```
> ####################### COMPUTATION OF THE PAPER
  ############################
  ################# Large values of n/phi(n) and sigma(n)/n
  ####################

> read "phisig.mpl": # contains the procedures
> read "phisig.m": # contains the registered data:
     # lisCA2, list of 6704 CA numbers of type 2, [eps,q,alfaq,p,
  xi,sigma(N)/N,logN]
       # allowing to compute all CA numbers with largest prime
  factor <=  2000530391
     # lispithetaphi, list of 6698 primorials [p,pi(p),log(M)=theta
  (p),M/phi(M)]
       # allowing to compute all primorials with largest prime
  factor <=  2000530391
     # lisresugeni, list of 6703 lists ["i=",i,comin,pmin,pmax,
  comax] used to
       # compute the minimum and the maximum of rho(X) between two
  consecutive
       # CA2 numbers, cf. below and Section 4.2.3

> Digits:=20:tau:=2+gamma-log(4*Pi): "tau=",evalf(tau); # cf (1.6)
```

"tau=", 0.0461914179322420676 **(1)**

```
> egam:=exp(evalf(gamma)): "egam=",egam,"alpha_0=",evalf(0.0094243*
  egam); #cf. (1.19)
```

"egam=", 1.7810724179901979852, "alpha_0=", 0.016785360788865022872 **(2)**

```
>   #### proof of Lemma 2.5 #####
> a:='a': alfa:='alfa': x0:='x0': K:='K':
  b:=(1-a/log(x0)^(K+1))*(a-2*alfa/log(x0)-alfa*a/log(x0)^(K+1)):
  b, evalf(subs(K=2,a=0.069,x0=10.^9,alfa=0.42065,b)), 0.0283*sqrt
  (10.^9)/log(10.^9)^2;
```

$$\left(1-\frac{a}{\ln(x0)^{K+1}}\right)\left(a-\frac{2\,alfa}{\ln(x0)}-\frac{alfa\,a}{\ln(x0)^{K+1}}\right),\ 0.028399635409225904119,\quad (3)$$
$$2.0838672014996973523$$

```
>   #### proof of Lemma 2.6 #####
> 2*sqrt(2.)-4; a:='a':t:='t': t0:='t0':
  h:=proc(a,t) a*exp(t/2)*t^2 -sqrt(8)*t end: # cf. (2.11)
  hh:=proc(a,t) a*exp(t/2)*(t^2/2+2*t)-sqrt(8) end:
  "h = ",h(a,t)," hh = ",hh(a,t);
  "diff(h(a,t),t)-hh(a,t)=", simplify(diff(h(a,t),t)-hh(a,t));
  A:=normal((sqrt(2)*t0-5/2)*(1+4/t0)-2*sqrt(2));
  racA:= solve(A=0,t0); evalf(racA);
```

$$-1.1715728752538099024$$

$$\text{"h = "},\ a\,e^{\frac{t}{2}}\,t^2-2\sqrt{2}\,t,\ \text{" hh = "},\ a\,e^{\frac{t}{2}}\left(\frac{1}{2}\,t^2+2\,t\right)-2\sqrt{2}$$

$$\text{"diff(h(a,t),t)-hh(a,t)=", } 0$$

$$A := \frac{2\sqrt{2}\, t0^2 + 4\sqrt{2}\, t0 - 5\, t0 - 20}{2\, t0}$$

$$racA := \frac{\sqrt{2}\left(-4\sqrt{2} + 5 + \sqrt{57 + 120\sqrt{2}}\right)}{8}, -\frac{\left(4\sqrt{2} - 5 + \sqrt{57 + 120\sqrt{2}}\right)\sqrt{2}}{8}$$

$$2.5455654369921980431, -2.7777984840258292321 \quad (4)$$

```
> ############# Section 3, Colossally abundant (CA) numbers
  ##########
> t:='t': u:='u': "F(t,u) = ", F(t,u); # cf. (3.1)
```

$$\text{"F(t,u) = ", } \frac{\ln\left(1. + \frac{t - 1.}{t^{u+1.} - 1.\, t}\right)}{\ln(t)} \quad (5)$$

```
> log(3./2)/log(2.), log(4./3)/log(3.); # cf. (3.3)
```

$$0.58496250072115618145, 0.26185950714291487417 \quad (6)$$

```
> Digits:=7: liseps(20); # The set E_i, cf. Section 3, (3.2) and
  (3.3)
  # liseps(xmax) computes the elements of E_i >= log(1+1/xmax)/log
  (xmax) [eps=F(p,k),p,k]
```

[[∞, 1, 0], [0.5849625, 2, 1], [0.2618593, 3, 1], [0.2223929, 2, 2], [0.1132828, 5, 1], [0.09953625, 2, 3], [0.07285775, 3, 2], [0.06862152, 7, 1], [0.04730525, 2, 4], [0.03628653, 11, 1], [0.02889254, 13, 1], [0.02308357, 2, 5], [0.02304524, 3, 3], [0.02037326, 5, 2], [0.02017457, 17, 1], [0.01742053, 19, 1]] **(7)**

```
> ############### Computation of N0 and xi0, cf. (3.16)--(3.20)
  ###############

  Digits:=50: calcullogN(10^9+7,2):
```

"k=", 1, "xi0[k] = ", $1.000000007\ 10^9$, "pp = ", 1000000007, "np = ", 1000000009

"k=", 2, "xi0[k] = ", 44023.5388108010, "pp = ", 44021, "np = ", 44027

"k=", 3, "xi0[k] = ", 1418.38081890335, "pp = ", 1409, "np = ", 1423

"k=", 4, "xi0[k] = ", 247.382205897601, "pp = ", 241, "np = ", 251

"k=", 5, "xi0[k] = ", 85.6248328688747, "pp = ", 83, "np = ", 89

"k=", 6, "xi0[k] = ", 41.9055559251791, "pp = ", 41, "np = ", 43

"k=", 7, "xi0[k] = ", 25.0431586873868, "pp = ", 23, "np = ", 29

"k=", 8, "xi0[k] = ", 16.9729134094723, "pp = ", 13, "np = ", 17

"k=", 9, "xi0[k] = ", 12.5175409933509, "pp = ", 11, "np = ", 13

"k=", 10, "xi0[k] = ", 9.79795805483709, "pp = ", 7, "np = ", 11
"k=", 11, "xi0[k] = ", 8.01061571828594, "pp = ", 7, "np = ", 11
"k=", 12, "xi0[k] = ", 6.76795496572222, "pp = ", 5, "np = ", 7
"k=", 13, "xi0[k] = ", 5.86506102880587, "pp = ", 5, "np = ", 7
"k=", 14, "xi0[k] = ", 5.18545963462664, "pp = ", 5, "np = ", 7
"k=", 15, "xi0[k] = ", 4.65896848724528, "pp = ", 3, "np = ", 5
"k=", 16, "xi0[k] = ", 4.24122002838259, "pp = ", 3, "np = ", 5
"k=", 17, "xi0[k] = ", 3.90301827869997, "pp = ", 3, "np = ", 5
"k=", 18, "xi0[k] = ", 3.62448665807836, "pp = ", 3, "np = ", 5
"k=", 19, "xi0[k] = ", 3.39169307626448, "pp = ", 3, "np = ", 5
"k=", 20, "xi0[k] = ", 3.19462248738160, "pp = ", 3, "np = ", 5
"k=", 21, "xi0[k] = ", 3.02591503456320, "pp = ", 3, "np = ", 5
"k=", 22, "xi0[k] = ", 2.88005599510308, "pp = ", 2, "np = ", 3
"k=", 23, "xi0[k] = ", 2.75284137279059, "pp = ", 2, "np = ", 3
"k=", 24, "xi0[k] = ", 2.64101669040753, "pp = ", 2, "np = ", 3
"k=", 25, "xi0[k] = ", 2.54202750288495, "pp = ", 2, "np = ", 3
"k=", 26, "xi0[k] = ", 2.45384368850374, "pp = ", 2, "np = ", 3
"k=", 27, "xi0[k] = ", 2.37483350469800, "pp = ", 2, "np = ", 3
"k=", 28, "xi0[k] = ", 2.30367186118659, "pp = ", 2, "np = ", 3
"k=", 29, "xi0[k] = ", 2.23927253465124, "pp = ", 2, "np = ", 3
"k=", 30, "xi0[k] = ", 2.18073740435309, "pp = ", 2, "np = ", 3
"k=", 31, "xi0[k] = ", 2.12731796681198, "pp = ", 2, "np = ", 3
"k=", 32, "xi0[k] = ", 2.07838582871057, "pp = ", 2, "np = ", 3
"k=", 33, "xi0[k] = ", 2.03340984655939, "pp = ", 2, "np = ", 3
"k=", 34, "xi0[k] = ", 1.99193824400876, "pp = ", 1, "np = ", 2

