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PARTITIONS WITHOUT SMALL PARTS

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1. INTRODUCTION

Let us denote by $p(n)$ the number of non restricted partitions of n. It is known from Hardy and Ramanujan (cf. [Ram], §36 and [Rad], p. 278) that:

$$(1) \qquad p(n) = \frac{1}{2\pi\sqrt{2}} \frac{d}{dn} \frac{\exp(C\sqrt{n-1/24})}{\sqrt{n-1/24}} + f_1(n)$$

where the constant C is equal to $\pi\sqrt{2/3} = 2.565...$ and:

$$(2) \qquad f_1(n) = O\left(\frac{1}{n} \exp \frac{C\sqrt{n}}{2}\right) .$$

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In [Nic], it is proved that $|f_1(n)| \le \frac{1}{n} \exp\left(\frac{C\sqrt{n}}{2}\right)$ for $n > 26$. Actually, looking at the values of $f_1(n)$ for $1 \le n \le 25$, this inequality holds for all $n \ge 1$. With more accurate estimations, it can be shown that $|f_1(n)| \le \frac{0.11..}{n} \exp\left(\frac{C\sqrt{n}}{2}\right)$, for all $n \ge 1$, with equality for $n = 12$. Let us set:

$$(3) \qquad g(\lambda,t) = \left(\exp\left(C\,\frac{\sqrt{1-\lambda t^2}-1}{t}\right)\right)(C(1-\lambda t^2)^{-1} - t(1-\lambda t^2)^{-3/2}).$$

Expanding the derivative, formula (1) writes:

$$(4) \qquad p(n) = \frac{\exp(C\sqrt{n})}{4\pi\sqrt{2}n}\, g(1/24,\ 1/\sqrt{n}) + f_1(n).$$

The function $g(\lambda,t)$, with λ fixed, is analytic at $t = 0$. Using its Taylor expansion, we can get an asymptotic expansion of $p(n)$ according to powers of $1/\sqrt{n}$. In particular:

$$(5) \qquad p(n) = \frac{\exp(C\sqrt{n})}{4\sqrt{3}n}\,(1 + O(1/\sqrt{n})).$$

Now, we define $p(n,m)$ as the number of partitions of n whose parts are $\le m$; or, what is equivalent, as the number of partitions of n in at most m parts.

Finally, we define $r(n,m)$ as the number of partitions of n whose parts are $\ge m$. These notations are borrowed from the Gupta-Gwyther-Miller tables (cf. [Gup 1]); these tables give the formulas:

$$r(n,m) = r(n,m+1) + r(n-m,m); \qquad r(n,n) = 1.$$

These formulas were used by Gupta (cf. [Gup 4]) to calculate by hand the values of $r(n,m)$ and of $p(n) = r(n,1)$ for $n \le 300$. Let us mention also:

$$(6) \qquad r(n,m) = \sum_{t=0}^{\lfloor n/m \rfloor} p(n - tm, t) \qquad \text{([Gup 1], p.xiii)}$$

and it is easy to see that:

$$(7) \qquad r(n,m) \le p(n, \lfloor n/m \rfloor).$$

By definition, $\lfloor x \rfloor = \max\limits_{n\in\mathbb{Z},\, n\le x} n$

Using (6), Gupta, in [Gup 2], gave an asymptotic estimate of $r(n,m)$, valid for $n = O(m \log m)$.

Recently, J. Herzog gave in his thesis (cf. [Her], p. 57) the following estimate, valid for $m = O(n^{3/8} (\log n)^{3/4})$:

$$\log r(n,m) = C\sqrt{n} - \frac{1}{2} m \log n + m \log m - m\left(1 + \log\frac{\sqrt{6}}{\pi}\right) + O(n^{1/4}\sqrt{\log n}).$$

This result is obtained by using a Tauberian theorem.

On the other hand, Erdős and Szalay studied the analogous problem for unequal partitions. Let $q(n)$ be the number of unequal partitions of n, and $\rho(n,m)$ the number of unequal partitions of n whose parts are $\geq m$; they proved (cf. [Erd 2], p. 433) that, for $m \leq n^{1/5}$,

$$\rho(n,m) = (1 + o(1))\, q(n)/2^{m-1}$$

by integrating the generating function in the complex domain.

