On functions connected with prime divisors of an integer

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Let n be an integer. We write its standard factorization into primes

$$n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k} \quad \text{with} \quad q_1 < q_2 < \ldots < q_k.$$

We define:

$$f(n) = \sum_{i=1}^{k-1} q_i/q_{i+1} \quad ; \qquad F(m) = \sum_{i=1}^{k-1} (1 - q_i/q_{i+1}).$$

$$h(n) = \sum_{i=1}^{k-1} \frac{1}{q_{i+1}-q_i} \quad ; \qquad \hat{h}(n) = \sum_{1 \le i < j \le k} \frac{1}{q_j - q_i}$$

and $\omega(n) = k$. When $k = 1$, the above empty sums are 0. Moreover, we say that n is a champion for the function f (or an f-champion) if

$$m < n \Rightarrow f(m) < f(n).$$

In [Erd 2], it was shown that $n(x) = \prod_{p \le x} p$ was a f-champion for x large enough, but was not a F-champion for all x large enough. We shall consider here the following problem. Is n(x) a h-champion? a $\hat{h}$-champion?

In [Erd 3] and [De K], function h is studied. It is shown that

$$\frac{\log n(x)}{(\log\log n(x))^2} << h(n(x)) << \frac{\log n(x) \log\log\log n(x)}{(\log\log n(x))^2} . \qquad (1)$$

For all n, we have:

R. A. Mollin (ed.), Number Theory and Applications, 381–391.

$$h(n) \leq \omega(n) \ll \frac{\log n}{\log \log n} .$$

Let $t_1 = 3$, $t_2 = 5$, $t_3 = 7$, $t_4 = 11$, $t_5 = 13$, ... be the sequence of twin primes, and let us assume that this sequence is infinite and that $t_k \ll k \log^2 k$. Then for the sequence $n_k = t_1 t_2 \ldots t_k$, it is not difficult to see that

$$h(n_k) \asymp \frac{\log n_k}{\log \log n_k} .$$

With (1), this relation shows that, for x large enough, n(x) is not a h-champion. But we have assumed a strong hypothesis about twin primes. Without any conjecture, we shall prove:

<u>Theorem 1</u>. Let $n(x) = \prod_{p \leq x} p$. For x large enough, n(x) is not a h-champion, i.e. there exists $m < n(x)$ with $h(m) > h(n(x))$.

<u>Proof</u>. It follows from Maier's result (cf [Mai]) that there exists an absolute constant $D > 1$, such that for all k and for x large enough, there exist between $x^{1/D}$ and x, k consecutive primes $p_1, \ldots, p_k$ and a constant depending on k, say $\alpha(k)$, with the property:

$$p_{i+1} - p_i \geq \alpha_k (\log x)\, \varphi(x), \quad 1 \leq i \leq k-1,$$

where $\varphi(x)$ is a function going to infinity with x.

We apply this result with $k = 2D + 3$. Moreover between x and 2x, there certainly exist 2 prime q_1 and q_2 such that the difference

$q_2 - q_1 \leq \frac{11}{10} \log x$. We consider

$$m = \frac{n(x)\; q_1 q_2}{p_2 \cdots p_{2D+2}} \leq \frac{4x^2}{x^{(2D+1)/D}}\, n(x).$$

Thus m is smaller than n(x) for x large enough. Further:

$$h(m) \geq h(n(x)) + \frac{1}{q_2 - q_1} - \sum_{i=1}^{2D+2} \frac{1}{p_{i+1} - p_i}$$

$$\geq h(n(x)) + \frac{10}{11 \log x} - \frac{(2D+2)}{\alpha_k \log x\, \varphi(x)}$$

which is bigger than $h(n(x))$ for x large enough.

Unfortunately we were not able to prove the same theorem than theorem 1 for the function $\hat{h}$. To get the same result we need 2 very strong conjectures:

(H1) $\forall \epsilon > 0$, $\forall \eta > 0$, $\exists x_0$ such that for $x \geq x_0$ and $y \geq x^{\epsilon}$,

$$(1-\eta)\frac{y}{\log x} \leq \pi(x) - \pi(x-y) \leq (1+\eta)\frac{y}{\log x}.$$

(H2) There exists a fixed $\beta < 1/100$ such that, for x large enough, it is always possible to find between x and $x + x^{\beta}$, four primes q_1, $q_2 = q_1 + 2$, $q_3 = q_1 + 6$, $q_4 = q_1 + 8$.

Hypothesis (H1) has been partially proved by Hoheisel for a fixed $\epsilon < 1$. The Riemann hypothesis implies (H1) for all $\epsilon > 1/2$. We shall prove:

<u>Theorem 2</u>. Under the assumption of (H1) and (H2), for x large enough, $n(x) = \prod_{p \leq x} p$ is never a $\hat{h}$-champion number.

