.3^2.5.7.11.13.17.19.23.29.31$

TABLE I–Continued

n	$G(n)$	Prime factors of $G(n)$
166	5745087588240	$2^4.3^2.5.7.11.13.17.19.23.29.37$
168	9626903526240	$2^5.3^2.5.7.11.13.17.19.23.29.31$
172	14440355289360	$2^4.3^3.5.7.11.13.17.19.23.29.31$
176	24067258815600	$2^4.3^2.5^2.7.11.13.17.19.23.29.31$
180	28880710578720	$2^5.3^3.5.7.11.13.17.19.23.29.31$
184	48134517631200	$2^5.3^2.5^2.7.11.13.17.19.23.29.31$
188	72201776446800	$2^4.3^3.5^2.7.11.13.17.19.23.29.31$
192	89048857617720	$2^3.3^2.5.7.11.13.17.19.23.29.3137$
196	178097715235440	$2^4.3^2.5.7.11.13.17.19.23.29.31.37$
200	197351522287920	$2^4.3^2.5.7.11.13.17.19.23.29.31.41$
202	206978425814160	$2^4.3^2.5.7.11.13.17.19.23.29.31.43$
204	356195430470880	$2^5.3^2.5.7.11.13.17.19.23.29.31.37$
208	534293145706320	$2^4.3^3.5.7.11.13.17.19.23.2931.37$
212	890488576177200	$2^4.3^2.5^2.7.11.13.17.19.23.29.31.37$
216	1068586291412640	$2^5.3^3.5.7.11.13.17.19.23.29.31.37$
220	1780977152354400	$2^5.3^2.5^2.7.11.13.17.19.23.29.31.37$
224	2671465728531600	$2^4.3^3.5^2.7.11.13.17.19.23.29.31.37$
228	2960272834318800	$2^4.3^3.5^2.7.11.13.17.19.2329.31.41$
230	3104676387212400	$2^4.3^3.5^2.7.11.13.17.19.23.29.31.43$
232	5342931457063200	$2^5.3^3.5^2.7.11.13.17.19.23.29.31.37$
236	7302006324653040	$2^4.3^2.5.7.11.13.17.19.23.29.31.37.41$
238	7658201755123920	$2^4.3^2.5.7.11.13.17.19.23.29.31.37.43$
242	8486115458380560	$2^4.3^2.5.7.11.13.17.19.23.29.31.41.43$
244	14604012649306080	$2^5.3^2.5.7.11.13.17.19.23.29.31.37.41$
246	15316403510247840	$2^5.3^2.5.7.11.13.17.19.23.29.31.37.43$
248	21906018973959120	$2^4.3^3.5.7.11.13.17.19.23.29.31.37.41$
250	22974605265371760	$2^4.3^3.5.7.11.13.17.19.23.29.31.3743$
252	36510031623265200	$2^4.3^2.5^2.7.11.13.17.19.23.29.31.37.41$
254	38291008775619600	$2^4.3^2.5^2.7.11.13.17.19.23.29.31.37.43$
256	43812037947918240	$2^5.3^3.5.7.11.13.17.19.23.29.31.37.41$
258	45949210530743520	$2^5.3^3.5.7.11.13.17.19.23.29.31.37.43$
260	73020063246530400	$2^5.3^2.5^2.7.11.13.17.19.23.29.31.37.41$
262	76582017551239200	$2^5.3^2.5^2.7.11.13.17.19.23.29.31.37.43$
264	109530094869795600	$2^4.3^3.5^2.7.11.13.17.19.23.29.31.37.41$
266	114873026326858800	$2^4.3^3.5^2.7.11.13.17.19.23.29.31.37.43$
270	127291731875708400	$2^4.3^3.5^2.7.11.13.17.19.23.29.31.4143$
272	219060189739591200	$2^5.3^3.5^2.7.11.13.17.19.23.29.31.37.41$
274	229746052653717600	$2^5.3^3.5^2.7.11.13.17.19.23.29.31.37.43$
278	313986271960080720	$2^4.3^2.5.7.11.13.17.19.23.29.31.37.41.43$
282	343194297258692880	$2^4.3^2.5.7.11.13.17.19.23.29.31.37.41.47$
284	359935482490824240	$2^4.3^2.5.7.11.13.17.19.23.29.31.37.43.47$
286	627972543920161440	$2^5.3^2.5.7.11.13.17.19.23.29.31.37.41.43$
290	941958815880242160	$2^4.3^3.5.7.11.13.17.19.23.29.31.37.41.43$
294	1569931359800403600	$2^4.3^2.5^2.7.11.13.17.19.23.29.31.37.41.43$
298	1883917631760484320	$2^5.3^3.5.7.11.13.17.19.23.29.31.37.41.43$

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