The sum of divisors function and the Riemann hypothesis

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Abstract

Let $\sigma(n) = \sum_{d|n} d$ be the sum of divisors function and $\gamma = 0.577\ldots$ the Euler constant. In 1984, Robin proved that, under the Riemann hypothesis, $\sigma(n)/n < e^\gamma \log\log n$ holds for $n > 5040$ and that this inequality is equivalent to the Riemann hypothesis. Under the Riemann hypothesis, Ramanujan gave the asymptotic upper bound

$$\frac{\sigma(n)}{n} \leqslant e^\gamma\Big(\log\log n - \frac{2(\sqrt{2}-1)}{\sqrt{\log n}} + S_1(\log n) + \frac{\mathcal{O}(1)}{\sqrt{\log n}\log\log n}\Big)$$

with $S_1(x) = \sum_\rho x^{\rho-1}/(\rho(1-\rho)) = \sum_\rho x^{\rho-1}/|\rho|^2$ where ρ runs over the non-trivial zeros of the Riemann ζ function.

In this paper, an effective form of the asymptotic upper bound of Ramanujan is given, which provides a slightly better upper bound for $\sigma(n)/n$ than Robin's inequality.

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1 Introduction

Let n be a positive integer, $\sigma(n) = \sum_{d|n} d$ the sum of its divisors and $\gamma = 0.577\ldots$ the Euler constant. In 1913, Gronwall proved that $\sigma(n)/n \leqslant (1 + o(1))e^\gamma \log\log n$ (cf. [19]). In 1982, under the Riemann hypothesis, Robin proved that

$$\text{for } n > 5040, \quad \frac{\sigma(n)}{n} < e^\gamma \log\log n \tag{1.1}$$

holds and, moreover, that (1.1) is equivalent to the Riemann hypothesis (cf. [36, 37]).

Much earlier, in his PHD thesis, Ramanujan worked on the large values taken by the function $\sigma(n)$. In the notes of the book *Collected Papers of Ramanujan*, about the paper *Highly Composite Numbers* (cf. [33]), it is mentioned "*The London Mathematical Society was in some financial difficulty at the time, and Ramanujan suppressed part of what he had written in order to save expense*". After the death of Ramanujan, all his manuscripts were sent to the University of Cambridge (England) where they slept in a closet for a long time. They reappeared in the 1980s and, among them, handwritten by Ramanujan, (see [35, 280–312]) the suppressed part of "Highly Composite Numbers". A typed version can be found in [34] or in [2, Chapter 8]. For the history of this manuscript, see the foreword of [34], the introduction of Chapter 8 of [2] and [29].

In this suppressed part, under the Riemann hypothesis, Ramanujan gave the asymptotic upper bound

$$\frac{\sigma(n)}{n} \leqslant e^\gamma\Big(\log\log n - \frac{2(\sqrt{2}-1)}{\sqrt{\log n}} + S_1(\log n) + \frac{\mathcal{O}(1)}{\sqrt{\log n}\log\log n}\Big) \tag{1.2}$$

with (cf. [34, Section 65]),

$$S_1(x) = -\sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)} = \frac{1}{x}\sum_\rho \frac{x^\rho}{|\rho|^2} \tag{1.3}$$

where ρ runs over the non-trivial zeros of the Riemann ζ function. Under the Riemann hypothesis, following Ramanujan in [33, Equation (226)], we may write

$$|S_1(x)| \leqslant \frac{1}{\sqrt{x}}\sum_\rho \frac{1}{\rho(1-\rho)} = \frac{1}{\sqrt{x}}\sum_\rho\Big(\frac{1}{\rho} + \frac{1}{1-\rho}\Big) = \frac{2}{\sqrt{x}}\sum_\rho \frac{1}{\rho} = \frac{\tau}{\sqrt{x}} \tag{1.4}$$

with

$$\tau = 2 + \gamma - \log(4\pi) = 0.046\,191\,417\,932\,242\,0\ldots \tag{1.5}$$

where γ is the Euler constant. The value of $\tau = 2\sum_\rho 1/\rho$ can also be found in several books, for instance [15, p. 67] or [12, p. 272]. See also [3, p. 9–10]. We prove:

Theorem 1.1. *(i) Under the Riemann hypothesis, for $n \geqslant 2$,*

$$\frac{\sigma(n)}{n} \leqslant e^\gamma\Big(\log\log n - \frac{2(\sqrt{2}-1)}{\sqrt{\log n}} + S_1(\log n) + \frac{3.789}{\sqrt{\log n}\log\log n} + \frac{0.026\log\log n}{\log^{2/3} n}\Big). \tag{1.6}$$

(ii) If the Riemann hypothesis is not true, there exist infinitely many n's for which (1.6) *does not hold. In other words, (i) is equivalent to the Riemann hypothesis.*

(iii) Independently of the Riemann hypothesis, there exist infinitely many n's such that

$$\frac{\sigma(n)}{n} \geqslant e^\gamma\Big(\log\log n - \frac{2(\sqrt{2}-1)}{\sqrt{\log n}} + S_1(\log n) + \frac{0.9567}{\sqrt{\log n}\log\log n} - \frac{0.48\log\log n}{\log^{2/3} n}\Big). \tag{1.7}$$

From (1.6) and (1.4) it follows

$$\frac{\sigma(n)}{n} \leqslant e^\gamma\Big(\log\log n + \frac{1}{\sqrt{\log n}}\Big(-2(\sqrt{2}-1) + \tau + \frac{3.789}{\log\log n} + \frac{0.026\log\log n}{\log^{1/6} n}\Big)\Big). \tag{1.8}$$

If $n_0 = 3.98\ldots 10^{80}$ is the root of

$$-2(\sqrt{2}-1) + \tau + \frac{3.789}{\log\log n_0} + \frac{0.026\log\log n_0}{\log^{1/6} n_0} = 0,$$

then (1.8) is better than Robin's result (1.1) only for $n > n_0$. So, to study the behaviour of large values of $\sigma(n)/n$, for $n \geqslant 2$, we define $\alpha(n)$ by

$$\frac{\sigma(n)}{n} = e^\gamma\Big(\log\log n - \frac{\alpha(n)}{\sqrt{\log n}}\Big). \tag{1.9}$$

k		ν_k	$\alpha(\nu_k)$
1	$**$	$2^4 3^2 5 \dots 7 \ = \ 5040$	$-0.0347834895\dots$
2	$*$	$2^5 3^2 5 \dots 7 \ = \ 10080$	$0.09563797587\dots$
3	$**$	$2^4 3^2 5 \dots 11 \ = \ 55440$	$0.1323247679\dots$
4	$*$	$2^5 3^2 5 \dots 11 \ = \ 110880$	$0.2169221889\dots$
5	$**$	$2^4 3^2 5 \dots 13 \ = \ 720720$	$0.2575558824\dots$
6	$**$	$2^5 3^2 5 \dots 13 \ = \ 1441440$	$0.2990357813\dots$
7	$*$	$2^4 3^3 5 \dots 13 \ = \ 2162160$	$0.3189756880\dots$
8	$**$	$2^5 3^3 5 \dots 13 \ = \ 4324320$	$0.3442304679\dots$
9	$**$	$2^5 3^3 5^2 7 \dots 19 \ = \ 6.98\dots 10^9$	$0.3912282440\dots$
10	$**$	$2^5 3^3 5^2 7 \dots 23 \ = \ 1.60\dots 10^{11}$	$0.4044234073\dots$
11	$**$	$2^6 3^3 5^2 7 \dots 23 \ = \ 3.21\dots 10^{11}$	$0.4167364286\dots$
12	$*$	$2^7 3^3 5^2 7 \dots 23 \ = \ 6.42\dots 10^{11}$	$0.4911990553\dots$
13	$**$	$2^6 3^3 5^2 7 \dots 29 \ = \ 9.31\dots 10^{12}$	$0.4939642676\dots$
14	$**$	$2^6 3^3 5^2 7^2 11 \dots 31 \ = \ 2.02\dots 10^{15}$	$0.5250314374\dots$
15	$**$	$2^6 3^4 5^2 7^2 11 \dots 31 \ = \ 6.06\dots 10^{15}$	$0.5436913001\dots$
16	$**$	$2^6 3^4 5^2 7^2 11 \dots 37 \ = \ 2.24\dots 10^{17}$	$0.5704418438\dots$

Figure 1: Values of ν_k

In Table 1, we give the value of $\alpha(\nu_k)$ for a sequence $(\nu_k)_{1\leqslant k\leqslant 16}$. The SA numbers (cf. below (4.2)) are marked by one star and the CA numbers (cf. Section 4.1) by two.

Corollary 1.2. *Let us assume the Riemann hypothesis.*

(i) For $n > \nu_{16}$,

$$\frac{\sigma(n)}{n} \leqslant e^\gamma \Big(\log\log n - \frac{0.582}{\sqrt{\log n}} \Big). \tag{1.10}$$

(ii) For $1 \leqslant k \leqslant 15$ and $n > \nu_k$,

$$\frac{\sigma(n)}{n} \leqslant e^\gamma \Big(\log\log n - \frac{\alpha(\nu_{k+1})}{\sqrt{\log n}} \Big).$$

Let $\varphi(n)$ denote the Euler function. It is known that, for all n, $\sigma(n)/n \leqslant n/\varphi(n)$ holds and that

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n\log\log n} = \limsup_{n\to\infty} \frac{n}{\varphi(n)\log\log n} = e^\gamma.$$

But, there are infinitely many n's such that $n/\varphi(n) > e^\gamma \log\log n$ (cf. [26, 28]) while there are infinitely many n's such that $\sigma(n)/n > e^\gamma \log\log n$ only if the Riemann hypothesis fails (cf. [37, 36]). The large values of $\sigma(n)/n$ and $n/\varphi(n)$ depend on the product $\prod_{p\leqslant x}(1-1/p)$. We shall use below the formula of [28, (2.16)], valid for $x \geqslant 10^9$

$$\frac{0.055}{\sqrt{x}\log^2 x} \leqslant \sum_{p\leqslant x} \log\left(1-\frac{1}{p}\right) + \gamma + \log\log\theta(x) + \frac{2}{\sqrt{x}\log x} + \frac{S_1(x)}{\log x} \leqslant \frac{2.062}{\sqrt{x}\log^2 x}, \tag{1.11}$$

where $\theta(x) = \sum_{p\leqslant x}\log p$ is the Chebyshev function. Other inequalities about $\sigma(n)$ and $\varphi(n)$ can be found in [27].

The works of Robin and Ramanujan have aroused several interesting papers, cf. [4, 5, 9, 10, 11, 20, 22, 23, 24, 40, 41]. I also recommend the reading of Chapters 6–9 of the book by Broughan [6].

1.1 Notation

- $\theta(x) = \sum\limits_{p\leqslant x}\log p$ and $\psi(x) = \sum_{p^k\leqslant x}\log p$ are the Chebyshev functions.
- $\pi(x) = \sum\limits_{p\leqslant x} 1$ is the prime counting function.
- $\mathcal{P} = \{2, 3, 5, \dots\}$ denotes the set of primes. $p_1 = 2,\ p_2 = 3, \dots, p_j$ is the jth prime. For $p \in \mathcal{P}$ and $n \in \mathbb{N}$, $v_p(n)$ denotes the largest exponent such that $p^{v_p(n)}$ divides n.
- If $\lim\limits_{n\to\infty} u_n = +\infty$, $v_n = \Omega_\pm(u_n)$ is equivalent to $\limsup_{n\to\infty} v_n/u_n > 0$ and $\liminf_{n\to\infty} v_n/u_n < 0$.

We use the following constants:

- $\xi^{(0)} = 10^9 + 7$ is the smallest prime exceeding 10^9, $\log\xi^{(0)} = 20.723265\ldots$,
- $N^{(0)}$ is defined in (4.37), and the numbers $\left(\xi_k^{(0)}\right)_{2\leqslant k\leqslant 33}$ in (4.36).
- For convenience, we sometimes write L for $\log N$, λ for $\log\log N$, L_0 for $\log N^{(0)}$ and λ_0 for $\log\log N^{(0)}$.

We often implicitly use the following results: for $u > 0, v > 0, w \in \mathbb{R}$,

$$t \mapsto \frac{(\log t - w)^u}{t^v} \text{ is decreasing for } t > \exp(w + u/v), \tag{1.12}$$

$$\max_{t \geqslant e^w} \frac{(\log t - w)^u}{t^v} = \left(\frac{u}{v}\right)^u \exp(-u - vw), \tag{1.13}$$

$$u(1 - u_0/2) \leqslant u - u^2/2 \leqslant \log(1 + u) \leqslant u, \quad 0 \leqslant u \leqslant u_0 < 1, \tag{1.14}$$

$$-\frac{u}{1 - u_0} \leqslant -\frac{u}{1 - u} \leqslant \log(1 - u) \leqslant -u, \quad 0 \leqslant u \leqslant u_0 < 1, \tag{1.15}$$

$$-u - \frac{u^2}{2(1 - u_0)} \leqslant -u - \frac{u^2}{2(1 - u)} \leqslant \log(1 - u) \leqslant -u, \quad 0 \leqslant u \leqslant u_0 < 1, \tag{1.16}$$

$$\frac{1}{t+1} < \frac{1}{t + 1/2} < \log\left(1 + \frac{1}{t}\right) \leqslant \frac{1}{t}, \quad t > 0, \tag{1.17}$$

$$1 - u \leqslant \exp(-u) \leqslant 1 - u + u^2/2, \quad u > 0 \tag{1.18}$$

and, for $0 \leqslant u \leqslant u_0 < 1, 0 \leqslant v \leqslant v_0 < 1$,

$$1 + u - v \leqslant \frac{1}{(1 - u)(1 + v)} \leqslant 1 + \frac{u}{1 - u_0} - (1 - v_0)v. \tag{1.19}$$

1.2 Plan of the article

The proof of Theorem 1.1 follows the proofs of (1.1) in [37] and of (1.2) in [34, Section 71], but in a more precise way.

In Section 2, we recall various results about prime numbers and functions $\theta(x)$, $\psi(x)$, $\pi(x)$ and, in Lemma 2.1, the sum $\sum_{p \geqslant y} \log(1 - 1/p^2)$ is estimated.

Section 3 is devoted to the study of $S_1(x)$ defined by (1.3). A formula allowing us to compute $S_1(x)$ is given in Lemma 3.2 while, in Lemma 3.4, we give an upper bound for the difference $S_1(x) - S_1(y)$ and in Lemma 3.5, it is proved that the mapping $x \mapsto 0.16 \log(x) + S_1(x)$ is increasing for $x \geqslant 3$.

Section 4 studies the colossally abundant numbers (CA). As these numbers look like very much to the superior highly composite (SHC) numbers, the presentation of SHC numbers given in [30] is followed. The computation of CA numbers is explained in Section 4.2. The notion of benefit, very convenient for computation on numbers with a large sum of divisors, is explained in Section 4.6. In Section 4.7, an argument of convexity is given, that allows to reduce certain computation for all integers only to CA numbers. To each CA number N is associated a number $\xi =$

$\xi(N)$ (see Definition 4.3) close to $\log N$. In Section 4.8, which is a little technical, are given upper and lower bounds of the difference $f(\xi) - f(\log N)$ for several functions f.

The proof of Theorem 1.1 is given in Section 5. First, in Proposition 5.1, it is proved that (1.6) and (1.7) hold when n is a CA number $N \geqslant N^{(0)}$ (defined in (4.37)). This proposition is the crucial part of the paper and the main tools to prove it are the estimates of the sums $\sum_{p \geqslant y} \log(1 - 1/p^2)$ (cf. (2.10) and (2.11)) and $\sum_{p \leqslant x} \log(1 - 1/p)$ (cf. (1.11)). Then, it follows easily that (1.6) also holds when n is between two consecutive CA numbers $\geqslant N^{(0)}$. The case $n < N^{(0)}$ is treated by computation. The argument of convexity given in Section 4.7 restricts the computation to CA numbers.

The proof (mainly computational) of Corollary 1.2 is exposed in Section 6. For $n \geqslant N^{(0)}$, (i) follows easily from Theorem 1.1 (i). For $n < N^{(0)}$, here also, the argument of convexity reduces the computation to CA numbers. To get the array of Table 1, the benefit method (cf. Section 4.6) is used to find the 161 integers n satisfying $5040 \leqslant n \leqslant v_{16} = 2.24 \ldots 10^{17}$ and $\sigma(n)/n \geqslant e^\gamma(\log\log n - 0.582/\sqrt{\log n})$.

The last Section presents two open problems.

2 Useful results

In [8, Theorem 2], Büthe has proved

$$\theta(x) = \sum_{p \leqslant x} \log p < x \quad \text{for } x \leqslant 10^{19} \tag{2.1}$$

and

$$\theta(x) > x - 1.94\sqrt{x} \quad \text{for } 1427 < x \leqslant 10^{19}. \tag{2.2}$$

The results (2.1) and (2.2) have been improved in [7]. It is shown that

$$|\theta(x) - x| \leqslant \sqrt{x}/(8\pi \log^2 x)$$

holds for all $599 \leqslant x \leqslant X$ where X is the largest number for which

$$4.92\sqrt{X/\log X} \leqslant H,$$

where H is the height to which the Riemann hypothesis has been verified. This, in conjunction with the paper by Platt and Trudgian [32], which establishes the Riemann hypothesis up to $3 \cdot 10^{12}$, should give a very good error term and would allow a better estimate in Lemma 2.1 below.

