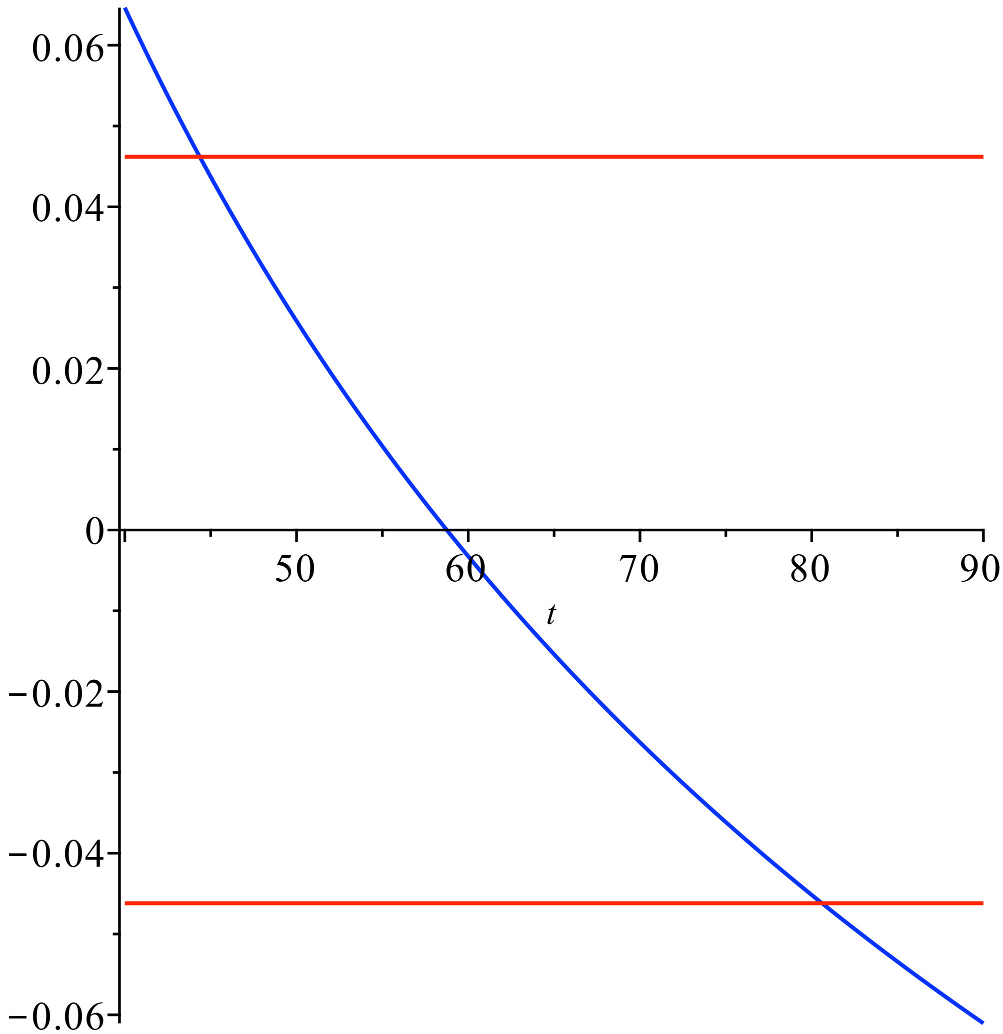
```
> ######################### COMPUTATION OF THE PAPER
  ##############################

  ########  The sum of divisors function and the Riemann hypothesis
  ############

> read "colHR.mpl": # contains the procedures
> read "colHR.m": # contains the registered data:
     # lisCA2, list of 6704 CA numbers of type 2, [eps,q,alfaq,p,xi,
  sigma(N)/N,logN]
     # lisCA,  list of the 47 first CA numbers, [eps,q,alfaq,p,xi,
  sigma(N)/N,N]
     # lisSA, list of SA numbers < 10^50, [M,sigma(M)/M]
     # lisctilda, computes the min and the max of ctilda(N) for all
  CA number N
                 # between the jth and the (j+1)th CA number of type 2

                 # [i,pmin,ctildamin,ctildamax,pmax] cf. Sect. 5.2
     # lisalpha,  computes the min and the max of alpha(N) for all CA
  number N
                 # between the jth and the (j+1)th CA number of type 2

                 # [i,pmin,alphamin,alphamax,pmax] cf. Sect. 6.2
     # the two real numbers ega:=exp(evalf(gamma)), tau:=evalf(2+
  gamma-log(4*Pi)); cf. (1.5)

> Digits:=20: # Computation of n_0, cf. (1.8)
  fsolve(-2*(sqrt(2)-1)+tau+3.789/log(log(n0))+0.026*log(log(n0))/log
  (n0)^(1/6),n0);
```

$$3.9852277562711315731\ 10^{80} \qquad (1)$$

```
> plot([-2*(sqrt(2)-1)+3.789/log(log(10^t))+0.026*log(log(10^t))/log
  (10^t)^(1/6),-tau,tau],t=40..90,color=["blue","red","red"]); # cf.
  (1.8)
```




```
> ######################## Proof of Lemma 2.1 ####################

n:='n': t:='t': int(1/(t^2*log(t)^n)+n/(t^2*log(t)^(n+1)),t);
```

$$-\frac{1}{t\ln(t)^n} \tag{2}$$

```
> Digits:=10: J:='J': A:='A': for n from 1 to 5 do J[n]:=A[n]-n*J
[n+1] od:
-A[1]+A[4]+2*J[1]+J[2]+2*J[4]+J[5], -3+17/log(10.^8);
```

$$A_1 - 3\,A_4 - A_2 + 2\,A_3 + 17\,A_5 - 85\,J_6,\ -2.077124226 \tag{3}$$

```
> J:='J': A:='A': for n from 1 to 4 do J[n]:=A[n]-n*J[n+1] od:
-A[1]-A[4]+2*J[1]+J[2]-2*J[4]-J[5];
```

$$A_1 - 9\,A_4 - A_2 + 2\,A_3 + 31\,J_5 \tag{4}$$

```
>  2*log(10.^8)^4/10^16;
```

$$2.302781322\ 10^{-11} \tag{5}$$

```
> y:='y': a:='a': f:=-1/(y*log(y))+1/(y*log(y)^2)-2/(y*log(y)^3)+a/
(y*log(y)^4);
f1:=normal(diff(f,y)): "df/dy = ",f1;
Y:='Y': hY:=subs(log(y)=Y,numer(f1)): h1Y:=diff(hY,Y):
"h(y) = ",hY, "dh/dy = ",h1Y;
subs(a=13,Y=log(19),h1Y), evalf(subs(a=13,Y=log(19),h1Y));
```

```
subs(a=13,Y=log(19),hY),evalf(subs(a=13,Y=log(19),hY));
2*log(10.^8)^4/10^16;
```

$$f := -\frac{1}{y\ln(y)} + \frac{1}{y\ln(y)^2} - \frac{2}{y\ln(y)^3} + \frac{a}{y\ln(y)^4}$$

$$\text{"df/dy = "}, \frac{\ln(y)^4 - a\ln(y) + 6\ln(y) - 4a}{y^2\ln(y)^5}$$

$$\text{"h(y) = "}, Y^4 - aY + 6Y - 4a, \text{"dh/dy = "}, 4Y^3 - a + 6$$

$$4\ln(19)^3 - 7, 95.1098566$$

$$\ln(19)^4 - 7\ln(19) - 52, 2.55298765$$

$$2.302781322\ 10^{-11} \qquad (6)$$

```
> ######################## Product g(y) (cf. Lemma 2.1)
  ########################

  # The negative step function y -> g(y) = sum(log(1-1/p^2),p >= y)
  # is constant on [p',p'') where p' < p'' are two consecutive primes
  and g(p'') > g(p')
  Digits:=50: fongy(24318); # for y = 24318, computes g(y) = g(p''),
  a(p'), a(y) and a(p'')
```

"p=", 24317, "np=", 24329

["a(p")=comin=", 9.121231627, "a(y)=", 9.583870096, "a(p')=comax=", 9.625920922, "g(y)=", $-3.710643530\ 10^{-6}$] (7)

```
> Digits:=50: ti:=time(): calcullogPy(100000,19300,19400,24250,24350)
  ; "time = ",time()-ti;
  # To prove (2.13) and (2.14), execute calcullogPy(10^8+7,19300,
  19400,24250,24350);
  # the running time is about 6 hours
```

"p' = ", 19301, "p" = ", 19309, "a(p") = comin = ", 3.5645756, "a(p') = comax = ", 3.9630903, "g(p") = gnp = ", $-4.80454331416\ 10^{-6}$

"p' = ", 19309, "p" = ", 19319, "a(p") = comin = ", 3.5577752, "a(p') = comax = ", 4.0557258, "g(p") = gnp = ", $-4.80186118039\ 10^{-6}$

"p' = ", 19319, "p" = ", 19333, "a(p") = comin = ", 3.3518364, "a(p') = comax = ", 4.0487741, "g(p") = gnp = ", $-4.79918182257\ 10^{-6}$

"p' = ", 19333, "p" = ", 19373, "a(p") = comin = ", 1.8508293, "a(p') = comax = ", 3.8426238, "g(p") = gnp = ", $-4.79650634386\ 10^{-6}$

"p' = ", 19373, "p" = ", 19379, "a(p") = comin = ", 2.0422065, "a(p') = comax = ", 2.3410138, "g(p") = gnp = ", $-4.79384190203\ 10^{-6}$

"p' = ", 19379, "p" = ", 19381, "a(p") = comin = ", 2.4327548, "a(p') = comax = ", 2.5323007, "g(p") = gnp = ", $-4.79117910984\ 10^{-6}$

"p' = ", 19381, "p" = ", 19387, "a(p") = comin = ", 2.6243507, "a(p') = comax = ", 2.9228189, "g(p") = gnp = ", $-4.78851686718\ 10^{-6}$

"p' = ", 19387, "p" = ", 19391, "a(p") = comin = ", 2.9154515, "a(p') = comax = ", 3.1143246, "g(p") = gnp = ", $-4.78585627213\ 10^{-6}$

"p' = ", 19391, "p" = ", 19403, "a(p") = comin = ", 2.8089975, "a(p') = comax = ", 3.4053653, "g(p") = gnp = ", $-4.78319677462\ 10^{-6}$

"p' = ", 24251, "p" = ", 24281, "a(p") = comin = ", 10.283947, "a(p') = comax = ", 11.545085, "g(p") = gnp = ", $-3.71403083208\ 10^{-6}$

"p' = ", 24281, "p" = ", 24317, "a(p") = comin = ", 9.1981693, "a(p') = comax = ", 10.712081, "g(p") = gnp = ", $-3.71233467190\ 10^{-6}$

"p' = ", 24317, "p" = ", 24329, "a(p") = comin = ", 9.1212316, "a(p') = comax = ", 9.6259209, "g(p") = gnp = ", $-3.71064353016\ 10^{-6}$

"p' = ", 24329, "p" = ", 24337, "a(p") = comin = ", 9.2124964, "a(p') = comax = ", 9.5488558, "g(p") = gnp = ", $-3.70895405627\ 10^{-6}$

"p' = ", 24337, "p" = ", 24359, "a(p") = comin = ", 8.7151904, "a(p') = comax = ", 9.6400356, "g(p") = gnp = ", $-3.70726569292\ 10^{-6}$

"*************** Table of minimal records for a(y) ************"

"for y >= p" = ", 19373, "a(y) >= comini = ", 2., "a(p") = ", 1.8508293

"for y >= p" = ", 3299, "a(y) >= comini = ", 1.8508293, "a(p") = ", 1.8041609

"for y >= p" = ", 2657, "a(y) >= comini = ", 1.8041609, "a(p") = ", 1.5679327

"for y >= p" = ", 1427, "a(y) >= comini = ", 1.5679327, "a(p") = ", 0.77877493

"for y >= p" = ", 1423, "a(y) >= comini = ", 0.77877493, "a(p") = ", −0.087998269

"*************** Table of maximal records for a(y) ************"

"for y > p" = ", 60317, "a(y) <= comaxi = ", 9., "p' = ", 60293, "a(p') = ", 9.0669708

"for y > p" = ", 60271, "a(y) <= comaxi = ", 9.0669708, "p' = ", 60259, "a(p') = ", 9.0852552

"for y > p" = ", 60251, "a(y) <= comaxi = ", 9.0852552, "p' = ", 60223, "a(p') = ", 9.1477572

"for y > p" = ", 60209, "a(y) <= comaxi = ", 9.1477572, "p' = ", 60169, "a(p') = ", 9.6068201

"for y > p" = ", 59863, "a(y) <= comaxi = ", 9.6068201, "p' = ", 59833, "a(p') = ", 9.9510113

"for y > p" = ", 59833, "a(y) <= comaxi = ", 9.9510113, "p' = ", 59809, "a(p') = ", 10.237190

"for y > p" = ", 59809, "a(y) <= comaxi = ", 10.237190, "p' = ", 59797, "a(p') = ", 10.257915

"for y > p" = ", 24317, "a(y) <= comaxi = ", 10.257915, "p' = ", 24281, "a(p') = ", 10.712081

"for y > p" = ", 24281, "a(y) <= comaxi = ", 10.712081, "p' = ", 24251, "a(p') = ", 11.545085

"for y > p" = ", 24151, "a(y) <= comaxi = ", 11.545085, "p' = ", 24137, "a(p') = ", 11.628057

"for y > p" = ", 6389, "a(y) <= comaxi = ", 11.628057, "p' = ", 6379, "a(p') = ", 11.649848

"for y > p" = ", 4327, "a(y) <= comaxi = ", 11.649848, "p' = ", 4297, "a(p') = ", 11.680765

"for y > p" = ", 2999, "a(y) <= comaxi = ", 11.680765, "p' = ", 2971, "a(p') = ", 11.771144

"for y > p" = ", 2879, "a(y) <= comaxi = ", 11.771144, "p' = ", 2861, "a(p') = ", 12.558278

"for y > p" = ", 2819, "a(y) <= comaxi = ", 12.558278, "p' = ", 2803, "a(p') = ", 12.785982

"for y > p" = ", 1693, "a(y) <= comaxi = ", 12.785982, "p' = ", 1669, "a(p') = ", 13.288559

"for y > p" = ", 1657, "a(y) <= comaxi = ", 13.288559, "p' = ", 1637, "a(p') = ", 13.680096 "for y > p" = ", 1637, "a(y) <= comaxi = ", 13.680096, "p' = ", 1627, "a(p') = ", 14.241810

"for y > p" = ", 293, "a(y) <= comaxi = ", 14.241810, "p' = ", 283, "a(p') = ", 14.885060

"for y > p" = ", 127, "a(y) <= comaxi = ", 14.885060, "p' = ", 113, "a(p') = ", 15.067897

"time = ", 63.251 **(8)**

```
> ############################# Proof of Lemma 2.2, (2.15)
#####################

Digits:=10: eta:=7.5*10.^(-7): c:=0.945:
(1+eta)*(3+1/log(1418.))/2-c,    0.624/log(1418.);
```

0.623900145, 0.08598591250 **(9)**

```
> ############################# Proof of Lemma 2.3, (2.16)
#####################

Digits:=10: 1+1/log(89693.)^3; lempipi(89693):
```