To prove theorem 2, we need 3 lemmas.

<u>Lemma 1</u>. There is an absolute constant K such that for all $x, y, d \in \mathbb{Z}$, $2 \leq y < x$,

$$\sum_{\substack{q \text{ prime} \\ x-y<q\leq x \\ |q-d| \text{ is prime}}} 1 \leq K \frac{y}{\log^2 y} \prod_{p|d} (1 + 1/p).$$

Moreover

$$\sum_{1\leq d\leq x} \frac{1}{d} \prod_{p|d} (1 + 1/p) \leq K' \log x.$$

<u>Proof</u>. The first part is a classical application of sieve's method, (cf[Hal], Cor. 2.4.1, or [Sie] for an effective value of K). For the second fact, let us call $w(d) = \frac{1}{d} \prod_{p|d} (1 + 1/p)$. It is a multiplicative function, and,

$$\sum_{d\leq x} w(d) \leq \prod_{p\leq x} (1 + w(p) + \ldots + w(p^k) + \ldots)$$

$$= \prod_{p\leq x} \left(1 + \frac{1}{p} + \frac{2}{p^2} + \frac{2}{p^3} + \dots\right)$$

$$\leq \prod_{p\leq x} \left(\frac{1}{1-1/p}\right) \prod_{p} \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \dots\right).$$

We complete the proof by using Mertens formula (cf [Har]) to estimate the first product and observing that the second product is convergent.

<u>Lemma 2</u>. Let $0 < \alpha < \beta \leq 1$ be fixed real numbers. We define

$$U(x,\alpha,\beta) = \sum_{x-x^{\beta}<p\leq x-x^{\alpha}} \frac{1}{x-p}.$$

Under the assumption of hypothesis (H1), we have for x going to infinity:

$$U(x,\alpha,\beta) = \beta - \alpha + o(1).$$

<u>Proof</u>. We apply (H1) with $\epsilon = \alpha$, η, x and $y = x - p$. We get for $p \leq x-x^{\alpha}$, and x large enough :

$$\frac{(1-\eta)(x-p)}{\log x} \leq \pi(x) - \pi(p) \leq (1+\eta)\,\frac{x-p}{\log x}$$

and

$$\frac{1-\eta}{\log x(\pi(x)-\pi(p))} \leq \frac{1}{x-p} \leq \frac{1+\eta}{\log x}\,\frac{1}{(\pi(x)-\pi(p))}.$$

Further, we apply (H1) with $\epsilon = \alpha$, η, x, $y = x^{\alpha}$:

$$\frac{1-\eta}{\log x}\, x^{\alpha} \leq \pi(x) - \pi(x - x^{\alpha}) \leq \frac{1+\eta}{\log x}\, x^{\alpha}. \tag{2}$$

The same inequality holds with β instead of α.

Then we have

$$U(x,\alpha,\beta) \leq \frac{1+\eta}{\log x} \sum_{x-x^{\beta}<p\leq x-x^{\alpha}} \frac{1}{\pi(x)-\pi(p)}$$

$$= \frac{1+\eta}{\log x} \sum_{\pi(x)-\pi(x-x^{\beta})<j\leq\pi(x)-\pi(x-x^{\alpha})} 1/j$$

$$\leq \frac{1+\eta}{\log x} \sum_j 1/j,$$

Where j runs between $\frac{(1-\eta)x^{\alpha}}{\log x}$ and $\frac{(1+\eta)x^{\beta}}{\log x}$.
We deduce:

$$U(x,\alpha,\beta) \leq (1+\eta)\ (\beta-\alpha+o(1)).$$

In the same way we can obtain the lower bound

$$U(x,\alpha,\beta) \geq (1-\eta)(\beta-\alpha+o(1)),$$

and choosing η as small as we want completes the proof of lemma 2.

Lemma 3. For q prime, and real x, we define:

$$V(q) = \sum_{p<q} \frac{1}{q-p} \text{ and } W(q,x) = \sum_{q<p\leq x} \frac{1}{p-q} .$$

Then we have under the assumption of (H1):

$$\text{(3)} \qquad \underline{\lim}\ V(q) \geq 1,$$

and for $0 < \alpha < 1$,

$$\text{(4)} \qquad \sum_{x-x^{\alpha}<q\leq x} V(q) + W(q,x) \leq (1+\alpha+o(1)) \frac{x^{\alpha}}{\log x} .$$

Proof. With the notation of lemma 2, we get:

$$V(q) \geq U(q,\alpha,1)$$

for all $\alpha > 0$, and thus $\underline{\lim}\ V(q) \geq 1$. We observe that replacing hypothesis (H1) by Hoheisel's theorem: will give

$$\underline{\lim}\ V(q) > 0.$$

We have now to prove (4). We choose $\epsilon > 0$, and $\epsilon < \alpha$. Then, we have:

$$\sum_{x-x^{\alpha}<q\leq x} U(q,\epsilon,1) + \sum_{d\leq x^{\epsilon}} \frac{1}{d} \Big(\sum_{\substack{x-x^{\alpha}<q\leq x \\ q-d \text{ prime}}} 1 \Big).$$