Platt and Trudgian in [31, Corollary 2] have also shown that

$$\theta(x) < (1+\eta)x \quad \text{with } \eta = 7.5\times 10^{-7} \quad \text{for } x > 0. \tag{2.3}$$

We also know that (cf. [38, Theorem 10])

$$\theta(x) \geqslant 0.89x \quad \text{for} \quad x \geqslant 227 \quad \text{and} \quad \theta(x) \geqslant 0.945x \quad \text{for} \quad x \geqslant 853, \tag{2.4}$$

that (cf. [38, (3.6)])

$$\pi(x) = \sum_{p\leqslant x} 1 < 1.26x/\log x \quad \text{for} \quad x > 1 \tag{2.5}$$

and that (cf. [13, Theorem 4.2])

$$|\theta(x) - x] < x/\log^3 x \quad \text{for} \quad x \geqslant 89\,967\,803. \tag{2.6}$$

Under the Riemann hypothesis, we use the upper bounds (cf. [39, Theorem 10, (6.3)])

$$|\psi(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}\log^2 x \quad \text{for} \quad x \geqslant 73.2 \tag{2.7}$$

and

$$|\theta(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}\log^2 x \quad \text{for} \quad x \geqslant 599. \tag{2.8}$$

It would be possible to slightly improve the numerical results of this paper by using the formula of Dusart (cf. [14, Proposition 2.5])

$$|\theta(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}(\log x - \log\log x - 2)\log x \quad \text{for} \quad x \geqslant 11977$$

improving on Schoenfeld's formula (cf. [39, Theorem 10, (6.1)])

$$|\theta(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}(\log x - 2)\log x \quad \text{for} \quad x \geqslant 23\cdot 10^8$$

but it would make the presentation of algebraic calculations more technical.

Lemma 2.1. *For $y \geqslant 10^8$,*

$$\begin{aligned}\frac{1}{y\log y} - \frac{1}{y\log^2 y} + \frac{2}{y\log^3 y} - \frac{9}{y\log^4 y} &\leqslant \sum_{p\geqslant y}\frac{1}{p^2}\\ &\leqslant \frac{1}{y\log y} - \frac{1}{y\log^2 y} + \frac{2}{y\log^3 y} - \frac{2.07}{y\log^4 y}.\end{aligned} \tag{2.9}$$

For $y > 24317$,

$$\sum_{p \geqslant y} \log\left(1 - \frac{1}{p^2}\right) \leqslant -\frac{1}{y \log y} + \frac{1}{y \log^2 y} - \frac{2}{y \log^3 y} + \frac{10.26}{y \log^4 y} \qquad (2.10)$$

and, for $y > 19373$,

$$\sum_{p \geqslant y} \log\left(1 - \frac{1}{p^2}\right) \geqslant -\frac{1}{y \log y} + \frac{1}{y \log^2 y} - \frac{2}{y \log^3 y} + \frac{2}{y \log^4 y}. \qquad (2.11)$$

Proof. By Stieltjes's integral,

$$\sum_{p \geqslant y} \frac{1}{p^2} = \int_y^\infty \frac{d[\theta(t)]}{t^2 \log t} = -\frac{\theta(y)}{y^2 \log y} + \int_y^\infty \frac{\theta(t)(2 \log t + 1)}{t^3 \log^2 t} dt$$

and, for $y \geqslant 10^8$, from (2.6),

$$\begin{aligned} \sum_{p \geqslant y} \frac{1}{p^2} &\leqslant -\frac{y(1 - 1/\log^3 y)}{y^2 \log y} + \int_y^\infty \frac{t(1 + 1/\log^3 t)(2 \log t + 1)}{t^3 \log^2 t} dt \\ &= -A_1 + A_4 + 2J_1 + J_2 + 2J_4 + J_5 \end{aligned}$$

by using the notation

$$A_n = \frac{1}{y \log^n y} \quad \text{and} \quad J_n = \int_y^\infty \frac{dt}{t^2 \log^n t} = A_n - n\, J_{n+1}.$$

We get (cf. [43])

$$-A_1 + A_4 + 2J_1 + J_2 + 2J_4 + J_5 = A_1 - A_2 + 2A_3 - 3A_4 + 17A_5 - 85J_6$$

and the upper bound of (2.9) follows from the positivity of J_6 and from

$$-3A_4 + 17A_5 = (-3 + 17/\log y)A_4 \leqslant (-3 + 17/\log 10^8)A_4 = -2.077\ldots A_4.$$

Similarly,

$$\sum_{p \geqslant y} \frac{1}{p^2} \geqslant -A_1 - A_4 + 2J_1 + J_2 - 2J_4 - J_5 = A_1 - A_2 + 2A_3 - 9A_4 + 31J_5,$$

which proves the lower bound of (2.9), since $J_5 > 0$ holds.

To prove (2.10) and (2.11), it is convenient to set

$$g(y) = \sum_{p \geqslant y} \log\left(1 - \frac{1}{p^2}\right) \text{ and } f(a, y) = -\frac{1}{y \log y} + \frac{1}{y \log^2 y} - \frac{2}{y \log^3 y} + \frac{a}{y \log^4 y}.$$

Note that, for y fixed > 1, $f(a, y)$ is increasing in a. Moreover, the derivative df/dy is equal to (cf. [43])

$$\frac{df}{dy} = \frac{h(a, Y)}{y^2 Y^5} \quad \text{with} \quad h(a, Y) = Y^4 - aY + 6Y - 4a \quad \text{and} \quad Y = \log y.$$

For $a \leqslant 13$ and $y \geqslant 19$, $\partial h/\partial Y = 4Y^3 - a + 6 \geqslant 4 \log^3 19 - 7 > 0$, $h(a, Y)$ is increasing in Y and $h(a, Y) \geqslant h(13, \log 19) = \log^4 19 - 7 \log 19 - 52 = 2.55 \ldots > 0$ (cf. [43]), so that, for a fixed $\leqslant 13$, $f(a, y)$ is increasing in y for $y \geqslant 19$.

One has

$$\frac{1}{p^2} \leqslant -\log\left(1 - \frac{1}{p^2}\right) = \sum_{k \geqslant 1} \frac{1}{k p^{2k}} < \frac{1}{p^2} + \sum_{k \geqslant 2} \frac{1}{2p^{2k}} = \frac{1}{p^2} + \frac{1}{2p^2(p^2 - 1)},$$

which, from the lower bound of (2.9), implies

$$g(y) \leqslant f(9, y) \quad \text{for} \quad y \geqslant 10^8. \tag{2.12}$$

Furthermore,

$$\begin{aligned} \sum_{p \geqslant y} \frac{1}{2p^2(p^2 - 1)} &\leqslant \frac{1}{2(y^2 - 1)} \sum_{p \geqslant y} \frac{1}{p^2} \leqslant \frac{1}{y^2} \sum_{n \geqslant y} \frac{1}{n^2} \leqslant \frac{1}{y^2} \int_{y-1}^{\infty} \frac{dt}{t^2} \\ &= \frac{1}{y^2(y - 1)} \leqslant \frac{2}{y^3} = \frac{2 \log^4 y}{y^2} A_4 \leqslant \frac{2 \log^4(10^8)}{10^{16}} A_4 < 0.07 A_4, \end{aligned}$$

which, from the upper bound of (2.9), proves

$$g(y) \geqslant f(2, y) \quad \text{for} \quad y \geqslant 10^8. \tag{2.13}$$

Let us assume that $19 \leqslant p' < y \leqslant p''$ where $p' < p''$ are two consecutive primes. As $p \geqslant y$ is equivalent to $p \geqslant p''$, $g(y)$ is equal to $g(p'')$. Now, $a(y)$, $a(p')$ and $a(p'')$ are defined by

$$g(y) = g(p'') = f(a(y), y) = f(a(p'), p') = f(a(p''), p'').$$

If $a(p'') \leqslant 13$, then

$$g(y) = f(a(y), y) = f(a(p''), p'') \geqslant f(a(p''), y),$$

whence $a(y) \geqslant a(p'')$. Similarly, if $a(p') \leqslant 13$, then

$$g(y) = f(a(y), y) = f(a(p'), p') < f(a(p'), y)$$

implies $a(y) < a(p')$, so that $a(p'') \leqslant a(y) < a(p')$ holds. The formula

$$g(y) = \sum_{p \geqslant p''} \log\left(1 - \frac{1}{p^2}\right) = \log\left(\frac{6}{\pi^2}\right) - \sum_{p \leqslant p'} \log\left(1 - \frac{1}{p^2}\right)$$

allows a numerical computation of $g(p'') = g(y)$, $a(p')$ and $a(p'')$. By computation (cf. [43]), one observes that, for $1669 < p' < 10^8$, $a(p'') < a(p') < 12.79 < 13$, that, for $24317 < y \leqslant 10^8$, $a(y) < 10.26$ while, for $19373 < y \leqslant 10^8$, $a(y) > 2$, which, with (2.12) and (2.13), completes the proof of (2.10) and (2.11). □

Lemma 2.2. *Let x be a real number $\geqslant 2$ and c be such that $\theta(t) \geqslant ct$ for $t \geqslant x$ (cf.* (2.4)*). Then*

$$\sum_{p \geqslant x} \frac{1}{p^3} \leqslant \frac{1}{2x^2 \log x}\Big((1+\eta)\big(3 + \frac{1}{\log x}\big) - 2c\Big), \tag{2.14}$$

where η is defined by (2.3)*. As an Application, if $x \geqslant 1418$, then,*

$$\sum_{p \geqslant x} \frac{1}{p^3} \leqslant \frac{0.624}{x^2 \log x} \leqslant \frac{0.086}{x^2}. \tag{2.15}$$

Proof. By Stieltjes's integral,

$$\begin{aligned}\sum_{p \geqslant x} \frac{1}{p^3} &= \int_x^\infty \frac{d[\theta(t)]}{t^3 \log t} = -\frac{\theta(x)}{x^3 \log x} + \int_x^\infty \frac{\theta(t)(3\log t + 1)}{t^4 \log^2 t} dt \\ &\leqslant -\frac{c}{x^2 \log x} + (1+\eta)\frac{3\log x + 1}{\log^2 x}\int_x^\infty \frac{dt}{t^3} = -\frac{c}{x^2 \log x} + (1+\eta)\frac{3\log x + 1}{2x^2 \log^2 x},\end{aligned}$$

which proves (2.14).

From (2.4), for $x \geqslant 1418$, $c = 0.945$ can be chosen and $(1+\eta)(3+1/\log(1418))/2 - c = 0.6239\ldots$, which proves (2.15).

It is possible to get a better upper bound for $\sum_{p \geqslant x} 1/p^3$ by copying the proof of Lemma 2.1 with an exponent 3 instead of 2. □

Lemma 2.3. *Let us denote by p_i the ith prime. Then, for $p_i \geqslant 127$, we have*

$$p_{i+1}/p_i \leqslant 149/139 = 1.0719\dots \tag{2.16}$$

We order the prime powers p^m, with $m \geqslant 1$, in a sequence $(a_i)_{i\geqslant 1} = (2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, \dots)$. Then, for $a_i \geqslant 127$,

$$a_{i+1}/a_i \leqslant 1.072. \tag{2.17}$$

Proof. In [13, Proposition 5.4], it is proved that, for $x \geqslant 89693$, there exists a prime p satisfying $x < p \leqslant x(1 + 1/\log^3 x)$. This implies that for $p_i \geqslant 89693$, we have $p_{i+1} \leqslant p_i + p_i/\log^3 p_i$ and

$$p_{i+1}/p_i \leqslant 1 + 1/\log^3 p_i \leqslant 1 + 1/\log^3 89693 = 1.000674\dots$$

For $2 \leqslant p_i < 89693$, the computation of p_{i+1}/p_i completes the proof of (2.16).

If p_j is the largest prime $\leqslant a_i$, then $a_{i+1} \leqslant p_{j+1}$ holds, so that, by (2.16), for $a_i \geqslant 127$,

$$a_{i+1}/a_i \leqslant p_{j+1}/p_j \leqslant 1.072$$

holds, which proves (2.17) (cf. [43]). □

3 Study of the function $S_1(x)$ defined by (1.3)

Lemma 3.1. *Let $\psi(x) = \sum_{p^k\leqslant x} \log p$ be the Chebyshev function. If a, b are fixed real numbers satisfying $1 \leqslant a < b < \infty$, and g any function with a continuous derivative on the interval $[a, b]$, then*

$$\sum_\rho \int_a^b \frac{g(t)t^\rho}{\rho}dt = \int_a^b g(t)\left[t - \psi(t) - \log(2\pi) - \frac{1}{2}\log\left(1 - \frac{1}{t^2}\right)\right]dt, \tag{3.1}$$

where ρ runs over the non-trivial zeros of the Riemann ζ function.

Proof. This is Théorème 5.8(b) of [17, p.169] or Theorem 5.8(b) of [16, p.162]. □

Lemma 3.2. *For $x > 1$, $S_1(x)$ defined by (1.3) satisfies*

$$\begin{aligned} S_1(x) &= -\sum_\rho \frac{x^{\rho-1}}{\rho(\rho-1)} \\ &= \int_1^x \frac{\psi(t)}{t^2}dt - \log x + 1 + \gamma - \frac{\log(2\pi)}{x} + \sum_{k=1}^\infty \frac{1}{2k(2k+1)x^{2k+1}}. \end{aligned}$$

Proof. Applying Lemma 3.1 with $g(t) = 1/t^2, a = 1, b = x$ and, (cf. (1.5)), $\tau = \sum_\rho 1/(\rho(1-\rho)) = 2 - \gamma - \log(4\pi)$ yields

$$S_1(x) - \tau = \sum_\rho \int_1^x \frac{t^{\rho-2}}{\rho} dt = \int_1^x \left[\psi(t) - t + \log(2\pi) + \frac{1}{2}\log\left(1 - \frac{1}{t^2}\right)\right]\frac{dt}{t^2}. \quad (3.2)$$

From the expansion in power series (cf. [43]),

$$\int_1^x \frac{\log(1-1/t^2)}{2t^2} dt = \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)x^{2k+1}} - \sum_{k=1}^{\infty}\left(\frac{1}{2k} - \frac{1}{2k+1}\right)$$
$$= \sum_{k=1}^{\infty} \frac{1}{2k(2k+1)x^{2k+1}} + \log 2 - 1,$$

(3.2) completes the proof of Lemma 3.2. Note that Lemma 3.2 allows the numerical computation of $S_1(x)$, cf. [43]. As the mapping $\rho \mapsto 1-\rho$ is a permutation of the non-trivial roots of the Riemann ζ function, $S_1(x) = S_1(1/x)$ allows the computation of $S_1(x)$ for $0 < x < 1$. $\square$

Lemma 3.3. *For $t \geqslant 2$, the function $Y(t) = \log(1-1/t^2)/(2t)$ is increasing and satisfies $0 < Y'(t) < 1/(2t^2)$.*

Proof. From $Y = -\sum_{k=1}^{\infty} 1/(2kt^{2k+1})$, for $t \geqslant 2$, one deduces

$$Y' = \sum_{k=1}^{\infty} \frac{2k+1}{2kt^{2k+2}} \leqslant \sum_{k=1}^{\infty} \frac{3}{2t^{2k+2}} \leqslant \frac{3}{2t^2}\sum_{k=1}^{\infty}\frac{1}{2^{2k}} = \frac{1}{2t^2},$$

which completes the proof of Lemma 3.3. $\square$

Lemma 3.4. *Let S_1 be defined by* (1.3) *and x and y be real numbers. Then, under the Riemann hypothesis,*

$$|S_1(x) - S_1(y)| \leqslant 0.0515\,\frac{|x-y|\log^2 y}{y^{3/2}} \quad \textit{for} \quad 73.2 \leqslant y \leqslant x \quad (3.3)$$

and

$$\left|\frac{S_1(x)}{\log x} - \frac{S_1(y)}{\log y}\right| \leqslant \frac{0.0521|x-y|\log y}{y^{3/2}} \quad \textit{for} \quad 73.2 \leqslant y \leqslant x. \quad (3.4)$$

Proof. From (2.7) and (1.12), for $73.2 \leqslant y \leqslant x$

$$\left|\int_y^x \frac{\psi(t)-t}{t^2}dt\right| \leqslant \int_y^x \frac{\log^2 t}{8\pi\, t^{3/2}}dt \leqslant \frac{|x-y|\log^2 y}{8\pi\, y^{3/2}}, \tag{3.5}$$

$$\int_y^x \frac{dt}{t^2} = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{x\,y} \leqslant \frac{|x-y|}{y^2} \tag{3.6}$$

and

$$\left|\int_y^x -\frac{\log(1-1/t^2)}{2t^2}dt\right| = \int_y^x \frac{1}{2t^2}\Big(\sum_{j=1}^{\infty} \frac{1}{jt^{2j}}\Big)dt \leqslant \frac{|x-y|}{2y^2} \sum_{j=1}^{\infty} \frac{1}{j\,y^{2j}}$$

$$\leqslant \frac{|x-y|}{2y^2} \sum_{j=1}^{\infty} \frac{1}{y^{2j}} = \frac{|x-y|}{2y^2(y^2-1)}, \tag{3.7}$$

whence, from (3.2), (3.5), (3.6) and (3.7), with $y_0 = 73.2$,

$$\begin{aligned}
|S_1(x) - S_1(y)| &\leqslant |x-y|\Big(\frac{\log^2 y}{8\pi\, y^{3/2}} + \frac{\log(2\pi)}{y^2} + \frac{1}{2y^2(y^2-1)}\Big) \\
&= \frac{|x-y|\log^2 y}{y^{3/2}}\Big(\frac{1}{8\pi} + \frac{1}{\sqrt{y}\log^2 y}\Big(\log(2\pi) + \frac{1}{2(y^2-1)}\Big)\Big) \\
&\leqslant \frac{|x-y|\log^2 y}{y^{3/2}}\Big(\frac{1}{8\pi} + \frac{1}{\sqrt{y_0}\log^2 y_0}\Big(\log(2\pi) + \frac{1}{2(y_0^2-1)}\Big)\Big) \\
&= 0.0514\ldots \frac{|x-y|\log^2 y}{y^{3/2}}
\end{aligned}$$

which proves (3.3). Finally, from (3.3) and (1.4),

$$\begin{aligned}
\left|\frac{S_1(x)}{\log x} - \frac{S_1(y)}{\log y}\right| &\leqslant \left|\frac{S_1(x)}{\log y} - \frac{S_1(y)}{\log y}\right| + \left|\frac{S_1(x)}{\log x} - \frac{S_1(x)}{\log y}\right| \\
&= \frac{|S_1(x) - S_1(y)|}{\log y} + |S_1(x)| \int_y^x \frac{dt}{t\log^2 t} \\
&\leqslant \frac{0.0515|x-y|\log y}{y^{3/2}} + \frac{\tau}{\sqrt{y}}\frac{|x-y|}{y\log^2 y} = \frac{|x-y|\log y}{y^{3/2}}\Big(0.0515 + \frac{\tau}{\log^3 y}\Big),
\end{aligned}$$

which, as from (1.5), $\tau/\log^3 y \leqslant 0.0462/\log^3 73.2 \leqslant 0.0006$, completes the proof of (3.4) and of Lemma 3.4. □

Lemma 3.5. *The function*

$$h_1(x) = 0.16 \log x + S_1(x), \tag{3.8}$$

where $S_1(t)$ is defined by (1.3)*, is increasing for $x \geqslant 3$.*

Proof. Let us use the sequence $(a_i)_{i\geqslant 1}$ introduced in Lemma 2.3. From (3.2), $h_1(x)$ is continuous for $x \geqslant 1$ and differentiable on $x \in (a_i, a_{i+1})$, so it suffices to prove that $h_1(x)$ is increasing on each interval (a_i, a_{i+1}) for $a_i \geqslant 3$.