"xi = ", 1000000007, "eps = F(xi,1) = ", $4.82549420554828\,10^{-11}$, "xiN = ", $1.000000007\,10^{9}$

"epsmax = ", [1000000007, 1], $4.82549420554828\,10^{-11}$, "epsmin = ", [1000000009, 1], $4.82549419543158\,10^{-11}$

"N = ", $6.42931756558252\,10^{434300801}$, "prime factors of N = ", $(2)^{33}$, $(3)^{21}$, $(5)^{14}$, $(7)^{11}$, $(11)^{9}$, $(13)^{8}$, $(17)^{7}$, $(19)^{7}$, $(23)^{7}$, $(29)^{6}$, " ... ", $(41)^{6}$, $(43)^{5}$, " ... ", $(83)^{5}$, $(89)^{4}$, " ... ", $(241)^{4}$, $(251)^{3}$, " ... ", $(1409)^{3}$, $(1423)^{2}$, " ... ", $(44021)^{2}$, (44027), " ... ", (1000000007)

"q = ", 44021, "expq = ", 2, "p = ", 1000000007, "K(N) = ", 33

"sigNsN = ", 36.9096185662946, "logN = ", $1.00001455211884\,10^{9}$, "loglogN = ", 20.7232803889594 **(8)**

```
>    ################ Proof of Lemma 3.2 ###################

> Digits:= 10: (1.9769*log(10.^9))/(sqrt(2.)*10.^(9/6))-0.323,
  0.5931*sqrt(2.),
  sqrt(2.)+0.8388/log(10.^9); # cf. (3.29)
```

$$0.5930683994, 0.8387700636, 1.454689808 \quad (9)$$

```
> (1+1.93378*10.^(-8))*1.4547, 0.42065+1.455*log(10.^9)^3/sqrt(10.
  ^9),1+0.8302/log(10.^9)^3; # cf. (3.30)
```

$$1.454700028, 0.8301346139, 1.000093284 \quad (10)$$

```
> 1-0.0806/log(44023.),1-0.42065/log(10.^9)^3; # cf. (3.31)
```

$$0.9924619832, 0.9999527343 \quad (11)$$

```
> 1/0.999952,1/1.000094; # proof of (3.24)
```

$$1.000048002, 0.9999060088 \quad (12)$$

```
> 1+0.000049/20.72,1+log(0.9999)/20.72; # proof of (3.25)
```

$$1.000002365, 0.9999951735 \quad (13)$$

```
> 0.8302*1.000049/0.999995^3; # proof of (3.26)
```

$$0.8302531335 \quad (14)$$

```
> t:='t': int((log(t)+2)/(2*t^(3/2)*log(t)^2),t),
  (0.8303*1.000094^(3/2))/(2*0.999995)*(1+2/log(10.^9)),
  0.456/log(10.^9)^2; # proof of (3.27)
```

$$-\frac{1}{\sqrt{t}\,\ln(t)}, 0.4552825424, 0.001061813998 \quad (15)$$

```
> t:='t': int((log(t)+4)/(2*t^(3/2)*log(t)^3),t),
  (0.8303*1.000094^(3/2))/(2*0.999995^2)*(1+4/log(10.^9)),
  0.496/log(10.^9)^3; # proof of (3.28)
```

$$-\frac{1}{\sqrt{t}\,\ln(t)^2}, 0.4953569477, 0.00005573231485 \quad (16)$$

```
>    ################ Proof of Lemma 3.3 ###################

> numtheory[sigma](4)/4,(7/6)/(1+1/7);
  numtheory[sigma](24)/24,numtheory[sigma](30)/30;
```

$$\frac{7}{4}, \frac{49}{48}$$

$$\frac{5}{2}, \frac{12}{5} \quad (17)$$

```
>    ######## An algorithm to compute SA numbers (Section 3.1)
  ##########

> lisSA:=superabondant(10^50,0): # [n,sigma(n)/n]
  # computation of SA numbers < 10^50 by the classical algorithm
> lisSA[1..10]; ifactor(lisSA[324][1]), " = ", evalf(lisSA[324][1],
```

```
5);
```

$$\left[[1,1],\left[2,\frac{3}{2}\right],\left[4,\frac{7}{4}\right],[6,2],\left[12,\frac{7}{3}\right],\left[24,\frac{5}{2}\right],\left[36,\frac{91}{36}\right],\left[48,\frac{31}{12}\right],\left[60,\frac{14}{5}\right],[120,3]\right]$$

$$(2)^8\,(3)^5\,(5)^3\,(7)^2\,(11)^2\,(13)\,(17)\,(19)\,(23)\,(29)\,(31)\,(37)\,(41)\,(43)\,(47)\,(53)\,(59)\,(61)\,(67)\,(71)\,(73)\,(79)\,(83)\,(89)\,(97)\,(101)\,(103)\,(107), \text{" = "}, 5.1221\;10^{49} \tag{18}$$

```
> # below, we check that the algorithm of Section 3.1 gives the same list of SA numbers
  # than the classical algorithm

  lisCA139:=ca(139,type1,0):# list of CA numbers whose largest prime factor is <= 139
  nops(lisCA139), evalf(lisCA139[51]);
```

$$51, \left[0.001452736395, 139., 1., 139., 139., 8.762439903, 2.598392208\;10^{63}\right] \tag{19}$$

```
> lisN:=lisCA139[43]: N:=lisN[7]: eps:=lisN[1]:
  "N=",ifactor(N),"eps=",eps;NN:=lisCA139[44][7]: "NN=",ifactor(NN)
  ;
```

$$\text{"N="}, (2)^8\,(3)^5\,(5)^3\,(7)^2\,(11)^2\,(13)^2\,(17)\,(19)\,(23)\,(29)\,(31)\,(37)\,(41)\,(43)\,(47)\,(53)\,(59)\,(61)\,(67)\,(71)\,(73)\,(79)\,(83)\,(89)\,(97)\,(101), \text{"eps="}, 0.002134786363$$

$$\text{"NN="}, (2)^8\,(3)^5\,(5)^3\,(7)^2\,(11)^2\,(13)^2\,(17)\,(19)\,(23)\,(29)\,(31)\,(37)\,(41)\,(43)\,(47)\,(53)\,(59)\,(61)\,(67)\,(71)\,(73)\,(79)\,(83)\,(89)\,(97)\,(101)\,(103) \tag{20}$$

```
> i1:=cherchez(lisSA,N,1,324,1):i2:=cherchez(lisSA,NN,1,324,1):"i1=
  ",i1,"i2=",i2;
```

$$\text{"i1="}, 305, \text{"i2="}, 317 \tag{21}$$

```
> map(w->w[1]/N,lisSA[i1..i2]);
  # list of n/N where n runs among the SA numbers >= N and <= NN,
  # computed by the classical algorithm
```

$$\left[1, 2, \frac{103}{39}, \frac{103}{26}, \frac{206}{39}, 7, \frac{103}{13}, \frac{206}{13}, \frac{412}{13}, \frac{103}{3}, \frac{103}{2}, \frac{206}{3}, 103\right] \tag{22}$$

```
> i:=cherchez(lisCA2,101,1,nops(lisCA2),4);eps:=F(101,1);
```

$$i := 16$$

$$eps := 0.002134786363 \tag{23}$$

```
> map(w->w[1],procNSA(i,eps,2*eps,101,103,0));
  # list of n/N where n runs among the SA numbers >= N and <= NN
  # computed with the algorithm exposed in Section 3.1
```