1.000674235

"x = ", 89693, "quomax = ", 1.000674

"p=", 89689, "pp=", 89753, "pp-p = ", 64, "pp/p = ", 1.00071358

"p=", 79699, "pp=", 79757, "pp-p = ", 58, "pp/p = ", 1.00072774

"p=", 60539, "pp=", 60589, "pp-p = ", 50, "pp/p = ", 1.00082591

"p=", 59281, "pp=", 59333, "pp-p = ", 52, "pp/p = ", 1.00087718

"p=", 58831, "pp=", 58889, "pp-p = ", 58, "pp/p = ", 1.00098587

"p=", 48679, "pp=", 48731, "pp-p = ", 52, "pp/p = ", 1.00106822

"p=", 45893, "pp=", 45943, "pp-p = ", 50, "pp/p = ", 1.00108949

"p=", 44293, "pp=", 44351, "pp-p = ", 58, "pp/p = ", 1.00130946

"p=", 43331, "pp=", 43391, "pp-p = ", 60, "pp/p = ", 1.00138469

"p=", 35677, "pp=", 35729, "pp-p = ", 52, "pp/p = ", 1.00145752

"p=", 35617, "pp=", 35671, "pp-p = ", 54, "pp/p = ", 1.00151613

"p=", 34061, "pp=", 34123, "pp-p = ", 62, "pp/p = ", 1.00182026

"p=", 31397, "pp=", 31469, "pp-p = ", 72, "pp/p = ", 1.00229321

"p=", 19609, "pp=", 19661, "pp-p = ", 52, "pp/p = ", 1.00265184

"p=", 15683, "pp=", 15727, "pp-p = ", 44, "pp/p = ", 1.00280559

"p=", 11743, "pp=", 11777, "pp-p = ", 34, "pp/p = ", 1.00289534

"p=", 10799, "pp=", 10831, "pp-p = ", 32, "pp/p = ", 1.00296324

"p=", 9973, "pp=", 10007, "pp-p = ", 34, "pp/p = ", 1.00340920

"p=", 9551, "pp=", 9587, "pp-p = ", 36, "pp/p = ", 1.00376924

"p=", 8467, "pp=", 8501, "pp-p = ", 34, "pp/p = ", 1.00401559

"p=", 7253, "pp=", 7283, "pp-p = ", 30, "pp/p = ", 1.00413622

"p=", 6917, "pp=", 6947, "pp-p = ", 30, "pp/p = ", 1.00433714

"p=", 6491, "pp=", 6521, "pp-p = ", 30, "pp/p = ", 1.00462178

"p=", 5953, "pp=", 5981, "pp-p = ", 28, "pp/p = ", 1.00470351

"p=", 5749, "pp=", 5779, "pp-p = ", 30, "pp/p = ", 1.00521830

"p=", 5591, "pp=", 5623, "pp-p = ", 32, "pp/p = ", 1.00572348
"p=", 4831, "pp=", 4861, "pp-p = ", 30, "pp/p = ", 1.00620989
"p=", 4297, "pp=", 4327, "pp-p = ", 30, "pp/p = ", 1.00698162
"p=", 3271, "pp=", 3299, "pp-p = ", 28, "pp/p = ", 1.00856007
"p=", 2971, "pp=", 2999, "pp-p = ", 28, "pp/p = ", 1.00942444
"p=", 2477, "pp=", 2503, "pp-p = ", 26, "pp/p = ", 1.01049657
"p=", 2179, "pp=", 2203, "pp-p = ", 24, "pp/p = ", 1.01101423
"p=", 1951, "pp=", 1973, "pp-p = ", 22, "pp/p = ", 1.01127627
"p=", 1669, "pp=", 1693, "pp-p = ", 24, "pp/p = ", 1.01437987
"p=", 1327, "pp=", 1361, "pp-p = ", 34, "pp/p = ", 1.02562170
"p=", 523, "pp=", 541, "pp-p = ", 18, "pp/p = ", 1.03441683
"p=", 317, "pp=", 331, "pp-p = ", 14, "pp/p = ", 1.04416404
"p=", 293, "pp=", 307, "pp-p = ", 14, "pp/p = ", 1.04778157
"p=", 211, "pp=", 223, "pp-p = ", 12, "pp/p = ", 1.05687204
"p=", 199, "pp=", 211, "pp-p = ", 12, "pp/p = ", 1.06030151
"p=", 139, "pp=", 149, "pp-p = ", 10, "pp/p = ", 1.07194245
"p=", 113, "pp=", 127, "pp-p = ", 14, "pp/p = ", 1.12389381
"p=", 47, "pp=", 53, "pp-p = ", 6, "pp/p = ", 1.12765957
"p=", 31, "pp=", 37, "pp-p = ", 6, "pp/p = ", 1.19354839
"p=", 23, "pp=", 29, "pp-p = ", 6, "pp/p = ", 1.26086957
"p=", 13, "pp=", 17, "pp-p = ", 4, "pp/p = ", 1.30769231
"p=", 7, "pp=", 11, "pp-p = ", 4, "pp/p = ", 1.57142857
"p=", 3, "pp=", 5, "pp-p = ", 2, "pp/p = ", 1.66666667 **(10)**

```
> ################################ Function S_1, Sect. 3
  ##################################


  assume(x > 1): f:=int(log(1-1/t^2)/(2*t^2),t=1..x): series(f,x=
  infinity,14);
  # proof of Lemma 3.2
```

$$\ln(2) - 1 + \frac{1}{6\,x\sim^{3}} + \frac{1}{20\,x\sim^{5}} + \frac{1}{42\,x\sim^{7}} + \frac{1}{72\,x\sim^{9}} + \frac{1}{110\,x\sim^{11}} + \frac{1}{156\,x\sim^{13}} + O\left(\frac{1}{x\sim^{14}}\right) \quad \textbf{(11)}$$

```
> Digits:=30:constrlisteS1(10000): nops(lisS1);
  # computes a list [p^m,p,psi(p^m),integral(psi(t)/t^2,t=2..p^m)]
  for p^m <= 9973
  # allows the computation of S_1(t) by Lemma 3.2 for t <= 9973
  for i from 1 to 5 do print(lisS1[i]); od:
```

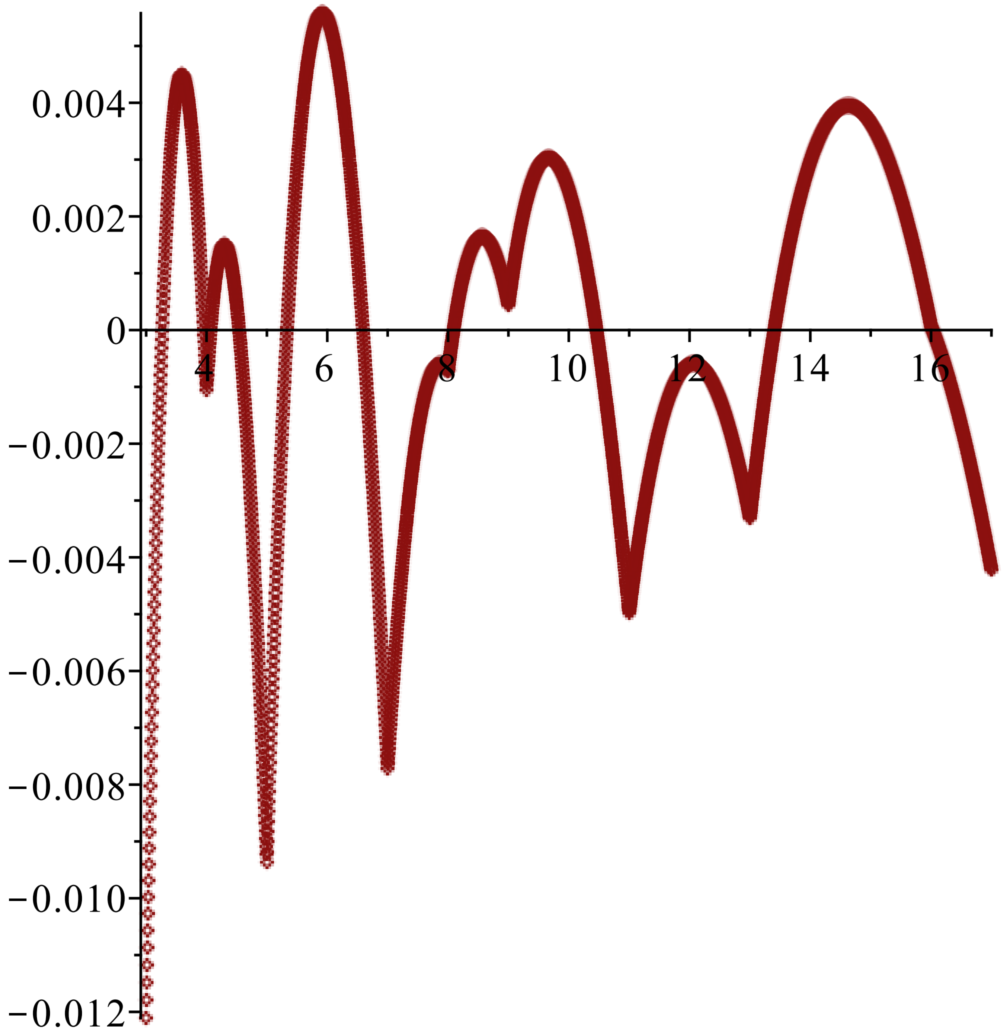
1280
[2, 2, 0.693147180559945309417232121458, 0.]
[3, 3, 1.79175946922805500081247735838, 0.115524530093324218236205353577]
[4, 2, 2.48490664978800031022970947984, 0.264837819195662134970578466775]
[5, 5, 4.09434456222210068483046881307, 0.389083151685062150482063940767]

[7, 7, 6.04025471127741398993582155651, 0.62304569809775361818666 2158657] **(12)**

```
> S1num(1000,0);# computation of S_1(1000)
```

−0.00030696356355675791527871692644 **(13)**

```
> liste:=[]: for x from 3. by 0.005 to 17 do liste:=[op(liste),[x,
  S1num(x,0)]]; od:
  plot(liste,style=point); # curve S_1(x), 3 <= x <= 17
```



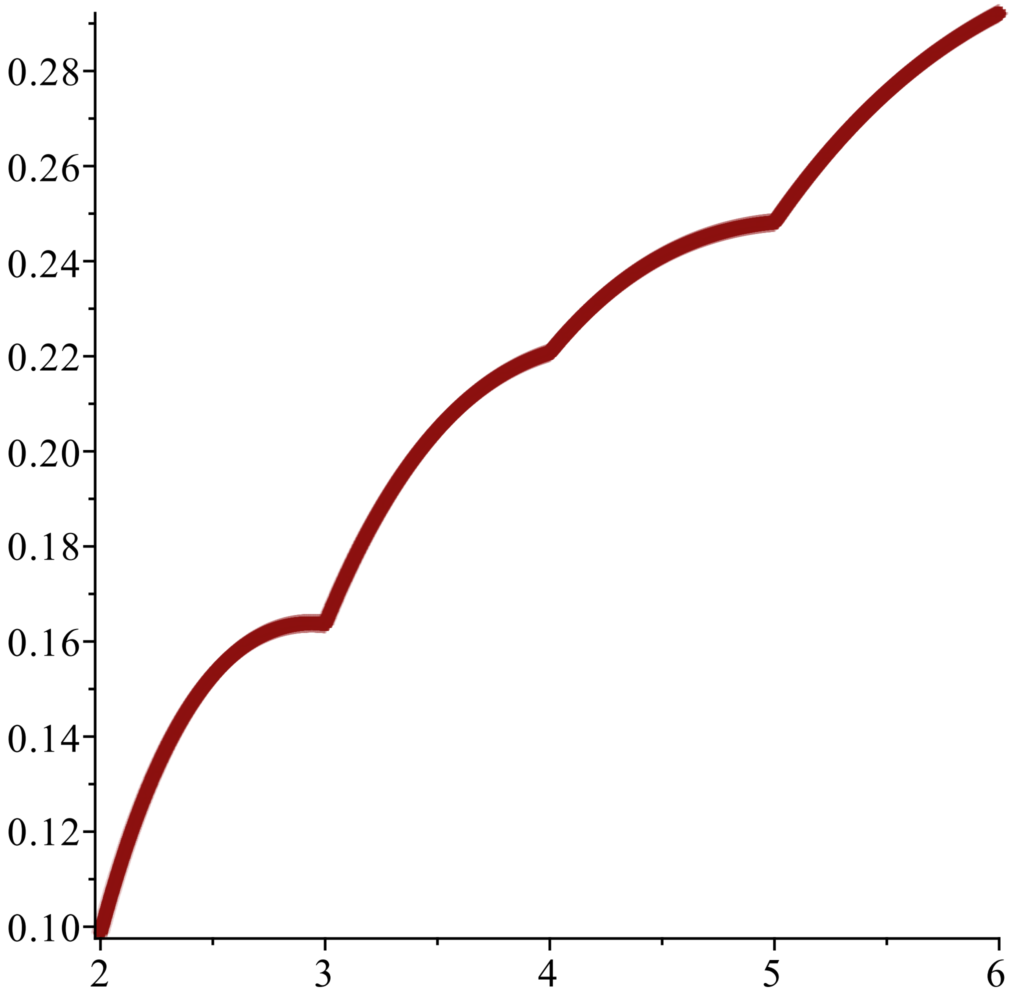

```
> Digits:=10:y0:=73.2: # proof of Lemma 3.4, (3.3)
  evalf(1/(8*Pi)+(1/sqrt(y0)/log(y0)^2)*(log(2*Pi)+1/2/(y0^2-1))),
  0.0462/log(73.2)^3; # proof of Lemma 3.4, (3.4)
```

0.05144398748, 0.0005838482773 **(14)**

```
> Digits:=10:(1/1.072)*(1-log(128.)^2/(8*evalf(Pi)*sqrt(128.))),log(1
  -1/128.^2)/256,
  0.16+0.855-1-2.4*10^(-7); # proof of Lemma 3.5
```

0.8556020264, $-2.384260262\ 10^{-7}$, 0.01499976000 **(15)**

```
> liste:=[]: for x from 2. by 0.001 to 6 do liste:=[op(liste),[x,
  0.16*log(x)+S1num(x,0)]]; od:
  plot(liste,style=point); # curve h_1(x), 2 <= x <= 6, cf. (3.8)
```




```
> ####################  Functions G and H : Proof of Lemma 3.6
##############

Digits:=10: u:='u': v:='v': t:='t': g1:=u*log(t)+v/sqrt(t)/log(t);
g11:=normal(diff(g1,t));
num1:=expand(numer(g11)/t^(3/2)/log(t)^2); evalf(subs(u=0.8,v=3.3,
t=6.,num1));
g12:=normal(diff(g11,t));
num2:=-expand(numer(g12)/t^(3/2)/log(t)^3); evalf(subs(u=0.95,v=
3.3,t=10.,num2));
```

$$g1 := u\ln(t) + \frac{v}{\sqrt{t}\,\ln(t)}$$

$$g11 := \frac{2\,u\,t^{3/2}\ln(t)^2 - v\,t\ln(t) - 2\,v\,t}{2\,t^{5/2}\ln(t)^2}$$

$$num1 := 2\,u - \frac{v}{\sqrt{t}\,\ln(t)} - \frac{2\,v}{\sqrt{t}\,\ln(t)^2}$$

$$0.0088186568$$

$$g12 := -\frac{4\,t^{3/2}\ln(t)^3\,u - 3\,t\ln(t)^2\,v - 8\,v\,t\ln(t) - 8\,v\,t}{4\,t^{7/2}\ln(t)^3}$$

$$num2 := 4\,u - \frac{3\,v}{\sqrt{t}\,\ln(t)} - \frac{8\,v}{\sqrt{t}\,\ln(t)^2} - \frac{8\,v}{\sqrt{t}\,\ln(t)^3}$$

$$0.1819217966 \qquad (16)$$

```
> a:='a': v:='v': t:='t':   g2:=-a/sqrt(t)+v/sqrt(t)/log(t);   g21:=
  normal(diff(g2,t));
  num1:=expand(numer(g21)/log(t)^2);evalf(subs(a=0.75,v=0.6,t=6.,
  num1));
  g22:=normal(diff(g21,t));
  num2:=-expand(numer(g22)/log(t)^3);    evalf(subs(a=0.75,v=0.6,t=
  10.,num2));
```

$$g2 := -\frac{a}{\sqrt{t}} + \frac{v}{\sqrt{t}\,\ln(t)}$$

$$g21 := \frac{a\ln(t)^2 - v\ln(t) - 2\,v}{2\,t^{3/2}\ln(t)^2}$$

$$num1 := a - \frac{v}{\ln(t)} - \frac{2\,v}{\ln(t)^2}$$

$$0.0413486582$$

$$g22 := -\frac{3\ln(t)^3 a - 3\ln(t)^2 v - 8\,v\ln(t) - 8\,v}{4\,t^{5/2}\ln(t)^3}$$

$$num2 := 3\,a - \frac{3\,v}{\ln(t)} - \frac{8\,v}{\ln(t)^2} - \frac{8\,v}{\ln(t)^3}$$

$$0.1697512951 \qquad (17)$$

```
> u:='u': c:='c': t:='t':   g3:=u*log(t)+c*log(t)/t^(2/3);   g31:=
  normal(diff(g3,t));
  g31modif:=(1/(3*t))*(3*u-2*c*(log(t)-3/2)/t^(2/3));
  "g31 - g31modif = ",   simplify(g31-g31modif);(1/(2/3))^1*exp(-1-
  (2/3)*(3/2)); # cf.(1.12)
  evalf(subs(u=0.04,c=0.1,3*u-2*c*3/2/exp(2.)));
```

$$g3 := u\ln(t) + \frac{c\ln(t)}{t^{2/3}}$$

$$g31 := -\frac{-3\,u\,t^{5/3} + 2\,c\ln(t)\,t - 3\,c\,t}{3\,t^{8/3}}$$

$$g31modif := \frac{3\,u - \frac{2\,c\left(\ln(t) - \frac{3}{2}\right)}{t^{2/3}}}{3\,t}$$