Let $x \in (a_i, a_{i+1})$. From Lemma 3.2 with the notation $Y(x) = \log(1-1/x^2)/(2x)$,

$$h_1'(x) = \frac{1}{x}\Big(0.16 + \frac{\psi(a_i) + \log(2\pi)}{x} - 1 + Y(x)\Big)$$

and, from Lemma 3.3,

$$h_1'(x) \geqslant \frac{1}{x}\Big(0.16 + \frac{\psi(a_i) + \log(2\pi)}{a_{i+1}} - 1 + Y(a_i)\Big). \tag{3.9}$$

For each a_i satisfying $3 \leqslant a_i \leqslant 127$, the smallest value of the parenthesis of (3.9) is > 0.0033 (cf. [43]) and thus positive. For $a_i \geqslant 128$, from (2.17) and (2.7),

$$\frac{\psi(a_i) + \log(2\pi)}{a_{i+1}} \geqslant \frac{a_i}{a_{i+1}}\Big(1 - \frac{\log^2 a_i}{8\pi\sqrt{a_i}}\Big) \geqslant \frac{1}{1.072}\Big(1 - \frac{\log^2 128}{8\pi\sqrt{128}}\Big) \geqslant 0.855.$$

From Lemma 3.3, $Y(a_i) \geqslant Y(128) = -2.38\ldots 10^{-7}$ so that (3.9) yields $h_1'(x) \geqslant 0.16 + 0.855 - 1 - 2.4 \times 10^{-7} > 0.0149 > 0$, which ends the proof of Lemma 3.5. □

Lemma 3.6. *Let $a \geqslant 0.75$, $b \leqslant 3.9$ and $c \leqslant 0.1$. The function*

$$G(t) = G(a, b, c, t) = \log t - \frac{a}{\sqrt{t}} + \frac{b}{\sqrt{t}\log t} + \frac{c \log t}{t^{2/3}} \tag{3.10}$$

is increasing for $t \geqslant 6$ and concave for $t \geqslant 10$, while the function

$$H(t) = H(a, b, c, t) = \log t - \frac{a}{\sqrt{t}} + \frac{b}{\sqrt{t}\log t} + S_1(t) + \frac{c \log t}{t^{2/3}}, \tag{3.11}$$

where $S_1(t)$ is defined by (1.3)*, is increasing for $t \geqslant 6$.*

Proof. Let us set

$$
\begin{aligned}
g_1(u,v,t) &= u\log t + \frac{v}{\sqrt{t}\log t}, &(3.12)\\
g_1'(u,v,t) &= \frac{dg_1(u,v,t)}{dt} = \frac{1}{2t}\Big(2u - \frac{v}{\sqrt{t}\log t} - \frac{2v}{\sqrt{t}\log^2 t}\Big),\\
g_1''(u,v,t) &= -\frac{1}{4t^2}\Big(4u - \frac{3v}{\sqrt{t}\log t} - \frac{8v}{\sqrt{t}\log^2 t} - \frac{8v}{\sqrt{t}\log^3 t}\Big).
\end{aligned}
$$

For $u \geqslant 0.8$, $v \leqslant 3.3$ and $t \geqslant 6$,

$$
g_1'(u,v,t) \geqslant \frac{1}{2t}\Big(1.6 - \frac{3.3}{\sqrt{6}\log 6} - \frac{6.6}{\sqrt{6}\log^2 6}\Big) = \frac{0.0088\ldots}{2t} > 0 \qquad (3.13)
$$

and, for $u \geqslant 0.95$, $v \leqslant 3.3$ and $t \geqslant 10$,

$$
\begin{aligned}
g_1''(u,v,t) &\leqslant -\frac{1}{4t^2}\Big(3.8 - \frac{9.9}{\sqrt{10}\log 10} - \frac{26.4}{\sqrt{10}\log^2 10} - \frac{26.4}{\sqrt{10}\log^3 10}\Big)\\
&= -0.18\ldots/(4t^2) < 0. &(3.14)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
g_2(a,v,t) &= -\frac{a}{\sqrt{t}} + \frac{v}{\sqrt{t}\log t}, &(3.15)\\
g_2'(a,v,t) &= \frac{dg_2(a,v,t)}{dt} = \frac{1}{2t^{3/2}}\Big(a - \frac{v}{\log t} - \frac{2v}{\log^2 t}\Big),\\
g_2''(a,v,t) &= -\frac{1}{4t^{5/2}}\Big(3a - \frac{3v}{\log t} - \frac{8v}{\log^2 t} - \frac{8v}{\log^3 t}\Big),
\end{aligned}
$$

and for $a \geqslant 0.75$, $v \leqslant 0.6$ and $t \geqslant 6$,

$$
g_2'(a,v,t) \geqslant \frac{1}{2t^{3/2}}\Big(0.75 - \frac{0.6}{\log 6} - \frac{1.2}{\log^2 6}\Big) = \frac{0.041\ldots}{2t^{3/2}} > 0, \qquad (3.16)
$$

while, for $a \geqslant 0.75$, $v \leqslant 0.6$ and $t \geqslant 10$,

$$
g_2''(a,v,t) \leqslant -\frac{1}{4t^{5/2}}\Big(2.25 - \frac{1.8}{\log 10} - \frac{4.8}{\log^2 10} - \frac{4.8}{\log^3 10}\Big) = -\frac{0.16\ldots}{4t^{5/2}} < 0. \qquad (3.17)
$$

Let us set

$$\begin{aligned} g_3(u,c,t) &= u\log t + \frac{c\log t}{t^{2/3}}, \\ g_3'(u,c,t) &= \frac{1}{3t}\Big(3u - 2c\frac{\log t - 3/2}{t^{2/3}}\Big), \\ g_3''(u,c,t) &= -\frac{1}{9t^2}\Big(9u - 10c\frac{\log t - 2.1}{t^{2/3}}\Big). \end{aligned} \tag{3.18}$$

For $t \geqslant 6$, $\log t - 3/2$ is positive. Thus, if $c \leqslant 0$, then $g_3'(0.04, c, t) > 0$ holds while, if $0 < c \leqslant 0.1$, from (1.13), $(\log t - 3/2)/t^{2/3} \leqslant 3/(2e^2)$ and

$$g_3'(0.04, c, t) \geqslant (0.12 - 0.2 \times 3/(2e^2))/(3t) = 0.079\ldots/(3t) > 0.$$

In both cases, for $c \leqslant 0.1$ and $t \geqslant 6$,

$$g_3'(0.04, c, t) \text{ is positive.} \tag{3.19}$$

For $t \geqslant 10$, $\log t - 2.1$ is positive. Thus, if $c \leqslant 0$, then $g_3''(0.04, c, t) < 0$ holds while, if $0 < c \leqslant 0.1$, from (1.13),

$$(\log t - 2.1)/t^{2/3} \leqslant 1.5\exp(-2.4)$$

and

$$g_3''(0.04, c, t) \leqslant -(0.36 - 10 \times 0.1 \times 1.5\exp(-2.4)))/(9t^2) = -0.22\ldots/(9t^2) < 0.$$

In both cases, for $c \leqslant 0.1$ and $t \geqslant 6$,

$$g_3''(0.04, c, t) \text{ is negative.} \tag{3.20}$$

From (3.10), (3.12), (3.15) and (3.18), one may write

$$G(a, b, c, t) = g_1(0.96, b - 0.6, t) + g_2(a, 0.6, t) + g_3(0.04, c, t).$$

Since $a \geqslant 0.75$, $b \leqslant 3.9$ and $c \leqslant 0.1$ are assumed, from (3.13), (3.16) and (3.19), $G(a, b, c, t)$ is increasing for $t \geqslant 6$, while (3.14), (3.17) and (3.20) prove the concavity of $G(a, b, c, t)$ for $t \geqslant 10$.

From (3.11), (3.8), (3.12), (3.15) and (3.18),

$$H(a, b, c, t) = h_1(t) + g_1(0.8, b - 0.6, t) + g_2(a, 0.6, t) + g_3(0.04, c, t)$$

and Lemma 3.5, (3.13), (3.16) and (3.19) prove that $H(a, b, c, t)$ is increasing in t for $t \geqslant 6$. □

4 Colossally abundant (CA) numbers

4.1 Definition of CA numbers

Definition 4.1. *A number N is said to be colossally abundant (CA) if there exists $\varepsilon > 0$ such that*

$$\frac{\sigma(M)}{M^{1+\varepsilon}} \leqslant \frac{\sigma(N)}{N^{1+\varepsilon}} \tag{4.1}$$

holds for all positive integers M. The number ε is called a parameter of the CA number N.

These numbers were introduced in 1944 (cf. [1]) by Alaoglu and Erdős who did not know that, earlier, in a manuscript not yet published, Ramanujan already defined these numbers and called them *generalized superior highly composite* (cf. [34, Section 59]). The CA numbers have been considered in many papers and more especially in [25, 18, 37, 9, 10, 6].

The study of CA numbers is close to the study of the *superior highly composite* numbers introduced by Ramanujan in [33, Section 32]. In this paper, we use a presentation of the CA numbers similar to that given in [30] for the superior highly composite numbers.

An integer n is said to be *superabundant* (SA for short) if

$$m \leqslant n \quad \text{implies} \quad \sigma(m)/m \leqslant \sigma(n)/n. \tag{4.2}$$

The SA numbers have been introduced and studied by Alaoglu and Erdős (cf. [1, Section 4]). They also were defined and studied by Ramanujan (cf. [34, Section 59]) who called them *generalized highly composite*. It is possible to adapt the algorithm described in [30, Section 3.4] to compute a table of SA numbers (cf. [43]).

If $n = \prod_{p\in\mathcal{P}} p^{v_p(n)}$ is superabundant, then $v_p(n)$ is non-increasing in p (cf. [1, Theorem 1]). From the definition of CA numbers, it follows that a CA number N is SA and thus

$$v_p(N) \ \text{ is non-increasing in } \ p. \tag{4.3}$$

From the definition (4.1), note that two CA numbers N of parameter ε and N' of parameter ε' satisfy $(N'/N)^{\varepsilon-\varepsilon'} \geqslant 1$ and consequently,

$$\text{if} \quad N < N', \quad \text{then} \quad \varepsilon \geqslant \varepsilon'. \tag{4.4}$$

For t real > 1 and k positive integer, one defines

$$F(t,k) = \frac{\log(1 + 1/(t^k + t^{k-1} + \ldots + t))}{\log t} = \frac{\log(1 + (t-1)/(t^{k+1} - t))}{\log t}. \tag{4.5}$$

Note that the second formula allows to calculate $F(t,u)$ for u real positive and that $F(t,u)$ is decreasing in t for u fixed and in u for t fixed.

For p prime, we consider the set

$$\mathcal{E}_p = \{F(p,k), k \text{ integer } \geqslant 1\}, \tag{4.6}$$

and the set

$$\mathcal{E} = \{F(p,k), p \text{ prime}, k \text{ integer } \geqslant 1\} = \bigcup_{p\in\mathcal{P}} \mathcal{E}_p. \tag{4.7}$$

It is convenient to order the elements of $\mathcal{E} \cup \{\infty\}$ defined in (4.7) in the decreasing sequence

$$\begin{aligned}\varepsilon_1 = \infty > \varepsilon_2 = F(2,1) = \frac{\log(3/2)}{\log 2} = 0.58\ldots \\ > \varepsilon_3 = \frac{\log(4/3)}{\log 3} = 0.26\ldots > \ldots > \varepsilon_i = F(p_i,k_i) > \ldots\end{aligned} \tag{4.8}$$

Each element of $\mathcal{E}$ is the quotient of the logarithm of a rational number by the logarithm of a prime so that the diophantine properties of $\mathcal{E}$ are similar to those of the set $\mathcal{E}$ studied in [30, Section 3.1]. From the Six Exponentials Theorem (cf. [42] or [21]) there could exist elements in the set $\mathcal{E}$ defined by (4.7) admitting two representations

$$\varepsilon_i = F(q_i,k_i) = F(q'_i,k'_i) \tag{4.9}$$

with $k_i > k'_i \geqslant 1$ and $q_i < q'_i$. An element $\varepsilon_i \in \mathcal{E}$ satisfying (4.9) is said to be *extraordinary*. If ε_i is not extraordinary, it is said to be *ordinary* and satisfies in only one way

$$\varepsilon_i = F(q_i,k_i). \tag{4.10}$$

For $\varepsilon > 0$, let us introduce

$$N_\varepsilon = \prod_{p\in\mathcal{P}} p^{\lfloor\mu\rfloor} \text{ with } \mu = \mu(p,\varepsilon) = \frac{\log((p^{1+\varepsilon}-1)/(p^{1+\varepsilon}-p))}{\log p} \tag{4.11}$$

which is CA of parameter ε (cf. [1, Theorem 10]). Note that (cf. [43])

$$F(p,\mu) = F(p,\mu(p,\varepsilon)) = \varepsilon. \tag{4.12}$$

We observe that N_ε is a non-increasing function of ε. More precisely,

$$\text{if } \varepsilon \leqslant \varepsilon', \text{ then } N_{\varepsilon'} \text{ divides } N_\varepsilon. \tag{4.13}$$

By convention, $N_{\varepsilon_1} = N_\infty = 1$, as (4.11) yields $N_\varepsilon = 1$ for $\varepsilon > \varepsilon_2 = \log(3/2)/\log 2$.

Proposition 4.2. *If ε_i (with $i \geqslant 2$) belongs to the sequence* (4.8) *and ε satisfies $\varepsilon_{i-1} > \varepsilon > \varepsilon_i$, there is only one CA number of parameter ε, namely $N_\varepsilon = N_{\varepsilon_{i-1}}$ defined by* (4.11).

If ε_i is ordinary and satisfies (4.10)*, there are two CA numbers of parameter ε_i, namely N_{ε_i} and $N_{\varepsilon_{i-1}}$ satisfying*

$$N_{\varepsilon_i} = q_i N_{\varepsilon_{i-1}} \tag{4.14}$$

with q_i defined by (4.10).

If ε_i in (4.8) *is extraordinary of the form* (4.9)*, there are four CA numbers of parameter ε_i, namely $N_{\varepsilon_{i-1}},\ q_i N_{\varepsilon_{i-1}},\ q_i' N_{\varepsilon_{i-1}},\ N_{\varepsilon_i} = q_i q_i' N_{\varepsilon_{i-1}}$.*

In conclusion, if there is no extraordinary ε_i, any CA number is of the form N_{ε_i} (with $i \geqslant 1$). If extraordinary ε_i's exist, for each of them, there are two extra CA numbers N_{ε_i}/q_i' and N_{ε_i}/q_i and they both have only one parameter ε_i. In both cases, the set of parameters of N_{ε_i} is $[\varepsilon_{i+1}, \varepsilon_i]$ and two consecutive CA numbers have one and only one common parameter.