(24)

$$\left[1, 2, \frac{103}{39}, \frac{103}{26}, \frac{206}{39}, 7, \frac{103}{13}, \frac{206}{13}, \frac{412}{13}, \frac{103}{3}, \frac{103}{2}, \frac{206}{3}, 103\right] \tag{24}$$

```
> numtheory[sigma](360360)/360360,numtheory[sigma](332640)/332640;
  # largely abundant number
  ifactor(332640), ifactor(360360);
```

$$\frac{48}{11}, \frac{48}{11}$$

$$(2)^5 (3)^3 (5) (7) (11), (2)^3 (3)^2 (5) (7) (11) (13) \tag{25}$$

```
> ########### Proof of Theorem 1.1, X > N^{(0)}, Section 4.1
  ##########
```

```
> Digits:=10: -2*0.0806,-0.347-0.1612; # proof of (4.8)
```

$$-0.1612, -0.5082 \tag{26}$$

```
> -0.323+0.1612,-0.1618+1.977*log(10.^9)/(sqrt(2.)*10.^(9/6)); #
  Proof of (4.9)
```

$$-0.1618, 0.7543147379 \tag{27}$$

```
> 0.755+log(10.^9)^2/sqrt(2.*10.^9), sqrt(2.)*(1+0.765/log(10.^9));
  # proof of (4.11)
```

$$0.7646028777, 1.466419299 \tag{28}$$

```
> 0.069/log(10.^9), 1+0.0034/log(10.^9); # proof of (4.13)
```

$$0.003329591028, 1.000164067 \tag{29}$$

```
> -0.522-0.00017, sqrt(2.)*(1-0.523/log(10.^9)); # proof of (4.14)
```

$$-0.52217, 1.378522581 \tag{30}$$

```
> 0.5264*sqrt(2.); # proof of (4.22)
```

$$0.7444420190 \tag{31}$$

```
> log(10.^9)/2/(10.^9-1); # upper bound of U_6
```

$$1.036163293\ 10^{-8} \tag{32}$$

```
> 0.779+0.0001+0.779*0.0001/log(10.^9),0.77911*sqrt(2.); # proof of
  (4.23)
```

$$0.7791037591, 1.101827928 \tag{33}$$

```
> 0.338*log(10.^9)^2/10.^(9/6); # proof of (4.25)
```

$$4.590215727 \tag{34}$$

```
> 4.6-2.5627+1.102; # proof of (4.26)
```

$$3.1393 \tag{35}$$

```
> 3.353+0.745; # proof of (4.27)
```

$$4.098 \tag{36}$$

```
> 2/(1-2*10.^(-9)), 2.0001*log(10.^9)^2/sqrt(10.^9); # proof of
  (4.28)
```

$$2.000000004, 0.02716239787 \tag{37}$$

```
> 3/(1-3*10.^(-9)),3.0001*log(10.^9)^2/sqrt(10.^9); # proof of
```

```
(4.29)
```
$$3.000000009,\ 0.04074291778 \tag{38}$$

```
> -4.098-0.0408, 3.1393+0.0272; # proof of (4.30)
```
$$-4.1388,\ 3.1665 \tag{39}$$

```
> 0.0011*2*sqrt(2.)+0.000056*3.1665+3.1665,2*sqrt(2.)+3.1698/log
  (10.^9),
  2.982/log(10.^9)/sqrt(10.^9); # upper bound of (4.31)
```
$$3.169788594,\ 2.981385640,\ 4.550398597\ 10^{-6} \tag{40}$$

```
> 0.0011*2*sqrt(2.)+0.000056*4.1388+4.1388; # lower bound of (4.31)
```
$$4.142143043 \tag{41}$$

```
> 2*(1-0.000005), 2.9814^2/1.99999/sqrt(10.^9), 3.1698+0.000141;
  # last computation of Sect. 4.1
```
$$1.999990,\ 0.0001405441166,\ 3.169941 \tag{42}$$

```
> ###### Proof of Theorem 1.1, 4 <= X <= 10^49, Section 4.2.1
  ########
```

```
> Digits:=20: ti:=time[real](): lisPHISIG:=proclisPHISIG(10^50,3):
  "nops(lisPHISIG)=",nops(lisPHISIG),"time =", time[real]()-ti;
```
"nops(lisprimo=", 31, [[2., 2., 0.69315, 2.], [3., 6., 1.7918, 3.], [5., 30., 3.4012, 3.7500], [7., 210., 5.3471, 4.3750], [11., 2310., 7.7450, 4.8125]], $\left[127., 4.0145\ 10^{48}, 111.91, 8.7822\right]$

"nops(lisSA)=", 324, $\left[[1, 1], \left[2, \frac{3}{2}\right], \left[4, \frac{7}{4}\right], [6, 2], \left[12, \frac{7}{3}\right], \left[24, \frac{5}{2}\right]\right]$, $\left[5.1221\ 10^{49}, 8.3075\right]$

"nops(lisprimoSA1)=", 355, [1, 2, 2, 4, 6, 6], $5.1221\ 10^{49}$

"nops(lisprimoSA2)=", 351, [4, 6, 12, 24, 30, 36]

"largest X considered = ", $(2)^8\ (3)^5\ (5)^3\ (7)^2\ (11)^2\ (13)\ (17)\ (19)\ (23)\ (29)\ (31)\ (37)\ (41)\ (43)\ (47)\ (53)\ (59)\ (61)\ (67)\ (71)\ (73)\ (79)\ (83)\ (89)\ (97)\ (101)\ (103)\ (107)$, " = ", $5.1221\ 10^{49}$

"nops(lisPHISIG)=", 350, "time =", 8.955 **(43)**

```
> evalf(lisPHISIG[349..350],5); evalf(lisPHISIG[1..8],5);
  print("n_350 = ",ifactor(lisPHISIG[350][1])," = ",evalf(lisPHISIG
  [350][1],5));
```
$\left[\left[3.4147\ 10^{49}, 0.059016, 0.74347, 0.76528, \text{"case1"}\right], \left[4.3562\ 10^{49}, 0.058536, 0.65007, 0.66442, \text{"case1"}\right]\right]$