"g31 - g31modif = ", 0

$$\frac{3\,\mathrm{e}^{-2}}{2}$$

$$0.07939941504 \qquad (18)$$

```
> g32:=normal(diff(g31,t)); g32modif:=-(1/(9*t^2))*(9*u-10*c*(log(t)
  -21/10)/t^(2/3));
  "g32 - g32modif = ",simplify(g32-g32modif);(1/(2/3))^1*exp(-1-(2/3)
```

```
*(21/10));
evalf(subs(u=0.04,c=0.1,9*u-10*c*3/2*exp(-2.4)));
```

$$g32 := \frac{-9\,u\,t^{5/3} + 10\,c\,\ln(t)\,t - 21\,c\,t}{9\,t^{11/3}}$$

$$g32modif := -\frac{9\,u - \frac{10\,c\left(\ln(t) - \frac{21}{10}\right)}{t^{2/3}}}{9\,t^{2}}$$

"g32 - g32modif = ", 0

$$\frac{3\,e^{-\frac{12}{5}}}{2}$$

$$0.2239230701 \quad \textbf{(19)}$$

```
> ################# Section 4.1, Definition of CA numbers #########

superabondant(10^3,0); # list of SA numbers <= 1000 [M,sigma(M)/M],
cf. (4.2)
nops(lisSA), lisSA[323..324]; # lisSA is the list of the 324 SA
numbers < 10^50
```

$$\left[[1, 1], \left[2, \frac{3}{2}\right], \left[4, \frac{7}{4}\right], [6, 2], \left[12, \frac{7}{3}\right], \left[24, \frac{5}{2}\right], \left[36, \frac{91}{36}\right], \left[48, \frac{31}{12}\right], \left[60, \frac{14}{5}\right], [120, 3], \left[180, \frac{91}{30}\right], \left[240, \frac{31}{10}\right], \left[360, \frac{13}{4}\right], \left[720, \frac{403}{120}\right], \left[840, \frac{24}{7}\right]\right]$$

$$324, \left[\left[435615818746928159305552715741044768677175074240000, \frac{44182896271258530223226880000}{5325436922711235449158398689}\right], \left[512207611054080363139496050376832859873161900480000, \frac{288762620528362675297526061465600}{34759126794536233776656868243103}\right]\right] \quad \textbf{(20)}$$

```
> t:='t': u:='u': F(t,u); # cf. (4.5)
```

$$\frac{\ln\left(1. + \frac{t - 1.}{t^{u + 1.} - 1.\,t}\right)}{\ln(t)} \quad \textbf{(21)}$$

```
> Digits:=7: liseps(20); # The set E_i, cf. Section 4.1, (4.7) and
(4.8)
  # liseps(xmax) computes the elements >= F(xmax,1) of E_i
```

[[∞, 1, 0], [0.5849625, 2, 1], [0.2618593, 3, 1], [0.2223929, 2, 2], [0.1132828, 5, 1], [0.09953625, 2, 3], [0.07285775, 3, 2], [0.06862152, 7, 1], [0.04730525, 2, 4], [0.03628653, 11, 1], [0.02889254, 13, 1], [0.02308357, 2, 5], [0.02304524, 3, 3], [0.02037326, 5, 2], [0.02017457, 17, 1], [0.01742053, 19, 1]] **(22)**

```
> p:='p': eps:='eps':
```

```
mu:=proc(p,eps) evalf(log((p^(1+eps)-1)/(p^eps-1))/log(p)-1) end: #
cf (4.11)
mu(p,eps);    A:=F(p,mu(p,eps)):
A:= simplify(A,assume(p > 2 and eps > 0)); # cf. (4.12)
```

$$\frac{\ln\left(\frac{p^{eps+1.}-1.}{p^{eps}-1.}\right)}{\ln(p)}-1.$$

$$A := eps \tag{23}$$

```
> Digits:=50: calcullogN(7.1,2):
# calcullogN(xi,2) computes the CA number N of parameter epsilon =
F(xi,1),
# cf. (4.11) and (4.19). xi can be real or integer. Choose Digits
large
# computes xiN, called xi(N) in the paper cf. Definition 4.3
# computes xi_0[k] with xi_0[1] = xiN cf. (4.15) and (4.16)
# epsmin and epsmax are the smallest and the largest parameter of N
# epsmax is also eps_i such that N_{eps_{i-1}} < N <= N_{eps_i},
cf. Definition 4.3
# F(xiN,1) = epsmax
```

"k=", 1, "xi0[k] = ", 7., "pp = ", 7, "np = ", 11

"k=", 2, "xi0[k] = ", 3.07030931523062, "pp = ", 3, "np = ", 5

"k=", 3, "xi0[k] = ", 2.20978372619847, "pp = ", 2, "np = ", 3

"k=", 4, "xi0[k] = ", 1.84794020465102, "pp = ", 1, "np = ", 2

"xi = ", 7.1, "eps = F(xi,1) = ", 0.0672259722864226, "xiN = ", 7.

"epsmax = ", [7, 1], 0.0686215613240665, "epsmin = ", [2, 4], 0.0473057147783567

"N = ", 2520, "prime factors of N = ", $(2)^3\ (3)^2\ (5)\ (7)$

"q = ", 3, "expq = ", 2, "p = ", 7, "K(N) = ", 3

"sigNsN = ", 3.71428571428571, "logN = ", 7.83201418050547, "loglogN = ", 2.05821971581205 **(24)**

```
> ca(47,false,2): # list of the CA numbers with largest prime factor
<= 17, cf. Figure 2
                # ca(17,true,2) returns only the CA numbers of type
2
```

"i=", 1, [1, 0], "epsq = ", ∞, "xi = ", 1., "p = ", 1, "sigma(N)/N = ", 1, "N = ", 1.

"i=", 2, [2, 1], "eps = ", 0.58496, "N = ", 2, (2), "sigma(N)/N = ", $\frac{3}{2}$, $\frac{(3)}{(2)}$, "xi = ", 2

"i=", 3, [3, 1], "eps = ", 0.26186, "N = ", 6, (2), (3), "sigma(N)/N = ", 2, (2), "xi = ", 3

"i=", 4, [2, 2], "eps = ", 0.22239, "N = ", 12, $(2)^2$, (3), "sigma(N)/N = ", $\frac{7}{3}$, $\frac{(7)}{(3)}$, "xi = ", 3.2940

"i=", 5, [5, 1], "eps = ", 0.11328, "N = ", 60, $(2)^2$, (3), (5), "sigma(N)/N = ", $\frac{14}{5}$, $\frac{(2)\ (7)}{(5)}$, "xi = ", 5

"i=", 6, [2, 3], "eps = ", 0.099536, "N = ", 120, $(2)^3$, (3), (5), "sigma(N)/N = ", 3, (3), "xi = ", 5.4434

$"i=", 7, [3, 2], "eps = ", 0.072858, "N = ", 360, (2)^3, (3)^2, (5), "sigma(N)/N = ", \frac{13}{4}, \frac{(13)}{(2)^2},$

$"xi = ", 6.7175$

$"i=", 8, [7, 1], "eps = ", 0.068622, "N = ", 2520, (2)^3, (3)^2, (5), (7), "sigma(N)/N = ", \frac{26}{7},$

$\frac{(2)\ (13)}{(7)}, "xi = ", 7$

$"i=", 9, [2, 4], "eps = ", 0.047306, "N = ", 5040, (2)^4, (3)^2, (5), (7), "sigma(N)/N = ", \frac{403}{105},$

$\frac{(13)\ (31)}{(3)\ (5)\ (7)}, "xi = ", 9.0874$

$"i=", 10, [11, 1], "eps = ", 0.036287, "N = ", 55440, (2)^4, (3)^2, (5), "...", (11),$

$"sigma(N)/N = ", \frac{1612}{385}, \frac{(2)^2\ (13)\ (31)}{(5)\ (7)\ (11)}, "xi = ", 11$

$"i=", 11, [13, 1], "eps = ", 0.028893, "N = ", 720720, (2)^4, (3)^2, (5), "...", (13),$

$"sigma(N)/N = ", \frac{248}{55}, \frac{(2)^3\ (31)}{(5)\ (11)}, "xi = ", 13$

$"i=", 12, [2, 5], "eps = ", 0.023084, "N = ", 1441440, (2)^5, (3)^2, (5), "...", (13),$

$"sigma(N)/N = ", \frac{252}{55}, \frac{(2)^2\ (3)^2\ (7)}{(5)\ (11)}, "xi = ", 15.362$

$"i=", 13, [3, 3], "eps = ", 0.023045, "N = ", 4324320, (2)^5, (3)^3, (5), "...", (13),$

$"sigma(N)/N = ", \frac{672}{143}, \frac{(2)^5\ (3)\ (7)}{(11)\ (13)}, "xi = ", 15.382$

$"i=", 14, [5, 2], "eps = ", 0.020373, "N = ", 21621600, (2)^5, (3)^3, (5)^2, (7), "...", (13),$

$"sigma(N)/N = ", \frac{3472}{715}, \frac{(2)^4\ (7)\ (31)}{(5)\ (11)\ (13)}, "xi = ", 16.875$

$"i=", 15, [17, 1], "eps = ", 0.020174, "N = ", 367567200, (2)^5, (3)^3, (5)^2, (7), "...", (17),$

$"sigma(N)/N = ", \frac{62496}{12155}, \frac{(2)^5\ (3)^2\ (7)\ (31)}{(5)\ (11)\ (13)\ (17)}, "xi = ", 17$

$"i=", 16, [19, 1], "eps = ", 0.017420, "N = ", 6983776800, (2)^5, (3)^3, (5)^2, (7), "...", (19),$

$"sigma(N)/N = ", \frac{249984}{46189}, \frac{(2)^7\ (3)^2\ (7)\ (31)}{(11)\ (13)\ (17)\ (19)}, "xi = ", 19$

$"i=", 17, [23, 1], "eps = ", 0.013573, "xi = ", 23, "N = ", (2)^5, (3)^3, (5)^2, (7), "...", (23),$

$"sigma(N)/N = ", \frac{5999616}{1062347}, \frac{(2)^{10}\ (3)^3\ (7)\ (31)}{(11)\ (13)\ (17)\ (19)\ (23)}, "xi = ", 23$

$"i=", 18, [2, 6], "eps = ", 0.011405, "N = ", (2)^6, (3)^3, (5)^2, (7), "...", (23), "sigma(N)/N = ",$

$\frac{6047232}{1062347}, \frac{(2)^9 (3) (31) (127)}{(11) (13) (17) (19) (23)}$, "xi = ", 26.316

"i=", 19, [29, 1], "eps = ", 0.010068, "xi = ", 29, "N = ", $(2)^6, (3)^3, (5)^2, (7)$, "...", (29),

"sigma(N)/N = ", $\frac{181416960}{30808063}, \frac{(2)^{10} (3)^2 (5) (31) (127)}{(11) (13) (17) (19) (23) (29)}$, "xi = ", 29

"i=", 20, [31, 1], "eps = ", 0.0092454, "xi = ", 31, "N = ", $(2)^6, (3)^3, (5)^2, (7)$, "...", (31),

"sigma(N)/N = ", $\frac{187269120}{30808063}, \frac{(2)^{15} (3)^2 (5) (127)}{(11) (13) (17) (19) (23) (29)}$, "xi = ", 31

"i=", 21, [7, 2], "eps = ", 0.0090958, "N = ", $(2)^6, (3)^3, (5)^2, (7)^2, (11)$, "...", (31),

"sigma(N)/N = ", $\frac{70225920}{11350339}, \frac{(2)^{12} (3)^3 (5) (127)}{(7) (11) (13) (17) (23) (29)}$, "xi = ", 31.399

"i=", 22, [3, 4], "eps = ", 0.0075539, "N = ", $(2)^6, (3)^4, (5)^2, (7)^2, (11)$, "...", (31),

"sigma(N)/N = ", $\frac{6437376}{1031849}, \frac{(2)^9 (3)^2 (11) (127)}{(7) (13) (17) (23) (29)}$, "xi = ", 36.346

"i=", 23, [37, 1], "eps = ", 0.0073854, "xi = ", 37, "N = ", $(2)^6, (3)^4, (5)^2, (7)^2, (11)$, "...",

(37), "sigma(N)/N = ", $\frac{244620288}{38178413}, \frac{(2)^{10} (3)^2 (11) (19) (127)}{(7) (13) (17) (23) (29) (37)}$, "xi = ", 37

"i=", 24, [41, 1], "eps = ", 0.0064890, "xi = ", 41, "N = ", $(2)^6, (3)^4, (5)^2, (7)^2, (11)$, "...",

(41), "sigma(N)/N = ", $\frac{1467721728}{223616419}, \frac{(2)^{11} (3)^3 (11) (19) (127)}{(13) (17) (23) (29) (37) (41)}$, "xi = ", 41

"i=", 25, [43, 1], "eps = ", 0.0061123, "xi = ", 43, "N = ", $(2)^6, (3)^4, (5)^2, (7)^2, (11)$, "...",

(43), "sigma(N)/N = ", $\frac{64579756032}{9615506017}, \frac{(2)^{13} (3)^3 (11)^2 (19) (127)}{(13) (17) (23) (29) (37) (41) (43)}$, "xi = ", 43

"i=", 26, [2, 7], "eps = ", 0.0056688, "N = ", $(2)^7, (3)^4, (5)^2, (7)^2, (11)$, "...", (43),

"sigma(N)/N = ", $\frac{3813765120}{565618001}, \frac{(2)^{12} (3)^4 (5) (11)^2 (19)}{(13) (23) (29) (37) (41) (43)}$, "xi = ", 45.665

"i=", 27, [47, 1], "eps = ", 0.0054682, "xi = ", 47, "N = ", $(2)^7, (3)^4, (5)^2, (7)^2, (11)$, "...", **(25)**

(47), "sigma(N)/N = ", $\frac{183060725760}{26584046047}, \frac{(2)^{16} (3)^5 (5) (11)^2 (19)}{(13) (23) (29) (37) (41) (43) (47)}$, "xi = ",

47

```
> ################ Enumeration of CA numbers, Section 4.2 ###########
  Digits:=7: liseps2(20,2); # computation of CA numbers N of type 2
  with xi(N) < 20
  # [eps,q,alfaq,p,xi,sigNsN,logN]
```

"ecartmin between xi of type 2 and the nearest integer = ", 0.087453, "qmini = ", [2, 4]

[[0.2223929, 2, 2, 3, 3.294007], [0.09953625, 2, 3, 5, 5.443355], [0.07285775, 3, 2, 5, **(26)**

6.717536], [0.04730525, 2, 4, 7, 9.087453], [0.02308357, 2, 5, 13, 15.36249], [0.02304524, 3, 3, 13, 15.38163], [0.02037326, 5, 2, 13, 16.87465]]

```
> Digits:=7: cantype2(20,1,0): # computes the list of CA numbers of type 2
            # for xi < 20      [epsilon,q,expq,p,xi,sigma(N)/N,log N]
            # To generate the list lisCA2 of the 6704 first CA numbers of type 2
            # execute lisCA2:=cantype2(2000700000,100,0);
            # with Digits:=70: (running time about 8 hours)
            # or read "colHR.m" that contains the list lisCA2
```

"j = ", 1, "T[j] = ", [0.2223929, 2, 2, 3, 3.294007, 2.333333, 2.484906], "time = ", 0.