Proof. The proof is similar to the one of [30, Proposition 3.7], see also [18, Proposition 4] and [30, Remark 3.8]. □

Definition 4.3. *Let N be a CA number satisfying $N_{\varepsilon_{i-1}} < N \leqslant N_{\varepsilon_i}$ (where ε_{i-1} and ε_i is are elements of the sequence* (4.8) *and N_ε is defined by* (4.11)*). From Proposition 4.2, either N is equal to N_{ε_i} or ε_i is extraordinary. In both cases, the largest parameter of N is ε_i. We define $\xi_1 = \xi = \xi(N)$ by (cf. [18, (8)] or [37, Section 2])*

$$F(\xi, 1) = \frac{\log(1 + 1/\xi)}{\log \xi} = \varepsilon_i, \tag{4.15}$$

for $k \geqslant 1$, the numbers $\xi_k = \xi_k(N)$ by

$$F(\xi_k, k) = \frac{\log(1 + 1/(\xi_k + \xi_k^2 + \ldots + \xi_k^k))}{\log \xi_k} = F(\xi, 1) = \varepsilon_i \tag{4.16}$$

and, from (4.3),

$$K = K(N) = K(N_{\varepsilon_i}) = v_2(N_{\varepsilon_i}) = \max_{p \geqslant 2} v_p(N_{\varepsilon_i}). \tag{4.17}$$

Lemma 4.4. *With the notation of Definition 4.3, for $k \geqslant 1$ and p prime, we have*

$$v_p(N_{\varepsilon_i}) = \begin{cases} k & \textit{for} \quad \xi_{k+1} < p \leqslant \xi_k \quad \textit{if} \quad k \geqslant 1 \\ 0 & \textit{for} \quad p > \xi_1 \end{cases} \tag{4.18}$$

and

$$N_{\varepsilon_i} = \prod_{k=1}^{K} \prod_{\xi_{k+1} < p \leqslant \xi_k} p^k. \tag{4.19}$$

Proof. Let $k \geqslant 1$ be an integer, p be a prime satisfying $\xi_{k+1} < p \leqslant \xi_k$ and $\mu = \mu(p, \varepsilon_i)$ be defined by (4.11). From (4.16) and (4.12), one has

$$F(\xi_k, k) = F(p, \mu) = \varepsilon_i. \tag{4.20}$$

As $F(t, u)$ is decreasing in t and u, from (4.20), $p \leqslant \xi_k$ implies $\mu \geqslant k$. Similarly, $F(\xi_{k+1}, k+1) = F(p, \mu)$ and $p > \xi_{k+1}$ imply $\mu < k + 1$, so that $\lfloor \mu \rfloor = k$ and, from (4.11), $v_p(N) = k$ which proves the first case of (4.18). If $p > \xi_1 = \xi$, $F(\xi, 1) = F(p, \mu)$ shows that $\mu < 1$ and $v_p(N_{\varepsilon_i}) = 0$, which completes the proof of (4.18) and (4.19) follows. □

Proposition 4.5. *Let N be a CA number and ε_i the element of* (4.8) *such that $N_{\varepsilon_{i-1}} < N \leqslant N_{\varepsilon_i}$. The numbers ξ, ξ_k and K are defined by Definition 4.3. Then*

$$\log N_{\varepsilon_i} - \log \xi = \sum_{k=1}^{K} \theta(\xi_k) - \log \xi \leqslant \log N \leqslant \log N_{\varepsilon_i} = \sum_{k=1}^{K} \theta(\xi_k) \tag{4.21}$$

and

$$\frac{\xi\, \sigma\big(N_{\varepsilon_i}\big)}{(\xi+1) N_{\varepsilon_i}} \leqslant \frac{\sigma(N)}{N} \leqslant \frac{\sigma\big(N_{\varepsilon_i}\big)}{N_{\varepsilon_i}} = \prod_{k=1}^{K} \prod_{\xi_{k+1} < p \leqslant \xi_k} \frac{1 - 1/p^{k+1}}{1 - 1/p}. \tag{4.22}$$

Proof. From (4.19), it follows that

$$\log N_{\varepsilon_i} = \sum_{1 \leqslant k \leqslant K} \theta(\xi_k) \ \text{ and } \ \frac{\sigma\big(N_{\varepsilon_i}\big)}{N_{\varepsilon_i}} = \prod_{k=1}^{K} \prod_{\xi_{k+1} < p \leqslant \xi_k} \frac{1 - 1/p^{k+1}}{1 - 1/p}. \tag{4.23}$$

• If ε_i is ordinary, from Proposition 4.2, $N_{\varepsilon_{i-1}}$ and N_{ε_i} are two consecutive CA numbers, $N_{\varepsilon_{i-1}} < N \leqslant N_{\varepsilon_i}$ implies $N = N_{\varepsilon_i}$, so that (4.23) prove (4.21) and (4.22).
• If ε_i is extraordinary and given by (4.9), from Proposition 4.2, N is equal to N_{ε_i}, N_{ε_i}/q_i, or N_{ε_i}/q_i' and so N does not exceed N_{ε_i}, which proves the upper bound of (4.21). As N_{ε_i} is CA, thus superabundant (cf. (4.2)), $\sigma(N)/N \leqslant \sigma(N_{\varepsilon_i})/N_{\varepsilon_i}$ holds, which proves the upper bound of (4.22). The two primes q_i and q_i' divide N_{ε_i} and

so, from (4.19), are both $\leqslant \xi$, which proves the lower bound of (4.21). From (4.9) and (4.16), we also have for $q = q_i$ or q_i' and $k = k_i$ or k_i',

$$\frac{\sigma(N_{\varepsilon_i})}{N_{\varepsilon_i}} = \frac{\sigma(N\,q)}{N\,q} = \Big(\frac{1+q+\ldots+q^k}{q+\ldots+q^k}\Big)\frac{\sigma(N)}{N} = \Big(1+\frac{1}{q+\ldots+q^k}\Big)\frac{\sigma(N)}{N}$$
$$= q^{F(q,k)}\frac{\sigma(N)}{N} = q^{\varepsilon_i}\frac{\sigma(N)}{N} = q^{F(\xi,1)}\frac{\sigma(N)}{N} \leqslant \xi^{F(\xi,1)}\frac{\sigma(N)}{N} = \Big(1+\frac{1}{\xi}\Big)\frac{\sigma(N)}{N},$$

which proves the lower bound of (4.22). □

Proposition 4.6. *Let $n \geqslant 2$ be an integer. There exists two consecutive CA numbers $N' < N$ such that*

$$\frac{N}{\xi} \leqslant N' < n \leqslant N \text{ and } \frac{\sigma(n)}{n} \leqslant \frac{\sigma(N)}{N} \leqslant \frac{(\xi+1)\sigma(N')}{\xi N'} \leqslant \frac{\sigma(N')}{N'} + \frac{\sigma(N)}{\xi N}, \quad (4.24)$$

where $\xi = \xi(N)$ is defined in Definition 4.3.

Proof. First, we determine the element ε_i of the sequence (4.8) such that $N_{\varepsilon_i - 1} < n \leqslant N_{\varepsilon_i}$.

• If ε_i is ordinary and given by (4.10), from Proposition 4.2, we choose $N = N_{\varepsilon_i}$, $N' = N_{\varepsilon_{i-1}} = N_{\varepsilon_i}/q_i$ and, from (4.2), $\sigma(n)/n \leqslant \sigma(N)/N$ follows. As q_i does not exceed ξ, we have $N_{\varepsilon_i}/q_i \geqslant N_{\varepsilon_i}/\xi$ and, from (4.16),

$$\frac{\sigma(N)}{N} = \frac{\sigma(N'\,q_i)}{N'\,q_i} = \Big(\frac{1+q_i+\ldots+q_i^{k_i}}{q_i+\ldots+q_i^{k_i}}\Big)\frac{\sigma(N')}{N'} = \Big(1+\frac{1}{q_i+\ldots+q_i^{k_i}}\Big)\frac{\sigma(N')}{N'}$$
$$= q_i^{\varepsilon_i}\frac{\sigma(N')}{N'} = q_i^{F(\xi,1)}\frac{\sigma(N')}{N'} \leqslant \xi^{F(\xi,1)}\frac{\sigma(N')}{N'} = \Big(1+\frac{1}{\xi}\Big)\frac{\sigma(N')}{N'}, \quad (4.25)$$

which proves (4.24) since, from (4.2), $\sigma(N')/N' < \sigma(N)/N$ holds.

• If ε_i is extraordinary and given by (4.9), from Proposition 4.2, there are four consecutive CA numbers in $[N_{\varepsilon_{i-1}}, N_{\varepsilon_i}]$, namely $M_1 = N_{\varepsilon_i - 1}$, $M_2 = q_i N_{\varepsilon_{i-1}} = q_i M_1$, $M_3 = q_i' N_{\varepsilon_{i-1}} = q_i' M_2/q_i$, $M_4 = q_i q_i' N_{\varepsilon_{i-1}} = q_i M_3$. For $j = 1$ or $j = 3$, $M_{j+1} = q M_j$ with $q = q_i$ or q_i' and $k = k_i$ or k_i'. By copying (4.25), one gets $\sigma(M_{j+1})/M_{j+1} \geqslant (1+1/\xi)\sigma(M_j)/M_j$. For $j = 2$,

$$\frac{\sigma(M_3)}{M_3} = \frac{1+1/q_i'+\ldots+1/(q_i')^{k_i'}}{1+1/q_i+\ldots+1/(q_i)^{k_i+1}}\frac{\sigma(M_2)}{M_2} \leqslant \big(1+1/q_i'+\ldots+1/(q_i')^{k_i'}\big)\frac{\sigma(M_2)}{M_2}$$
$$= (q_i')^{\varepsilon_i}\frac{\sigma(M_2)}{M_2} \leqslant \xi^{\varepsilon_i}\frac{\sigma(M_2)}{M_2} = \xi^{F(\xi,1)}\frac{\sigma(M_2)}{M_2} = \frac{\xi}{\xi+1}\frac{\sigma(M_2)}{M_2}$$

so that, for $1 \leqslant j \leqslant 3$, $\sigma(M_{j+1})/M_{j+1} < (\xi+1)/\xi)\sigma(M_j)/M_j$ holds. From the choice of ε_i, $n \in (M_{j_0}, M_{j_0+1}]$ for some $j_0 \in \{1,2,3\}$. Then, by choosing $N' = M_{j_0}$, $N = M_{j_0+1}$, (4.24) is satisfied. □

The first CA numbers are (for a longer table, cf. [1, p. 468] or [43]):

i	ε_i	$N = N_{\varepsilon_i}$	$\sigma(N)/N$	parameter	$\xi(N)$
1	∞	1	1	$[\varepsilon_2, \varepsilon_1)$	1
2	$F(2,1) = 0.58$	2	$3/2$	$[\varepsilon_3, \varepsilon_2]$	2
3	$F(3,1) = 0.26$	$6 = 2 \cdot 3$	2	$[\varepsilon_4, \varepsilon_3]$	3
4	$F(2,2) = 0.22$	$12 = 2^2 \cdot 3$	$7/3$	$[\varepsilon_5, \varepsilon_4]$	3.29
5	$F(5,1) = 0.11$	$60 = 2^2 \cdot 3 \cdot 5$	$14/5$	$[\varepsilon_6, \varepsilon_5]$	5
6	$F(2,3) = 0.09$	$120 = 2^3 \cdot 3 \cdot 5$	3	$[\varepsilon_7, \varepsilon_6]$	5.44
7	$F(3,2) = 0.07$	$360 = 2^3 \cdot 3^2 \cdot 5$	$13/4$	$[\varepsilon_8, \varepsilon_7]$	6.71
8	$F(7,1) = 0.06$	$2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$	$26/7$	$[\varepsilon_9, \varepsilon_8]$	7

Figure 2: The first colossally abundant numbers

4.2 Enumeration of CA numbers

How to compute CA numbers? For a short table, one determines the sequence ε_i (cf. (4.8)) and, if ε_i satisfies (4.10), then $N_{\varepsilon_i} = q_i N_{\varepsilon_{i-1}}$ (cf. (4.14). In the proof of Lemma 5.3, we have to compute the CA numbers up to $N^{(0)}$ (defined below by (4.37)). Let us say that N_{ε_i} is a CA number of type 2 if ε_i satisfies (4.10) with $k_i \geqslant 2$ and of type 1 if $k_i = 1$. We have precomputed the table of the 6704 CA numbers N of type 2 (with $\xi = \xi(N)$ defined by Definition 4.3) satisfying $\xi < 20007 \times 10^5$ (cf. [43]). If N is of type 2 with its largest prime factor equal to the rth prime p_r, then the following CA numbers are $N p_{r+1}$, $N p_{r+1} p_{r+2}$, etc. up to the next CA number of type 2.

Note that we have not found any extraordinary case (cf. Section 4.1). The smallest difference $\varepsilon_i - \varepsilon_{i+1} = 2.57 \ldots 10^{-23}$ has been obtained with $\varepsilon_i = F(54371, 2)$ and $\varepsilon_{i+1} = F(1524427141, 1)$ (cf. [43]).

4.3 Estimates of ξ_k defined by (4.16)

Proposition 4.7. *Let $k \geqslant 2$ be an integer, $\xi > 1$ a real number and ξ_k be defined by $F(\xi_k, k) = F(\xi, 1)$. Then*

$$\xi^{1/k} \leqslant \xi_k \leqslant (k\xi)^{1/k} \quad \textit{for} \quad k \geqslant 2 \textit{ and } \xi > 1, \tag{4.26}$$

when ξ tends to infinity, the asymptotic expansion of ξ_2 is

$$\xi_2 = \sqrt{2\xi}\Big(1 - \frac{a}{2\log\xi} + \frac{a/2 + 3a^2/8}{\log^2\xi} - \frac{a/2 + a^2 + 5a^3/16}{\log^3\xi} + \frac{\mathcal{O}(1)}{\log^4\xi}\Big) \tag{4.27}$$

with $a = \log 2$,

$$\xi_2 \geqslant \sqrt{2\xi}\Big(1 - \frac{\log 2}{2\log\xi}\Big) \quad \textit{for} \quad \xi \geqslant 1530 \tag{4.28}$$

and

$$\xi_2 \leqslant \sqrt{2\xi}\Big(1 - \frac{0.323}{\log\xi}\Big) \quad \textit{for} \quad \xi \geqslant 10^9. \tag{4.29}$$

Proof. The proofs of the lower bound of (4.26) and of (4.28) are given in [37, p. 190].

To prove the upper bound of (4.26), we first observe that F is decreasing and $F(\xi_k, k) = F(\xi, 1)$ holds. So, we have to show that $F((k\xi)^{1/k}, k) \leqslant F(\xi_k, k) = F(\xi, 1)$. Setting $z = (k\xi)^{1/k}$, from (1.14), one has

$$F(z, k) = \frac{k\log(1 + 1/(z + z^2 + \ldots + z^k))}{\log\xi + \log k} \leqslant \frac{k/z^k}{\log\xi + \log k} = \frac{1/\xi}{\log\xi + \log k}$$

and, from (1.17),

$$F(\xi, 1) = \frac{\log(1 + 1/\xi)}{\log\xi} \geqslant \frac{1}{(\xi + 1/2)\log\xi}.$$

Therefore, it suffices to show $1/(\xi(\log\xi + \log k)) \leqslant 1/((\xi + 1/2)\log\xi)$, i.e. $\xi(\log\xi + \log k) \geqslant (\xi + 1/2)\log\xi$ which is true, since $\xi \geqslant \log\xi$ and $\log k \geqslant \log 2 > 1/2$ hold.

To compute the asymptotic expansion of ξ_2, we observe that, from (4.26), ξ_2 tends to infinity with ξ. From (4.16), one has

$$F(\xi_2, 2) = \frac{\log(1 + 1/(\xi_2 + \xi_2^2))}{\log\xi_2} = F(\xi, 1) = \frac{\log(1 + 1/\xi)}{\log\xi}. \tag{4.30}$$

As $\log(1+1/u) \sim 1/u$ when $u \to \infty$, it follows that $1/(\xi_2^2 \log \xi_2) \sim 1/(\xi \log \xi)$, which implies $\xi_2^2 \log \xi_2 \sim \xi \log \xi$, $2\log \xi_2 \sim \log \xi$, $\xi_2^2 \sim (\xi \log \xi)/\log \xi_2 \sim 2\xi$ and $\xi_2 \sim \sqrt{2\xi}$. Furthermore (cf. [43]),

$$\log\Big(1 + \frac{1}{\xi_2 + \xi_2^2}\Big) = \frac{1}{\xi_2^2} + \frac{\mathcal{O}(1)}{\xi_2^3} = \frac{1}{\xi_2^2} + \frac{\mathcal{O}(1)}{\xi^{3/2}}$$

and

$$F(\xi_2, 2) = \frac{1}{\xi_2^2 \log \xi_2} + \frac{\mathcal{O}(1)}{\xi^{3/2} \log \xi_2} = \frac{1}{\xi_2^2 \log \xi_2}\Big(1 + \frac{\mathcal{O}(1)}{\sqrt{\xi}}\Big),$$

whence, from (4.30), as $F(\xi, 1) = (1/(\xi \log \xi))(1 + \mathcal{O}(1)/\sqrt{\xi})$ holds,

$$\xi_2^2 = \frac{\xi \log \xi}{\log \xi_2}\Big(1 + \frac{\mathcal{O}(1)}{\sqrt{\xi}}\Big). \tag{4.31}$$

In (4.31), the change of variables $t = \log \xi$ and $\xi_2 = \sqrt{2\xi}w$ yields

$$w^2 = \frac{\xi_2^2}{2\xi} = \frac{\log \xi}{2\log \xi_2}\Big(1 + \frac{\mathcal{O}(1)}{\sqrt{\xi}}\Big) = \frac{t}{\log 2 + t + 2\log w}\Big(1 + \frac{\mathcal{O}(1)}{\sqrt{\xi}}\Big)$$

and

$$w = \Big(1 + \frac{\log 2}{t} + \frac{2\log w}{t}\Big)^{-1/2}\Big(1 + \mathcal{O}\Big(\exp\Big(-\frac{t}{2}\Big)\Big)\Big). \tag{4.32}$$

Iterating (4.32) from $w = 1 + o(1)$ gives the expansion (4.27). It is possible to get the asymptotic expansion of ξ_k for $k > 2$ in a similar way (cf. [43]).