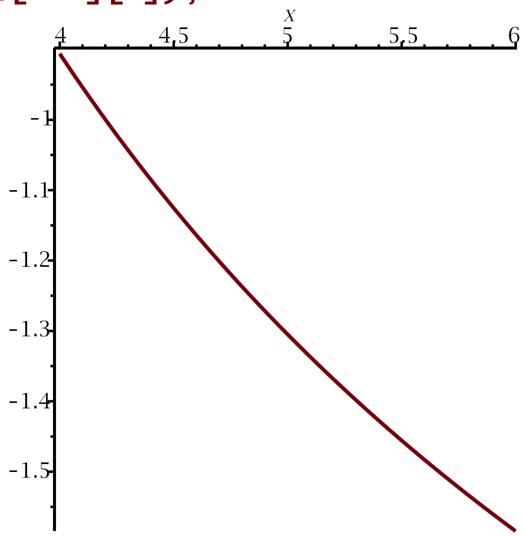
[[4., 0.14286, −1.5845, −0.90592, "case2"], [6., 0.50000, −1.9215, −1.4219, "case2"], [12., 0.28571, −2.5894, −2.2014, "case2"], [24., 0.20000,

$-2.9096, -2.7937$, "case2"], [30., 0.50000, $-2.0806, -2.0681$, "case1"],
[36., 0.48352, $-2.1190, -2.0855$, "case1"], [48., 0.45161, -2.2005,
-2.1712, "case1"], [60., 0.33929, $-2.6288, -2.6088$, "case3"]]

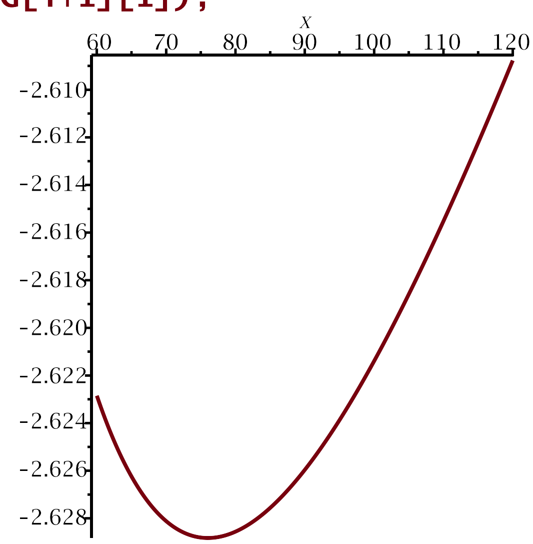
"n_350 = ", **(44)**

$(2)^8 (3)^5 (5)^3 (7)^3 (11)^2 (13)^2 (17) (19) (23) (29) (31) (37) (41) (43) (47) (53) (59) (61) (67) (71) (73) (79) (83) (89) (97) (101) (103)$, " = ",
$4.3562\ 10^{49}$

```
> i:=1: plot(subs(a=lisPHISIG[i][2],h(a,log(log(X)))),X=lisPHISIG
  [i][1]..lisPHISIG[i+1][1]);
```




```
> i:=8: plot(subs(a=lisPHISIG[i][2],h(a,log(log(X)))),X=lisPHISIG
  [i][1]..lisPHISIG[i+1][1]);
```




```
> for i from 1 to nops(lisPHISIG) do # case 2 and case 3
     if lisPHISIG[i][5]="case2" then
        print("i=",i,evalf(lisPHISIG[i],5),lisPHISIG[i+1][1]):
     fi:
  od:
  print():
```

```
for i from 1 to nops(lisPHISIG) do
   if lisPHISIG[i][5]="case3" then
      print("i=",i,evalf(lisPHISIG[i],5),lisPHISIG[i+1][1]):
   fi:
od:
```

"i=", 1, [4., 0.14286, −1.5845, −0.90592, "case2"], 6

"i=", 2, [6., 0.50000, −1.9215, −1.4219, "case2"], 12

"i=", 3, [12., 0.28571, −2.5894, −2.2014, "case2"], 24

"i=", 4, [24., 0.20000, −2.9096, −2.7937, "case2"], 30

"i=", 9, [120., 0.25000, −3.1133, −3.0879, "case2"], 180

"i=", 10, [180., 0.23626, −3.2064, −3.1983, "case2"], 210

"i=", 8, [60., 0.33929, −2.6288, −2.6088, "case3"], 120 **(45)**

```
> mini:=infinity:
  for k from 1 to nops(lisPHISIG) do
     if lisPHISIG[k][3] < mini then kmini:=k; mini:=lisPHISIG[k][3]
  ; fi;
  od:
  print("kmini=",kmini, lisPHISIG[kmini]);
  print("27720=",ifactor(27720));
```

"kmini=", 24, $\left[27720, \frac{937}{4992}, -3.3308646074445910296, -3.3183523704272329069, \text{"case1"}\right]$

"27720=", $(2)^3\,(3)^2\,(5)\,(7)\,(11)$ **(46)**

```
> maxi:=-infinity:
  for k from 1 to nops(lisPHISIG) do
     if lisPHISIG[k][4] > maxi then kmaxi:=k; maxi:=lisPHISIG[k][4]
  ; fi;
  od:
  print("kmaxi=",kmaxi, lisPHISIG[kmaxi]);
  print("M_127 = ",ifactor(lisPHISIG[kmaxi][1])," = ",evalf
  (lisPHISIG[kmaxi][1],7));
  print("N^{(1)} = ",ifactor(lisPHISIG[kmaxi+1][1])," = ",evalf
  (lisPHISIG[kmaxi+1][1]));
  print();
```

"kmaxi=", 342,

[4014476939333036189094441199026045136645885247730,

600516001163351366549280803276946155687748080304014949572740249/
94913383203858673490550372142989492074972831365862522880000\
00000, 1.553475510457070700, 1.556689679106292355, "case1"]

"M_127 = ",

$(2)\,(3)\,(5)\,(7)\,(11)\,(13)\,(17)\,(19)\,(23)\,(29)\,(31)\,(37)\,(41)\,(43)\,(47)\,(53)$
$(59)\,(61)\,(67)\,(71)\,(73)\,(79)\,(83)\,(89)\,(97)\,(101)\,(103)\,(107)\,(109)\,(113)$
(127), " = ", $4.014477\,10^{48}$

"N^{(1)} = ",

$(2)^9\,(3)^4\,(5)^3\,(7)^2\,(11)^2\,(13)^2\,(17)\,(19)\,(23)\,(29)\,(31)\,(37)\,(41)\,(43)\,(47)$
$(53)\,(59)\,(61)\,(67)\,(71)\,(73)\,(79)\,(83)\,(89)\,(97)\,(101)\,(103)$, " = ",
$4.148722083304077707\,10^{48}$

(47)

```
> ###### Proof of Theorem 1.1, 10^49 < X <= 10^7648, Section 4.2.2
  ########

  # N is a CA number of parameter eps, NN is the CA number
  following N
  # pN is the largest prime factor of N, NNsN is equal to NN/N
  # N is >= (i)th and < (i+1)th CA2 numbers
  # N is caracterized by i and pN
  # procNSA computes the list
  #    [n_k/N,(sigma(n_k)/n_k)/(sigma(N)/N),benefit(n_k)], k=1,2,..
  .
  #    of SA numbers $n_k$ satisfying  N <= n_k <= NN
  # difmin,difmax,difkMmin,... are the distances smaller than 0.001
  between
  # a SA number and a primorial
```

```
> Digits:=5: exp(lisCA2[16][7]),exp(lisCA2[81][7]),exp(lisCA2[17]
  [7]);
```

$$5.9671\,10^{44},\ 8.4300\,10^{7647},\ 5.1921\,10^{63} \quad \textbf{(48)}$$

```
> Digits:=20:ti:=time[real]():lisPHISIG70:=proclisPHISIG(10^70,0):
  time[real]()-ti;
```

$$14.627 \quad \textbf{(49)}$$

```
> # comparison of the methods of computation of rho(X) of the
  Sections 4.2.1 and 4.2.2
  for i from 4 to 16 do print("i=",i,compar(i)); od:
```

