Highly composite numbers and champions

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1 Introduction

Let $d(n) = \sum_{a|n} 1$ be the number of divisors of n. In his PHD thesis entitled *Highly composite numbers* (cf. [23, 18]), Srinivasa Ramanujan defines an integer n to be *hignly composite* (hc for short) if $m < n$ implies $d(m) < d(n)$.

More generally, if the sequence $\big(f(n)\big)_{n\geqslant 1}$ satisfies

$$\limsup_{n\to\infty} f(n) = \infty \quad \text{and for all } \varepsilon > 0, \qquad \lim_{n\to\infty} f(n)/n^{\varepsilon} = 0, \tag{1}$$

one says that n is a f-champion if $m < n$ implies $f(m) < f(n)$ so that a hc number is a d-champion.

Here some properties of hc numbers are presented first, following which certain f-champions will be discussed.

Let us set $p_1 = 2, p_2 = 3, \ldots, p_k$ the k-th prime. An integer n is said *with non increasing exponent* (wnie for short) if $v_{p_k}(n)$, the largest number such that $p^{v_{p_k}(n)}$ divides n, is a non-increasing function on p_k. Ramanujan proved that a hc number is a wnie number (cf. [23, Section 8]). He was consequently interested by these wnie numbers and wrote with Hardy a nice article (cf. [12]) on the number $Q_{\text{wnie}}(x)$ of wnie numbers up to x where they prove that for x tending to infinity

$$Q_{\text{wnie}}(x) = \exp\left((1+o(1))\frac{2\pi}{\sqrt{3}}\sqrt{\frac{\log x}{\log\log x}}\right). \tag{2}$$

Let us call $Q_f(x)$ the number of f-champions up to x. Therefore $Q_d(x)$ is the number of hc numbers $\leqslant x$. From the inequality $Q_d(x) \leqslant Q_{\text{wnie}}(x)$ it follows that (2) provides an upper bound for $Q_d(x)$. In [23, Section 28], Ramanujan proves that

$$Q_d(x) = o\left(\frac{(\log x)\sqrt{\log\log x}}{(\log\log\log x)^{3/2}}\right)$$

does not hold.

Let us define $c(x)$ by $Q_d(x) = (\log x)^{c(x)}$. It is now known that (cf. [18, Section V] and [10, 17, 19])

$$c_1 = \limsup_{x\to\infty} c(x) \leqslant 1.71 \quad \text{and} \quad c_2 = \liminf_{x\to\infty} c(x) \in [1.136, 1.44] \tag{3}$$

and conjectured that $\lim_{x\to\infty} c(x) = (\log 30)/(\log 16) = 1.226\ldots$ No effective upper or lower bounds for $Q_d(x)$ of the form $A\log^\gamma x$ is known.

Let n be a hc number. As n is wnie, it can be written

$$n = \prod_{k=1}^{K} p_k^{a_k} \quad \text{with} \quad a_1 \geqslant a_2 \geqslant \ldots \geqslant a_K.$$

Ramanujan proved (cf. [23, Equation (43)]) that if $n \notin \{4, 36\}$, then $a_{p_K} = 1$ holds and, if k is fixed and n tends to infinity, then (cf. [23, Equation (86)] $a_{p_k}(\log p_k) = (1 + o(1))(\log p_K/(\log 2)$. He also proved that $\lim_{n\to\infty} p_K/(\log n) = 1$. Other inequalities about the factorization of a hc number into primes are given in [23, Sections 8-27], while, in [23, Section 28], it is proved that, as hc $\to \infty$, the quotient of two consecutive hc numbers tends to 1.

For f satisfying (1), an integer N is called a super f-champion if there exists $\varepsilon > 0$ such that

$$\frac{f(M)}{M^\varepsilon} \leqslant \frac{f(N)}{N^\varepsilon} \quad \text{for} \quad M \leqslant N \quad \text{and} \quad \frac{f(M)}{M^\varepsilon} < \frac{f(N)}{N^\varepsilon} \quad \text{for} \quad M > N. \tag{4}$$

The super d-champions were introduced by Ramanujan who called them *superior highly composite* (shc) numbers (cf. [23, Section 32]), and this was a very great idea. The number ε is called a *parameter* of the super f-champion N. Clearly, a super f-champion is an f-champion.

For $\varepsilon > 0$ fixed, the conditions (1) imply that the sequence $f(n)/n^\varepsilon$ has a maximum which is attained at a finite number of points. If it is attained at only one point, say N, from (4), N is a super f-champion. If it is attained at several points, the largest one is super f-champion.