To prove (4.29), let us set $\beta = 0.323$, $t = \log \xi$, $a = \log 2$, $x_0 = 10^9$ and $z = \sqrt{2\xi}(1 - \beta/\log \xi)$. We have to prove $F(z, 2) \leqslant F(\xi_2, 2)$, i.e., from (4.30), $F(z, 2) - F(\xi, 1) \leqslant 0$. From (1.14) and (1.17), it follows that

$$\begin{aligned} F(z, 2) - F(\xi, 1) &= \frac{\log(1 + 1/(z + z^2))}{\log z} - \frac{\log(1 + 1/\xi)}{\log \xi} \\ &\leqslant \frac{1}{(z + z^2)\log z} - \frac{1}{(\xi + 1/2)\log \xi} < \frac{1}{z^2 \log z} - \frac{1}{(\xi + 1/2)\log \xi}. \end{aligned}$$

So, it suffices to show that

$$z^2 \log z - (\xi + 1/2)\log \xi \geqslant 0 \quad \text{for} \quad \xi \geqslant x_0. \tag{4.33}$$

From (1.15), one has

$$\log z = \frac{1}{2}(a+t) + \log\left(1 - \frac{\beta}{t}\right) \geqslant \frac{1}{2}(a+t) - \frac{\beta}{t-\beta},$$

by using Maple (cf. [43]) ,

$$\begin{aligned} z^2 \log z \geqslant \frac{2\xi}{t^2}(t-\beta)^2\Big(\frac{1}{2}(a+t) - \frac{\beta}{t-\beta}\Big) \\ = \frac{\xi}{t^2}\Big(t^3 + (a-2\beta)t^2 + \beta(\beta - 2a - 2)t + \beta^2(a+2)\Big) \end{aligned}$$

and, for $\xi \geqslant x_0 = 10^9$,

$$(\xi + 1/2)\log \xi = \frac{\xi}{t^2}\Big(t^3 + \frac{t^3}{2\xi}\Big) \leqslant \frac{\xi}{t^2}\Big(t^3 + \frac{\log^3 x_0}{2x_0}\Big).$$

Therefore, one gets

$$z^2 \log z - (\xi + 1/2)\log \xi \geqslant \frac{\xi}{t^2}\Big((a-2\beta)t^2 + \beta(\beta - 2a - 2)t + \beta^2(a+2) - \frac{\log^3(x_0)}{2x_0}\Big). \tag{4.34}$$

The roots of the trinomial on t in (4.34) are $0.2879\ldots$ and $20.698\ldots$ so that it is positive for $t \geqslant \log x_0 = 20.723\ldots$ which proves (4.33) and completes the proof of (4.29). □

4.4 Study of CA numbers $\geqslant N^{(0)}$

By (4.19), we define $N^{(0)} = N_{\varepsilon^{(0)}}$ with $\varepsilon^{(0)} = F(\xi^{(0)}, 1) \in \mathcal{E}$ and

$$\xi^{(0)} = \xi_1^{(0)} = 10^9 + 7, \tag{4.35}$$

the smallest prime exceeding 10^9. For $k \geqslant 2$, we define $\xi_k^{(0)}$ by (4.15) and (4.16), obtaining (cf. [43])

$$\xi_2^{(0)} = 44023.5\ldots, \quad \xi_3^{(0)} = 1418.3\ldots, \quad \xi_4^{(0)} = 247.3\ldots, \quad \xi_5^{(0)} = 85.6\ldots, \quad \ldots, \quad \xi_{33}^{(0)} = 2.033\ldots, \tag{4.36}$$

from (4.19),

$$N^{(0)} = N_{\varepsilon^{(0)}} = 2^{33}3^{21}5^{14}7^{11}11^{9}13^{8}17^{7}19^{7}23^{7}\prod_{p=29}^{41}p^6\prod_{p=43}^{83}p^5\prod_{p=89}^{241}p^4$$

$$\prod_{p=251}^{1409}p^3\prod_{p=1423}^{44021}p^2\prod_{p=44027}^{1000000007}p, \tag{4.37}$$

$$L_0 = \log N^{(0)} = 1000014552.11\ldots,\quad \lambda_0 = \log\log N^{(0)} = 20.7232803\ldots, \tag{4.38}$$

$\sigma(N^{(0)})/N^{(0)} = 36.909618566\ldots$ and from (4.17), $K = K(N^{(0)}) = 33$.

Lemma 4.8. *Let N be a CA number $> N^{(0)} = N_{\varepsilon^{(0)}}$ and $\varepsilon_i, \xi, \xi_k$ and K be defined in Definition 4.3. Then $\varepsilon_i \leqslant \varepsilon^{(0)}$, $\xi \geqslant \xi^{(0)}$, $\xi_k \geqslant \xi_k^{(0)}$ for $k \geqslant 1$, and $K = K(N) \geqslant 33$.*

Proof. By (4.21), $N \leqslant N_{\varepsilon_i}$ holds. Since $N^{(0)} < N$ is assumed, this implies $N^{(0)} = N_{\varepsilon^{(0)}} < N_{\varepsilon_i}$ and, from (4.4), $\varepsilon_i \leqslant \varepsilon^{(0)}$. Moreover, from (4.15), one has $F(\xi, 1) = \varepsilon_i \leqslant \varepsilon^{(0)} = F(\xi^{(0)}, 1)$ and as F is decreasing, $\xi = \xi(N) \geqslant \xi^{(0)} = 10^9 + 7$. Similarly, from (4.16), for $k \geqslant 2$, $F(\xi_k, k) = \varepsilon_i \leqslant \varepsilon^{(0)} = F(\xi_k^{(0)}, k)$ implies $\xi_k = \xi_k(N) \geqslant \xi_k^{(0)}$ given by (4.36). Finally, as $N^{(0)} = N_{\varepsilon^{(0)}}$ divides N_{ε_i} (cf. (4.13)), from (4.17), it follows that $K(N) \geqslant K(N^{(0)}) = 33$. □

Lemma 4.9. *Let N be a CA number $> N^{(0)} = N_{\varepsilon^{(0)}}$ with ξ, ξ_k and K be defined in Definition 4.3. Then*

$$K = K(N) \leqslant 1.71 \log \xi. \tag{4.39}$$

Proof. From (4.17) and (4.18), we have $\xi_{K+1} < 2 \leqslant \xi_K$ so that, from (4.26), $2 \leqslant (K\xi)^{1/K}$ holds, which implies

$$\log \xi \geqslant K \log 2 - \log K = K(\log 2 - (\log K)/K).$$

But, from Lemma 4.8, $K \geqslant 33$ and thus $\log \xi \geqslant K(\log 2 - (\log 33)/33) \geqslant 0.587K$ which yields $K \leqslant (1/0.587)\log \xi < 1.71 \log \xi$ and completes the proof of Lemma 4.9. □

4.5 Some properties of numbers ξ_k

Lemma 4.10. *Let N be a CA number satisfying $N \geqslant N^{(0)}$ defined by* (4.37). *The numbers $\xi = \xi(N)$, ξ_k and K are defined by Definition 4.3, so that, from Lemma 4.8, $\xi \geqslant \xi^{(0)} = 10^9 + 7$ and $K \geqslant 33$ hold. Then*

$$T_3 = \sum_{k=3}^{K} \xi_k \leqslant 1.9769\, \xi^{1/3}. \tag{4.40}$$

Proof. If k_0 satisfies $4 \leqslant k_0 \leqslant K$, as ξ_k is non-increasing in k, from (4.26) and (4.39), we have

$$\begin{aligned} T_3 &\leqslant \Big(\sum_{k=3}^{k_0-1} \xi_k\Big) + (K-k_0+1)\xi_{k_0} \leqslant \Big(\sum_{k=3}^{k_0-1} (k\xi)^{1/k}\Big) + (K-k_0+1)(k_0\xi)^{1/k_0} \\ &= \xi^{1/3}\left(\Big(\sum_{k=3}^{k_0-1} \frac{k^{1/k}}{\xi^{1/3-1/k}}\Big) + \frac{K-(k_0-1))k_0^{1/k_0}}{\xi^{1/3-1/k_0}}\right) \\ &\leqslant \xi^{1/3}\Big(\Big(\sum_{k=3}^{k_0-1} \frac{k^{1/k}}{(\xi^{(0)})^{1/3-1/k}}\Big) + \frac{1.71\, k_0^{1/k_0}(\log\xi - w)}{\xi^{v}}\Big) \qquad (4.41) \end{aligned}$$

with $w = (k_0-1)/1.71$ and $v = 1/3 - 1/k_0$. An upper bound of (4.41) can be obtained from (1.12) or (1.13), with $u = 1$. The best choice is $k_0 = 31$, which gives $T_3/\xi^{1/3} \leqslant 1.976836\ldots$ (cf. [43]). □

Lemma 4.11. *Let N be a CA number. The numbers $\xi = \xi(N)$ and $\xi_2 = \xi_2(N)$ are defined in Lemma 4.8. Let us assume that $\xi \geqslant \xi^{(0)} = 10^9 + 7$. Then*

$$\frac{\sqrt{2}}{\sqrt{\xi}\log\xi} - \frac{0.524}{\sqrt{\xi}\log^2\xi} \leqslant \frac{1}{\xi_2\log\xi_2} \leqslant \frac{\sqrt{2}}{\sqrt{\xi}\log\xi} - \frac{0.4}{\sqrt{\xi}\log^2\xi} \qquad (4.42)$$

and

$$\frac{2.67}{\sqrt{\xi}\log^2\xi} \leqslant \frac{1}{\xi_2\log^2\xi_2} \leqslant \frac{2\sqrt{2}}{\sqrt{\xi}\log^2\xi} \leqslant \frac{2.829}{\sqrt{\xi}\log^2\xi}. \qquad (4.43)$$

Proof. From (4.29), one may write $\xi_2 \leqslant \sqrt{2\xi}(1-0.323/\log\xi)$ and $\log\xi_2 \leqslant \log(\sqrt{2\xi}) = (\log\xi)(1+(\log 2)/\log\xi)/2$, whence, from (1.19),

$$\begin{aligned} \frac{1}{\xi_2\log\xi_2} &\geqslant \frac{2}{(\sqrt{2\xi}\log\xi)(1-0.323/\log\xi)(1+(\log 2)/\log\xi)} \\ &\geqslant \frac{\sqrt{2}}{\sqrt{\xi}\log\xi}\Big(1-\frac{\log 2 - 0.323}{\log\xi}\Big) \geqslant \frac{\sqrt{2}}{\sqrt{\xi}\log\xi} - \frac{0.524}{\sqrt{\xi}\log^2\xi}, \end{aligned}$$

which proves the lower bound of (4.42). Next,

$$\frac{1}{\xi_2\log\xi_2} \geqslant \frac{1}{\sqrt{\xi}\log\xi}\Big(\sqrt{2} - \frac{0.524}{\log\xi^{(0)}}\Big) \geqslant \frac{1.38}{\sqrt{\xi}\log\xi}$$

and, as $\log \xi_2 \leqslant (\log \xi)(1 + (\log 2)/\log \xi)/2$ follows from (4.26),

$$\frac{1}{\xi_2 \log^2 \xi_2} \geqslant \frac{2.76}{(\sqrt{\xi} \log^2 \xi)(1 + (\log 2)/\log \xi)}$$
$$\geqslant \frac{2.76}{(\sqrt{\xi} \log^2 \xi)(1 + (\log 2)/\log \xi^{(0)})} \geqslant \frac{2.67}{\sqrt{\xi} \log^2 \xi},$$

which proves the lower bound of (4.43).

From (4.28), one has

$$\xi_2 \geqslant \sqrt{2\xi}\Big(1 - \frac{\log 2}{2 \log \xi}\Big) \geqslant \sqrt{2\xi}\Big(1 - \frac{\log 2}{2 \log \xi^{(0)}}\Big) \geqslant 1.39\sqrt{\xi},$$

which implies $\xi_2 \geqslant \sqrt{2\xi}(1 - u)$ with

$$u = (\log 2)/(2 \log \xi) \leqslant u_0 = (\log 2)/(2 \log \xi^{(0)}) = 0.016\ldots$$

and

$$\log \xi_2 \geqslant (\log \xi + 2 \log 1.39)/2 \geqslant (\log \xi + 0.658)/2 = (\log \xi)(1 + v)/2$$

with $v = 0.658/\log \xi \leqslant v_0 = 0.658/\log \xi^{(0)} = 0.031\ldots$. Therefore, from (1.19),

$$\frac{1}{\xi_2 \log \xi_2} \leqslant \frac{2}{\sqrt{2\xi} \log \xi (1 - u)(1 + v)} \leqslant \frac{\sqrt{2}(1 + u/(1 - u_0) - v(1 - v_0))}{\sqrt{\xi} \log \xi}$$
$$= \frac{\sqrt{2}}{\sqrt{\xi} \log \xi}\Big(1 - \frac{0.284\ldots}{\log \xi}\Big) \leqslant \frac{\sqrt{2}}{\sqrt{\xi} \log \xi} - \frac{0.4}{\sqrt{\xi} \log^2 \xi},$$

which proves the upper bound of (4.42). Then, as from (4.26), $\xi_2 > \sqrt{\xi}$ holds,

$$\frac{1}{\xi_2 \log^2 \xi_2} = \Big(\frac{1}{\xi_2 \log \xi_2}\Big)\Big(\frac{1}{\log \xi_2}\Big) \leqslant \Big(\frac{\sqrt{2}}{\sqrt{\xi} \log \xi}\Big)\Big(\frac{2}{\log \xi}\Big) = \frac{2\sqrt{2}}{\sqrt{\xi}(\log^2 \xi)},$$

which proves the upper bound of (4.43). $\square$

4.6 Benefit

Definition 4.12. *Let ε be a positive real number and N a CA number of parameter ε. For a positive integer n, we introduce the benefit of n*

$$\operatorname{ben}_\varepsilon(n) = \log\Big(\frac{\sigma(N)}{N^{1+\varepsilon}}\Big) - \log\Big(\frac{\sigma(n)}{n^{1+\varepsilon}}\Big) = \log\Big(\frac{\sigma(N)}{\sigma(n)}\Big) + (1+\varepsilon)(\log n - \log N). \tag{4.44}$$

If $\widetilde{N}$ is another CA number of parameter ε, then note that the value of the right-hand side of (4.44) does not change when replacing N by $\widetilde{N}$. Indeed (4.1) yields $\sigma(N)/N^{1+\varepsilon} \leqslant \sigma(\widetilde{N})/\widetilde{N}^{1+\varepsilon}$ and $\sigma(\widetilde{N})/\widetilde{N}^{1+\varepsilon} \leqslant \sigma(N)/N^{1+\varepsilon}$, so that $\sigma(N)/N^{1+\varepsilon} = \sigma(\widetilde{N})/\widetilde{N}^{1+\varepsilon}$, which implies

$$\log\Big(\frac{\sigma(N)}{N^{1+\varepsilon}}\Big) - \log\Big(\frac{\sigma(n)}{n^{1+\varepsilon}}\Big) = \log\Big(\frac{\sigma(\widetilde{N})}{\widetilde{N}^{1+\varepsilon}}\Big) - \log\Big(\frac{\sigma(n)}{n^{1+\varepsilon}}\Big).$$

From (4.1), it follows that, for any n,

$$\operatorname{ben}_\varepsilon(n) \geqslant 0 \tag{4.45}$$

holds. Let us write

$$N = \prod_{p\in\mathcal{P}} p^{a_p} \quad \text{and} \quad n = \prod_{p\in\mathcal{P}} p^{b_p}, \tag{4.46}$$

so that $\sigma(N)/\sigma(n) = \prod_{p\in\mathcal{P}}(p^{a_p+1}-1)/(p^{b_p+1}-1)$. We define

$$\operatorname{Ben}_{p,\varepsilon}(n) = \operatorname{ben}_\varepsilon(Np^{b_p-a_p}) = \log\Big(\frac{p^{a_p+1}-1}{p^{b_p+1}-1}\Big) + (1+\varepsilon)(b_p - a_p)\log p \geqslant 0 \tag{4.47}$$

and (4.44) gives

$$\operatorname{ben}_\varepsilon(n) = \sum_{p\in\mathcal{P}} \operatorname{Ben}_{p,\varepsilon}(n). \tag{4.48}$$

This notion of benefit has been used in [25, 18] for theoretical results on numbers n having a large value of $\sigma(n)$ and in [30, Section 3.5], it is defined and used for the divisor function $d(n) = \sum_{d|n} 1$.

For p, a_p, ε fixed and $t \geqslant 0$, let us introduce the mapping

$$t \mapsto B_{p,a_p,\varepsilon}(t) = \log\Big(\frac{p^{a_p+1}-1}{p^{t+1}-1}\Big) + (1+\varepsilon)(t-a_p)\log p \tag{4.49}$$

so that, if t is an integer, then $B(t)$ is equal to $\operatorname{ben}_\varepsilon(Np^{t-a_p}) = \operatorname{Ben}_{p,\varepsilon}(Np^{t-a_p})$.

Lemma 4.13. *Let p be a prime, a_p a non-negative integer, ε a positive real number and, for $t \geqslant 0$, $B_{p,a_p,\varepsilon}(t) = B(t)$ be defined by* (4.49). *Let us assume that, if t is an integer, then $B(t) \geqslant 0$ holds. Then,*

(i) $\lim_{t\to\infty} B(t) = \infty$ *and* $\lim_{t\to\infty} B'(t) = \varepsilon \log p > 0$.

(ii) For $t \geqslant 0$, $B(t)$ is convex, i.e. $B'(t)$ is increasing.

(iii) $B(t)$ is increasing for $t \geqslant a_p + 1$ and, if $a_p \geqslant 2$, decreasing for $0 \leqslant t \leqslant a_p - 1$. *Moreover, $B(a_p + 1) \geqslant 0 = B(a_p)$ and $B(a_p - 1) \geqslant 0 = B(a_p)$.*

(iv) $\lim_{p\to\infty} B_{p,0,\varepsilon}(1) = \infty$.

Proof of (i). The inequality

$$B(t) > \log\left(\frac{p^{a_p+1}-1}{p^{t+1}}\right) + (1+\varepsilon)(t-a_p)\log p = \log(p^{a_p+1}-1) + (\varepsilon t - 1 - a_p(1+\varepsilon))\log p$$

shows that $\lim_{t\to\infty} B(t) = \infty$. The derivative $B'(t)$ is equal to $\varepsilon\log(p) - (\log p)/(p^{t+1} - 1)$ and tends to $\varepsilon \log p$ when t tends to infinity.