A more complicate asymptotic estimation of $\rho(n,m)$ has been given in [Erd 3], which is valid for $m \leq n^{3/8-\varepsilon}$

Finally, in [Odl], A. Odlyzko has studied $\Delta^k(p(n))$ where Δ^k is the k^{th} forward difference operator, with a method somewhat similar to that one which we shall use below. We shall prove:

THEOREM 1. *The relation*

$$r(n,m) = p(n)\left(\frac{C}{2\sqrt{n}}\right)^{m-1} (m-1)!\ \left(1 + O(m^2/\sqrt{n})\right)$$

holds uniformly for $1 \leq m \leq n^{1/4}$.

We hope to be able to improve theorem 1, by giving an asymptotic expansion of $r(n,m)$ for m small, and lower bounds and upper bounds for $r(n,m)$ for m slightly larger than $n^{1/4}$.

Let $n = n_1 + \dots + n_s$ be a partition of n. Let $k = 1,2,\dots n$. We say that the partition represents k if k can be written as a subsum $k = n_{i_1} + \dots + n_{i_t}$ $(1 \leq i_1 < \dots < i_t \leq s)$. We say that the partition is "practical" if it represents all integers $1, 2, \dots, n$.

P. Erdős and M. Szalay denoted by $M(n)$ the number of partitions of n which are not practical. In [Erd 1], the number of non practical partitions is proved to be:

$$M(n) = \left(1 + O\left(\frac{\log^{30} n}{\sqrt{n}}\right)\right) \frac{\pi}{\sqrt{6n}}\, p(n),$$

which shows that almost all partitions are practical. We shall denote $\tilde{p}(n) = p(n) - M(n)$ the number of practical partitions. We improve the result of P. Erdős and M. Szalay:

THEOREM 2. *Let* $b \in \mathbb{N}$, $b \geq 1$. *There exist real numbers* $\alpha_1, \dots, \alpha_b$, *such that the following asymptotic expansion is valid, as* $n \to +\infty$:

$$M(n) = \left(\sum_{a=1}^{b} \alpha_a\, n^{-a/2} + O(n^{-(b+1)/2})\right) p(n).$$

We have in particular:

$$\alpha_1 = \pi/\sqrt{6};\ \alpha_2 = \pi^2/4 - 1;$$

$$\alpha_3 = \frac{1}{\pi\sqrt{6}} \left(\frac{7}{36}\pi^4 - \frac{179}{48}\pi^2 + \frac{3}{2}\right);$$

$$\alpha_4 = \frac{1}{\pi^2} \left(\frac{197}{288}\pi^6 - \frac{83}{96}\pi^4 + \frac{89}{24}\pi^2 + \frac{3}{4}\right)$$

We were led to these problems by studying the number of fundamental invariants of binary forms: a lower bound for this number is the number of solutions of a diophantine equation related to those equations which are considered in the theory of partitions (cf. [Dix]).

2. PROOF OF THEOREM 1. First we recall the definition of the m^{th} Bessel polynomial:

$$y_m(x) = 1 + \sum_{k=1}^{m} a_k^{(m)} x^k$$

with

(8) $$a_k^{(m)} = \frac{1}{2^k\, k!} \prod_{j=-k+1}^{k} (m+j) .$$

In particular:

$$a_1^{(m)} = \frac{m(m+1)}{2} \quad \textit{and} \quad a_2^{(m)} = \frac{(m-1)\ m(m+1)(m+2)}{8} .$$

One finds in E. Grosswald's book (cf.[Gro]) many properties of Bessel polynomials. One has

$$y_0(x) = 1$$

$$y_1(x) = 1 + x$$

$$y_2(x) = 1 + 3x + 3x^2$$

$$y_3(x) = 1 + 6x + 15x^2 + 15x^3, \textit{ etc...}$$

They are characterized by several recurrence relations, in particular (cf. [Gro], p. 19):

(9) $$y_m(x) = (1 + mx)y_{m-1}(x) + x^2 y'_{m-1}(x).$$

With $y_0(x) = 1$, these relations can be used for defining y_m. It is rather easy to get (8) from (9).