"j = ", 2, "T[j] = ", [0.09953625, 2, 3, 5, 5.443355, 3.000000, 4.787491], "time = ", 0.003

"j = ", 3, "T[j] = ", [0.07285775, 3, 2, 5, 6.717536, 3.250000, 5.886103], "time = ", 0.007

"j = ", 4, "T[j] = ", [0.04730525, 2, 4, 7, 9.087453, 3.838097, 8.525160], "time = ", 0.014

"j = ", 5, "T[j] = ", [0.02308357, 2, 5, 13, 15.36249, 4.581821, 14.18116], "time = ", 0.019

"j = ", 6, "T[j] = ", [0.02304524, 3, 3, 13, 15.38163, 4.699304, 15.27977], "time = ", 0.025 **(27)**

```
> u:=lisCA2[6704]:  nops(lisCA2), "log(N) = ",u[7];
  Digits:=12: [evalf(u[1]),op(u[2..4]),evalf(op(u[5..7]))];
```

6704, "log(N) = ",

$2.000538105793703852349682445053652672670513348656314985773618694201452\ 10^{9}$

$[2.33401008331\ 10^{-11}, 62297, 2, 2000530391, 2.00053042579\ 10^{9}, 38.1446306163,$ **(28)**

$2.00053810579\ 10^{9}]$

```
> ############## Smallest difference between two consecutive epsilon_i's (cf. Sect. 4.2)

  Digits:=50: mineps21(1,6702,lisCA2); F(54371,2)-F(1524427141,1);
  # Difference between an epsilon_i of type 2 and an epsilon_i of type 1
```

"mineps21=", $2.577682050\ 10^{-23}$, "epsmin = ", [54371, 2], " - ", [1524427141, 1]

$2.5776820504686902084901443275911145306\ 10^{-23}$ **(29)**

```
> mineps22(1,6701,lisCA2); # difference between two epsilons of type 2
```

"mineps22=", $1.631095481\ 10^{-16}$, " = ", [3, 1559], " - ", [2, 50723] **(30)**

```
> # difference between two epsilon_i's of type 1
  Fp1:=log(1+1/t)/log(t):      Fp11:=normal(diff(Fp1,t)):
  print("Fp1 = ",Fp1,"Fp11 = ",Fp11);
  Fp12:=normal(diff(Fp11,t)); # Fp11 is negative and incresing
  -2*evalf(subs(t=2.*10^9,Fp11)); # Mean value theorem
```

$$\text{"Fp1 = "}, \frac{\ln\left(1+\frac{1}{t}\right)}{\ln(t)}, \text{"Fp11 = "}, -\frac{\ln\left(\frac{t+1}{t}\right) t + \ln(t) + \ln\left(\frac{t+1}{t}\right)}{t\,(t+1)\ln(t)^2}$$

$$Fp12 := \frac{1}{t^2\,(t+1)^2\ln(t)^3}\left(\ln\left(\frac{t+1}{t}\right) t^2 \ln(t) + 2\ln(t)^2 t + 2\ln\left(\frac{t+1}{t}\right) t \ln(t) + 2\ln\left(\frac{t+1}{t}\right) t^2 + \ln(t)^2 + \ln(t)\ln\left(\frac{t+1}{t}\right) + 2\,t\ln(t) + 4\ln\left(\frac{t+1}{t}\right) t + 2\ln(t)\right)$$

$$+ 2\ln\left(\frac{t+1}{t}\right)\Big)$$

$$2.44367055923173711159109093458166115951410221105 08\ 10^{-20} \quad \textbf{(31)}$$

```
> p:=prevprime(2*10^9):p1:=nextprime(p): di:=p1-p: "p = ",p,"p1 = ",
  p1,"di = ",di;
  while di > 2 do
      p1:=p; p:=prevprime(p); di:=p1-p;
  od: # p and p1 are the largest twin primes < 2*10^9
  print("p=",p, "F(p,1) = ",evalf(F(p,1)));
  print("p1=",p1, "F(p1,1) = ",evalf(F(p1,1)));
  print("F(p,1) - F(p1,1) = ",evalf(F(p,1)-F(p1,1)));
```

"p = ", 1999999973, "p1 = ", 2000000011, "di = ", 38

"p=", 1999999871, "F(p,1) = ",

$$2.3346581578443740557780187848356621683046465398533\ 10^{-11}$$

"p1=", 1999999873, "F(p1,1) = ",

$$2.3346581554007031761280770496517654815836121713394\ 10^{-11}$$

"F(p,1) - F(p1,1) = ", $2.4436708796499417351838966867210343685139\ 10^{-20}$ **(32)**

```
> ############ Asymptotic expansion of xi_2, Section 4.3 ########

  devxi2(3,1);
  # computes the asymptotic expansion of xi_2 (cf. Proposition 4.7
  and (4.27))
  # sqrt(2*xi)*(1+co[1]/t+ ... +co[nmax]/t^nmax) + O(1/t^(nmax+1))
  # "t" is equal to log(xi), "a" to log(2), "imp" is a printing
  parameter
```

$$\text{"asy=", } 1 - \frac{a}{2\,t} + \mathrm{O}\left(\frac{1}{t^2}\right)$$

$$\text{"asy=", } 1 - \frac{a}{2\,t} + \frac{-\left(-\frac{1}{2}\,a\right) + \frac{3\,a^2}{8}}{t^2} + \mathrm{O}\left(\frac{1}{t^3}\right)$$

$$\text{"asy=", } 1 - \frac{a}{2\,t} + \frac{-\left(-\frac{1}{2}\,a\right) + \frac{3\,a^2}{8}}{t^2} + \frac{-\left(\frac{1}{2}\,a + \frac{3}{8}\,a^2\right) + \frac{\left(-\frac{1}{2}\,a\right)^2}{2} + \frac{3\,a\left(-\frac{1}{2}\,a\right)}{2} - \frac{5\,a^3}{16}}{t^3} + \mathrm{O}\left(\frac{1}{t^4}\right)$$

$$1 + \frac{-\frac{1}{2}\,a}{t} + \frac{\frac{1}{2}\,a + \frac{3}{8}\,a^2}{t^2} + \frac{-\frac{1}{2}\,a - a^2 - \frac{5}{16}\,a^3}{t^3} \quad \textbf{(33)}$$

```
> devxik(3,k,0);
  # computes the asymptotic expansion of xi_k (not given in the
  paper)
  # (k*xi)^(1/k)*(1+co[1]/t+ ... +co[nmax]/t^nmax) + O(1/t^(nmax+1))
  # "t" is equal to log(xi), "a" to log(k)
```

```
# "imp" is a printing parameter
```

$$1-\frac{a}{k\,t}+\frac{\frac{a}{k}+\frac{(k+1)\,a^2}{2\,k^2}}{t^2}+\frac{-\frac{a}{k}-\frac{(3\,k+2)\,a^2}{2\,k^2}-\frac{\left(2\,k^2+3\,k+1\right)\,a^3}{6\,k^3}}{t^3} \qquad \textbf{(34)}$$

```
> ##################  upper bound of xi_2, Section 4.3
##################

Digits:=10: xi:='xi': beta:='beta':
z:=sqrt(2*xi)*(1-beta/log(xi)); # cf. Proposition 4.7, (4.29)
```

$$z := \sqrt{2}\,\sqrt{\xi}\,\left(1-\frac{\beta}{\ln(\xi)}\right) \qquad \textbf{(35)}$$

```
> t:='t': A:=collect(normal(2*(t-beta)^2*((a+t)/2-beta/(t-beta))),[t,
beta]);
```

$$A := t^3+\left(a-2\,\beta\right)\,t^2+\left(\beta^2+\left(-2\,a-2\right)\,\beta\right)\,t+\left(a+2\right)\,\beta^2 \qquad \textbf{(36)}$$

```
> B:=evalf(subs(beta=0.323,a=log(2.),x0=10.^9,A-t^3-log(x0)^3/(2*x0))
);
```

$$B := 0.0471471806\,t^2-0.9894440787\,t+0.2809689024 \qquad \textbf{(37)}$$

```
> Brac:=solve(B = 0,t); exp(Brac[1]),exp(Brac[2]);
```

$$Brac := 20.69836665,\ 0.2879164322$$

$$9.754082410\ 10^8,\ 1.333645850 \qquad \textbf{(38)}$$

```
> Digits:=30:    xi:=10.^9+7:   t:='t':    xi2:=fsolve(F(t,2)=F(xi,1),
t);
solve(xi2=sqrt(2*xi)*(1-beta/log(xi)),beta);
```

$$\xi 2 := 44023.5388108009996569921476413$$

$$0.323360649773548837912893005787 \qquad \textbf{(39)}$$

```
> ############### Computation of N0 and xi0, cf. Section 4.4
###############

Digits:=50: calcullogN(10^9+7,2): # cf. (4.36) and (4.37)
```

"k=", 1, "xi0[k] = ", 1.000000007 10^9, "pp = ", 1000000007, "np = ", 1000000009

"k=", 2, "xi0[k] = ", 44023.5388108010, "pp = ", 44021, "np = ", 44027

"k=", 3, "xi0[k] = ", 1418.38081890335, "pp = ", 1409, "np = ", 1423

"k=", 4, "xi0[k] = ", 247.382205897601, "pp = ", 241, "np = ", 251

"k=", 5, "xi0[k] = ", 85.6248328688747, "pp = ", 83, "np = ", 89

"k=", 6, "xi0[k] = ", 41.9055559251791, "pp = ", 41, "np = ", 43

"k=", 7, "xi0[k] = ", 25.0431586873868, "pp = ", 23, "np = ", 29

"k=", 8, "xi0[k] = ", 16.9729134094723, "pp = ", 13, "np = ", 17

"k=", 9, "xi0[k] = ", 12.5175409933509, "pp = ", 11, "np = ", 13

"k=", 10, "xi0[k] = ", 9.79795805483709, "pp = ", 7, "np = ", 11

"k=", 11, "xi0[k] = ", 8.01061571828594, "pp = ", 7, "np = ", 11

"k=", 12, "xi0[k] = ", 6.76795496572222, "pp = ", 5, "np = ", 7

"k=", 13, "xi0[k] = ", 5.86506102880587, "pp = ", 5, "np = ", 7

"k=", 14, "xi0[k] = ", 5.18545963462664, "pp = ", 5, "np = ", 7
"k=", 15, "xi0[k] = ", 4.65896848724528, "pp = ", 3, "np = ", 5
"k=", 16, "xi0[k] = ", 4.24122002838259, "pp = ", 3, "np = ", 5
"k=", 17, "xi0[k] = ", 3.90301827869997, "pp = ", 3, "np = ", 5
"k=", 18, "xi0[k] = ", 3.62448665807836, "pp = ", 3, "np = ", 5
"k=", 19, "xi0[k] = ", 3.39169307626448, "pp = ", 3, "np = ", 5
"k=", 20, "xi0[k] = ", 3.19462248738160, "pp = ", 3, "np = ", 5
"k=", 21, "xi0[k] = ", 3.02591503456320, "pp = ", 3, "np = ", 5
"k=", 22, "xi0[k] = ", 2.88005599510308, "pp = ", 2, "np = ", 3
"k=", 23, "xi0[k] = ", 2.75284137279059, "pp = ", 2, "np = ", 3
"k=", 24, "xi0[k] = ", 2.64101669040753, "pp = ", 2, "np = ", 3
"k=", 25, "xi0[k] = ", 2.54202750288495, "pp = ", 2, "np = ", 3
"k=", 26, "xi0[k] = ", 2.45384368850374, "pp = ", 2, "np = ", 3
"k=", 27, "xi0[k] = ", 2.37483350469800, "pp = ", 2, "np = ", 3
"k=", 28, "xi0[k] = ", 2.30367186118659, "pp = ", 2, "np = ", 3
"k=", 29, "xi0[k] = ", 2.23927253465124, "pp = ", 2, "np = ", 3
"k=", 30, "xi0[k] = ", 2.18073740435309, "pp = ", 2, "np = ", 3
"k=", 31, "xi0[k] = ", 2.12731796681198, "pp = ", 2, "np = ", 3
"k=", 32, "xi0[k] = ", 2.07838582871057, "pp = ", 2, "np = ", 3
"k=", 33, "xi0[k] = ", 2.03340984655939, "pp = ", 2, "np = ", 3
"k=", 34, "xi0[k] = ", 1.99193824400876, "pp = ", 1, "np = ", 2

"xi = ", 1000000007, "eps = F(xi,1) = ", 4.82549420554828 10^{-11}, "xiN = ", 1.000000007 10^{9}

"epsmax = ", [1000000007, 1], 4.82549420554828 10^{-11}, "epsmin = ", [1000000009, 1], 4.82549419543158 10^{-11}

"N = ", 6.42931756558252 $10^{434300801}$, "prime factors of N = ", $(2)^{33}$, $(3)^{21}$, $(5)^{14}$, $(7)^{11}$, $(11)^{9}$, $(13)^{8}$, $(17)^{7}$, $(19)^{7}$, $(23)^{7}$, $(29)^{6}$, " ... ", $(41)^{6}$, $(43)^{5}$, " ... ", $(83)^{5}$, $(89)^{4}$, " ... ", $(241)^{4}$, $(251)^{3}$, " ... ", $(1409)^{3}$, $(1423)^{2}$, " ... ", $(44021)^{2}$, (44027), " ... ", (1000000007)

"q = ", 44021, "expq = ", 2, "p = ", 1000000007, "K(N) = ", 33

"sigNsN = ", 36.9096185662946, "logN = ", 1.00001455211884 10^{9}, "loglogN = ", 20.7232803889594 **(40)**

```
> Digits:=10: log(2.)-log(33.)/33, 1/0.587; # Proof of Lemma 4.9
```

0.5871924060, 1.703577513 **(41)**

```
> ############################## Section 4.5
  ###############################

  Digits:=20:xi0:=10^9+7: T3(xi0,33,1.71,1): # Proof of Lemma 4.10,
  (4.40)
```

"k0 = ", 4, "T[k0] = ", 9.59966902532, " = ", 1.4422496, " + ", 8.15742, "exp(w+u/v) = ", 940719.

"k0 = ", 5, "T[k0] = ", 4.43046730712, " = ", 1.6937363, " + ", 2.73673, "exp(w+u/v) = ", 18754.7
"k0 = ", 6, "T[k0] = ", 3.07824135687, " = ", 1.7807913, " + ", 1.29745, "exp(w+u/v) = ", 7509.84
"k0 = ", 7, "T[k0] = ", 2.57388833657, " = ", 1.8234190, " + ", 0.750469, "exp(w+u/v) = ", 6366.34
"k0 = ", 8, "T[k0] = ", 2.34068860669, " = ", 1.8489133, " + ", 0.491775, "exp(w+u/v) = ", 7285.05
"k0 = ", 9, "T[k0] = ", 2.21644240823, " = ", 1.8662069, " + ", 0.350235, "exp(w+u/v) = ", 9685.26
"k0 = ", 10, "T[k0] = ", 2.14333993845, " = ", 1.8789721, " + ", 0.264368, "exp(w+u/v) = ", 14029.0
"k0 = ", 11, "T[k0] = ", 2.09709318304, " = ", 1.8889721, " + ", 0.208121, "exp(w+u/v) = ", 21438.6
"k0 = ", 12, "T[k0] = ", 2.06618867319, " = ", 1.8971540, " + ", 0.169035, "exp(w+u/v) = ", 33955.3
"k0 = ", 13, "T[k0] = ", 2.04464154228, " = ", 1.9040712, " + ", 0.140570, "exp(w+u/v) = ", 55132.8
"k0 = ", 14, "T[k0] = ", 2.02910662990, " = ", 1.9100691, " + ", 0.119038, "exp(w+u/v) = ", 91171.7
"k0 = ", 15, "T[k0] = ", 2.01760155185, " = ", 1.9153745, " + ", 0.102227, "exp(w+u/v) = ", 152833.
"k0 = ", 16, "T[k0] = ", 2.00889364064, " = ", 1.9201433, " + ", 0.0887503, "exp(w+u/v) = ", 258901.
"k0 = ", 17, "T[k0] = ", 2.00218498785, " = ", 1.9244860, " + ", 0.0776990, "exp(w+u/v) = ", 442236.
"k0 = ", 18, "T[k0] = ", 1.99694123102, " = ", 1.9284835, " + ", 0.0684577, "exp(w+u/v) = ", 760324.
"k0 = ", 19, "T[k0] = ", 1.99279392180, " = ", 1.9321966, " + ", 0.0605973, "exp(w+u/v) = ", $1.31428\ 10^{6}$
"k0 = ", 20, "T[k0] = ", 1.98948252283, " = ", 1.9356719, " + ", 0.0538107, "exp(w+u/v) = ", $2.28186\ 10^{6}$
"k0 = ", 21, "T[k0] = ", 1.98681870231, " = ", 1.9389457, " + ", 0.0478730, "exp(w+u/v) = ", $3.97645\ 10^{6}$
"k0 = ", 22, "T[k0] = ", 1.98466366106, " = ", 1.9420469, " + ", 0.0426168, "exp(w+u/v) = ", $6.95103\ 10^{6}$
"k0 = ", 23, "T[k0] = ", 1.98291333576, " = ", 1.9449988, " + ", 0.0379145, "exp(w+u/v) = ",