For $\varepsilon > 0$, the number

$$N_\varepsilon = \prod_p p^{v_p(N)} \quad \text{with} \quad v_p(N) = \lfloor 1/(p^\varepsilon - 1) \rfloor \tag{5}$$

is a super d-champion (i.e. is shc) of parameter ε (cf. [21, Equation (3.8)]). Let us consider the set

$$\mathcal{E} = \left\{ \frac{\log(1 + 1/k)}{\log p}, \quad k \geqslant 1, \quad p \text{ prime} \right\} \tag{6}$$

and order the elements of $\mathcal{E} \cup \{\infty\}$ in the decreasing sequence

$$\varepsilon_1 = \infty > \varepsilon_2 = 1 > \varepsilon_3 = \frac{\log 2}{\log 3} > \varepsilon_4 = \frac{\log 3/2}{\log 2} > \ldots > \varepsilon_i > \ldots \qquad (7)$$

In [21, Sections 3.1–3.2] (see also the last lines of Section 3 of [2]), it is explained that there could exist elements in the set $\mathcal{E}$ defined by (6) admitting two representations

$$\varepsilon_i = \frac{\log(1+1/k_i)}{\log q_i} = \frac{\log(1+1/k_i')}{\log q_i'} \qquad (8)$$

with $k_i > k_i' \geqslant 1$ and $q_i < q_i'$, but not three. An element $\varepsilon_i \in \mathcal{E}$ satisfying (8) is said to be *extraordinary*. If ε_i is not extraordinary, it is said to be *ordinary* and satisfies in only one way

$$\varepsilon_i = \frac{\log(1+1/k_i)}{\log q_i}. \qquad (9)$$

Let ε_{i-1} and ε_i be two consecutive elements of the sequence (7) and ε a number satisfting $\varepsilon_{i-1} \geqslant \varepsilon > \varepsilon_i$. Then, from (5), $N_\varepsilon = N_{\varepsilon_{i-1}}$, and there is only one shc number of parameter ε (cf. [21, Lemma 3.6]), namely

$$N_\varepsilon = N_{\varepsilon_{i-1}}.$$

For $i \geqslant 1$, the number N_{ε_i} is shc and its parameters are all the ε's belonging to the interval $[\varepsilon_i, \varepsilon_{i-1}]$. Moreover, if ε_i is ordinary and satisfies (9), then, from (5),

$$N_{\varepsilon_i} = q_i N_{\varepsilon_{i-1}},$$

while if ε_i is extraordinary and satisfies (8), then, from (5),

$$N_{\varepsilon_i} = q_i q_i' N_{\varepsilon_{i-1}},$$

which allows an easy computation of shc numbers. If ε_i is ordinary, then the maximum of $d(n)/n^{\varepsilon_i}$ is attained at two numbers $N_{\varepsilon_{i-1}}$ and N_{ε_i}. If ε_i is extraordinary and satisfies (8), then the maximum of $d(n)/n^{\varepsilon_i}$ is attained at four numbers $N_{\varepsilon_{i-1}}$, $q_i N_{\varepsilon_{i-1}}$, $q_i' N_{\varepsilon_{i-1}}$, N_{ε_i}.

All $\varepsilon_i > (\log 2)/\log 10^9 = 0.0334\ldots$ are ordinary. One may conjecture that extraordinary numbers do not exist, but it seems very difficult to prove it.

Ramanujan defined the hc numbers in order to study how large the number of divisors of an integer n can be. Let $\beta = \log(3/2)/\log 2 = 0.584\ldots$, $\mathrm{li}(t)$ the logarithmic integral of t and $R(t) = \left(2\sqrt{t} + \sum_\rho t^\rho/\rho^2\right)/\log^2 t$, where ρ runs over the non-trivial zeros of the Riemann ζ function. In [23, Equation (236], for n tending to infinity, it is proved

$$\frac{\log d(n)}{\log 2} \leqslant \mathrm{li}(\log n) + \beta\, \mathrm{li}(\log^\beta n) - \frac{\log^\beta n}{\log\log n} - R(\log n) + \mathcal{O}\left(\frac{\sqrt{\log n}}{(\log\log n)^3}\right). \tag{10}$$

Inequality (10) is an equality when n is shc. An effective form of (10) is given in [21].