Proof of (ii). The second derivative (cf. [43]) $d^2B(t)/dt^2 = (\log^2 p)p^{t+1}/(p^{t+1} - 1)^2$ is positive.

Proof of (iii). Note that $B(a_p) = 0$.

• If $B'(0) \geqslant 0$, then, as, from (ii), $B'(t)$ is increasing, one has $B'(t) \geqslant 0$ for $t \geqslant 0$, so that $B(t)$ is increasing for $t \geqslant 0$. This implies $a_p = 0$ since, if a_p were positive, we should have $B(a_p) > 0$, so contradicting $B(a_p) = 0$.

• If $B'(0) < 0$, then, as $B'(t)$ is increasing and tends to $\varepsilon \log p > 0$ when t tends to infinity, B' has one and only one zero, say t_0. We have

$$a_p - 1 \leqslant t_0 \leqslant a_p + 1. \tag{4.50}$$

Indeed, if $t_0 > a_p + 1$, then $B(t)$ would decrease on $[a_p, a_p + 1]$ and, since $B(a_p) = 0$, $B(a_p + 1)$ would be negative. Similarly, if $t_0 < a_p - 1$, then $B(t)$ would increase on $[a_p - 1, a_p]$ and $B(a_p - 1)$ would be negative. From (4.50), $B(t)$ is increasing for $t \geqslant a_p + 1$ and if $a_p \geqslant 2$, decreasing for $t \leqslant a_p - 1$.

Proof of (iv). $B_{p,0,\varepsilon}(1)$ is equal to $\varepsilon \log p - \log(1 + 1/p)$ that tends to infinity with p. □

Proposition 4.14. *Let $\varepsilon > 0$, N a CA number of parameter ε and β a positive real number. Then, the set of integers n satisfying* $\mathrm{ben}_\varepsilon(n) \leqslant \beta$ *is finite.*

Proof. We use the notation (4.46) and assume $\mathrm{ben}_\varepsilon(n) \leqslant \beta$. In view of applying Lemma 4.13, one remarks that, if $t \geqslant 0$ is an integer, then, from (4.45), $B_{p,a_p,\varepsilon}(t) = \mathrm{ben}_\varepsilon(Np^{t-a_p}) \geqslant 0$ holds.

Let p' be the smallest prime not dividing N, so that $a_p = 0$ for $p \geqslant p'$. From Lemma 4.13 (iv), there exists $p'' \geqslant p'$ such that, for $p \geqslant p''$, $B_{p,0,\varepsilon}(1) = \mathrm{Ben}_{p,\varepsilon}(Np) > \beta$. From Lemma 4.13 (ii), if $b_p \geqslant 1$, then $\mathrm{Ben}_{p,\varepsilon}(n) = B_{p,0,\varepsilon}(b_p) \geqslant B_{p,0,\varepsilon}(1) > \beta$, which, from (4.48), implies that $\mathrm{ben}_\varepsilon(n) \geqslant \mathrm{Ben}_{p,\varepsilon}(n) > \beta$. Consequently, as $\mathrm{ben}_\varepsilon(n) \leqslant \beta$ is assumed, $b_p = 0$ for $p \geqslant p''$.

For $p < p''$, from Lemma 4.13 (i), there exists an integer b'_p such that, for $b_p \geqslant b'_p$, $\mathrm{Ben}_{p,\varepsilon}(n) = B_{p,a_p,\varepsilon}(b_p)$ exceeds β, which implies that, as $\mathrm{ben}_\varepsilon(n) \leqslant \beta$ is assumed, then $0 \leqslant b_p \leqslant b'_p - 1$. Therefore, $|\{n,\ \mathrm{ben}_\varepsilon(n) \leqslant \beta\}| \leqslant \prod_{p<p''} b'_p$. □

Proposition 4.14 and Lemma 4.13 allow to get an algorithm to compute all integers n such that $\mathrm{ben}_\varepsilon(n) \leqslant \beta$. This algorithm is efficient if β is not too large (not much larger than ε), cf. [43].

4.7 Convexity

Lemma 4.15. *Let $N' \geqslant 2$ and N be two consecutive CA numbers and f a function of class C_2, increasing and concave on the interval* $[\log N', \log N]$ *such that*

$$\sigma(N')/N' \leqslant f(\log N') \quad \textit{and} \quad \sigma(N)/N \leqslant f(\log N). \tag{4.51}$$

Let n be an integer satisfying $N' \leqslant n \leqslant N$. Then

$$\sigma(n)/n \leqslant f(\log n). \tag{4.52}$$

Proof. From Proposition 4.2, N' and N share a common parameter, say ε. From the definition (4.1) of CA numbers, one deduces that $\log(\sigma(N')/N') - \varepsilon \log N' = \log(\sigma(N)/N) - \varepsilon \log N$. For $n \in [N', N]$, from (4.1), one has

$$\log \frac{\sigma(n)}{n} - \varepsilon \log n \leqslant \log \frac{\sigma(N')}{N'} - \varepsilon \log N' = \log \frac{\sigma(N)}{N} - \varepsilon \log N. \tag{4.53}$$

In view of using a convexity argument, one writes

$$\log n = \lambda \log N' + \mu \log N \quad \text{with} \quad 0 \leqslant \lambda \leqslant 1 \quad \text{and} \quad \mu = 1 - \lambda.$$

From (4.53) and (4.51), it follows that

$$\begin{aligned}
\log \frac{\sigma(n)}{n} &\leqslant \varepsilon \log n + \lambda\Big(\log \frac{\sigma(N')}{N'} - \varepsilon \log N'\Big) + \mu\Big(\log \frac{\sigma(N)}{N} - \varepsilon \log N\Big) \\
&= \varepsilon(\lambda \log N' + \mu \log N) + \lambda \log \frac{\sigma(N')}{N'} + \mu \log \frac{\sigma(N)}{N} \\
&\qquad - \lambda\varepsilon \log N' - \mu\varepsilon \log N = \lambda \log \frac{\sigma(N')}{N'} + \mu \log \frac{\sigma(N)}{N}.
\end{aligned} \tag{4.54}$$

The inequality $\sigma(N')/N' > 1$ and (4.51) imply $\log f(\log N') > 0$. As f is supposed increasing and concave, $f'(t) \geqslant 0$ and $f''(t) \leqslant 0$ hold for $t \in [\log N', \log N]$. Therefore, as the derivatives of $\log f$ are f'/f and $(ff'' - f'^2)/f^2$, $\log f$ is increasing, concave and positive on $[\log N', \log N]$ and (4.54) and (4.51) give

$$\log \frac{\sigma(n)}{n} \leqslant \lambda \log(f(\log N')) + \mu \log(f(\log N))$$
$$\leqslant \log(f(\lambda \log N' + \mu \log N)) = \log(f(\log n)),$$

which yields (4.52) and completes the proof of Lemma 4.15. □

4.8 Estimates of a CA number N in terms of ξ

In this Section, N is a CA number $\geqslant N^{(0)} = N_{\varepsilon^{(0)}}$ defined by (4.37). The numbers $\varepsilon_i = \varepsilon_i(N)$, $\xi = \xi(N)$, $\xi_k = \xi_k(N)$ and $K = K(N)$ are defined by Definition 4.3, so that, from Lemma 4.8, $\xi \geqslant \xi^{(0)} = 10^9 + 7$, $\xi_k \geqslant \xi_k^{(0)}$ and $K \geqslant 33$ hold. We give estimates of such an N and of a few functions of N in terms of ξ. Some of these estimates are valid if the Riemann hypothesis fails.

To get shorter formulas, we use the notation $L = \log N$, $\lambda = \log\log N$, $L_0 = \log N^{(0)} > 10^9$, and $\lambda_0 = \log\log N^{(0)} > \log(10^9)$ (cf. (4.38)).

Lemma 4.16. *Let $N \geqslant N^{(0)}$ be a CA number. Then, under the Riemann hypothesis,*

$$\theta(\xi) + \xi_2 + 0.534\xi^{1/3} \leqslant L = \log N \leqslant \theta(\xi) + \xi_2 + 2.043\xi^{1/3}, \tag{4.55}$$

$$L/1.0006 \leqslant \xi \leqslant 1.0006L, \tag{4.56}$$

$$\lambda/1.00003 \leqslant \log\xi \leqslant 1.00003\lambda, \tag{4.57}$$

$$|L - \xi| \leqslant 0.0433\sqrt{\xi}\log^2\xi \leqslant 0.044\sqrt{L}\lambda^2, \tag{4.58}$$

$$\left|\frac{1}{\sqrt{L}\lambda} - \frac{1}{\sqrt{\xi}\log\xi}\right| \leqslant \frac{0.00052}{L^{2/3}}, \tag{4.59}$$

$$\left|\frac{1}{\sqrt{L}\lambda^2} - \frac{1}{\sqrt{\xi}\log^2\xi}\right| \leqslant \frac{0.000027}{L^{2/3}}, \tag{4.60}$$

$$\left|\frac{1}{L\lambda} - \frac{1}{\xi\log\xi}\right| \leqslant \frac{0.00099}{L^{7/6}} \leqslant \frac{0.001}{\xi^{7/6}} \tag{4.61}$$

and

$$\left|\frac{S_1(L)}{\log L} - \frac{S_1(\xi)}{\log\xi}\right| \leqslant \frac{0.021}{L^{2/3}}. \tag{4.62}$$

Proof. To prove the upper bound of (4.55), from (4.21), (2.3) and (4.40), we may write

$$L-\theta(\xi)-\theta(\xi_2) \leqslant \sum_{k=3}^{K} \theta(\xi_k) \leqslant (1+\eta) \sum_{k=3}^{K} \xi_k \leqslant (1+\eta)1.9769\xi^{1/3} \leqslant 1.977\xi^{1/3}. \quad (4.63)$$

Next, if $\xi^{(0)} \leqslant \xi \leqslant 5\times 10^{37}$, then, from (4.26), $\xi_2 \leqslant \sqrt{2\xi} \leqslant 10^{19}$ and, from (2.1), $\theta(\xi_2) < \xi_2$, while, if $\xi > 5\times 10^{37}$, then, from (2.7),

$$\begin{aligned} |\theta(\xi_2)-\xi_2| &\leqslant \frac{\sqrt{\xi_2}\log^2 \xi_2}{8\pi} \leqslant \frac{(2\xi)^{1/4}\log^2(2\xi)}{32\pi} \\ &= \frac{2^{1/4}\xi^{1/3}\log^2(2\xi)}{32\pi\xi^{1/12}} \leqslant \frac{2^{1/4}\log^2(10^{38})\xi^{1/3}}{32\pi(5\times 10^{37})^{1/12}} \leqslant 0.066\xi^{1/3}. \quad (4.64) \end{aligned}$$

So, for $\xi \geqslant \xi^{(0)}$, we have $\theta(\xi_2) \leqslant \xi_2 + 0.066\xi^{1/3}$ which, together with (4.63), proves the upper bound of (4.55).

From (4.21), we deduce $\log N \geqslant \sum_{k=1}^{4} \theta(\xi_k) - \log \xi$. But, from (4.36), ξ_4 exceeds 247, so that from (2.4) and (4.26), $\theta(\xi_4) \geqslant 0.89\xi_4 \geqslant 0.89\xi^{1/4}$, which is $> \log \xi$ for $\xi > 10^9$. Therefore, since from (4.36), ξ_3 exceeds 1418, from (2.4) and (4.26),

$$L \geqslant \sum_{k=1}^{3} \theta(\xi_k) \geqslant \theta(\xi_1)+\theta(\xi_2)+0.945\xi_3 \geqslant \theta(\xi_1)+\theta(\xi_2)+0.945\xi^{1/3}. \quad (4.65)$$

- If $\xi^{(0)} \leqslant \xi \leqslant 5\times 10^{37}$, then, from (4.26), $1427 < \sqrt{\xi^{(0)}} \leqslant \sqrt{\xi} \leqslant \xi_2 \leqslant \sqrt{2\xi} \leqslant 10^{19}$ and, from (2.2),

$$\begin{aligned} \theta(\xi_2)-\xi_2 \geqslant -1.94\sqrt{\xi_2} \geqslant -1.94(2\xi)^{1/4} \geqslant -2.31\xi^{1/4} = -2.31\xi^{1/3}/\xi^{1/12} \\ \geqslant -2.31\xi^{1/3}/(10^9)^{1/12} \geqslant -0.411\xi^{1/3}. \end{aligned}$$

- If $\xi > 5\times 10^{37}$, then, from (4.64), $\theta(\xi_2)-\xi_2 \geqslant -0.066\xi^{1/3}$.

In both cases, $\theta(\xi_2) \geqslant \xi_2 - 0.411\xi^{1/3}$, which, from (4.65), implies the lower bound of (4.55).

Furthermore, from (4.55), (2.8) and (4.26),

$$\begin{aligned} L &\leqslant \theta(\xi)+\xi_2+2.043\xi^{1/3} \leqslant \xi + \frac{\sqrt{\xi}\log^2\xi}{8\pi} + \sqrt{2\xi} + 2.043\xi^{1/3} \\ &= \xi + \sqrt{\xi}\log^2\xi\Big(\frac{1}{8\pi} + \frac{\sqrt{2}}{\log^2\xi} + \frac{2.043}{\xi^{1/6}\log^2\xi}\Big) \\ &\leqslant \xi + \sqrt{\xi}\log^2\xi\Big(\frac{1}{8\pi} + \frac{\sqrt{2}}{\log^2 10^9} + \frac{2.043}{(10^9)^{1/6}\log^2 10^9}\Big) \leqslant \xi + 0.0433\sqrt{\xi}\log^2\xi \end{aligned}$$

and

$$L \geqslant \theta(\xi) + \xi_2 + 0.534\xi^{1/3} \geqslant \xi - \frac{\sqrt{\xi}\log^2 \xi}{8\pi} \geqslant \xi - 0.04\sqrt{\xi}\log^2 \xi,$$

whence

$$|L - \xi| \leqslant 0.0433\sqrt{\xi}\log^2 \xi \leqslant \xi\Big(\frac{0.0433\log^2 10^9}{\sqrt{10^9}}\Big) \leqslant 0.00059\xi, \tag{4.66}$$

which implies

$$\frac{\xi}{1.0006} \leqslant (1 - 0.00059)\xi \leqslant L \leqslant 1.0006\xi$$

and proves (4.56). Next,

$$\log \xi \leqslant \lambda + 0.0006 = \lambda\Big(1 + \frac{0.0006}{\lambda}\Big) \leqslant \lambda\Big(1 + \frac{0.0006}{\log 10^9}\Big) \leqslant 1.00003\lambda$$

and

$$\log \xi \geqslant \lambda\Big(1 - \frac{\log(1.0006)}{\lambda}\Big) \geqslant \lambda\Big(1 - \frac{\log(1.0006)}{\log 10^9}\Big) \geqslant \frac{\lambda}{1.00003}$$

prove (4.57). Then, from (4.66), (4.56) and (4.57), one may write

$$|L - \xi] \leqslant 0.0433\sqrt{\xi}\log^2 \xi \leqslant 0.0434\sqrt{L}\lambda^2,$$

which ends the proof of (4.58).

To prove (4.59), it is convenient to introduce $\rho = \min(\xi, L) \geqslant \min(\xi^{(0)}, L_0)$. From (4.56), (4.35) and (4.38), it follows that

$$L \geqslant \rho = \min(\xi, L) \geqslant \frac{L}{1.0006} \quad \text{and also} \quad L \geqslant \rho \geqslant 10^9. \tag{4.67}$$

One may write

$$\left|\frac{1}{\sqrt{L}\lambda} - \frac{1}{\sqrt{\xi}\log \xi}\right| = \left|\int_\xi^L \frac{\log t + 2}{2t^{3/2}\log^2 t}dt\right| \leqslant \frac{|L - \xi|}{2\rho^{3/2}\log \rho}\Big(1 + \frac{2}{\log \rho}\Big)$$

and, from (4.58) and (4.67), this is

$$\leqslant \frac{0.044(1.0006)^{3/2}\sqrt{L}\lambda^2}{2L^{3/2}\log 10^9}\Big(1 + \frac{2}{\log 10^9}\Big) \leqslant \frac{0.0012}{L^{2/3}}\frac{\lambda^2}{L^{1/3}} \leqslant \frac{0.0005153\ldots}{L^{2/3}},$$

which proves (4.59).