$1.21824\ 10^{7}$

"k0 = ", 24, "T[k0] = ", 1.98148850466, " = ", 1.9478205, " + ", 0.0336680, "exp(w + u/v) = ", $2.14000\ 10^{7}$

"k0 = ", 25, "T[k0] = ", 1.98032802622, " = ", 1.9505277, " + ", 0.0298004, "exp(w + u/v) = ", $3.76634\ 10^{7}$

"k0 = ", 26, "T[k0] = ", 1.97938412773, " = ", 1.9531333, " + ", 0.0262508, "exp(w + u/v) = ", $6.63995\ 10^{7}$

"k0 = ", 27, "T[k0] = ", 1.97861906386, " = ", 1.9556485, " + ", 0.0229705, "exp(w + u/v) = ", $1.17236\ 10^{8}$

"k0 = ", 28, "T[k0] = ", 1.97800270874, " = ", 1.9580827, " + ", 0.0199200, "exp(w + u/v) = ", $2.07263\ 10^{8}$

"k0 = ", 29, "T[k0] = ", 1.97751079519, " = ", 1.9604438, " + ", 0.0170670, "exp(w + u/v) = ", $3.66827\ 10^{8}$

"k0 = ", 30, "T[k0] = ", 1.97712361000, " = ", 1.9627387, " + ", 0.0143849, "exp(w + u/v) = ", $6.49946\ 10^{8}$

"k0 = ", 31, "T[k0] = ", 1.97683617392, " = ", 1.9649735, " + ", 0.0118627, "exp(w + u/v) = ", $1.15262\ 10^{9}$

"k0 = ", 32, "T[k0] = ", 1.97686421205, " = ", 1.9671534, " + ", 0.00971082, "exp(w + u/v) = ", $2.04569\ 10^{9}$

"mink = ", 31, "Tmin = ", 1.9768361739216531769 **(42)**

```
> ################### proof of Lemma 4.11 ##################

  Digits:=10: xi0:=10^9+7.:
  sqrt(2.)*(log(2.)-0.323),    sqrt(2.)-0.524/log(xi0),  # Proof of
  the lower bound of (4.43)
  2.76/(1+log(2.)/log(xi0)); # Proof of the lower bound of (4.43)
```

0.5234671627, 1.388927972, 2.670671960 **(43)**

```
> sqrt(2.)*(1-log(2.)/2/log(xi0)),   2*log(1.39); # Proof of the
  upper bound of (4.43)
  xi:='xi':   u:=log(2.)/2/log(xi):   u0:=log(2.)/2/log(xi0);
  v:=0.658/log(xi):      v0:=0.658/log(xi0);
  1+u/(1-u0)-(1-v0)*v,    sqrt(2.)*numer(u/(1-u0)-(1-v0)*v);
```

1.390562412, 0.6586074942

$$u0 := 0.01672388864$$

$$v0 := 0.03175175212$$

$$1 - \frac{0.2846391174}{\ln(\xi)}, -0.4025405001 \qquad \textbf{(44)}$$

```
> ##################### Section 4.6, Benefit ###################

  p:='p': ap:='ap': eps:='eps':
  B:=log((p^(ap+1)-1)/(p^(t+1)-1))+(1+eps)*(t-ap)*log(p);  # cf.
  (4.49)
```

```
"dB/dt = ",normal(diff(B,t));    # proof of Lemma 4.13
"d^2 B/dt^2 = ",normal(diff(B,t$2));
```

$$B := \ln\left(\frac{p^{ap+1}-1}{p^{t+1}-1}\right) + (eps+1)\,(t-ap)\,\ln(p)$$

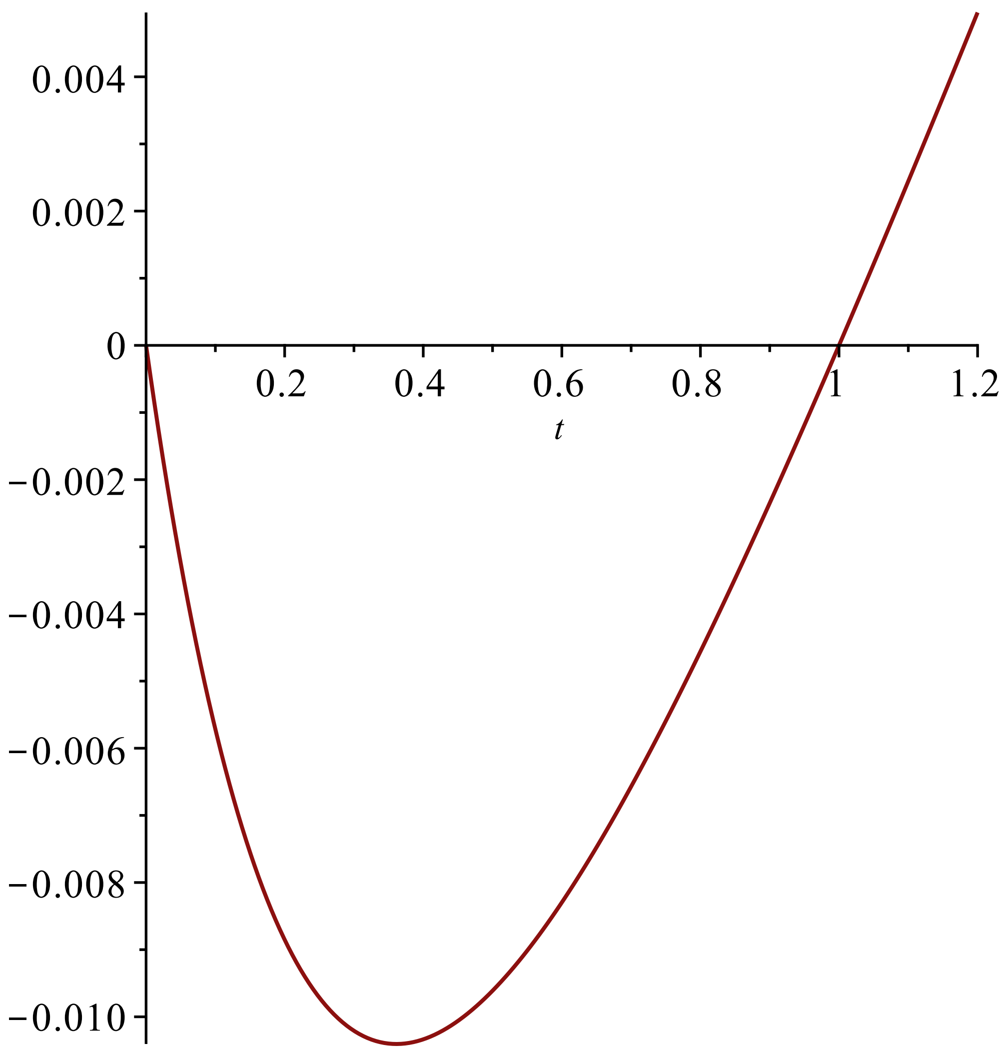
$$\text{"dB/dt = ", } \frac{\ln(p)\left(eps\,p^{t+1}-eps-1\right)}{p^{t+1}-1}$$

$$\text{"d^2 B/dt^2 = ", } \frac{p^{t+1}\ln(p)^2}{\left(p^{t+1}-1\right)^2} \tag{45}$$

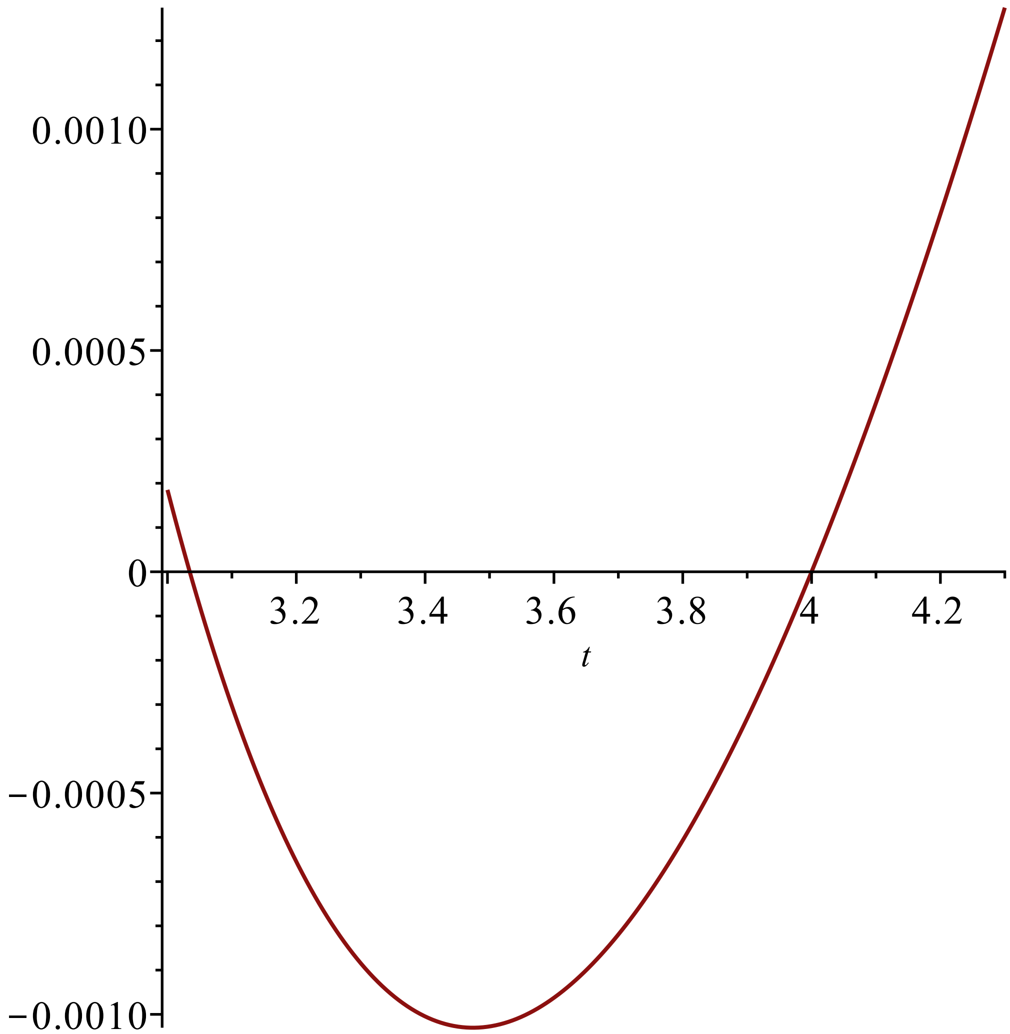
```
> Digits:=10: N:=lisCA[23][7]: eps0:=lisCA[23][1]:
  "N = ",imprifac(N),"eps0 =",evalf(eps0,5);
```

$$\text{"N = ", } (2)^6, (3)^4, (5)^2, (7)^2, (11), \text{"...", } (37), \text{"eps0 =", } 0.0073854 \tag{46}$$

```
> plot(subs(p=37,ap=1,eps=eps0,B), t=0..1.2); # plot of B_{37,1,eps0}
  (t)
```




```
> plot(subs(p=3,ap=4,eps=eps0,B), t=3..4.3);  # plot of B_{3,4,eps0}
  (t)
```




```
> N:=lisCA[23][7]: eps:=lisCA[23][1]; benmax:=0.01;
  bensmall(N,eps,benmax,N/2,N,4); # cf. proposition 4.14
  # computation of integers n s.t. ben_{eps}(n) <= benmax
  # here, there are 172 such n's and 14 satisfy N/2 <= n <= N [n,
  sigma(n),ben_eps(n)]
```

$$eps := 0.007385448161$$

$$benmax := 0.01$$

"N = ", 224403121196654400, "nops(lisben)=", 172, "nops(lisbenshort) = ", 14

$[[112201560598327200, 7.132476067\ 10^{17}, 0.0027859770], [121298984430624000, 7.661256126\ 10^{17}, 0.009805792], [124331459041389600, 7.883263020\ 10^{17}, 0.0061148200], [128230354969516800, 8.119621877\ 10^{17}, 0.007678421], [130396408262920800, 8.258656498\ 10^{17}, 0.0075746080], [149602080797769600, 9.543651562\ 10^{17}, 0.0013749824], [160287943711896000, 1.015503747\ 10^{18}, 0.008783688], [165775278721852800, 1.054824646\ 10^{18}, 0.0047038254], [173861877683894400, 1.105054391\ 10^{18}, 0.0061636134], [187002600997212000, 1.186527610\ 10^{18}, 0.0084265640], [190035075607977600, 1.205513882\ 10^{18},$ **(47)**

$0.0087566394], [192345532454275200, 1.218908480\ 10^{18}, 0.0098807944],$

$[219060189739591200, 1.389886593\ 10^{18}, 0.0096280360], [224403121196654400,$

$1437816604247654400, 0.]]$

```
> map(w->w[1]/N,lisbenshort);
  # list of the quotients n/N with N/2 <= n <= N and ben_{eps}(n) <=
  0.01
```

$$\left[\frac{1}{2}, \frac{20}{37}, \frac{41}{74}, \frac{4}{7}, \frac{43}{74}, \frac{2}{3}, \frac{5}{7}, \frac{82}{111}, \frac{86}{111}, \frac{5}{6}, \frac{94}{111}, \frac{6}{7}, \frac{41}{42}, 1\right] \tag{48}$$

```
> map(w->w[1]/N,lisben);
  # list of all the quotients n/N with ben_{eps}(n) <= 0.01
```

$$\left[\frac{1}{24087}, \frac{1}{8029}, \frac{1}{6882}, \frac{1}{3441}, \frac{1}{3219}, \frac{1}{2294}, \frac{2}{3441}, \frac{1}{1554}, \frac{1}{1147}, \frac{1}{1073}, \frac{1}{777}, \frac{1}{651}, \right. \tag{49}$$

$$\frac{2}{1147}, \frac{1}{518}, \frac{2}{777}, \frac{1}{259}, \frac{1}{222}, \frac{1}{217}, \frac{4}{777}, \frac{1}{186}, \frac{5}{777}, \frac{2}{259}, \frac{1}{111}, \frac{1}{93}, \frac{1}{87}, \frac{3}{259},$$

$$\frac{41}{3441}, \frac{1}{74}, \frac{4}{259}, \frac{1}{62}, \frac{2}{111}, \frac{5}{259}, \frac{2}{93}, \frac{5}{222}, \frac{6}{259}, \frac{1}{42}, \frac{41}{1554}, \frac{1}{37}, \frac{1}{31}, \frac{1}{29},$$

$$\frac{41}{1147}, \frac{4}{111}, \frac{10}{259}, \frac{3}{74}, \frac{5}{111}, \frac{1}{21}, \frac{41}{777}, \frac{2}{37}, \frac{43}{777}, \frac{2}{31}, \frac{5}{74}, \frac{1}{14}, \frac{8}{111}, \frac{41}{518}, \frac{3}{37},$$

$$\frac{10}{111}, \frac{2}{21}, \frac{82}{777}, \frac{4}{37}, \frac{86}{777}, \frac{5}{37}, \frac{1}{7}, \frac{41}{259}, \frac{6}{37}, \frac{43}{259}, \frac{1}{6}, \frac{20}{111}, \frac{41}{222}, \frac{4}{21}, \frac{43}{222},$$

$$\frac{8}{37}, \frac{5}{21}, \frac{10}{37}, \frac{2}{7}, \frac{82}{259}, \frac{12}{37}, \frac{86}{259}, \frac{1}{3}, \frac{41}{111}, \frac{43}{111}, \frac{47}{111}, \frac{3}{7}, \frac{41}{93}, \frac{1}{2}, \frac{20}{37}, \frac{41}{74}, \frac{4}{7},$$