In the last section of [23], Ramanujan considers large values of the iterated function $d(d(n))$ and proves that there exist infinitely many intgers n satisfying

$$\log d(d(n)) > (\sqrt{2}\log 4)\frac{\sqrt{\log n}}{\log\log n} = 1.96\ldots\frac{\sqrt{\log n}}{\log\log n}.$$

In [6], this result has been improved :

$$\limsup_{n\to\infty} \frac{(\log d(d(n)))\log\log n}{\sqrt{\log n}} = \left(8\sum_{j=1}^{\infty}\log^2(1+1/j)\right)^{1/2} = 2.79\ldots$$

Since some fundamental properties of hc numbers (i.e. d-champions) have been discussed, now more generally, certain f-champions will be considered.

2 Champion numbers and the Riemann hypothesis

Let $\sigma(n) = \sum_{\delta | n}\delta$ be the sum of the divisors of n and $\gamma = 0.577\ldots$ the Euler constant. In 1982, Guy Robin proved that, under the Riemann hypothesis,

$$\text{for} \quad n > 5040, \quad \sigma(n)/n < e^\gamma \log\log n \tag{11}$$

holds and, moreover, that (11) is equivalent to the Riemann hypothesis (cf. [26, 27]). The main tool of his proof is the use of the super champion numbers for the function $\sigma(n)/n$. These numbers were introduced in 1944 (cf. [2]) and called *colossally abundant* (CA for short) by Alaoglu and Erdős who did not know that, earlier, in a manuscript not yet published, Ramanujan already defined these numbers and called them *generalized superior highly composite* (cf. [24, Section 59]). The CA numbers have been used in many papers and more especially in [16, 11, 27] and in the book [5]. The inequality (11) and its equivalence with the Riemann hypothesis have aroused many papers, cf. [4, 7, 8, 9, 13, 14, 15, 28, 29].

In his PHD thesis, Srinivasa Ramanujan worked on the large values taken by the function $\sigma(n)$. In the notes of the book *Collected Papers of Srinivasa Ramanujan*, about the paper *Highly Composite Numbers* (cf. [23]), it is mentionned " *The London Mathematical Society was in some financial difficulty at the time, and Ramanujan suppressed part of what he had written in order to save expense* ". After the death of Ramanujan, all his manuscripts were sent to the University of Cambridge (England) where they slept in a closet during a long time. They reappeared in the eighthy's and, among them, handwritten by Ramanujan, (see [25, p. 280–312]) the suppressed part of "Highly Composite Numbers". A typed version can be found in [24] or in [3, Chap. 8]. For the history of this manuscript, see the foreword of [24], the introduction of Chapter 8 of [3] and [20].

In this suppressed part, under the Riemann hypothesis, Ramanujan gave the asymptotic upper bound (cf. [24, Equation (382)])

$$\frac{\sigma(n)}{n} \leqslant e^{\gamma}\Big(\log\log n - \frac{2(\sqrt{2}-1)}{\sqrt{\log n}} + S_1(\log n) + \frac{\mathcal{O}(1)}{\sqrt{\log n}\log\log n}\Big) \qquad (12)$$

with (cf. [24, Section 65]),

$$S_1(x) = -\sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho-1)} = \frac{1}{x}\sum_{\rho}\frac{x^{\rho}}{|\rho|^2}$$

where ρ runs over the non-trivial zeros of the Riemann ζ function. In [22], an effective form of (12) is given that yields a slight improvment of (11), namely,

$$\text{for} \quad n > 5040, \quad \frac{\sigma(n)}{n} < e^{\gamma}\Big(\log\log n - \frac{0.095}{\sqrt{\log n}}\Big).$$

Let us set $f(n) = \sigma(n)/n$. The f-champion numbers were called *superabundant* by Alaoglu and Erdős in [2, Section 2] and earlier *generalized highly composite* by Ramanujan (cf. [24, Section 59]). In [11], it is proved that the number $Q_f(x)$ of superabundant numbers up to x satisfies $Q_f(x) > \log^{1+a} x$ for $a < 5/48$ and x large enough. It is not known whether there exists a positive constant b such that $Q_f(x) = \mathcal{O}\big(\log^b x\big)$.

The σ-champions were called *hignly abundant* by Alaoglu and Erdős in [2, Section 5]. They give some properties of these numbers and ask questions about the size of $Q_\sigma(x)$ or about the factorization into primes of σ- champion numbers which, up to now, have not been answered.

3 Highly composite polynomials

Let $\mathbb{F}_q$ be the finite field with q elements. It is well known that the arithmetic of polynomials of $\mathbb{F}_q[t]$ looks like the arithmetic of integers. In [1], Ardavan Afshar considers the question of maximising the divisor function in $\mathbb{F}_q[t]$ with a special attention to the case $q = 2$. Here, for simplicity, only the case $q = 2$ is considered, but the results for $q > 2$ are similar.