LEMMA 1. *Let* $F(x) = (\exp(\sqrt{x}))/\sqrt{x}$; *for the* m^{th} *derivative of* F, *we have:*

$$F^{(m)}(x) = \frac{\exp\sqrt{x}}{2^m\, x^{(m+1)/2}}\, y_m(-1/\sqrt{x}).$$

PROOF. The formula is true for $m = 0$. Then one gives a recurrence proof, using (9). We are glad to thank A. Salinier, who observed to us that the polynomials occuring in $F^{(m)}(x)$ were the Bessel's polynomials. (Also cf. [Gro], p. 43. formula (6)).

LEMMA 2. *We have, for* $0 \leq x \leq \frac{2}{m(m+1)}$:

$$1 - \frac{m(m+1)}{2} x \leq y_m(-x) \leq 1.$$

PROOF. We have

$$y_m(-x) = \sum_{k=0}^{m} (-1)^k a_k^{(m)} x^k ,$$

with $a_0^{(m)} = 1$. Then, we get:

$$\frac{a_{k+1}^{(m)} x^{k+1}}{a_k^{(m)} x^k} = \frac{x}{2(k+1)} (m+k+1)(m-k) \leq x \frac{m(m+1)}{2} \leq 1.$$

So $y_m(-x)$ is smaller than its first term 1, and larger than the sum $1 - \frac{m(m+1)}{2} x$ of its two first terms. (Also cf. [Gro], p. 125).

LEMMA 3. *Let* $f:\mathbb{R}\to\mathbb{R}$, $m \geq 1$, *and* $u_1, u_2, \ldots, u_m \geq 0$. *We define inductively the operator* $D^{(m)}(u_1,\ldots,u_m;\ f;\ x)$ *by:*

$$D^{(1)} (u_1;\ f,\ x) = f(x) - f(x - u_1)$$

$$D^{(m)} (u_1,\ldots,u_m;\ f,\ x) =$$

$$= D^{(m-1)}(u_1,\ldots,u_{m-1};f,x) - D^{(m-1)}(u_1,\ldots,u_{m-1};f,x-u_m).$$

Let $S_m = u_1 + u_2 + \ldots + u_m$.

(i) Assume that the function f *verifies* $|f(y)| \leq M$ *for* $y \in [x - S_m, x]$*; then*

$$|D^{(m)} (u_1,\ldots,u_m;\ f,x)| \leq 2^m M.$$

(ii) Assume that the function f *is* C^m *on* $[x - S_m,\ x]$*; then there exists* $\xi \in [x - S_m,\ x]$ *such that:*

$$D^{(m)} (u_1,\ldots,u_m;\ f,x) = \left(\prod_{i=1}^{m} u_i\right) f^{(m)}(\xi)$$

PROOF: (i) is clear. We prove (ii) by recurrence on m. The assertion is true for m = 1 by the mean value theorem. We assume that the assertion is true for every C^m function and all values of $u_1,\ldots,u_m$. Now we apply the induction hypothesis to the function $\varphi(x) = f(x) - f(x-u_1)$. We get:

$$D^{m-1}(u_2,\ldots,u_m;\varphi,x) = \left(\prod_{i=2}^{m} u_i\right)\varphi^{(m-1)}(\xi_1) . \tag{10}$$

But the left hand side of (10) is $D^{(m)}(u_1,\ldots,u_m;f,x)$ and

$$\varphi^{(m-1)}(\xi_1) = f^{(m-1)}(\xi_1) - f^{(m-1)}(\xi_1-u_1) = u_1 f^{(m)}(\xi).$$

PROOF OF THEOREM 1. Set $p(0) = r(0,m) = 1$. The generating function of $p(n)$ is $\sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1}$. The generating function of $r(n,m)$ is

$$\sum_{n=0}^{\infty} r(n,m)x^n = \prod_{n=m}^{\infty} (1-x^n)^{-1} =$$

$$= \left(\prod_{n=1}^{m-1} (1 - x^n)\right) \left(\sum_{n=0}^{\infty} p(n)x^n\right) .$$

We see that

$$r(n,2) = p(n) - p(n-1),$$
$$r(n,3) = r(n,2) - r(n-2,2) =$$
$$= p(n) - p(n-1) - p(n-2) + p(n-3)$$

and, in general

(11) $$r(n,m) = D^{(m-1)} (1,2,\ldots,m-1;\ p,n).$$

Let $F(x) = (\exp\sqrt{x})/\sqrt{x}$. Then formula (1) can be written:

$$p(n) = \frac{C^3}{2\pi\sqrt{2}} F'(C^2(n-1/24)) + f_1(n).$$

Then, due to the linearity of the operator $D^{(m)}$, to lemma 3, and to formula (2), there exists $\xi \in \left[n - \frac{m(m-1)}{2},\ n\right]$ such that:

(12) $$r(n,m) = (m-1)!\ \frac{C^{2m+1}}{2\pi\sqrt{2}}\ F^{(m)}(C^2(\xi - 1/24)) +$$
$$+ O(2^{m-1} \exp(C\sqrt{n}/2)) .$$

Let $\zeta = C^2(\xi - 1/24)$. One has:

$$C^2\left(n - \frac{m^2}{2}\right) \le \zeta \le C^2 n ,$$

$$1/(C^2 n) \le 1/\zeta \le (1/(C^2 n))(1 + m^2/n)$$

for $m \le \sqrt{n}$; and:

(13) $$1/(C\sqrt{n}) \le 1/\sqrt{\zeta} \le (1/C\sqrt{n})(1 + m^2/(2n)).$$

So we have, for $1 \le m \le n^{1/4}$,

$$\frac{1}{\sqrt{\zeta}} \le \frac{1 + m^2/(2m^4)}{C\ m^2} \le \frac{3}{2\,C\,m^2} \le \frac{1}{m^2} \le \frac{2}{m(m+1)} .$$

Using lemma 2, we deduce that:

$$y_m\left(-\frac{1}{\sqrt{\zeta}}\right) = 1 + O(m^2/\sqrt{n}).$$

Then, we get:

$$\sqrt{\zeta} = C\sqrt{n} + O(m^2/\sqrt{n}),$$
$$\exp(\sqrt{\zeta}) = \exp(C\sqrt{n})\ (1 + O(m^2/\sqrt{n})),$$
$$\zeta^{(m+1)/2} = C^{m+1}\ n^{(m+1)/2}\ (1 + O(m^3/n)),$$

and using lemma 1:

$$F^{(m)}(\zeta) = \frac{\exp(C\sqrt{n})}{2^m (C\sqrt{n})^{m+1}} \left(1 + O(m^2/\sqrt{n})\right).$$

Then formulas (12) and (5) complete the proof of theorem 1.

3. PROOF OF THEOREM 2

LEMMA 4. *Let* $n_1, n_2, \ldots, n_j$ *be positive integers, all smaller than* a *with the following property: for all* k, $1 \le k \le a-1$, *there exists a subset* $n_{i_1}, \ldots, n_{i_r}$, $1 \le i_1 < i_2 < \ldots < i_r \le j$, *such that* $n_{i_1} + \ldots + n_{i_r} = k$; *but there is no subset of sum* a. *Then* $n_1 + n_2 + \ldots + \ldots + n_j = a-1$.

PROOF. First we observe that $n_1 + \ldots + n_j \ge a-1$, because $k = a-1$ can be represented as a subsum. So we shall have to prove that $n_1 + \ldots + n_j \ge a$ is impossible.

Let us suppose that $n_1 + \ldots + n_j \ge a$. We take off $n_{i_1}, \ldots, n_{i_r}$ whose sum is a-1. Let us denote by u the smallest remaining n_i; u cannot be equal to one (in that case $n_{i_1} + \ldots + n_{i_r} + u$ would be equal to a). So $u \ge 2$. From our hypothesis, there is a subsum of the n_i's whose sum is $u-1$. But from the definition of u, clearly, this subsum must be a subsum of $n_{i_1} + \ldots + n_{i_r}$, let us say $u - 1 = n_{i_1} + \ldots + n_{i_l}$.

Then $\{u, n_{i_{l+1}}, \ldots, n_{i_r}\}$ would be a subset of $\{n_1, \ldots, n_j\}$ whose sum would be a.

LEMMA 5. *With the notations of the introduction, we have:*

$$M(n) = \sum_{1 \le a \le n/2} X(a) \textit{ with}$$

$$X(a) = \tilde{p}(a-1)\, r(n-a+1, a+1).$$

PROOF. Let us first notice that lemma 5 allows the computation of M(n) by induction, since $\tilde{p}(a) = p(a) - M(a)$, and r(n,m) is easy to compute. A table of M(n), for $1 \le n \le 100$, is joined at the end of this paper. We are pleased to thank J.P. Massias for carrying out these calculations.