$$\frac{43}{74}, \frac{2}{3}, \frac{5}{7}, \frac{82}{111}, \frac{86}{111}, \frac{5}{6}, \frac{94}{111}, \frac{6}{7}, \frac{41}{42}, 1, \frac{41}{37}, \frac{43}{37}, \frac{47}{37}, \frac{41}{31}, \frac{4}{3}, \frac{10}{7}, \frac{164}{111}, \frac{3}{2},$$

$$\frac{172}{111}, \frac{5}{3}, \frac{205}{111}, \frac{41}{21}, 2, \frac{43}{21}, \frac{82}{37}, \frac{86}{37}, \frac{5}{2}, \frac{94}{37}, \frac{8}{3}, \frac{41}{14}, 3, \frac{123}{37}, \frac{10}{3}, \frac{82}{21}, 4, \frac{86}{21},$$

$$\frac{164}{37}, \frac{172}{37}, 5, \frac{205}{37}, \frac{41}{7}, 6, \frac{43}{7}, \frac{246}{37}, \frac{20}{3}, \frac{41}{6}, \frac{43}{6}, 8, 10, \frac{410}{37}, \frac{82}{7}, 12, \frac{86}{7}, \frac{41}{3},$$

$$\frac{43}{3}, \frac{47}{3}, \frac{1763}{111}, 20, \frac{41}{2}, \frac{43}{2}, \frac{82}{3}, \frac{86}{3}, \frac{94}{3}, \frac{3526}{111}, 41, 43, 47, \frac{1763}{37}, \frac{164}{3}, \frac{172}{3},$$

$$\left.\frac{205}{3}, 82, 86, 94, \frac{3526}{37}, 123, 164, 172, 205, 246, 410, \frac{1763}{3}, \frac{3526}{3}, 1763, 3526\right]$$

```
> ############################ Section 4.8 Lemma 4.16
  #################

  Digits:= 10: eta:=7.5*10^(-7): 1.9769*(1+eta), # cf. (4.63),(4.64)
  and (4.55)
  2.^(1/4)*log(10.^38)^2/(5*10.^37)^(1/12)/32/evalf(Pi),    1.977 +
  0.066;
```

$$1.976901483, 0.06536953335, 2.043 \tag{50}$$

```
> "log(t)/t^(1/4) is decreasing for t > ",exp(1./(1/4)); # cf. (1.12)
  0.89*(10.^9)^(1/4),    log(10.^9);    # proof of the lower bound of
  (4.55)
  1.94*2.^(1/4),    2.31/(10.^9)^(1/12),    0.945-0.411;
```

"log(t)/t^(1/4) is decreasing for t > ", 54.59815003

158.2668675, 20.72326584

2.307061803, 0.4107825437, 0.534 **(51)**

```
> 1/8/evalf(Pi)+sqrt(2.)/log(10.^9)^2 +2.043/(10.^9)^(1/6)/log(10.^9)
  ^2,
  1/8/evalf(Pi),    0.0433*log(10.^9)^2/sqrt(10.^9); # proof of
  (4.66)
```

0.04323222393, 0.03978873576, 0.0005880365121 **(52)**

```
> (1-0.00059)-1/1.0006; # this difference is positive, proof of the
  lower bound of (4.56)
```

$9.6402\ 10^{-6}$ **(53)**

```
> 1+0.0006/log(10.^9), # proof of (4.57)
  1-log(1.0006)/log(10.^9)-1/1.00003; # this difference is positive
```

$1.000028953, 1.05481696\ 10^{-6}$ **(54)**

```
> 0.0433*sqrt(1.0006)*1.00003^2;# proof of (4.58)
```

0.04331558687 **(55)**

```
> t:='t': int((log(t)+2)/(2*t^(3/2)*log(t)^2),t), # proof of (4.59)
  0.044*1.0006^(3/2)/2/log(10.^9)*(1+2/log(10.^9)),
  0.0012*log(10.^9)^2/(10.^9)^(1/3);
```

$-\frac{1}{\sqrt{t}\,\ln(t)}, 0.001165112285, 0.0005153444965$ **(56)**

```
> t:='t': int((log(t)+4)/(2*t^(3/2)*log(t)^3),t), # proof of (4.60)
  0.044*(1.0006)^(3/2)/2/log(10.^9)^2*(1+4/log(10.^9)),
  0.000062*log(10.^9)^2/1000;
```

$-\frac{1}{\sqrt{t}\,\ln(t)^2}, 0.00006117087221, 0.00002662613232$ **(57)**

```
> t:='t': int((log(t)+1)/(t^2*log(t)^2),t), # proof of (4.61)
  0.044*1.0006^2/log(10.^9)*(1+1/log(10.^9)),
  0.0023*log(10.^9)^2/(10.^9)^(1/3),    0.00099*1.0006^(7/6);
```

$-\frac{1}{\ln(t)\,t}, 0.002228344812, 0.0009877436183, 0.0009906930346$ **(58)**

```
> 1.0006^(1/3)*log(10.^9)^3/1000, # proof of (4.62)
  0.0521*0.044*(1.0006)^(7/6)*8.91;
```

8.901463748, 0.02043958241 **(59)**

```
> ################ Proof of Lemma 4.17 ################

  Digits:=10:   sqrt(2.)*log(2.)/2,   0.001*sqrt(2.); # proof of
  (4.68)
```

0.4901290716, 0.001414213562 **(60)**

```
> sqrt(2.)+2.043/10.^(9/6),    1.48*sqrt(1.0006),    # estimate of u
  1.49/sqrt(10.^9),   2*(1-0.00005),   # estimate of u_0
  1.49^2/1.9999;
```

1.478818895, 1.480443933, 0.00004711793713, 1.99990, 1.110105505 **(61)**

```
> 2.043+1.12/1000,    # estimate of utilda
  sqrt(2.)+2.05/10.^(9/6),    1.48*sqrt(1.0006),
  1.49/sqrt(10.^9)/log(10.^9),  2*(1-0.000003);   # estimate of
  utilda_0
```

$2.044120000, 1.479040254, 1.480443933, 2.273673343\ 10^{-6}, 1.999994$ **(62)**

```
> 2.05+1.49^2/1.999994/log(10.^9)/1000,   2.06*1.0006/log(10.^9),
  0.1+0.001*sqrt(2.),  sqrt(2.)*0.323; # proof of (4.69)
```
2.050053566, 0.09946482451, 0.1014142136, 0.4567909805 **(63)**

```
> ######################## Section 5, proof of Proposition 5.1
  #################

  Digits:=15:  1.26*3.^(1/3)/(1-1/(10.^9+7))/log(1418.); # estimate
  of U_1, cf. (5.5)
```
0.250411159228125 **(64)**

```
> 0.086/(1-1/1418.^3); # estimate of U_2, cf. (5.6)
```
0.0860000000301627 **(65)**

```
> 2*log(44023.)^4/44023; # estimate of U_4
```
0.593829976589628 **(66)**

```
> 10.26+0.6+0.6, log(44023.)-11.46/2; # estimate of U_3 - U_4
```
11.46, 4.96246750359880 **(67)**

```
> 2.829+0.524; # proof of (5.7)
```
3.353 **(68)**

```
> 2/log(44023.),0.4+0.81*2.67; # proof of (5.8)
```
0.187047564028308, 2.5627 **(69)**

```
> 2.062-0.456; # proof of (5.9)
```
1.606 **(70)**

```
> 0.055-0.491; # proof of (5.10)
```
−0.436 **(71)**

```
> 3.353+0.436; # proof of (5.11)
```
3.789 **(72)**

```
> 2.5627-1.606, 0.251+0.087+0.11; # proof of (5.12)
```
0.9567, 0.448 **(73)**

```
> 0.0015*(1.0006)^(2/3), -0.448*1.0006^(2/3); # proof of (5.13) and
  (5.14)
```
0.00150059994001599, −0.448179182084776 **(74)**

```
> (2*sqrt(2.)-2)*0.00052+3.789*0.000027+0.021,  # proof of (5.15)
  (2*sqrt(2.)-2)*0.00052+0.9567*0.000027+0.021; # proof of (5.16)
```
0.0215330851048680, 0.0214566130048680 **(75)**

```
> 0.00151+0.0216; # proof of (5.17)-(5.18)
```
0.02311 **(76)**

```
> -0.449-0.0215;  # proof of (5.19)
```
−0.4705 **(77)**

```
> 2*sqrt(2.)-2+tau, # proof of (5.20)
  2*sqrt(2.)-2-tau-3.789/log(10.^9)-0.2311*log(10.^9)/10.^(9/6); #
  proof of (5.20)
```
0.874618542678442, 0.447951612621684 **(78)**

```
> t:='t': normal(diff(log(log(t))/t^(1/3),t)), log(log(18.))*log(18.)
  ;  # proof of (5.22)
  1.0006*(evalf(gamma)+log(log(10.^9)))/10^(9/3),    0.4705+0.0037;
```
$$-\frac{\ln(\ln(t))\ln(t)-3}{3\ln(t)\,t^{4/3}}, 3.06779760342970$$

0.00361063777109819, 0.4742 **(79)**

```
> 0.875^2/2/(1000*log(10.^9)^2); # end of the proof of Proposition
  5.1
```

$8.91394015583440\ 10^{-7}$ **(80)**

```
> ########################### Proof of Corollary 5.2
  ###################

  1.0006*exp(evalf(gamma))/10^(9/3);
```

0.00178214106144099 **(81)**

```
> ########################### Proof of Lemma 5.3
  #######################

  Digits:=10: log(55440.);
```

10.92305663 **(82)**

```
> Digits:=50: calculdec(1,10,1,1): # computation of ctilda(N) for N
  CA >= 12
  # If M < M' are two consecutive CA numbers of type 2 then displays
  c = ctilda(N)
  # for all CA numbers N s.t. M <= N < M'
  # To prove Lemma 5.3, execute calculdec(1,6703,100,0): running time
  about 72 h.
  # The result is the list lisctilda.
```

"q = ", 2, " al = ", 2, " p = ", 3, " sigNsN = ", 2.333333333, " logN = ", 2.484906650, " c = ", −3.398180969

"q = ", 2, " al = ", 2, " p = ", 5, " sigNsN = ", 2.800000000, " logN = ", 4.094344562, " c = ", −1.332112699

"******j=", 1, "pmin=", 3, "cmin=", −3.398180969, "cmax=", −1.332112699, "pmax=", 5, "temps=", 0.029

"q = ", 2, " al = ", 3, " p = ", 5, " sigNsN = ", 3.000000000, " logN = ", 4.787491743, " c = ", −1.066027196

"******j=", 2, "pmin=", 5, "cmin=", −1.066027196, "cmax=", −1.066027196, "pmax=", 5, "temps=", 0.044

"q = ", 3, " al = ", 2, " p = ", 5, " sigNsN = ", 3.250000000, " logN = ", 5.886104031, " c = ", −0.8614476326

"q = ", 3, " al = ", 2, " p = ", 7, " sigNsN = ", 3.714285714, " logN = ", 7.832014181, " c = ", −0.6094796865

"******j=", 3, "pmin=", 5, "cmin=", −0.8614476326, "cmax=", −0.6094796865, "pmax=", 7, "temps=", 0.068

"q = ", 2, " al = ", 4, " p = ", 7, " sigNsN = ", 3.838095238, " logN = ", 8.525161361, " c = ", −0.5726809252

"q = ", 2, " al = ", 4, " p = ", 11, " sigNsN = ", 4.187012987, " logN = ", 10.92305663, " c = ", −0.5248805923

"q = ", 2, " al = ", 4, " p = ", 13, " sigNsN = ", 4.509090909, " logN = ", 13.48800599, " c = ", −0.4976624367

"******j=", 4, "pmin=", 7, "cmin=", −0.5726809252, "cmax=", −0.4976624367, "pmax=", 13, "temps=", 0.112

"q = ", 2, " al = ", 5, " p = ", 13, " sigNsN = ", 4.581818182, " logN = ", 14.18115317, " c = ",

-0.5005433268
"******j=", 5, "pmin=", 13, "cmin=", -0.5005433268, "cmax=", -0.5005433268, "pmax=", 13, "temps=", 0.129
"q = ", 3, " al = ", 3, " p = ", 13, " sigNsN = ", 4.699300699, " logN = ", 15.27976546, " c = ", -0.4964581984
"******j=", 6, "pmin=", 13, "cmin=", -0.4964581984, "cmax=", -0.4964581984, "pmax=", 13, "temps=", 0.146
"q = ", 5, " al = ", 2, " p = ", 13, " sigNsN = ", 4.855944056, " logN = ", 16.88920337, " c = ", -0.4974482771
"q = ", 5, " al = ", 2, " p = ", 17, " sigNsN = ", 5.141587824, " logN = ", 19.72241672, " c = ", -0.4508392373
"q = ", 5, " al = ", 2, " p = ", 19, " sigNsN = ", 5.412197709, " logN = ", 22.66685570, " c = ", -0.3938637821
"q = ", 5, " al = ", 2, " p = ", 23, " sigNsN = ", 5.647510653, " logN = ", 25.80234991, " c = ", -0.3678054302
"******j=", 7, "pmin=", 13, "cmin=", -0.4974482771, "cmax=", -0.3678054302, "pmax=", 23, "temps=", 0.201
"q = ", 2, " al = ", 6, " p = ", 23, " sigNsN = ", 5.692332166, " logN = ", 26.49549709, " c = ", -0.3679646547
"q = ", 2, " al = ", 6, " p = ", 29, " sigNsN = ", 5.888619483, " logN = ", 29.86279292, " c = ", -0.3810822121
"q = ", 2, " al = ", 6, " p = ", 31, " sigNsN = ", 6.078574950, " logN = ", 33.29678013, " c = ", -0.3789034836
"******j=", 8, "pmin=", 29, "cmin=", -0.3810822121, "cmax=", -0.3679646547, "pmax=", 23, "temps=", 0.254
"q = ", 7, " al = ", 2, " p = ", 31, " sigNsN = ", 6.187120931, " logN = ", 35.24269028, " c = ", -0.3629582112
"******j=", 9, "pmin=", 31, "cmin=", -0.3629582112, "cmax=", -0.3629582112, "pmax=", 31, "temps=", 0.272
"q = ", 3, " al = ", 4, " p = ", 31, " sigNsN = ", 6.238680272, " logN = ", 36.34130256, " c = ", -0.3665518795
"q = ", 3, " al = ", 4, " p = ", 37, " sigNsN = ", 6.407293252, " logN = ", 39.95222048, " c = ", -0.3626491413
"q = ", 3, " al = ", 4, " p = ", 41, " sigNsN = ", 6.563568698, " logN = ", 43.66579254, " c = ", -0.3640020265
"q = ", 3, " al = ", 4, " p = ", 43, " sigNsN = ", 6.716209830, " logN = ", 47.42699266, " c = ", -0.3526741340
"******j=", 10, "pmin=", 31, "cmin=", -0.3665518795, "cmax=", -0.3526741340, "pmax=", 43, "temps=", 0.325
"jmaxabs = ", 10, "pmaxabs = ", 43, "cmaxabs = ", -0.3526741340, "total time = ", 0.331 **(83)**

```
> Digits:=50: maxi:=-infinity: # find the maximum of ctilda(N)
  for j from 1 to 6703 do
     lisj:=lisctilda[j];
     if lisj[4] > maxi then maxi:=lisj[4]; pmaxi:=lisj[5]; fi;
  od: "maxi = ",evalf(maxi,20),"pmaxi = ",pmaxi; calcullogN(pmaxi,1):
```

"maxi = ", -0.098377958771985568059, "pmaxi = ", 504457601

"xi = ", 504457601, "eps = F(xi,1) = ", $9.89234846896803\ 10^{-11}$, "xiN = ", $5.04457601\ 10^{8}$

"epsmax = ", [504457601, 1], $9.89234846896803\ 10^{-11}$, "epsmin = ", [504457661, 1], $9.89234723366078\ 10^{-11}$

"N = ", $6.34735810272381\ 10^{219087008}$, "prime factors of N = ", $(2)^{32}$, $(3)^{20}$, $(5)^{13}$, $(7)^{11}$, $(11)^{9}$, $(13)^{8}$, $(17)^{7}$, $(19)^{7}$, $(23)^{6}$, " ... ", $(37)^{6}$, $(41)^{5}$, " ... ", $(73)^{5}$, $(79)^{4}$, " ... ", $(199)^{4}$, $(211)^{3}$, " ... ", $(1123)^{3}$, $(1129)^{2}$, " ... ", $(31249)^{2}$, (31253), " ... ", (504457601)