"i=", 4, ["nops(lisabsi)=", 15, "nops(lisabs)=", 15, "maxi=", $2.2\,10^{-18}$]

"i=", 5, ["nops(lisabsi)=", 3, "nops(lisabs)=", 3, "maxi=", $9.\,10^{-19}$]

"i=", 6, ["nops(lisabsi)=", 5, "nops(lisabs)=", 5, "maxi=", $2.3\,10^{-18}$]

"i=", 7, ["nops(lisabsi)=", 27, "nops(lisabs)=", 27, "maxi=", $9.0\,10^{-18}$]

"i=", 8, ["nops(lisabsi)=", 30, "nops(lisabs)=", 30, "maxi=", $8.1\,10^{-18}$]

"i=", 9, ["nops(lisabsi)=", 3, "nops(lisabs)=", 3, "maxi=", $2.7\,10^{-18}$]

"i=", 10, ["nops(lisabsi)=", 37, "nops(lisabs)=", 37, "maxi=", $1.1\ 10^{-17}$]

"i=", 11, ["nops(lisabsi)=", 43, "nops(lisabs)=", 43, "maxi=", $1.8\ 10^{-17}$]

"i=", 12, ["nops(lisabsi)=", 68, "nops(lisabs)=", 68, "maxi=", $1.4\ 10^{-17}$]

"i=", 13, ["nops(lisabsi)=", 17, "nops(lisabs)=", 17, "maxi=", $1.3\ 10^{-17}$]

"i=", 14, ["nops(lisabsi)=", 17, "nops(lisabs)=", 17, "maxi=", $2.2\ 10^{-17}$]

"i=", 15, ["nops(lisabsi)=", 34, "nops(lisabs)=", 34, "maxi=", $1.5\ 10^{-17}$]

"i=", 16, ["nops(lisabsi)=", 123, "nops(lisabs)=", 123, "maxi=", $3.0\ 10^{-17}$] **(50)**

```
> # computation of rho(X) for 5.98*10^44 < X < 1.19*10^7648
  ti:=time[real]():lisminmax:=coPHISIGXminmax(16,80):"time = ",time
  [real]()-ti;
  lisminmax[50];
```

"difmax=", 0.0006159223457

"difkMmin=", 0.0006161120648

"difmax=", 0.0005956425276

"difkMmin=", 0.0005958199578

"imin=", 16, "imax=", 80, "nops(lisminmax)=", 60879

"time = ", 353.696

$$\left[16, 107, \frac{109}{3}, -0.258466648043468944, -0.226967970849419955, \text{"[N *"}, \frac{109}{3}, \text{"N *"}, \frac{109}{2}, \text{")"}\right] \qquad \textbf{(51)}$$

```
> mino:=sort(lisminmax, proc(u,v) u[4] < v[4] end):
  for j from 1 to 20 do
     print("j=",j,mino[j]);
  od:
  print(): # printing of the 20 first intervals where the minimum
  of rho(X) is small
```

"j=", 1, [16, 113, 1, −0.977176102680438349, −0.963305555134205415,
"[N *", 1, "M_", 139, ")"]

"j=", 2, [16, 127, 1, −0.818215098172024835, −0.793619797985071087,
"[N *", 1, "M_", 149, ")"]

"j=", 3, [17, 139, 1, −0.785682687086502087, −0.779180548829406670,
"[N *", 1, "M_", 163, ")"]

"j=", 4, [16, 131, 1, −0.741120506402729202, −0.707575400994591641,
"[N *", 1, "M_", 151, ")"]

"j=", 5, [17, 149, 1, −0.645963400650629248, −0.632526490539834021,
"[N *", 1, "M_", 167, ")"]

$$"j=", 6, \left[16, 97, \frac{101}{2}, -0.630945099399147766, -0.627195882146890163, "[N *", \frac{101}{2}, "M\_", 113, ")"\right]$$

$$"j=", 7, [16, 137, 1, -0.602992787721552176, -0.561019990829209530, "[N *", 1, "M\_", 157, ")"]$$

$$"j=", 8, [17, 151, 1, -0.534924190617846510, -0.513470196584416349, "[N *", 1, "M\_", 173, ")"]$$

$$"j=", 9, [16, 139, 1, -0.522373687077309148, -0.477719053305797321, "[N *", 1, "N *", 2, ")"]$$

$$"j=", 10, \left[16, 109, \frac{226}{3}, -0.516785988278014548, -0.487198115704426225, "[N *", \frac{226}{3}, "N *", 113, ")"\right]$$

$$"j=", 11, \left[16, 109, \frac{113}{2}, -0.486628932633383679, -0.465533268515318012, "[N *", \frac{113}{2}, "N *", \frac{226}{3}, ")"\right]$$

$$"j=", 12, \left[16, 97, \frac{101}{3}, -0.476130619194485931, -0.441937331691513300, "[N *", \frac{101}{3}, "N *", \frac{101}{2}, ")"\right]$$

$$"j=", 13, \left[16, 107, \frac{218}{3}, -0.465571991689435922, -0.435417581887727297, "[N *", \frac{218}{3}, "M\_", 137, ")"\right]$$

$$"j=", 14, \left[16, 103, \frac{214}{3}, -0.442040899204389966, -0.428740530777029051, "[N *", \frac{214}{3}, "M\_", 131, ")"\right]$$

$$"j=", 15, \left[16, 101, \frac{103}{2}, -0.438552338275766795, -0.417804296789413087, "[N *", \frac{103}{2}, "M\_", 127, ")"\right]$$

$$"j=", 16, \left[16, 107, \frac{109}{2}, -0.437901272023744994, -0.416007845971743495, "[N *", \frac{109}{2}, "N *", \frac{218}{3}, ")"\right]$$

$$"j=", 17, \left[18, 151, \frac{157}{7}, -0.417749442414295595, -0.388814408007021825,\right.$$

$"[N *", \frac{157}{7}, "M\_", 179, ")"]$

$"j=", 18, [16, 103, \frac{107}{2}, -0.416852265055133591, -0.394169859313320274,$
$"[N *", \frac{107}{2}, "N *", \frac{214}{3}, ")"]$

$"j=", 19, [17, 139, \frac{149}{2}, -0.372752970335373183, -0.343014300733354147,$
$"[N *", \frac{149}{2}, "N *", 119, ")"]$

$"j=", 20, [24, 317, 2, -0.372273279613880417, -0.370487232733946430,$
$"[N *", 2, "M\_", 359, ")"]$

(52)

```
> # estimation of X for the smallest rho(X) in the range 10^49.
  .10^7648
  lisexpo:=procexpo(16,0);
  pmin:=nextprime(lisexpo[nops(lisexpo)][1]):
  val:=1:
  for ter in lisexpo do val:=val*(ter[1]^ter[2]):od:
  p:=pmin:
  while p <= 113 do
     val:=val*p;
     p:=nextprime(p):
  od:
  lisexpo,pmin,"...",113, " = ",val,evalf(val),"rho = ",evalf(mino
  [1][4]);
```

$$lisexpo := [[2, 8], [3, 5], [5, 3], [7, 2], [11, 2], [13, 2]]$$

$[[2, 8], [3, 5], [5, 3], [7, 2], [11, 2], [13, 2]], 17, "...", 113, " = ",$ **(53)**
$820151948895904018262592470823888543557505566675808000,$
$8.201519488959040182610^{54}, "rho = ", -0.977176102680438349$

```
> # another estimation of X for the smallest rho(X) in the range
  10^49..10^7648
  evalf(lisCA2[16]);logN:=lisCA2[16][7]+log(101*103*107*109*113);
  Digits:=30: evalf(exp(logN)); Digits:=20: evalf(exp(logN));
```