We shall denote by X(a) the number of partitions of n which represent all $k \le a-1$, but which do not represent a. Clearly, if a partition represents k, it does represent also $n-k$ and then, for $a > n/2$, we shall have $X(a) = 0$ and thus we have:

$$M(n) = \sum_{1 \le a \le n/2} X(a).$$

We shall now estimate X(a). Let $n = n_1 + \ldots + n_s$ with $n_1 \le n_2 \le \ldots \le n_s$ be a partition of n which represents all k up to a − 1 but does not represent a. Let us define j by $n_j < a$ and $n_{j+1} > a$. The set $\{n_1, \ldots, n_j\}$ verifies the hypothesis of lemma 4, and thus $\sum_{1\le i\le j} n_i = a-1$. Moreover, $(n_1,\ldots,n_j)$ is a practical partition of a − 1. Finally $n_{j+1} + \ldots + n_s$ is a partition of n − a + 1 all the parts of which are ≥ a + 1.

LEMMA 6. *For all* $n \ge 0$, $p(n+1) \le 2p(n)$ *holds*.

PROOF. This is obvious for n = 0 if we set p(0) = 1. Let n be ≥ 1. The number of partitions of n + 1 with a part equal to 1 is p(n). Let $n_1 + \ldots + n_s = n+1$, with $2 \le n_1 \le \ldots \le n_s$. We associate to this partition of n + 1 the following partition of n : $n_1 - 1, n_2, \ldots, n_s$.

LEMMA 7. *For* $\alpha \ge 0$ *and for* $m \le \alpha\sqrt{n}$, *we have:*

$$p(n,m) \le \exp((\alpha^3/2 + 2\alpha(1-\log\alpha))\sqrt{n}).$$

PROOF. We shall use the following classical inequalities valid for all n and $1 \le m \le n$ (cf. [Com], p. 126 and [Gup 3], p. 232):

$$(14) \qquad \binom{n-1}{m-1} \le m!\ p(n,m) \le \binom{n+m(m+1)/2-1}{m-1}.$$

Observing that $n(n-1)\ldots(n-k+1) \le \left(n - \frac{k-1}{2}\right)^k$, we deduce from (14):

$$p(n,m) \le \frac{n^{m-1}}{m!(m-1)!}\left(1+\frac{m^2}{2n}\right)^{m-1} \le \frac{n^m}{(m!)^2}\left(1+\frac{m^2}{2n}\right)^m.$$

By Stirling's formula: $m! \ge m^m e^{-m}\sqrt{2\pi m}$, we obtain:

$$p(n,m) \le \exp(m^3/2n + m(\log n + 2 - 2\log m)).$$

But the above right hand side is, for n fixed, increasing in m, for m ≥ 0, and this completes the proof.

Much more precise estimations for p(n,m) have been given by G. Szekeres: (cf. [Sze 1], [Sze 2], and [Gup 3], ch. 9).

PROOF OF THEOREM 2: From lemma 5, we have:

$$(15) \qquad M(n) = S_1 + S_2 + S_3 + S_4$$

with

$$S_1 = \sum_{a=1}^{b} X(a);\ S_2 = \sum_{a=b+1}^{b_0-1} X(a);$$

$$S_3 = \sum_{a=b_0}^{b_1} X(a);\ S_4 = \sum_{a=b_1+1}^{\lfloor n/2\rfloor} X(a),$$

where we set $b_0 = \lfloor n^{1/4} - 1\rfloor$ and $b_1 = \lfloor 10\sqrt{n}\rfloor$. We shall first show that each of the sums S_2, S_3 and S_4 is $O(p(n)n^{-(b+1)/2})$. For that we shall replace X(a) by the upper bound p(a) r(n,a+1). Then, from theorem 1, there exists an absolute constant γ such that, for $1 \le m \le n^{1/4}$, we have:

$$r(n,m) \le \gamma p(n)(m-1)!\ (C/(2\sqrt{n}))^{m-1}.$$

Finally, from (5) there exists an absolute constant δ, such that, for all $n \ge 1$, we have:

$$p(n) \le \delta \exp(C\sqrt{n}).$$

UPPER BOUND OF S_2. For $b+1 \le a \le b_0 - 1$, we have $X(a) \le f_2(a)$, with

$$f_2(a) = \gamma p(a)\ p(n)\ a!(C/2\sqrt{n})^a ,$$

which implies, by lemma 6:

$$\frac{f_2(a+1)}{f_2(a)} = \frac{p(a+1)}{p(a)} \frac{C}{2\sqrt{n}} (a+1) \le C \frac{n^{1/4}}{\sqrt{n}} \le \frac{10}{11}$$

as soon as $n \ge 64$. Thus, for $n \ge 64$, we have:

(16) $$S_2 \le 11\ f_2(b+1) = O(p(n)n^{-(b+1)/2}).$$

UPPER BOUND OF S_3. For $b_0 \le a \le b_1$, we have:

$$X(a) \le p(b_1)\ r(n,\ b_0+1)$$
$$\le \delta\ \exp(C\sqrt{10}\ n^{1/4})\ \gamma p(n)\ b_0!\ (C/2\sqrt{n})^{b_0}.$$

But $b_0! \le b_0^{b_0}\ e^{-b_0+1}\ \sqrt{b_0}$, and we obtain:

$$S_3 \le 10\ e\gamma\delta(n)^{5/8} \exp\left[C\sqrt{10}\ n^{1/4} - b_0 \log(\frac{2e\sqrt{n}}{Cb_0})\right] p(n).$$

The quantity inside brackets is: $-\frac{1}{4} n^{1/4} \log n(1+o(1))$, and thus we have:

(17) $$S_3 = O(p(n)\ n^{-(b+1)/2}).$$

UPPER BOUND OF S_4. For $b_1 + 1 \le a \le n/2$, we use the obvious upper bound:

$$r(n,m) \le p(n,\ \lfloor n/m \rfloor).$$

Then we have:

$$X(a) \le p(\lfloor n/2 \rfloor)\ r(n,\ a+1)$$
$$\le \delta\ \exp(C\sqrt{n/2})\ p(n,m)$$

with $m = \lfloor n/(a+1) \rfloor \le \frac{1}{10}\sqrt{n}$. We use then the upper bound of p(n,m) given by lemma 7, and we get:

$$S_4 \le \delta\ \frac{n}{2} \exp((C/\sqrt{2}+0.67)\sqrt{n}) \le \frac{\delta n}{2} \exp((C-0.07)\sqrt{n}).$$

By (5), this shows:

$$(18) \qquad S_4 = O(p(n)\, n^{-(b+1)/2}).$$

From (15), (16), (17), (18) we conclude:

$$(19) \qquad M(n)/p(n) = S_1/p(n) + O(n^{-(b+1)/2}).$$

ESTIMATION OF S_1. We have:

$$S_1 = \sum_{a=1}^{b} \tilde{p}(a-1)\, r(n-a+1, a+1).$$

Lemma 5 allows us to calculate $\tilde{p}$, and we shall use the estimation of $r(n,m)$ given by (11). So, if we define the polynomial:

$$W_b(x) = \sum_{a=1}^{b} \tilde{p}(a-1)\, x^{a-1}\, (1-x) \ldots (1-x^a)$$

$$= \sum_{\mu=0}^{B} w_\mu^{(b)}\, x^\mu \text{ with } B = (b^2+3b-2)/2,$$

we have:

$$(20) \qquad S_1 = \sum_{\mu=0}^{B} w_\mu^{(b)}\, p(n-\mu)\ .$$

The first values of W_b are:

$$W_1(x) = 1 - x$$
$$W_2(x) = 1 - x^2 - x^3 + x^4$$
$$W_3(x) = 1 - 2x^3 + x^6 + x^7 - x^8$$
$$W_4(x) = 1 - 2x^4 - 2x^5 + x^6 + x^7 + 3x^8 - 2x^{11} - 2x^{12} + 2x^{13}.$$

Using the same argument as for (4), we get:

$$(21) \qquad p(n-\mu) = \frac{\exp(C\sqrt{n})}{4\pi\sqrt{2}\,n}\, g(\mu + 1/24,\ 1/\sqrt{n}) + f_1(n-\mu).$$

Let $T_b(\lambda, t)$ be the Taylor polynomial of $g(\lambda, t)$ with respect to t, of degree b, near $t = 0$. For instance, we have,

$$(22) \qquad T_3(\lambda,t) = C - \left(1 + \frac{C^2\lambda}{2}\right)t + \left(\frac{3C\lambda}{2} + \frac{C^3\lambda^2}{8}\right)t^2 -$$

$$- \left(\frac{3\lambda}{2} + \frac{3C^2\lambda^2}{4} + \frac{C^4\lambda^3}{48}\right) t^3\ .$$