"q = ", 31249, "expq = ", 2, "p = ", 504457601, "K(N) = ", 32

"sigNsN = ", 35.6908715732266, "logN = ", $5.04466480537506\ 10^{8}$, "loglogN = ", 20.0390119545943 **(84)**

```
> nops(lisctilda), lisctilda[3694];
```

6703, [3694, 504413407, $-0.098379723870700230905406315925520229988418716365788$, $-0.098377958771985568058597609108698278088850350228342$, 504457601] **(85)**

```
> calcc:=proc(n) local signsn,logn,lln,sqlogn,a,b,c; global ega,tau;
     signsn:=numtheory[sigma](n)/n; a:=2*(sqrt(2.)-1)+tau; b:=3.789;
     logn:=log(evalf(n));   lln:=log(logn);   sqlogn:=sqrt(logn);
     c:=solve(signsn/ega=lln-a/sqlogn+b/sqlogn/lln+t*lln/logn^(2/3),
  t);
  end: # computation of ctilda(n)
> cmax:=-infinity:   ti:=time():   # computation of ctilda(n) for 2
  <= n <= 55440
  for n from 2 to 55440 do
     c:=calcc(n);
     if c > cmax then cmax:=c; nmax:=n; fi;
  od: "cmax = ",evalf(cmax,10), "nmax = ",nmax,time()-ti;
```

"cmax = ", -0.5248805923, "nmax = ", 55440, 230.754 **(86)**

```
> ############################## Proof of Lemma 6.1
  #####################

  a:='a': t:='t': f:=log(t)-a/sqrt(t):  f1:=diff(f,t):  f2:=diff(f1,
  t): f1,f2;
```

$$\frac{1}{t}+\frac{a}{2\,t^{3/2}},\ -\frac{1}{t^2}-\frac{3\,a}{4\,t^{5/2}} \qquad \textbf{(87)}$$

```
> ############################## Proof of Lemma 6.2
  #####################

> Digits:=10:   log(log(8.))-1/sqrt(log(8.));
```

0.0386310224 **(88)**

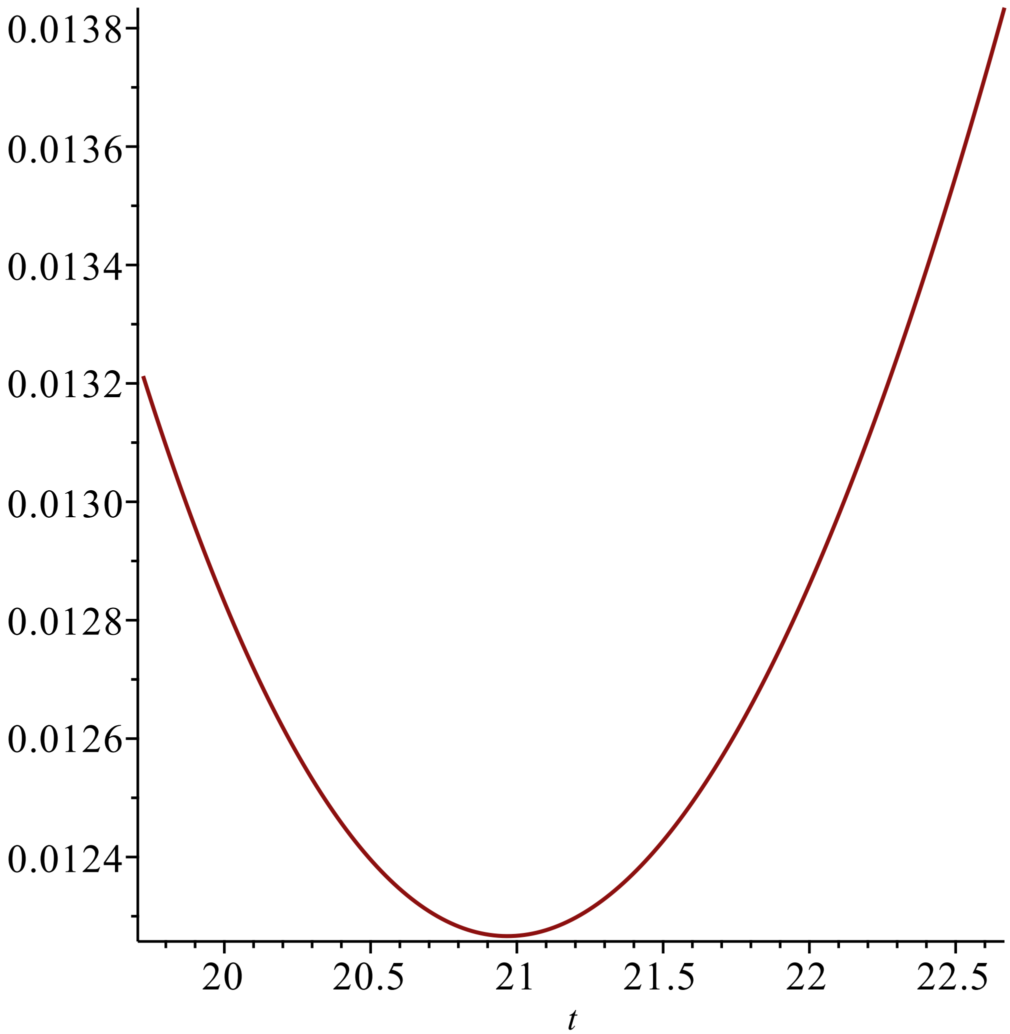
```
> a:='a':eps:='eps': g:=eps*t-log(log(t)-a/sqrt(t)): g1:=diff(g,t):
  g2:=diff(g1,t):
  g,g1,g2;
```

$$eps\, t - \ln\left(\ln(t) - \frac{a}{\sqrt{t}}\right), eps - \frac{\frac{1}{t} + \frac{a}{2\, t^{3/2}}}{\ln(t) - \frac{a}{\sqrt{t}}}, -\frac{-\frac{1}{t^2} - \frac{3\, a}{4\, t^{5/2}}}{\ln(t) - \frac{a}{\sqrt{t}}} + \frac{\left(\frac{1}{t} + \frac{a}{2\, t^{3/2}}\right)^2}{\left(\ln(t) - \frac{a}{\sqrt{t}}\right)^2} \tag{89}$$

```
> # example of computation of an upper bound of ben(n) by (6.2)
  Np:=lisCA[15][7]:  Npf:=evalf(Np): # Np < N are two consecutive CA
  numbers
  N:=lisCA[16][7]: sigNsN:=lisCA[16][6]: eps0:=lisCA[16][1]: # eps0
  is a parameter of N
  Nf:=evalf(N):  sigNsNf:=evalf(sigNsN):  a0:=0.59:
  "Np = ",Np,"N = ",N,"sigNsN = ",sigNsN,"a = ",a0,"eps = ",evalf
  (eps0,6);
  B:=subs(a=a0,eps=eps0,g)-gamma-eps0*log(Nf)+log(sigNsNf):  # cf.
  (6.2)
  plot(B,t=log(Np)..log(N)); # B(log(n) is an upper bound of ben_eps0
  (n)
                             # for an n such that alpha(n) <= a0
```

"Np = ", 367567200, "N = ", 6983776800, "sigNsN = ", $\frac{249984}{46189}$, "a = ", 0.59, "eps = ",

0.0174204




```
> t0:=fsolve(subs(a=a0,eps=eps0,g1)=0,t=log(Np)..log(N));
  "mingt0 = ",evalf(subs(t=t0,B)),
  "maxgNp = ",evalf(subs(t=log(Npf),B)),   "maxgN = ",evalf(subs(t=
  log(Nf),B));
```

$$t0 := 20.96756253$$

"mingt0 = ", 0.0122664121, "maxgNp = ", 0.0132123741, "maxgN = ", 0.0138346331 **(90)**

```
> ################# proof of Lemma 6.3 ####################

  Digits:=10:   -2*(sqrt(2.)-1)+tau+3.789/log(10.^9)+0.026*log(10.^9)
  /10.^(9/6);
```

$$-0.5823592218$$ **(91)**

```
> Digits:=50: maxi:=-infinity: # find the maximum of alpha(N)
  for j from 1 to 6703 do
     lisj:=lisalpha[j];
     if lisj[4] > maxi then maxi:=lisj[4]; pmaxi:=lisj[5]; fi;
  od: "maxi = ",evalf(maxi,20),"pmaxi = ",pmaxi;    calcullogN(pmaxi,
  1):
```

"maxi = ", 0.92019027627094120029, "pmaxi = ", 1019

"xi = ", 1019, "eps = F(xi,1) = ", 0.000141610069575099, "xiN = ", 1019.

"epsmax = ", [1019, 1], 0.000141610069575099, "epsmin = ", [1021, 1],

0.000141292813097974

"N = ", $1.35299473675479\ 10^{450}$, "prime factors of N = ", $(2)^{12}$, $(3)^{7}$, $(5)^{5}$, $(7)^{4}$, $(11)^{3}$,
$(13)^{3}$, $(17)^{2}$, " ... ", $(41)^{2}$, (43), " ... ", (1019)

"q = ", 41, "expq = ", 2, "p = ", 1019, "K(N) = ", 12

"sigNsN = ", 12.3160966345326, "logN = ", 1036.46561230645, "loglogN = ", 6.94357175458302 **(92)**

```
> Digits:=50: calculdealpha(1,12,1,2):
  # computes alpha(N) for the 34 first CA numbers N but N = 1,2,6
  # To prove Lemma 6.3 execute calculdealpha(1,6703,100,0): running
  time about 76h
```

"N = ", 12, " p = ", 3, " sigNsN = ", 2.333333333, " logN = ", 2.484906650, " alpha(N) = ", −0.6302865758

"N = ", 60, " p = ", 5, " sigNsN = ", 2.800000000, " logN = ", 4.094344562, " alpha(N) = ", −0.3287697545

"******j=", 1, "pmin=", 3, "alphamin=", −0.6302865758, "alphamax=", −0.3287697545, "pmax=", 5, "temps=", 0.023

"N = ", 120, " p = ", 5, " sigNsN = ", 3.000000000, " logN = ", 4.787491743, " alpha(N) = ", −0.2590015407

"******j=", 2, "pmin=", 5, "alphamin=", −0.2590015407, "alphamax=", −0.2590015407, "pmax=", 5, "temps=", 0.040

"N = ", 360, " p = ", 5, " sigNsN = ", 3.250000000, " logN = ", 5.886104031, " alpha(N) = ", −0.1265202179

"N = ", 2520, " p = ", 7, " sigNsN = ", 3.714285714, " logN = ", 7.832014181, " alpha(N) = ", −0.07612462493

"******j=", 3, "pmin=", 5, "alphamin=", −0.1265202179, "alphamax=", −0.07612462493, "pmax=", 7, "temps=", 0.065

"N = ", 5040, " p = ", 7, " sigNsN = ", 3.838095238, " logN = ", 8.525161361, " alpha(N) = ", −0.03478348953

"N = ", 55440, " p = ", 11, " sigNsN = ", 4.187012987, " logN = ", 10.92305663, " alpha(N) = ", 0.1323247680

"N = ", 720720, " p = ", 13, " sigNsN = ", 4.509090909, " logN = ", 13.48800599, " alpha(N) = ", 0.2575558825

"******j=", 4, "pmin=", 7, "alphamin=", −0.03478348953, "alphamax=", 0.2575558825, "pmax=", 13, "temps=", 0.098

"N = ", 1441440, " p = ", 13, " sigNsN = ", 4.581818182, " logN = ", 14.18115317, " alpha(N) = ", 0.2990357813

"******j=", 5, "pmin=", 13, "alphamin=", 0.2990357813, "alphamax=", 0.2990357813, "pmax=", 13, "temps=", 0.113

"N = ", 4324320, " p = ", 13, " sigNsN = ", 4.699300699, " logN = ", 15.27976546, " alpha(N) = ", 0.3442304680

"******j=", 6, "pmin=", 13, "alphamin=", 0.3442304680, "alphamax=", 0.3442304680, "pmax=", 13, "temps=", 0.125

"N = ", 21621600, " p = ", 13, " sigNsN = ", 4.855944056, " logN = ", 16.88920337, " alpha(N) = ", 0.4120280169

"N = ", 367567200, " p = ", 17, " sigNsN = ", 5.141587824, " logN = ", 19.72241672, " alpha(N) = ", 0.4217284365

"N = ", 6983776800, " p = ", 19, " sigNsN = ", 5.412197709, " logN = ", 22.66685570, " alpha(N) = ", 0.3912282441

"N = ", $1.606268664\ 10^{11}$, " p = ", 23, " sigNsN = ", 5.647510653, " logN = ", 25.80234991, " alpha(N) = ", 0.4044234073

"******j=", 7, "pmin=", 19, "alphamin=", 0.3912282441, "alphamax=", 0.4217284365, "pmax=", 17, "temps=", 0.193

"N = ", $3.212537328\ 10^{11}$, " p = ", 23, " sigNsN = ", 5.692332166, " logN = ", 26.49549709, " alpha(N) = ", 0.4167364287

"N = ", $9.316358251\ 10^{12}$, " p = ", 29, " sigNsN = ", 5.888619483, " logN = ", 29.86279292, " alpha(N) = ", 0.4939642677

"N = ", $2.888071058\ 10^{14}$, " p = ", 31, " sigNsN = ", 6.078574950, " logN = ", 33.29678013, " alpha(N) = ", 0.5342589702

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"N = ", $2.021649741\ 10^{15}$, " p = ", 31, " sigNsN = ", 6.187120931, " logN = ", 35.24269028, " alpha(N) = ", 0.5250314374

"******j=", 9, "pmin=", 31, "alphamin=", 0.5250314374, "alphamax=", 0.5250314374, "pmax=", 31, "temps=", 0.259

"N = ", $6.064949222\ 10^{15}$, " p = ", 31, " sigNsN = ", 6.238680272, " logN = ", 36.34130256, " alpha(N) = ", 0.5436913001

"N = ", $2.244031212\ 10^{17}$, " p = ", 37, " sigNsN = ", 6.407293252, " logN = ", 39.95222048, " alpha(N) = ", 0.5704418439

"N = ", $9.200527969\ 10^{18}$, " p = ", 41, " sigNsN = ", 6.563568698, " logN = ", 43.66579254, " alpha(N) = ", 0.6038870108

"N = ", $3.956227027\ 10^{20}$, " p = ", 43, " sigNsN = ", 6.716209830, " logN = ", 47.42699266, " alpha(N) = ", 0.6081793561

"******j=", 10, "pmin=", 31, "alphamin=", 0.5436913001, "alphamax=", 0.6081793561, "pmax=", 43, "temps=", 0.321

"N = ", $7.912454053\ 10^{20}$, " p = ", 43, " sigNsN = ", 6.742651601, " logN = ", 48.12013984, " alpha(N) = ", 0.6102717490

"N = ", $3.718853405\ 10^{22}$, " p = ", 47, " sigNsN = ", 6.886112273, " logN = ", 51.97028744,

" alpha(N) = ", 0.6084367998

"N = ", $1.970992305\ 10^{24}$, " p = ", 53, " sigNsN = ", 7.016038920, " logN = ", 55.94057936,
" alpha(N) = ", 0.6362562023

"N = ", $1.162885460\ 10^{26}$, " p = ", 59, " sigNsN = ", 7.134954834, " logN = ", 60.01811680,
" alpha(N) = ", 0.6868487398

"******j=", 11, "pmin=", 47, "alphamin=", 0.6084367998, "alphamax=", 0.6868487398,
"pmax=", 59, "temps=", 0.371

"N = ", $5.814427299\ 10^{26}$, " p = ", 59, " sigNsN = ", 7.180986801, " logN = ", 61.62755471,
" alpha(N) = ", 0.7008449240

"N = ", $3.546800652\ 10^{28}$, " p = ", 61, " sigNsN = ", 7.298707896, " logN = ", 65.73842858,
" alpha(N) = ", 0.7115100315

"N = ", $2.376356437\ 10^{30}$, " p = ", 67, " sigNsN = ", 7.407643834, " logN = ", 69.94312120,
" alpha(N) = ", 0.7408999030