$[0.00213628614096633606 08, 13., 2., 97., 100.94148259448124735,$
$8.1159044998046122074, 103.10250042183940021]$

$logN :=$
$103.10250042183940021095083107720244162627716429096225508277\backslash$
$49582671030 + \ln(13710311357)$

$$8.2015194889590401826259247081010^{54}$$

(54)

$$8.20151948895904018261\,10^{54} \tag{54}$$

```
> # printing of the 20 first intervals where rho(X) is large
  maxo:=sort(lisminmax, proc(u,v) u[4] > v[4] end):
  for j from 1 to 20 do print("j=",j,maxo[j]); od: print():
```

"j=", 1, [36, 1087, 1054.732, 2.418823673039286194,
2.419267242848729088, "[M_", 1153, "N *", 1091, ")"]
"j=", 2, [45, 2749, 70.17367, 2.409325740120772487,
2.409392143845890112, "[M_", 2843, "N *", 71, ")"]
"j=", 3, [36, 1069, 994.3575, 2.406783515375338840,
2.407958278924984186, "[M_", 1151, "N *", 1087, ")"]
"j=", 4, [45, 2741, 67.85347, 2.402282072200993202,
2.402539777836059263, "[M_", 2837, "N *", 71, ")"]
"j=", 5, [36, 1063, 923.5171, 2.396968269929997317,
2.398907687077284950, "[M_", 1129, "N *", 1069, ")"]
"j=", 6, [45, 2731, 65.55741, 2.394345351071818012,
2.394799780564840160, "[M_", 2833, "N *", 71, ")"]
"j=", 7, [45, 2729, 63.19706, 2.388114340079021677,
2.388779246158118794, "[M_", 2819, "N *", 71, ")"]
"j=", 8, [36, 1061, 869.5294, 2.384090943675116204,
2.386768008421799518, "[M_", 1123, "N *", 1063, ")"]
"j=", 9, [45, 2719, 61.17942, 2.380330854018327248,
2.381182990248187402, "[M_", 2803, "N *", 71, ")"]
"j=", 10, [36, 1051, 821.5233, 2.371285078384709745,
2.374711453462852143, "[M_", 1117, "N *", 1061, ")"]
"j=", 11, [45, 2713, 59.34600, 2.366562303154897827,
2.367590630623971742, "[M_", 2801, "N *", 71, ")"]
"j=", 12, [45, 2711, 57.48151, 2.356258227614850920,
2.357472241488961932, "[M_", 2797, "N *", 71, ")"]
"j=", 13, [36, 1049, 772.9821, 2.352752917228844850,
2.356888360689082159, "[M_", 1109, "N *", 1051, ")"]
"j=", 14, [45, 2707, 55.71411, 2.346787573343977992,
2.348184148078654733, "[M_", 2791, "N *", 71, ")"]
"j=", 15, $\left[36, 1039, \frac{1102499}{1118}\right.$, 2.337733597236376302,
2.338569465299479086, "[N *", $\frac{1102499}{1118}$, "N *", 1049, ")"]
"j=", 16, [36, 1039, 731.1616, 2.337515340465967122,

2.341562543773115612, ["M_", 1103, "N *", $\frac{1102499}{1118}$, ")"]]

"j=", 17, [45, 2699, 54.03730, 2.335511149400195046,
2.337087082211843501, ["M_", 2789, "N *", 71, ")"]]

"j=", 18, [35, 1009, 20.57303, 2.333178311693808459,

2.333800215671532344, ["M_", 1061, "N *", $\frac{43}{2}$, ")"]]

"j=", 19, [45, 2693, 52.29353, 2.325014460954607854,
2.326783556601233522, ["M_", 2777, "N *", 71, ")"]]

"j=", 20, [40, 1619, 3.791276, 2.320825403616500576,
2.321312536080755494, ["M_", 1699, "N *", 4, ")"]]

(55)

(56)

```
> # estimation of X for the largest rho(X) in the range 10^49.
.10^7648
lisexpo:=procexpo(36,0): pmin:=nextprime(lisexpo[nops(lisexpo)]
[1]);
val:=1.: for ter in lisexpo do val:=val*ter[1]^ter[2]: od:
p:=pmin; while p <= nextprime(1087) do val:=val*p; p:=nextprime
(p): od:
lisexpo,pmin," ... ",nextprime(1087),evalf(val),"rho = ",evalf
(maxo[1][5]);
```

$$pmin := 47$$

$$p := 47$$

[[2, 12], [3, 7], [5, 5], [7, 4], [11, 3], [13, 3], [17, 2], [19, 2], [23, 2], [29, 2], **(57)**
[31, 2], [37, 2], [41, 2], [43, 2]], 47, " ... ", 1091,
$1.0361525299032959924\ 10^{485}$, "rho = ", 2.419267242848729088

```
> # another estimation of X for the smallest rho(X) in the range
10^49..10^7648
evalf(lisCA2[36]);logN:=lisCA2[36][7]+log(1031*1033*1039*1049*
1051*1061*1063*1069*1087);
Digits:=30: evalf(exp(logN)*1091); Digits:=20: evalf(exp(logN)*
1091); print();
```

[0.0001404874990018512892 8, 43., 2., 1021., 1026.1150597555265492,
12.334675353316057810, 1047.1553502403042842]

$logN :=$
1047.15535024030428422035349635554371502047160858499072568 07\
81928416625 + ln(1598853368702008111243172987)

$$1.03615252990329599240773556701\ 10^{485}$$

$$1.0361525299032959924\ 10^{485}$$ (58)

```
> ###### Proof of Theorem 1.1, 10^7648 < X  <= N^{(0)}, Section
  4.2.3 ########


  # Let N2[i] denote the ith CA2 number. A CA number N satisfying
  N2[i] <= N < N2[i+1] is       # characterized by its index i and
  by its largest prime factor say p.
  # comin[i] is a lower bound of rho(X) for N2[i] <= X < N2[i+1]
  # comax[i] is an upper bound of rho(X) for N2[i] <= X < N2[i+1]
  # Nmin[i] is the CA number such that rho(Nmin[i]) = comin[i]
  # its largest prime factor is pmin[i]
  # Nmax[i] is the CA number of index i and its largest prime
  factor is pmax[i]
  # NNmax[i] is the CA number following Nmax[i]
  # comax[i] is an upper bound of rho(X) for Nmax[i] <= X < NNmax
  [i]
  # The procedure genNpM(i,imp) returns the list
  # resu[i] = [comin[i],pmin[i],pmax[i],comax[i]]
  # The procedure BCD(imin,imax,pas) return the list : [resu[i], i
  = imin..imax]
  # "pas" is a printing parameter.
  # to compute lisresugeni, execute BCD(1,6703,100); (the running
  time is about 4 days)
  # and modify the result by adding to each sublist "i=",i
```

```
> Digits:=50: BCD7981:=BCD(79,81,1); # computation of lisresugeni
  [i] for 79 <= i <= 81
  for j from 1 to 3 do print("i=",78+j,BCD7981[j]); od:
  # to compute lisresugeni, execute BCD(1,6703,100); The running
  time is about 4 days
```

"i = ", 79, "time = ", 0.532

"i = ", 80, "time = ", 0.785

"i = ", 81, "time = ", 1.391

BCD7981 := [[0.8145603152, 16193, 17027, 2.338674048], [0.4945365470, 17137, 17509, 2.008431730], [0.2275517798, 17987, 17977, 2.376684552]]

"i=", 79, [0.8145603152, 16193, 17027, 2.338674048]

"i=", 80, [0.4945365470, 17137, 17509, 2.008431730]