For μ fixed, we deduce from (21) and (2) the following asymptotic expansion, for $n \to +\infty$

$$(23) \qquad p(n-\mu) = \frac{\exp(C\sqrt{n})}{4\pi\sqrt{2}\,n}\, (T_b(\mu + 1/24,\ 1/\sqrt{n}) + O(n^{-(b+1)/2})),$$

and thus, from (20) we have:

$$(24) \qquad S_1 = \frac{\exp(C\sqrt{n})}{4\pi\sqrt{2}\,n} \left[\sum_{\mu=0}^{B} w_\mu^{(b)}\, T_b(\mu+1/24, 1/\sqrt{n}) + O(n^{-(b+1)/2})\right].$$

Dividing by $p(n)$ and using (21) with $\mu = 0$, we get:

$$S_1/p(n) = \frac{\sum_{\mu=0}^{B} w_\mu^{(b)}\, T_b(\mu+1/24,\ 1/\sqrt{n})}{T_b(1/24,\ 1/\sqrt{n})} + O(n^{-(b+1)/2}).$$

We now calculate the Taylor expansion of the above quotient with an error term $O(n^{-(b+1)/2})$, and with (19) this ends the proof of theorem 2.

CALCULATION OF THE COEFFICIENTS α_i. We shall now suppose $b = 3$ and $B = 8$. We set:

$$\sigma_i = \sum_{\mu=0}^{B} w_\mu^{(b)} \mu^i ; \quad \tau_i = \sum_{\mu=0}^{B} w_\mu^{(b)} \ (\mu + 1/24)^i .$$

One has

$$\sigma_0 = 0; \ \sigma_1 = -1; \ \sigma_2 = 3; \ \sigma_3 = -7$$

$$\tau_0 = 0; \ \tau_1 = \frac{1}{24} \sigma_0 + \sigma_1 = -1;$$

$$\tau_2 = 35/12; \ \tau_3 = -1273/192.$$

From (22) and (24), we have:

(25)
$$S_1 = \frac{\exp C\sqrt{n}}{4\pi\sqrt{2}\, n} \left[\tau_0 C - (\tau_0 + \frac{C^2}{2} \tau_0) \frac{1}{\sqrt{n}} + \left(\frac{3C\tau_1}{2} + \frac{C^3\tau_2}{8} \right) \frac{1}{n} - \right.$$
$$\left. - \left(\frac{3}{2} \tau_1 + \frac{3C^2}{4} \tau_2 + \frac{C^4}{48} \tau_3 \right) \frac{1}{n^{3/2}} + O(n^{-2}) \right] =$$
$$= \frac{\exp C\sqrt{n}}{4\pi\sqrt{2}\, n} \left[\frac{C^2}{2} \frac{1}{\sqrt{n}} + \left(\frac{3C}{2} + \frac{35}{96} C^3 \right) \frac{1}{n} + \right.$$
$$\left. + \left(\frac{3}{2} - \frac{35}{16} C^2 + \frac{1273}{9216} C^4 \right) \frac{1}{n^{3/2}} + O(n^{-2}) \right] .$$

From (22), with $\lambda = 1/24$, we deduce:

(26)
$$T_3(1/24,t))^{-1} = \frac{1}{C} \left[1 + \left(\frac{1}{C} + \frac{C}{48} \right) t + \right.$$
$$\left. + \left(\frac{1}{C^2} - \frac{1}{48} + \frac{C^2}{4608} \right) t^2 + O(t^3) \right] ,$$

and, multiplying (25) by (26) with $t = 1/\sqrt{n}$, we obtain:

$$S_1/p(n) = \frac{C}{2\sqrt{n}} + \left(- 1 + \frac{3}{8} C^2 \right) \frac{1}{n} +$$
$$+ \left(\frac{1}{2C} - \frac{179}{96} C + \frac{7}{48} C^3 \right) \frac{1}{n^{3/2}} + O(n^{-2}).$$