The proofs of (4.60) and (4.61) are similar :

$$\left|\frac{1}{\sqrt{L}\lambda^2}-\frac{1}{\sqrt{\xi}\log^2\xi}\right| = \left|\int_\xi^L \frac{\log t+4}{2t^{3/2}\log^3 t}dt\right|$$
$$\leqslant \frac{0.044(1.0006)^{3/2}\sqrt{L}\lambda^2}{2L^{3/2}\log^2 10^9}\Big(1+\frac{4}{\log 10^9}\Big) \leqslant \frac{0.000062\log^2 10^9}{(10^9)^{1/3}L^{2/3}} = \frac{0.0000266\ldots}{L^{2/3}}$$

and

$$\left|\frac{1}{L\lambda}-\frac{1}{\xi\log\xi}\right| = \left|\int_\xi^L \frac{\log t+1}{t^2\log^2 t}dt\right| \leqslant \frac{|L-\xi|}{\rho^2\log\rho}\Big(1+\frac{1}{\log\rho}\Big)$$
$$\leqslant \frac{0.044(1.0006)^2\sqrt{L}\lambda^2}{L^2\log 10^9}\Big(1+\frac{1}{\log 10^9}\Big) \leqslant 0.0023\frac{\log^2 L}{L^{3/2}}$$
$$\leqslant \frac{0.0023\log^2 10^9}{(10^9)^{1/3}L^{7/6}} \leqslant \frac{0.00099}{L^{7/6}} \leqslant \frac{0.0099\times 1.0006^{7/6}}{\xi^{7/6}} \leqslant \frac{0.001}{\xi^{7/6}}.$$

Finally, from (3.4), (4.58) and (4.67),

$$\left|\frac{S_1(L)}{\log L}-\frac{S_1(\xi)}{\log\xi}\right| \leqslant \frac{0.0521|L-\xi|\log\rho}{\rho^{3/2}} \leqslant \frac{0.0521\times 0.044\sqrt{L}\lambda^2\log\rho}{\rho^{7/6+1/3}}$$

and, observing from (4.67) that

$$\frac{\lambda^2\log\rho}{\rho^{1/3}} \leqslant \frac{(1.0006)^{1/3}\lambda^3}{L^{1/3}} \leqslant \frac{(1.0006)^{1/3}\log^3 10^9}{(10^9)^{1/3}} \leqslant 8.91,$$

we get

$$\left|\frac{S_1(L)}{\log L}-\frac{S_1(\xi)}{\log\xi}\right| \leqslant \frac{0.0521\times 0.044\times 1.0006^{7/6}\times 8.91\sqrt{L}}{L^{7/6}} = \frac{0.0204\ldots}{L^{2/3}}$$

which completes the proof of (4.62) and of Lemma 4.16. □

Lemma 4.17. *Let $N \geqslant N^{(0)}$ be a CA number. The numbers $\xi = \xi(N)$ and $\xi_2 = \xi_2(N)$ are defined by Definition 4.3. Then*

$$\log\log\theta(\xi) \leqslant \log\log\log N - \frac{\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{0.491}{\sqrt{\xi}\log^2\xi} + \frac{0.0015}{\xi^{2/3}} \qquad (4.68)$$

and

$$\log\log\theta(\xi) \geqslant \log\log\log N - \frac{\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{0.456}{\sqrt{\xi}\log^2\xi} - \frac{0.11}{\xi^{2/3}}. \tag{4.69}$$

Proof. From (4.55), we have $L = \log N \geqslant \theta(\xi)+\xi_2$, which implies $\theta(\xi) \leqslant L-\xi_2 = L(1-\xi_2/L)$,

$$\log\theta(\xi) \leqslant \lambda + \log(1-\xi_2/L) \leqslant \lambda - \xi_2/L = \lambda(1-\xi_2/(L\lambda))$$

and, from (4.61),

$$\log\log\theta(\xi) \leqslant \log\lambda - \frac{\xi_2}{L\lambda} \leqslant \log\lambda - \frac{\xi_2}{\xi\log\xi} + \frac{0.001\xi_2}{\xi^{7/6}}.$$

But, from (4.26), $\xi_2/\xi^{7/6} \leqslant \sqrt{2}/\xi^{2/3}$, and applying (4.28) ends the proof of the upper bound (4.68).

The lower bound (4.69) is less simple, because the lower bound of (1.16) is more complicated. From (4.55), we have $\theta(\xi) \geqslant L(1-\xi_2/L-2.043\xi^{1/3}/L)$, which, from (1.16), yields

$$\log\theta(\xi) \geqslant \lambda + \log(1-u) \geqslant \lambda - u - \frac{u^2}{2(1-u_0)}$$

with, from (4.26), (4.35) and (4.56) ,

$$u = \frac{\xi_2}{L} + \frac{2.043\xi^{1/3}}{L} \leqslant \frac{\sqrt{2\xi}}{L} + \frac{2.043\xi^{1/3}}{L} = \frac{\sqrt{\xi}}{L}\Big(\sqrt{2} + \frac{2.043}{\xi^{1/6}}\Big)$$
$$\leqslant \frac{\sqrt{\xi}}{L}\Big(\sqrt{2}+\frac{2.043}{10^{9/6}}\Big) \leqslant 1.48\frac{\sqrt{\xi}}{L} \leqslant \frac{1.48\times\sqrt{1.0006}}{\sqrt{L}} \leqslant \frac{1.49}{\sqrt{L}} \leqslant \frac{1.49}{\sqrt{10^9}} \leqslant 0.00005,$$

$u_0 = 0.00005$ and $2(1-u_0) = 1.9999$. Therefore, we may write

$$\log\theta(\xi) \geqslant \lambda - \frac{\xi_2}{L} - \frac{2.043\xi^{1/3}}{L} - \frac{(1.49)^2}{1.9999L} \geqslant \lambda - \frac{\xi_2}{L} - \frac{2.043\xi^{1/3}}{L} - \frac{1.12}{L}.$$

But

$$\frac{2.043\xi^{1/3}}{L} + \frac{1.12}{L} = \frac{\xi^{1/3}}{L}\Big(2.043 + \frac{1.12}{\xi^{1/3}}\Big) \leqslant \frac{\xi^{1/3}}{L}\Big(2.043 + \frac{1.12}{1000}\Big) \leqslant 2.05\frac{\xi^{1/3}}{L}$$

and so,

$$\log\theta(\xi) \geqslant \lambda\Big(1 - \frac{\xi_2}{\lambda L} - \frac{2.05\xi^{1/3}}{\lambda L}\Big) = \lambda(1-\widetilde{u})$$

with $\widetilde{u} = \xi_2/(\lambda L) + 2.05\xi^{1/3}/(\lambda L)$. Note that $\widetilde{u}$ is close to u/λ. A similar computation shows that (cf. [43])

$$\widetilde{u} \leqslant 1.49/(\lambda\sqrt{L}) \leqslant 1.49/(\log(10^9)\sqrt{10^9}) \leqslant \widetilde{u}_0 = 0.000003,$$

$$2(1-\widetilde{u}_0) = 1.999994$$

and

$$\log\log\theta(\xi) \geqslant \log\lambda - \widetilde{u} - \frac{\widetilde{u}^2}{2(1-\widetilde{u}_0)} \geqslant \log\lambda - \frac{\xi_2}{\lambda L} - \frac{2.05\xi^{1/3}}{\lambda L} - \frac{(1.49)^2}{1.999994\lambda^2 L}.$$

But

$$2.05\xi^{1/3} + \frac{(1.49)^2}{1.999994\lambda} \leqslant \xi^{1/3}\Big(2.05 + \frac{(1.49)^2}{1.999994\log(10^9)10^{9/3}}\Big) \leqslant 2.06\xi^{1/3}$$

and, from (4.56), (4.61), (4.26) and (4.29),

$$\begin{aligned}\log\log\theta(\xi) &\geqslant \log\lambda - \frac{\xi_2}{\lambda L} - \frac{2.06\xi^{1/3}}{\lambda L} \geqslant \log\lambda - \frac{\xi_2}{\lambda L} - \frac{2.06\times 1.0006}{\log(10^9)\xi^{2/3}}\\ &\geqslant \log\lambda - \frac{\xi_2}{\xi\log\xi} - 0.001\frac{\xi_2}{\xi^{7/6}} - \frac{0.1}{\xi^{2/3}} \geqslant \log\lambda - \frac{\xi_2}{\xi\log\xi} - \frac{0.001\sqrt{2}}{\xi^{2/3}} - \frac{0.1}{\xi^{2/3}}\\ &\geqslant \log\lambda - \frac{\xi_2}{\xi\log\xi} - \frac{0.11}{\xi^{2/3}} \geqslant \log\lambda - \frac{\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{\sqrt{2}\times 0.323}{\sqrt{\xi}\log\xi} - \frac{0.11}{\xi^{2/3}},\end{aligned}$$

which completes the proof of (4.69) and of Lemma 4.17. □

5 Proof of Theorem 1.1

5.1 The case n large

First, we prove (1.6) for a large CA number.

Proposition 5.1. *Let N be a CA number $N \geqslant N^{(0)}$ defined by* (4.37)*, and S_1 be defined by* (1.3)*. Then, under the Riemann hypothesis,*

$$\frac{\sigma(N)}{N} \leqslant e^{\gamma}\Big(\log\log N - \frac{2(\sqrt{2}-1)}{\sqrt{\log N}} + S_1(\log N) + \frac{3.789}{\sqrt{\log N}\log\log N} + \frac{0.024\log\log N}{\log^{2/3} N}\Big) \tag{5.1}$$

and,

$$\frac{\sigma(N)}{N} \geqslant e^{\gamma}\Big(\log\log N - \frac{2(\sqrt{2}-1)}{\sqrt{\log N}} + S_1(\log N) + \frac{0.9567}{\sqrt{\log N}\log\log N} - \frac{0.48\log\log N}{\log^{2/3} N}\Big). \tag{5.2}$$

Proof. The numbers $\xi = \xi(N)$, $\xi_k = \xi_k(N)$ and $K = K(N)$ are defined by Definition 4.3, and, from Lemma 4.8 and (4.36), $\xi \geqslant \xi^{(0)} = 10^9 + 7$, $\xi_2 \geqslant \xi_2^{(0)} > 44023$ and $\xi_3 \geqslant \xi_3^{(0)} > 1418$ hold. From (4.22), it follows that

$$U - \frac{U}{\xi+1} = \frac{\xi}{\xi+1}U \leqslant \log\Big(\frac{\sigma(N)}{N}\Big) \leqslant U \tag{5.3}$$

with

$$U = \log\Big(\prod_{k=1}^{K}\prod_{\xi_{k+1}<p\leqslant\xi_k}\frac{1-1/p^{k+1}}{1-1/p}\Big) = U_1 + U_2 + U_3 - U_4 - U_5 \tag{5.4}$$

with

$$U_1 = \sum_{k=3}^{K}\sum_{\xi_{k+1}<p\leqslant\xi_k}\log\Big(1-\frac{1}{p^{k+1}}\Big),\ U_2 = \sum_{\xi_3<p\leqslant\xi_2}\log\Big(1-\frac{1}{p^3}\Big),$$

$$U_3 = \sum_{p>\xi_2}\log\Big(1-\frac{1}{p^2}\Big),\ U_4 = \sum_{p>\xi}\log\Big(1-\frac{1}{p^2}\Big) \text{ and } U_5 = \sum_{p\leqslant\xi}\log\Big(1-\frac{1}{p}\Big).$$

Observing that, from (4.26), $p > \xi_{k+1}$ implies $p^{k+1} > \xi_{k+1}^{k+1} \geqslant \xi$, we have, from (1.15) with $u = 1/\xi$ and $u_0 = 1/\xi^{(0)}$), (2.5), (4.26) and (4.36),

$$\begin{aligned} 0 \geqslant U_1 &\geqslant \pi(\xi_3)\log\Big(1-\frac{1}{\xi}\Big) \geqslant -\frac{1.26\xi_3}{\xi(1-1/\xi^{(0)})\log\xi_3} \\ &\geqslant -\frac{1.26(3\xi)^{1/3}}{(1-1/\xi^{(0)})(\log\xi_3^{(0)})\xi} \geqslant -\frac{1.26(3\xi)^{1/3}}{(1-1/\xi^{(0)})(\log 1418)\xi} \geqslant -\frac{0.251}{\xi^{2/3}}. \end{aligned} \tag{5.5}$$

Next, from (1.15) with $u = 1/p^3$ and $u_0 = 1/(\xi_3^{(0)})^3$, (2.15) and (4.26),

$$0 \geqslant U_2 = \sum_{\xi_3 < p \leqslant \xi_2} \log\left(1 - \frac{1}{p^3}\right) \geqslant -\frac{1}{1 - 1/(\xi_3^{(0)})^3} \sum_{p \geqslant \xi_3} \frac{1}{p^3}$$
$$\geqslant -\frac{0.086}{\left(1 - 1/1418^3\right)\xi_3^2} \geqslant -\frac{0.087}{\xi^{2/3}}. \quad (5.6)$$

As the trinomial $2/\log^2 y - 2/\log y + 1$ is positive, from (2.11), it follows that $0 > U_4 \geqslant \sum_{p \geqslant \xi} \log(1 - 1/p^2) \geqslant -1/(\xi \log \xi)$ and that, from (4.26) and $\log \xi > 1$,

$$0 < -U_4 < \frac{1}{\xi} \leqslant \frac{2}{\xi_2^2} = \frac{2\log^4 \xi_2}{\xi_2(\xi_2 \log^4 \xi_2)} \leqslant \left(\frac{2\log^4 \xi_2^{(0)}}{\xi_2^{(0)}}\right)\frac{1}{\xi_2 \log^4 \xi_2} < \frac{0.6}{\xi_2 \log^4 \xi_2}.$$

In the same way, from (1.15),

$$-\log\left(1 - \frac{1}{\xi_2^2}\right) \leqslant \frac{1}{\xi_2^2\left(1 - 1/(\xi_2^{(0)})^2\right)} \leqslant \frac{2}{\xi_2^2} < \frac{0.6}{\xi_2 \log^4 \xi_2}$$

whence, from (2.10),

$$U_3 - U_4 \leqslant \sum_{p \geqslant \xi_2} \log\left(1 - \frac{1}{p^2}\right) - \log\left(1 - \frac{1}{\xi_2^2}\right) - U_4$$
$$\leqslant -\frac{1}{\xi_2 \log \xi_2} + \frac{1}{\xi_2 \log^2 \xi_2} - \frac{2}{\xi_2 \log^3 \xi_2} + \frac{11.46}{\xi_2 \log^4 \xi_2}.$$

As $\log \xi_2 > \log \xi_2^{(0)} > 11.46/2$ holds, from (4.42) and (4.43), it follows that

$$U_3 - U_4 \leqslant -\frac{1}{\xi_2 \log \xi_2} + \frac{1}{\xi_2 \log^2 \xi_2} \leqslant -\frac{\sqrt{2}}{\sqrt{\xi} \log \xi} + \frac{3.353}{\sqrt{\xi} \log^2 \xi}. \quad (5.7)$$

From (2.11), we get

$$U_3 - U_4 > U_3 \geqslant \sum_{p \geqslant \xi_2} \log\left(1 - \frac{1}{p^2}\right) \geqslant -\frac{1}{\xi_2 \log \xi_2} + \frac{1}{\xi_2 \log^2 \xi_2} - \frac{2}{\xi_2 \log^3 \xi_2}.$$

But, $2/\log \xi_2 \leqslant 2/\log \xi_2^{(0)} < 0.19$ and therefore, from (4.42) and (4.43),

$$U_3 - U_4 \geqslant -\frac{1}{\xi_2 \log \xi_2} + \frac{0.81}{\xi_2 \log^2 \xi_2} \geqslant -\frac{\sqrt{2}}{\sqrt{\xi} \log \xi} + \frac{2.5627}{\sqrt{\xi} \log^2 \xi}. \quad (5.8)$$

From (1.11) and (4.69), $U_5 = \sum_{p \leqslant \xi} \log(1 - 1/p)$ satisfies

$$U_5 \leqslant -\gamma - \log\lambda - \frac{2-\sqrt{2}}{\sqrt{\xi}\log\xi} + \frac{1.606}{\sqrt{\xi}\log^2\xi} - \frac{S_1(\xi)}{\log\xi} + \frac{0.11}{\xi^{2/3}} \tag{5.9}$$

while , from (1.11) and (4.68),

$$U_5 \geqslant -\gamma - \log\lambda - \frac{2-\sqrt{2}}{\sqrt{\xi}\log\xi} - \frac{0.436}{\sqrt{\xi}\log^2\xi} - \frac{S_1(\xi)}{\log\xi} - \frac{0.0015}{\xi^{2/3}}. \tag{5.10}$$

Therefore, from (5.4), (5.5), (5.6), (5.7) and (5.10), we deduce

$$U \leqslant \gamma + \log\lambda - \frac{2\sqrt{2}-2}{\sqrt{\xi}\log\xi} + \frac{3.789}{\sqrt{\xi}\log^2\xi} + \frac{S_1(\xi)}{\log\xi} + \frac{0.0015}{\xi^{2/3}} \tag{5.11}$$

while, from (5.4), (5.5), (5.6), (5.8) and (5.9), we get

$$U \geqslant \gamma + \log\lambda - \frac{2\sqrt{2}-2}{\sqrt{\xi}\log\xi} + \frac{0.9567}{\sqrt{\xi}\log^2\xi} + \frac{S_1(\xi)}{\log\xi} - \frac{0.448}{\xi^{2/3}}. \tag{5.12}$$

In (5.11), from (4.56), we have

$$0.0015/\xi^{2/3} \leqslant 0.0015 \times (1.0006)^{2/3}/L^{2/3} \leqslant 0.00151/L^{2/3} \tag{5.13}$$

and, similarly, in (5.12),

$$-0.448/\xi^{2/3} \geqslant -0.448 \times (1.0006)^{2/3}/L^{2/3} \geqslant -0.449/L^{2/3}. \tag{5.14}$$

Also, from (4.59), (4.60) and (4.62),

$$(2\sqrt{2}-2) \times 0.00052 + 3.789 \times 0.000027 + 0.021 \leqslant 0.0216$$

so that

$$-\frac{2\sqrt{2}-2}{\sqrt{\xi}\log\xi} + \frac{3.789}{\sqrt{\xi}\log^2\xi} + \frac{S_1(\xi)}{\log\xi} \leqslant -\frac{2\sqrt{2}-2}{\sqrt{L}\log L} + \frac{3.789}{\sqrt{L}\log^2 L} + \frac{S_1(L)}{\log L} + \frac{0.0216}{L^{2/3}} \tag{5.15}$$

and similarly

$$-\frac{2\sqrt{2}-2}{\sqrt{\xi}\log\xi}+\frac{0.9567}{\sqrt{\xi}\log^2\xi}+\frac{S_1(\xi)}{\log\xi}\geqslant-\frac{2\sqrt{2}-2}{\sqrt{L}\log L}+\frac{0.9567}{\sqrt{L}\log^2 L}+\frac{S_1(L)}{\log L}-\frac{0.0215}{L^{2/3}}. \tag{5.16}$$