"i=", 81, [0.2275517798, 17987, 17977, 2.376684552] (59)

```
> lisresugeni[79..81];log(2.);
```

[["i=", 79, 0.8145603152, 16193, 17027, 2.338674048], ["i=", 80, 0.4945365470, 17137, 17509, 2.008431730], ["i=", 81, 0.2275517798, 17987, 17977, 2.376684552]]

0.69314718055994530941723212145817656807550013436026 **(60)**

```
> ti:=time[real](): genNpM(6703,0); "time = ",time[real]()-ti;
```

[−1.903432139, 1998991369, 2000530361, −1.881973007]

"time = ", 448.464 **(61)**

```
> nops(lisresugeni),lisresugeni[6703];
```

6703, ["i=", 6703, −1.903432139, 1998991369, 2000530361, −1.881973007] **(62)**

```
> mini:=infinity: # the records for comin
  for i from 6703 by -1 to 1 do
     if lisresugeni[i][3] < mini then
        mini:=lisresugeni[i][3];
        print(lisresugeni[i]);
     fi;
  od:
```

["i=", 6703, −1.903432139, 1998991369, 2000530361, −1.881973007]

["i=", 6702, −1.909821862, 1996583803, 1998991363, −1.887886023]

["i=", 24, −2.196042283, 313, 457, 2.675273914]

["i=", 17, −2.883941579, 139, 149, 1.674401904]

["i=", 16, −3.134242153, 113, 103, 3.545605151]

["i=", 12, −3.838127393, 61, 71, 1.608117456]

["i=", 11, −3.970904613, 43, 53, 1.352294978]

["i=", 10, −4.274011134, 31, 41, 1.346307121] **(63)**

```
> exp(lisCA2[25][7]); # conclusion, for 1.64*10^205 <= X < N^{(0)},
  rho(X) > -1.91 holds
```

$1.63572689375021148217167270174116597903681362078 10\ 10^{205}$ **(64)**

```
> maxi:=-infinity: lismaxi:=[]: # there are 650 records for comax
  for i from 6703 by -1 to 1 do
     if lisresugeni[i][6] > maxi then
        maxi:=lisresugeni[i][6];
        lismaxi:=[op(lismaxi),lisresugeni[i]];
     fi;
  od: nops(lismaxi);
```

650 **(65)**

```
> for j from 1 by 20 to 650 do print("j=",j,lismaxi[j]); od:
```

"j=", 1, ["i=", 6703, −1.903432139, 1998991369, 2000530361,
−1.881973007]

"j=", 21, ["i=", 6481, −1.892716463, 1843787929, 1843787909,
−1.860914468]

"j=", 41, ["i=", 6239, −1.862165181, 1692083759, 1692909737,
−1.839616587]

"j=", 61, ["i=", 5944, −1.852000185, 1520817713, 1520817667,

−1.817613057]
"j=", 81, ["i=", 5726, −1.824524961, 1393098713, 1393312853, −1.800586208]
"j=", 101, ["i=", 5483, −1.818633901, 1262135473, 1262135447, −1.781550045]
"j=", 121, ["i=", 5261, −1.804024774, 1151600851, 1151600833, −1.765540831]
"j=", 141, ["i=", 5013, −1.782816218, 1025440259, 1025440181, −1.742485609]
"j=", 161, ["i=", 4757, −1.756481124, 908122967, 908122903, −1.714125431]
"j=", 181, ["i=", 4520, −1.734376631, 806062321, 806062261, −1.689937190]
"j=", 201, ["i=", 4314, −1.712457100, 724500319, 724500307, −1.666069165]
"j=", 221, ["i=", 4078, −1.685481761, 634661641, 634661603, −1.636560680]
"j=", 241, ["i=", 3830, −1.637606817, 549148819, 549216061, −1.602944352]
"j=", 261, ["i=", 3603, −1.630507285, 475782803, 475782763, −1.575600636]
"j=", 281, ["i=", 3359, −1.575400780, 403545223, 403605913, −1.536198949]
"j=", 301, ["i=", 3146, −1.563521536, 346692989, 346692977, −1.501221637]
"j=", 321, ["i=", 2915, −1.499473522, 288632147, 288729671, −1.454392080]
"j=", 341, ["i=", 2696, −1.453740615, 240545161, 240571127, −1.405629997]
"j=", 361, ["i=", 2480, −1.440842992, 197373181, 197373119, −1.362937483]
"j=", 381, ["i=", 2321, −1.384365732, 167441299, 167887607, −1.325681716]
"j=", 401, ["i=", 2101, −1.318431873, 132035887, 132064897, −1.257280105]
"j=", 421, ["i=", 1863, −1.253900831, 99216629, 99474383, −1.182858957]
"j=", 441, ["i=", 1660, −1.185488017, 75876391, 75878029, −1.109868888]
"j=", 461, ["i=", 1490, −1.169390758, 57576091, 57576019, −1.043154799]
"j=", 481, ["i=", 1324, −1.080524571, 43195853, 43195843, −0.9394371212]
"j=", 501, ["i=", 1128, −0.9882785774, 29345047, 29345023, −0.8245490906]
"j=", 521, ["i=", 959, −0.8928338433, 19528907, 19528889, −0.7015216779]
"j=", 541, ["i=", 782, −0.7711801203, 11456719, 11456681, −0.5370304649]
"j=", 561, ["i=", 606, −0.5817974773, 6034411, 6029021, −0.2829161528]
"j=", 581, ["i=", 450, −0.2847971899, 2745307, 2754893, −0.01733999711]
"j=", 601, ["i=", 338, −0.01778350496, 1255811, 1262281, 0.3422216493]
"j=", 621, ["i=", 195, 0.3681038507, 258809, 262583, 0.9834946110]
"j=", 641, ["i=", 78, 0.9331051122, 16007, 16067, 2.431309123] **(66)**

```
> for j from 631 to 650 do print("j=",j,lismaxi[j]); od:
```

"j=", 631, ["i=", 133, 0.6962166433, 81131, 83423, 1.629957658]
"j=", 632, ["i=", 127, 0.6737611656, 73699, 75707, 1.650850803]

"j=", 633, ["i=", 118, 0.2642377323, 58573, 58391, 1.764495611]
"j=", 634, ["i=", 113, 0.2437676965, 50651, 50287, 1.815913541]
"j=", 635, ["i=", 108, 0.8443345035, 40433, 40063, 1.979303849]
"j=", 636, ["i=", 105, 0.8629835889, 38461, 38651, 2.013418759]
"j=", 637, ["i=", 99, 0.3536454938, 33503, 33487, 2.133160557]
"j=", 638, ["i=", 91, 0.3322382019, 26293, 26263, 2.248960986]
"j=", 639, ["i=", 87, 0.2695144624, 22699, 22691, 2.273628543]
"j=", 640, ["i=", 82, 0.2228727518, 18947, 18493, 2.382130986]
"j=", 641, ["i=", 78, 0.9331051122, 16007, 16067, 2.431309123]
"j=", 642, ["i=", 73, 0.07867557372, 12037, 11971, 2.489912550]
"j=", 643, ["i=", 67, 0.6186437183, 9293, 9829, 2.493001663]
"j=", 644, ["i=", 66, 0.07037107247, 9283, 9277, 2.670482450]
"j=", 645, ["i=", 63, −0.03216670760, 7717, 7681, 2.712784300]
"j=", 646, ["i=", 58, −0.03056088481, 5351, 5333, 2.983489980]
"j=", 647, ["i=", 57, 0.01406932062, 5113, 5099, 3.086827875]
"j=", 648, ["i=", 54, 0.7440916700, 4567, 4549, 3.141454892]
"j=", 649, ["i=", 47, 0.9712475845, 2857, 2861, 3.389069062]
"j=", 650, ["i=", 36, −0.7743298613, 1123, 1087, 3.744556130] **(67)**

```
> exp(lisCA2[79][7]);
```