Replacing C by its value $\pi\sqrt{2/3}$, we get the announced values α_1, α_2, α_3. The above calculation has been checked by the computer algebra systems MACSYMA and MAPLE. We are pleased to thank J.P. Massias and F. Morain for computing the α_i's for $1 \le i \le 20$. It can be seen that α_i is a polynomial in π divided by a power of π. We give below an approximate value of α_i.

i	1	2	3	4	5	6	7	8
α_i	1.28	1.47	-2.13	61.9	-600.0	$9.88\ 10^3$	$-1.25\ 10^5$	$1.83\ 10^6$

i	9	10	11	12	13	14
α_i	$-2.94\ 10^7$	$6.09\ 10^8$	$-1.46\ 10^{10}$	$3.91\ 10^{11}$	$-1.11\ 10^{13}$	$3.26\ 10^{14}$

i	15	16	17	18	19	20
α_i	$-9.54\ 10^{15}$	$2.83\ 10^{17}$	$-8.90\ 10^{18}$	$3.12\ 10^{20}$	$-1.23\ 10^{22}$	$5.20\ 10^{23}$

TABLE OF M(n) AND $\tilde{p}(n)$ FOR $n \leq 100$

n	p(n)	M(n)	$\tilde{p}(n)$	n	p(n)	M(n)	$\tilde{p}(n)$
1	1	0	1	51	239943	57695	182248
2	2	1	1	52	281589	67628	213961
3	3	1	2	53	329931	77300	252631
4	5	3	2	54	386155	90242	295913
5	7	3	4	55	451276	103131	348145
6	11	6	5	56	526823	119997	406826
7	15	7	8	57	614154	136866	477288
8	22	12	10	58	715220	158990	556230
9	30	14	16	59	831820	180968	650852
10	42	22	20	60	966467	209586	756881
11	56	25	31	61	1121505	238533	882972
12	77	38	39	62	1300156	275425	1024731
13	101	46	55	63	1505499	312897	1192602
14	135	64	71	64	1741630	360806	1380824
15	176	76	100	65	2012558	409237	1603321
16	231	106	125	66	2323520	470659	1852861
17	297	124	173	67	2679689	533593	2146096
18	385	167	218	68	3087735	612257	2475478
19	490	199	291	69	3554345	693097	2861248
20	627	261	366	70	4087968	794162	3293806
21	792	309	483	71	4697205	897718	3799487
22	1002	402	600	72	5392783	1026447	4366336
23	1255	471	784	73	6185689	1159767	5025922
24	1575	604	971	74	7089500	1323290	5766210
25	1958	714	1244	75	8118264	1493120	6625144
26	2436	898	1538	76	9289091	1701634	7587457
27	3010	1053	1957	77	10619863	1917673	8702190
28	3718	1323	2395	78	12132164	2181368	9950796
29	4565	1542	3023	79	13848650	2456810	11391840
30	5604	1911	3693	80	15796476	2789799	13006677
31	6842	2237	4605	81	18004327	3138471	14865856
32	8349	2745	5604	82	20506255	3559678	16946577
33	10143	3201	6942	83	23338469	4000029	19338440
34	12310	3913	8397	84	26543660	4529625	22014035
35	14883	4536	10347	85	30167357	5086932	25080425
36	17977	5506	12471	86	34262962	5751522	28511440
37	21637	6402	15235	87	38887673	6452197	32435476
38	26015	7706	18309	88	44108109	7287542	36820567
39	31185	8918	22267	89	49995925	8167149	41828776
40	37338	10719	26619	90	56634173	9211235	47422938

TABLE OF M(n) AND $\tilde{p}(n)$ FOR $n \leq 100$

n	p(n)	M(n)	$\tilde{p}(n)$	n	p(n)	M(n)	$\tilde{p}(n)$
41	44583	12364	32219	91	64112359	10316448	53795911
42	53174	14760	38414	92	72533807	11619747	60914060
43	63261	17045	46216	93	83010177	13001601	69008576
44	75175	20234	54941	94	92669720	14629286	78040434
45	89134	23296	65838	95	104651419	16353590	88297829
46	105558	27600	77958	96	118114304	18378039	99736265
47	124754	31678	93076	97	133230930	20531393	112699537
48	147273	37365	109908	98	150198136	23044982	127153154
49	173525	42910	130615	99	169229875	25722031	143507844
50	204226	50371	153855	100	190569292	28844400	161724892

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