Let $\mathcal{M} = \{f \in \mathbb{F}_2[t]\}$, $\mathcal{M}_n = \{f \in \mathcal{M}, \deg f = n\}$, $\mathcal{I} = \{f \in \mathcal{M}$ irreducible$\}$, $\mathcal{I}_s = \{f \in \mathcal{I}, \deg f = s\}$ and $\pi(s) = |\mathcal{I}_s| = \frac{1}{s}\sum_{\delta|s}\mu(\delta)2^{s/\delta}$, where $\mu(\delta)$ is the Möbius function. For $f \in \mathcal{M}$, let $\tau(f)$ be the number of divisors of f, and observe that a generic polynomial f in $\mathcal{M}$ is of the form

$$f = \prod_{p\in\mathcal{I}} p^{a_p} \quad \text{with} \quad \deg f = \sum_{p\in\mathcal{I}} a_p \deg p \quad \text{and} \quad \tau(f) = \prod_{p\in\mathcal{I}} (1+a_p)$$

where only finitely many a_p's are non-zero.

If $\deg f = n$, then f is said to be highly composite (hc) if, for all polynomials g of degree $\leqslant n$, $\tau(g) \leqslant \tau(f)$ holds. At the end of [1], a table of hc polynomials of degrees $\leqslant 39$ is given.

A polynomial $h = h(x) \in \mathcal{M}$ is called a x-superior highly composite, or just x-shc, if for all $f \in \mathcal{M}$

$$\frac{\tau(h)}{q^{(\deg h)/x}} \begin{cases} \geqslant \tau(f)/q^{(\deg f)/x} & \text{if } \deg h \geqslant \deg f \\ > \tau(f)/q^{(\deg f)/x} & \text{if } \deg h < \deg f. \end{cases}$$

The polynomial

$$h(x) = \prod_{k\geqslant 1} \prod_{p\in\mathcal{I}_k} p^{a_k} \quad \text{with} \quad a_k = \left\lfloor \frac{1}{2^{k/x}-1} \right\rfloor. \tag{13}$$

is x-shc. A polynomial h is called x-semi-superior highly composite or x-sshc, if for all $f \in \mathcal{M}$

$$\frac{\tau(h)}{q^{\deg h/x}} \geqslant \frac{\tau(f)}{q^{\deg f/x}}.$$

A polynomial that is x-shc or x-sshc for some $x > 0$ is called superior highly composite (shc) or semi-superior highly composite (sshc), respectively. It is easy to see that a shc polynomial is sshc and that a sshc polynomial is hc.

Here, one defines the set

$$\mathcal{S} = \left\{ \frac{s \log 2}{\log(1+1/r)}, \quad s, r \geqslant 1 \right\}.$$

Note that if $x \in \mathcal{S}$, then there is a unique pair (s, r) such that $x = (s \log 2)/\log(1+1/r)$ (cf. [1, Lemma 2.4]). The set $\mathcal{S}$ is ordered in a sequence $(x_i)_{i\geqslant 1}$:

$$x_1 = 1 < x_2 = \frac{\log 2}{\log(3/2)} < x_3 = 2 < \ldots < x_i = \frac{s_i \log 2}{\log(1+1/r_i)} < \ldots$$

If $x_i = (s_i \log 2)/\log(1+1/r_i)$ and x satisfies $x_{i-1} \leqslant x < x_i$ then, from (13), the x-shc polynomial $h(x)$ is equal to $h(x_{i-1})$. Moreover, $h(x_i) = h(x_{i-1}) \prod_{p\in\mathcal{I}_{s_i}} p$.

If $x \notin \mathcal{S}$, then the maximum of $\tau(f)/2^{(\deg f)/x}$ for $f \in \mathcal{M}$ is attained at only one polynomial. If $x = x_i \in \mathcal{S}$, then this maximum is attained at $2^{\pi(s)}$

polynomials, namely

$$\frac{h(x_i)}{p_{j_1}p_{j_2}\dots p_{j_v}} \text{ with } 0 \leqslant v \leqslant \pi(s),\ 1 \leqslant j_1 < \dots < j_v \text{ and } j_1,\dots,j_v \in \mathcal{I}_s.$$

The situation is different from the one of shc numbers, where the maximum of $d(n)/n^\varepsilon$ is attained at 1, 2 and possibly 4 (if extraordinary ε's exist) numbers. As $\pi(s)$ tends to infinity with s, the maximum of $\tau(f)/2^{(\deg f)/x}$ can be attained at an unbounded number of polynomials.

It would be interesting to have an estimate of the number of hc polynomials of degree $\leqslant n$.

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