Consequently, (5.3), (5.11), (5.13) and (5.15) yield

$$\log\left(\frac{\sigma(N)}{N}\right)\leqslant U\leqslant\gamma+\log\lambda-u \tag{5.17}$$

with

$$u=\frac{2\sqrt{2}-2}{\sqrt{L}\lambda}-\frac{3.789}{\sqrt{L}\lambda^2}-\frac{S_1(L)}{\lambda}-\frac{0.02311}{L^{2/3}} \tag{5.18}$$

and, from (5.12), (5.14) and (5.16),

$$U\geqslant\gamma+\log\lambda-\frac{2\sqrt{2}-2}{\sqrt{L}\lambda}+\frac{0.9567}{\sqrt{L}\lambda^2}+\frac{S_1(L)}{\lambda}-\frac{0.4705}{L^{2/3}}. \tag{5.19}$$

From (5.18), (1.4) and (1.5), one remarks that

$$\begin{aligned}0<\frac{0.44}{\sqrt{L}\lambda}<\frac{2\sqrt{2}-2-3.789/\lambda_0-\tau-0.2311\lambda_0/L_0^{1/6}}{\sqrt{L}\lambda}&\leqslant u\\ &\leqslant\frac{2\sqrt{2}-2+\tau}{\sqrt{L}\lambda}<\frac{0.875}{\sqrt{L}\lambda}\end{aligned} \tag{5.20}$$

and thus, (5.17) implies

$$U<\gamma+\log\lambda. \tag{5.21}$$

The mapping $t\mapsto(\log\log t)/t^{1/3}$ is decreasing for $t>18$, so we may write

$$\frac{U}{\xi+1}<\frac{\gamma+\log\lambda}{\xi}\leqslant\frac{1.0006}{L^{2/3}}\left(\frac{\gamma}{10^{9/3}}+\frac{\log\log 10^9}{10^{9/3}}\right)\leqslant\frac{0.0037}{L^{2/3}}.$$

Therefore, it follows from (5.3) that

$$0\geqslant\log\left(\frac{\sigma(N)}{N}\right)-U\geqslant-\frac{U}{\xi+1}\geqslant-\frac{0.0037}{L^{2/3}},$$

which, from (5.19), yields

$$\log\left(\frac{\sigma(N)}{N}\right) \geqslant \gamma + \log\lambda - v \tag{5.22}$$

with

$$v = \frac{2\sqrt{2}-2}{\sqrt{L}\lambda} - \frac{0.9567}{\sqrt{L}\lambda^2} - \frac{S_1(L)}{\lambda} + \frac{0.4742}{L^{2/3}}.$$

Furthermore, from (5.20),

$$\frac{u^2}{2} \leqslant \frac{0.875^2}{2L\lambda^2} \leqslant \frac{0.875^2}{2 \times 10^{9/3}(\log^2 10^9)L^{2/3}} \leqslant \frac{10^{-6}}{L^{2/3}}$$

which, with (1.18) and (5.17), yields (5.1). Finally, (1.18) and (5.22) prove (5.2). □

Corollary 5.2. *Let n be an integer satisfying $n > N^{(0)}$ defined by* (4.37). *Then*

$$\frac{\sigma(n)}{n} \leqslant e^\gamma\Bigg(\log\log n - \frac{2(\sqrt{2}-1)}{\sqrt{\log n}} + S_1(\log n) + \frac{3.789}{\sqrt{\log n}\log\log n} + \frac{0.026\log\log n}{\log^{2/3} n}\Bigg). \tag{5.23}$$

Proof. By applying Proposition 4.6, we define the two consecutive CA numbers N' and N satisfying $N' < n \leqslant N$, so that (4.24) gives

$$\frac{\sigma(n)}{n} \leqslant \frac{\sigma(N')}{N'} + \frac{\sigma(N)}{\xi N} \tag{5.24}$$

where $\xi = \xi(N)$ is defined by Definition 4.3. As $\log N > \log N^{(0)} > 20$ holds, (1.1) or (5.1) implies $\sigma(N)/N < e^\gamma \log\log N$ and, from (4.56),

$$\frac{\sigma(N)}{\xi N} \leqslant \frac{1.0006e^\gamma\lambda}{L} \leqslant \frac{1.0006e^\gamma\lambda}{10^{9/3}L^{2/3}} \leqslant \frac{0.002\log\log N}{\log^{2/3} N} \leqslant \frac{0.002\log\log n}{\log^{2/3} n}. \tag{5.25}$$

Applying (5.1) to N' with the notation (3.11) with $a = 2(\sqrt{2}-1)$, $b = 3.789$ and $c = 0.024$ yields $\sigma(N')/N' \leqslant H(a,b,c,\log N')$ and, from Lemma 3.6 proving that H in increasing, as $N' < n$, $\sigma(N')/N' \leqslant H(a,b,c,\log n)$. Therefore, from (5.24) and (5.25), it follows that

$$\frac{\sigma(n)}{n} \leqslant H(a,b,c,\log n) + \frac{0.002\log\log n}{\log^{2/3} n} = H(a,b,c+0.002,\log n),$$

which proves (5.23). □

5.2 The case n small

Lemma 5.3. *Let n be an integer satisfying $2 \leqslant n \leqslant N^{(0)}$ defined by* (4.37)*. Then*

$$\frac{\sigma(n)}{n} \leqslant e^{\gamma}\Big(\log\log n - \frac{2(\sqrt{2}-1)}{\sqrt{\log n}} + S_1(\log n) + \frac{3.789}{\sqrt{\log n}\log\log n} - \frac{0.0983\log\log n}{\log^{2/3} n}\Big). \tag{5.26}$$

Proof. We use the notation of Lemma 3.6 so that, from (1.4), it suffices to show that, for $2 \leqslant n \leqslant N^{(0)}$,

$$\sigma(n)/n \leqslant e^{\gamma} G\big(2(\sqrt{2}-1)+\tau, 3.789, -0.0983, \log n\big). \tag{5.27}$$

• If $55440 \leqslant n \leqslant N^{(0)}$, as $\log 55440 > 10$, Lemma 3.6 implies that the mapping $t \mapsto G\big(2(\sqrt{2}-1)+\tau, 3.789, -0.0983, t\big)$ is increasing and concave for $t \geqslant \log 55440$. Therefore, as 55440 is CA, from Lemma 4.15, to prove (5.27), it suffices to prove it for all CA numbers N satisfying $55440 \leqslant N \leqslant N^{(0)}$. For $n \geqslant 2$, it is convenient to define $\widetilde{c}(n)$ by

$$e^{\gamma} G\big(2(\sqrt{2}-1)+\tau, 3.789, \widetilde{c}(n), \log n\big) = \sigma(n)/n.$$

For all CA numbers N satisfying $55440 \leqslant N \leqslant N^{(0)}$, we have computed $\widetilde{c}(N)$ (cf. [43]). The largest value, $-0.098377958\ldots$, is obtained for

$$N = 2^{32}3^{20}5^{13}7^{11}11^{9}13^{8}17^{7}19^{7}\prod_{p=23}^{37} p^6 \prod_{p=41}^{73} p^5 \prod_{p=79}^{199} p^4 \prod_{p=211}^{1123} p^3 \prod_{p=1129}^{31249} p^2 \prod_{p=31253}^{504457601} p$$

and $\log N = 5.04466\ldots 10^8$.

• If $n \leqslant 55440$, then one computes $\widetilde{c}(n)$ for all n's satisfying $2 \leqslant n \leqslant 55440$ (cf. [43]). The largest value of $\widetilde{c}(n)$ is obtained for $n = 55440$ and $\widetilde{c}(55440) = -0.52488\ldots < -0.0983$, which completes the proof of Lemma 5.3. □

5.3 Proof of Theorem 1.1 (i)

Part (i) follows from Corollary 5.2 and Lemma 5.3.

5.4 Proof of Theorem 1.1 (ii)

G. Robin has proved (cf. [37, Section 4]) that, if the Riemann hypothesis fails, then there exists $b < 1/2$ such that the CA number N satisfies

$$\sigma(N)/(N \log\log N) = e^{\gamma}\left(1 + \Omega_{\pm}(\log^{-b} N)\right). \tag{5.28}$$

Consequently, (5.28) implies the existence of infinitely many n's that do not satisfy (1.6), which proves (ii).

5.5 Proof of Theorem 1.1 (iii)

If the Riemann hypothesis is true, then (iii) follows from Proposition 5.1, (5.2). If it is false, then (iii) follows from (5.28). □

6 Proof of Corollary 1.2

6.1 Preliminary lemmas

Lemma 6.1. *Let $N' \geqslant 55440$ and N be two consecutive CA numbers with $\alpha(N')$ (defined in (1.9)) and $\alpha(N)$ positive. Let n such that $N' \leqslant n \leqslant N$. Then,*

$$\alpha(n) \geqslant \min(\alpha(N'), \alpha(N)). \tag{6.1}$$

Proof. First, note that the positivity of $\alpha(N')$ and $\alpha(N)$ follows from (1.1), but in each computation, it will be checked. Let us set $a = \min(\alpha(N'), \alpha(N)) > 0$ and $f(t) = e^{\gamma}(\log t - a/\sqrt{t})$. This function is increasing and concave for $t > 1$. Moreover, $\sigma(N')/N' = e^{\gamma}\left(\log\log N' - \alpha(N')/\sqrt{\log N'}\right) \leqslant f(\log N')$ and, similarly, $\sigma(N)/N \leqslant f(\log N)$. So, we may apply Lemma 4.15 that yields $e^{\gamma}(\log\log n - \alpha(n)/\sqrt{\log n}) = \sigma(n)/n \leqslant f(\log n)$, and $\alpha(n) \geqslant a$ follows. □

Lemma 6.2. *Let a be a real number belonging to $[0, 1]$, N a CA number of parameter ε, M' and M two integers such that $8 \leqslant M' \leqslant M$, and an integer n satisfying $M' \leqslant n \leqslant M$ and $\alpha(n) \leqslant a$. Then $\mathrm{ben}_{\varepsilon}(n)$ defined in (4.44) satisfies*

$$\mathrm{ben}_{\varepsilon}(n) \leqslant \max\left(g(\log M'), g(\log M)\right) - \gamma + \log(\sigma(N)/N) - \varepsilon \log N \tag{6.2}$$

with

$$g(t) = \varepsilon t - \log\left(\log t - a/\sqrt{t}\right).$$

Proof. First, for $t \geqslant \log 8$ and $0 \leqslant a \leqslant 1$, $\log t - a/\sqrt{t}$ is positive and

$$\frac{d^2 g(t)}{dt^2} = \frac{1/t^2 + 3a/(4t^{5/2})}{\log t - a/\sqrt{t}} + \frac{(1/t + a/(2t^{3/2}))^2}{\left(\log t - a/\sqrt{t}\right)^2} > 0,$$

thus $g(t)$ is convex. Next, from (4.44),

$$\mathrm{ben}_\varepsilon(n) = \log(\sigma(N)/N) - \log(\sigma(n)/n) + \varepsilon \log n - \varepsilon \log N.$$

But, from (1.9), as $\alpha(n) \leqslant a$ is assumed,

$$\frac{\sigma(n)}{n} = e^\gamma\Big(\log\log n - \frac{\alpha(n)}{\sqrt{\log n}}\Big) \geqslant e^\gamma\Big(\log\log n - \frac{a}{\sqrt{\log n}}\Big)$$

so that $\varepsilon \log n - \log(\sigma(n)/n) \leqslant g(\log n) - \gamma$ and

$$\mathrm{ben}_\varepsilon(n) \leqslant g(\log n) - \gamma + \log(\sigma(N)/N) - \varepsilon \log N.$$

As $g(t)$ is convex, its maximum on the interval $[\log M', \log M]$ is attained at one of its extremities, whence (6.2). □

Lemma 6.3. *If $n \geqslant N_{\varepsilon_{24}}$ the 24th CA number of parameter $\varepsilon_{24} = F(41, 1)$,*

$$N_{\varepsilon_{24}} = 9200527969062830400 = 2^6 3^4 5^2 7^2 11 \dots 41 = 9.2 \dots 10^{18}, \tag{6.3}$$

then $\alpha(n) > 0.582$ holds.

Proof. First, we consider the case $n \geqslant N^{0)}$ (defined by (4.37)) with $L_0 = \log N^{(0)} > 10^9$ and $\lambda_0 = \log\log N^{(0)} > \log 10^9$ (cf. (4.38)). From (1.8) and (1.12),

$$\begin{aligned} \frac{\sigma(n)}{n} &\leqslant e^\gamma\left(\log\log n + \frac{1}{\sqrt{\log n}}\Big(-2(\sqrt{2}-1) + \tau + \frac{3.789}{\lambda_0} + \frac{0.026\lambda_0}{L_0^{1/6}}\Big)\right) \\ &< e^\gamma\Big(\log\log n - \frac{0.582}{\sqrt{\log n}}\Big). \end{aligned} \tag{6.4}$$

Next, we consider the case $N_{\varepsilon_{24}} \leqslant n \leqslant N^{(0)}$. One computes $\alpha(N)$ for all CA numbers $N \leqslant N^{(0)}$. The largest value of $\alpha(N)$ found is $0.92019\dots$ for the CA number whose largest prime factor is 1019 (cf. [43]). For all CA numbers N's satisfying $N_{\varepsilon_{24}} \leqslant N \leqslant N^{(0)}$, $\alpha(N) \geqslant \alpha(N_{\varepsilon_{24}}) = 0.603\dots$ holds, which, from Lemma 6.1, completes the proof of Lemma 6.3. □

6.2 Proof of Corollary 1.2 (i).

If $N' \geqslant 12$ and N are two consecutive CA numbers, then Lemma 6.2, with ε the common parameter of N' and N (cf. Proposition 4.2), $M' = N'$, $M = N$ and $a \in [0,1]$, allows us to compute an upper bound $\beta = \beta(N,a)$ of $\mathrm{ben}_\varepsilon(n)$ when $N' \leqslant n \leqslant N$ and $\alpha(n) \leqslant a$. From Section 4.6, there exists an algorithm to determine the list $\mathcal{L}_{\mathrm{ben}} = \mathcal{L}_{\mathrm{ben}}(N,a)$ of those integers $n \in [N',N]$ such that $\mathrm{ben}_\varepsilon(n) \leqslant \beta$. By pruning $\mathcal{L}_{\mathrm{ben}}$, i.e. by taking off the n's with $\alpha(n) > a$, we determine the list $\mathcal{L}_\alpha = \mathcal{L}_\alpha(N,a)$ of integers n satisfying $N' < n \leqslant N$, $\alpha(n) \leqslant a$ and $n \geqslant 5040$.

With the notation of Table 1, $\nu_{16} = N_{\varepsilon_{23}}$ is the CA number preceding $N_{\varepsilon_{24}}$. For $n \geqslant N_{\varepsilon_{24}}$, (1.10) follows from Lemma 6.3. To complete the proof of (1.10), we have to check that there are no n satisfying $\alpha(n) \leqslant 0.582$ and $\nu_{16} < n < N_{\varepsilon_{24}}$, in other words, we have to check that the set $\mathcal{L}_\alpha(N_{\varepsilon_{24}}, 0.582)$ is empty, which is true (cf. [43]).

6.3 Proof of Corollary 1.2 (ii).

For each CA number N such that $N_{\varepsilon_9} = 5040 \leqslant N \leqslant N_{\varepsilon_{24}}$ (cf. (6.3)), one computes $\mathcal{L}_\alpha(N, 0.582)$. By bringing together these sets, one can get $\widehat{\mathcal{L}}$, the set of 161 numbers n satisfying $5040 \leqslant n \leqslant N_{\varepsilon_{24}}$ and $\alpha(n) \leqslant 0.582$ (cf. [43]). Among them, 15 are CA and 58 are SA (but not CA).

Let us write $\widehat{\mathcal{L}} = \{n_1 = \nu_1 < n_2 < \ldots < n_{161} = \nu_{16}\}$, To get the ν_k's of Table 1, we prune $\widehat{\mathcal{L}}$ in the following way: for k from 16 downwards to 2, ν_{k-1} is the largest $n_i < \nu_k$ such that $\alpha(n_i) < \alpha(\nu_k)$. □

7 Open questions

Find an asymptotic expansion of the upper bound of $\sigma(n)/n$ more precise than (1.2). In particular, find the asymptotic coefficient of $1/(\sqrt{\log n}\log\log n)$ in (1.2). Probably, for that, it would be necessary to improve the estimate (1.11) of $\prod_{p\leqslant x}(1-1/p)$.

In [34] (see also [2, Chapter 8]), under the Riemann hypothesis, Ramanujan considers

$$\sigma_{-s}(n) = \sum_{d\mid n} \frac{1}{d^s}, \quad \text{for} \quad 0 < s \leqslant 1$$

(note that $\sigma_{-1}(n) = \sigma(n)/n$), defines

$$S_s(x) = -s \sum_{\rho} \frac{x^{\rho-1}}{\rho(\rho - s)}$$

and gives an asymptotic expansion of the maximal order $\Sigma_{-s}(n)$ of $\sigma_{-s}(n)$. For example, for $s = 1/2$, (cf. [34, Equation (380)]),

$$\Sigma_{-1/2}(n) = -\frac{\sqrt{2}}{2}\zeta\left(\frac{1}{2}\right)\exp\left(\mathrm{li}(\sqrt{\log n}) + \frac{2\log 2 - 1 + S_{1/2}(\log n)}{\log\log n} + \frac{\mathcal{O}(1)}{(\log\log n)^2} \right), \tag{7.1}$$

where li denotes the logarithmic integral. It would be interesting to get an effective form to the asymptotic expansion of $\Sigma_{-s}(n)$. The general case $0 < s < 1$ might be difficult to deal with. But in the case $s = 1/2$, it is maybe possible to get an effective form to the expansion (7.1).

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