"N = ", $1.687213070\ 10^{32}$, " p = ", 71, " sigNsN = ", 7.511976846, " logN = ", 74.20580107,
" alpha(N) = ", 0.7681497915

"N = ", $1.231665541\ 10^{34}$, " p = ", 73, " sigNsN = ", 7.614880639, " logN = ", 78.49626051,
" alpha(N) = ", 0.7761552501

"******j=", 12, "pmin=", 59, "alphamin=", 0.7008449240, "alphamax=", 0.7761552501,
"pmax=", 73, "temps=", 0.446

"jmaxabs = ", 12, "pmaxabs = ", 73, "alphamaxabs = ", 0.7761552501, "total time = ", 0.452 **(93)**

```
> lisalpha[11]; # lisalpha[j] = [j,pmin,alphamin,alphamax,pmax]
                # alphamin is the minimum of alpha(N) for lisCA2[j]
  <= N < lisCA2[j+1]
                # alphamax is the maximum of alpha(N) for lisCA2[j]
  <= N < lisCA2[j+1]
                # N is a CA number. lisCA2[j] is the jth CA number of
  type 2
```

[11, 47, 0.60843679983847274863932478827521260192380048034959, **(94)**
0.68684873978232983923997858799284619471553635588955, 59]

```
> calculdealpha(34,35,1,2); # displays the largest value of alpha(N)
  for a CA number N <= N0
                            # this value has been obtained by
  calculdealpha(1,6703,100,0):
```

"N = ", $4.810464883\ 10^{406}$, " p = ", 919, " sigNsN = ", 12.13320312, " logN = ", 936.4203415,
" alpha(N) = ", 0.9107618420

"N = ", $4.468921876\ 10^{409}$, " p = ", 929, " sigNsN = ", 12.14626362, " logN = ", 943.2544502,
" alpha(N) = ", 0.9121958641

"******j=", 34, "pmin=", 919, "alphamin=", 0.9107618420, "alphamax=", 0.9121958641,
"pmax=", 929, "temps=", 0.018

"N = ", $1.832257969\ 10^{411}$, " p = ", 929, " sigNsN = ", 12.15331720, " logN = ", 946.9680223,

" alpha(N) = ", 0.9130342331

"N = ", 1.716825717 10^{414}, " p = ", 937, " sigNsN = ", 12.16628766, " logN = ", 953.8107056,
" alpha(N) = ", 0.9137797139

"N = ", 1.615533000 10^{417}, " p = ", 941, " sigNsN = ", 12.17921676, " logN = ", 960.6576487,
" alpha(N) = ", 0.9137589944

"N = ", 1.529909751 10^{420}, " p = ", 947, " sigNsN = ", 12.19207760, " logN = ", 967.5109478,
" alpha(N) = ", 0.9135225521

"N = ", 1.458003992 10^{423}, " p = ", 953, " sigNsN = ", 12.20487097, " logN = ", 974.3705627,
" alpha(N) = ", 0.9130715450

"N = ", 1.409889861 10^{426}, " p = ", 967, " sigNsN = ", 12.21749234, " logN = ", 981.2447612,
" alpha(N) = ", 0.9145279524

"N = ", 1.369003055 10^{429}, " p = ", 971, " sigNsN = ", 12.23007472, " logN = ", 988.1230877,
" alpha(N) = ", 0.9152394938

"N = ", 1.337515985 10^{432}, " p = ", 977, " sigNsN = ", 12.24259271, " logN = ", 995.0075743,
" alpha(N) = ", 0.9157326808

"N = ", 1.314778213 10^{435}, " p = ", 983, " sigNsN = ", 12.25504703, " logN = ", 1001.898183,
" alpha(N) = ", 0.9160087452

"N = ", 1.302945209 10^{438}, " p = ", 991, " sigNsN = ", 12.26741337, " logN = ", 1008.796898,
" alpha(N) = ", 0.9165793916

"N = ", 1.299036373 10^{441}, " p = ", 997, " sigNsN = ", 12.27971770, " logN = ", 1015.701649,
" alpha(N) = ", 0.9169330864

"N = ", 1.310727701 10^{444}, " p = ", 1009, " sigNsN = ", 12.29188789, " logN = ",
1022.618364, " alpha(N) = ", 0.9185673900

"N = ", 1.327767161 10^{447}, " p = ", 1013, " sigNsN = ", 12.30402203, " logN = ",
1029.539035, " alpha(N) = ", 0.9194876162

"N = ", 1.352994737 10^{450}, " p = ", 1019, " sigNsN = ", 12.31609663, " logN = ",
1036.465612, " alpha(N) = ", 0.9201902763

"N = ", 1.381407626 10^{453}, " p = ", 1021, " sigNsN = ", 12.32815941, " logN = ",
1043.394150, " alpha(N) = ", 0.9197001193

"******j=", 35, "pmin=", 929, "alphamin=", 0.9130342331, "alphamax=", 0.9201902763,
"pmax=", 1019, "temps=", 0.197

"jmaxabs = ", 35, "pmaxabs = ", 1019, "alphamaxabs = ", 0.9201902763, "total time = ", 0.203

[[34, 919, 0.91076184195392898648346320327782052459342719285114, **(95)**
0.91219586413522343596286513552897764580685101456257, 929], [35, 929,
0.91303423312912968562194027786121982011541907103530,

0.920190276270941200287050217222270562231562816664467, 1019]]

```
> ############### proof of Corollary 1.2 (i), (1.10), Section 6.2
  ###########

  Digits:=15: N_eps_24:=lisCA[24][7]: "The 24th CA number is",
  N_eps_24, " = ";
  ifactor(N_eps_24)," =",evalf(N_eps_24,4);
  "the common parameter of N_eps_24 and N_eps_23 is F(41,1) = ",evalf
  (F(41,1));
  "L_alpha(N_eps_24,0.582) = ",Lalpha(24,0.582,0);
```

"The 24th CA number is", 9200527969062830400, " = "

$(2)^6\,(3)^4\,(5)^2\,(7)^2\,(11)\,(13)\,(17)\,(19)\,(23)\,(29)\,(31)\,(37)\,(41)$, " =", $9.201\ 10^{18}$

"the common parameter of N_eps_24 and N_eps_23 is F(41,1) = ", 0.00648904912742070

"L_alpha(N_eps_24,0.582) = ", [ ] **(96)**

```
> ############### proof of Corollary 1.2 (ii), Figure 1, Section 6.3
  ###########

  Digits:=15: L_hat:=Lhat(9,24,0.582,2):
  "L^hat = ", map(u->u[2],sort(L_hat,proc(u,v) u[2]< v[2] end));
  # compute the list L^{hat} defined in 6.2
  # L_ben is the list L_ben(N,0.582) and L_alpha is L_alpha(N,0.582)
```

"i = ", 24, "Np = ", 224403121196654400, "N = ", 9200527969062830400, "alp = ", 0.582

"nops(L_ben) = ", 3, "nops(L_alpha) = ", 0

"i = ", 23, "Np = ", 6064949221531200, "N = ", 224403121196654400, "alp = ", 0.582

"nops(L_ben) = ", 5, "nops(L_alpha) = ", 3

"i = ", 22, "Np = ", 2021649740510400, "N = ", 6064949221531200, "alp = ", 0.582

"nops(L_ben) = ", 3, "nops(L_alpha) = ", 2

"i = ", 21, "Np = ", 288807105787200, "N = ", 2021649740510400, "alp = ", 0.582

"nops(L_ben) = ", 5, "nops(L_alpha) = ", 4

"i = ", 20, "Np = ", 9316358251200, "N = ", 288807105787200, "alp = ", 0.582

"nops(L_ben) = ", 18, "nops(L_alpha) = ", 9

"i = ", 19, "Np = ", 321253732800, "N = ", 9316358251200, "alp = ", 0.582

"nops(L_ben) = ", 21, "nops(L_alpha) = ", 12

"i = ", 18, "Np = ", 160626866400, "N = ", 321253732800, "alp = ", 0.582

"nops(L_ben) = ", 2, "nops(L_alpha) = ", 1

"i = ", 17, "Np = ", 6983776800, "N = ", 160626866400, "alp = ", 0.582

"nops(L_ben) = ", 16, "nops(L_alpha) = ", 11

"i = ", 16, "Np = ", 367567200, "N = ", 6983776800, "alp = ", 0.582

"nops(L_ben) = ", 14, "nops(L_alpha) = ", 12

"i = ", 15, "Np = ", 21621600, "N = ", 367567200, "alp = ", 0.582

"nops(L_ben) = ", 24, "nops(L_alpha) = ", 16

"i = ", 14, "Np = ", 4324320, "N = ", 21621600, "alp = ", 0.582

"nops(L_ben) = ", 17, "nops(L_alpha) = ", 11

"i = ", 13, "Np = ", 1441440, "N = ", 4324320, "alp = ", 0.582

"nops(L_ben) = ", 9, "nops(L_alpha) = ", 8

"i = ", 12, "Np = ", 720720, "N = ", 1441440, "alp = ", 0.582

"nops(L_ben) = ", 9, "nops(L_alpha) = ", 7

"i = ", 11, "Np = ", 55440, "N = ", 720720, "alp = ", 0.582

"nops(L_ben) = ", 54, "nops(L_alpha) = ", 32

"i = ", 10, "Np = ", 5040, "N = ", 55440, "alp = ", 0.582

"nops(L_ben) = ", 64, "nops(L_alpha) = ", 32

"i = ", 9, "Np = ", 2520, "N = ", 5040, "alp = ", 0.582

"nops(L_ben) = ", 13, "nops(L_alpha) = ", 1

"nops(L_hat) = ", 161, "number of CA, SA, other = ", 15, 58, 88

"L^hat = ", [5040, 6300, 6720, 7560, 7920, 8400, 9240, 9360, 10080, 10920, 12600, 13860, 15120, 15840, 16380, 16800, 17640, 18480, 20160, 21840, 22680, 25200, 27720, 30240, 32760, 35280, 36960, 37800, 40320, 42840, 45360, 50400, 55440, 60480, 65520, 75600, 83160, 85680, 90720, 95760, 98280, 100800, 110880, 131040, 138600, 151200, 163800, 166320, 196560, 221760, 262080, 277200, 327600, 332640, 360360, 388080, 393120, 415800, 443520, 471240, 498960, 554400, 655200, 665280, 720720, 831600, 942480, 997920, 1053360, 1081080, 1108800, 1441440, 1663200, 1801800, 1884960, 2162160, 2827440, 2882880, 3603600, 4324320, 5405400, 5765760, 6486480, 7207200, 8648640, 10810800, 12252240, 12972960, 13693680, 14414400, 21621600, 24504480, 27387360, 36756720, 41081040, 43243200, 49008960, 61261200, 73513440, 82162080, 122522400, 136936800, 147026880, 183783600, 205405200, 245044800, 367567200, 410810400, 465585120, 698377680, 735134400, 1102701600, 1163962800, 1396755360, 2327925600, 2793510720, 3491888400, 4655851200, 6983776800, 8454045600, 13967553600, 20951330400, 27935107200, 32125373280, 41902660800, 48886437600, 53542288800, 80313433200, 97772875200, 160626866400, 321253732800, 481880599200, 642507465600, 963761198400, 1124388064800, 1927522396800, 2248776129600, 3373164194400, 4497552259200, 4658179125600, 4979432858400, 6746328388800, 9316358251200, 9958865716800, 13974537376800, 18632716502400, 27949074753600, 32607253879200, 65214507758400, 144403552893600, 195643523275200, 288807105787200, 577614211574400, 866421317361600, 1010824870255200, 2021649740510400, 4043299481020800, 6064949221531200, 12129898443062400, 74801040398884800, 224403121196654400] **(97)**

```
> L_hat[1..10];
```

$$\left[\left["\ **n = ", 5040, \frac{403}{105}, -0.0347834895254546\right], \left["\ n = ", 6300, \frac{806}{225}, 0.466104419770151\right], \left["\ n = ", 6720, \frac{127}{35}, 0.412387009732545\right], \left["\ n = ", 7560, \frac{80}{21}, \right.\right. \tag{98}$$

$0.151193721072842\Big], \Big[$" n = ", 7920, $\frac{403}{110}$, $0.412571246633370\Big], \Big[$" n = ", 8400, $\frac{1922}{525}$,
$0.438089419348736\Big], \Big[$" n = ", 9240, $\frac{288}{77}$, $0.337545919211002\Big], \Big[$" n = ", 9360, $\frac{217}{60}$,
$0.551893049840840\Big], \Big[$" *n = ", 10080, $\frac{39}{10}$, $0.0956379758739210\Big], \Big[$" n = ", 10920, $\frac{48}{13}$,
$0.477998959236476\Big]\Big]$

```
> Lnu(0.582): # computes the elements of Figure 1
```

"nops(L_hat) = ", 161, "nops(L_nu) = ", 16

"k = ", 1, " **n = ", 5040, $(2)^4$, $(3)^2$, (5), (7), "signsn = ", $\frac{403}{105}$, "alpha(n) = ",
-0.0347834895254546

"k = ", 2, " *n = ", 10080, $(2)^5$, $(3)^2$, (5), (7), "signsn = ", $\frac{39}{10}$, "alpha(n) = ",
0.0956379758739210

"k = ", 3, " **n = ", 55440, $(2)^4$, $(3)^2$, (5), "...", (11), "signsn = ", $\frac{1612}{385}$, "alpha(n) = ",
0.132324767962730

"k = ", 4, " *n = ", 110880, $(2)^5$, $(3)^2$, (5), "...", (11), "signsn = ", $\frac{234}{55}$, "alpha(n) = ",
0.216922188958564

"k = ", 5, " **n = ", 720720, $(2)^4$, $(3)^2$, (5), "...", (13), "signsn = ", $\frac{248}{55}$, "alpha(n) = ",
0.257555882476618

"k = ", 6, " **n = ", 1441440, $(2)^5$, $(3)^2$, (5), "...", (13), "signsn = ", $\frac{252}{55}$, "alpha(n) = ",
0.299035781319322

"k = ", 7, " *n = ", 2162160, $(2)^4$, $(3)^3$, (5), "...", (13), "signsn = ", $\frac{1984}{429}$, "alpha(n) = ",
0.318975688027496

"k = ", 8, " **n = ", 4324320, $(2)^5$, $(3)^3$, (5), "...", (13), "signsn = ", $\frac{672}{143}$, "alpha(n) = ",
0.344230467952388

"k = ", 9, " **n = ", 6983776800, $(2)^5$, $(3)^3$, $(5)^2$, (7), "...", (19), "signsn = ", $\frac{249984}{46189}$,
"alpha(n) = ", 0.391228244059934

"k = ", 10, " **n = ", 160626866400, $(2)^5$, $(3)^3$, $(5)^2$, (7), "...", (23), "signsn = ", $\frac{5999616}{1062347}$,
"alpha(n) = ", 0.404423407327998

"k = ", 11, " **n = ", 321253732800, $(2)^6$, $(3)^3$, $(5)^2$, (7), "...", (23), "signsn = ", $\frac{6047232}{1062347}$,

"alpha(n) = ", 0.416736428677165

"k = ", 12, " *n = ", 642507465600, $(2)^7$, $(3)^3$, $(5)^2$, (7), "...", (23), "signsn = ", $\frac{357120}{62491}$,

"alpha(n) = ", 0.491199055388484

"k = ", 13, " **n = ", 9316358251200, $(2)^6$, $(3)^3$, $(5)^2$, (7), "...", (29), "signsn = ",

$\frac{181416960}{30808063}$, "alpha(n) = ", 0.493964267675243

"k = ", 14, " **n = ", 2021649740510400, $(2)^6$, $(3)^3$, $(5)^2$, $(7)^2$, (11), "...", (31),

"signsn = ", $\frac{70225920}{11350339}$, "alpha(n) = ", 0.525031437410294

"k = ", 15, " **n = ", 6064949221531200, $(2)^6$, $(3)^4$, $(5)^2$, $(7)^2$, (11), "...", (31),

"signsn = ", $\frac{6437376}{1031849}$, "alpha(n) = ", 0.543691300126343

"k = ", 16, " **n = ", 224403121196654400, $(2)^6$, $(3)^4$, $(5)^2$, $(7)^2$, (11), "...", (37), **(99)**

"signsn = ", $\frac{244620288}{38178413}$, "alpha(n) = ", 0.570441843864993

```
> ############################# the end #########################
```