$$2.03035002649301361302823848720663735954477046715 16\ 10^{7081} \tag{68}$$

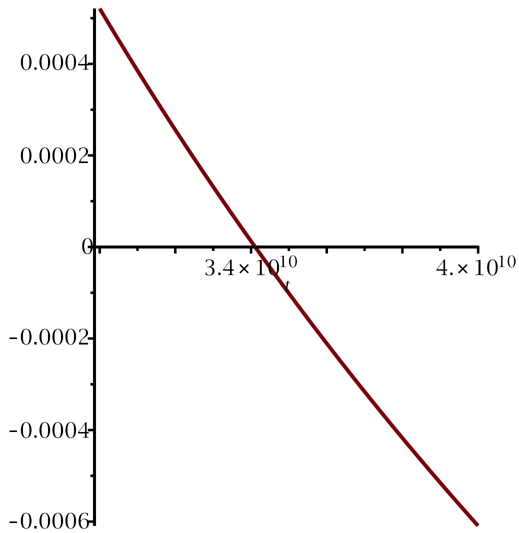
```
>                     # for 2.04*10^7081 < X < N^{(0)}, rho(X) < 2.382131 holds
                      # Note also that for i=36, the upper bound 3.74... is less
                      # accurate that the one 2.419 obtained in Section 4.2.2
```

```
> ################### Proof of Theorem 1.2, Section 5.2 ###################
```

```
> Digits:=20: f:=(2*sqrt(2)*log(t)^2+3.17*log(t))/sqrt(t); # proof of (5.3) with t = log(x)
```

$$f := \frac{2\sqrt{2}\,\ln(t)^2 + 3.17\ln(t)}{\sqrt{t}} \tag{69}$$

```
> plot(f-0.0094243,t=3*10^10..4*10^10); fsolve(f=0.0094243,t=3*10^10..4*10^10);
```

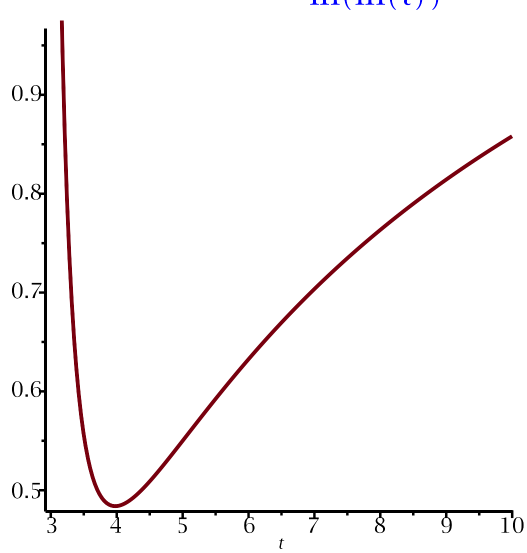


$$3.41100693931806578 18\ 10^{10} \quad \textbf{(70)}$$

```
> #################### Proof of Theorem 1.3, Section 5.4
  ###################
```

```
> Digits:=20:egam:=exp(evalf(gamma)): al0num:=0.0094243*egam:
  H:=log(log(t))+al0num/(log(log(t)))^2; plot(H,t=3..10); H1:=diff
  (H,t):
  H1,fsolve(H1=0,t=3..4); # function H is defined in (5.4)
```

$$H := \ln(\ln(t)) + \frac{0.016785360788865022872}{\ln(\ln(t))^2}$$



$$\frac{1}{t\ln(t)} - \frac{0.033570721577730045744}{\ln(\ln(t))^3\, t\ln(t)},\ 3.9776924884132211832 \quad \textbf{(71)}$$

```
> #################### Proof of Theorem 1.3, Section 6
  ###################
```

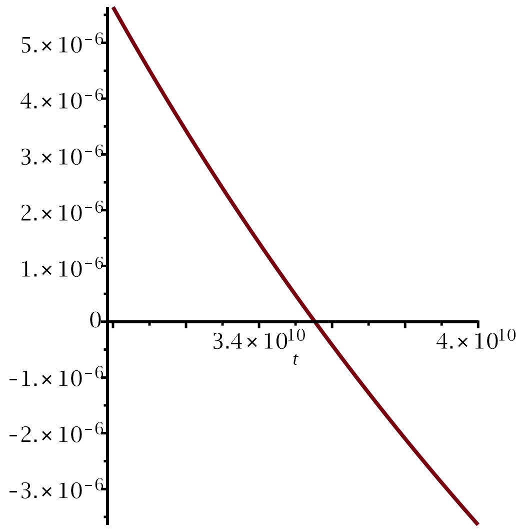
```
> om:=omega: expand((1-om+om^2)*(1+om)); # proof of (6.1)
```

$$\omega^3 + 1 \quad \textbf{(72)}$$

```
> fr:=-2*sqrt(2)*al0/log(t)^2+4.143*al0/log(t)^3+8*al0/sqrt(t)/log
  (t)^3
  +8*exp(gamma)/sqrt(t); # fr(log(x)) is equal to r(x) defined at
  the end of Section 6.1
```

$$fr := -\frac{2\sqrt{2}\ al0}{\ln(t)^2} + \frac{4.143\ al0}{\ln(t)^3} + \frac{8\ al0}{\sqrt{t}\ \ln(t)^3} + \frac{8\ e^{\gamma}}{\sqrt{t}} \tag{73}$$

```
> plot(subs(al0=al0num,fr),t=3*10^10..4*10^10);
  Digits:=20:fsolve(subs(al0=al0num,fr)=0,t=3*10^10..4*10^10); #
  computation of Y^{(1)}
```

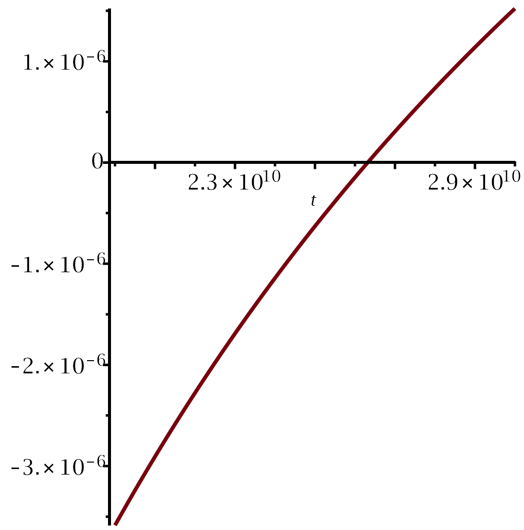


$$3.5528457898312757894\ 10^{10} \tag{74}$$

```
> fs:=exp(gamma)*(a0/log(t)^2-sqrt(8)/sqrt(t)+4.143/(sqrt(t)*log(t)
  ));
  # cf. (1.25) and Sect. 6.2
```

$$fs := e^{\gamma}\left(\frac{a0}{\ln(t)^2} - \frac{2\sqrt{2}}{\sqrt{t}} + \frac{4.143}{\sqrt{t}\ \ln(t)}\right) \tag{75}$$

```
> plot(subs(a0 = 0.94243e-2, fs), t = 2*10^10 .. 3*10^10); #
  computation of Y^{(0)}
  Digits := 20: fsolve(subs(a0 = 0.94243e-2, fs) = 0, t = 2*10^10 .
  . 3*10^10);
```



$$2.6318064419939283594\ 10^{10} \tag{76}$$

```
> ############################## the end
  ###############################
>
>
```