Lemma 2 tells us that first sum is

$$(\pi(x) - \pi(x-x^{\alpha}))\ (1-\epsilon+o(1))$$

which, by (H1) is smaller than $(1+\eta)\ \frac{x^{\alpha}}{\log x}\ (1+o(1))$. Applying lemma 1 to the second sum shows that it is bounded above by

$$K \sum_{d\leq x^{\epsilon}} \frac{x^{\alpha}}{d\alpha^2 \log^2 x} \prod_{p|d} (1+\frac{1}{p}) \leq KK' \frac{\epsilon}{\alpha^2} \frac{x^{\alpha}}{\log x}\ .$$

And, since we can choose ϵ as small as we want, this completes the proof of

$$\sum_{x-x^{\alpha}<q\leq x} V(q) = (1+o(1))\ \frac{x^{\alpha}}{\log x}\ .$$

It remains to evaluate

$$\sum_{x-x^{\alpha}<q\leq x} W(q,x) = \sum_{x-x^{\alpha}<q\leq x} \Big(\sum_{q<p<q+x^{\epsilon}} \frac{1}{p-q} + \sum_{q+x^{\epsilon}<p<x} \frac{1}{p-q} \Big)$$

$$\leq \sum_{d\leq x^{\epsilon}} \frac{1}{d} \Big(\sum_{\substack{x-x^{\alpha}<q\leq x \\ q+d \text{ prime}}} 1 \Big) + \Big(\sum_{x-x^{\alpha}<p\leq x}\ \sum_{x-x^{\alpha}<q<p-x^{\epsilon}} \frac{1}{p-q} \Big).$$

We treat the first sum by lemma 1 as above. The second sum is smaller than

$$\sum_{x-x^{\alpha}<p\leq x} U\ (p,\epsilon,\alpha)$$

by observing that $p-p^{\alpha} \leq x-x^{\alpha}$ and $p-x^{\epsilon} \leq p-p^{\epsilon}$. This sum is, as above, smaller than $\alpha\ \frac{x^{\alpha}}{\log x}\ (1+\eta+o(1))$, which ends the proof of lemma 3.

<u>Proof of theorem 2</u>.

We first choose $\alpha = 1/100$. Let $T = \pi(x) - \pi(x-x^{\alpha})$ and N the number of primes q verifying $x-x^{\alpha}<q\leq x$ and $V(q) + W(q,x) \geq 1+2\alpha$. It follows from lemma 2 that

$$N(1+2\alpha) + (1+o(1))(T-N) \leq (1+\alpha+o(1))T$$

which implies

$$N \leq (1/2 + o(1))T$$

and then it is possible to find 5 primes p_i, $1 \leq i \leq 5$ between $x - x^{\alpha}$ and x and such that

$$W(p_i) + W(p_i,x) \leq 1 + 2\alpha.$$

Since $V(p_i) \geq 1 + o(1)$, this implies $W(p_i,x) \leq 2\alpha + o(1)$.

We set $n = \prod_{p \leq x} p$ and $m = \dfrac{n}{p_1 p_2 p_3 p_4 p_5}$.

We have:

$$\hat{h}(n) = \hat{h}(m) + \sum_{i=1}^{5} (V(p_i) + W(p_i,x)) + \sum_{1 \leq i < j \leq 5} \frac{1}{p_j - p_i}$$

$$\leq \hat{h}(m) + \sum_{i=1}^{5} (V(p_i) + 2W(p_i,x))$$

(5) $\hat{h}(n) \leq \hat{h}(m) + 5 + 20\alpha + o(1)$.

Further, we use hypothesis (H2) to get four primes $q_1, \dots, q_4$ such that $x + x^{\alpha} \leq q_i \leq x + x^{2\alpha}$ and $q_2 = q_1 + 2$, $q_3 = q_1 + 6$, $q_4 = q_1 + 8$: We set

$$n' = m q_1 q_2 q_3 q_4.$$

Then

$$\hat{h}(n') = \hat{h}(m) + \sum_{i=1}^{4} \Big(\sum_{p \leq x} \frac{1}{q_i - p} \Big) - \sum_{\substack{1 \leq i \leq 5 \\ 1 \leq j \leq 4}} \frac{1}{q_j - p_i} + \frac{41}{24}$$

$$\leq \hat{h}(m) + 4 \sum_{p \leq x} \Big(\frac{1}{x + x^{2\alpha} - p} \Big) + \frac{41}{24} + o(1)$$

$$\geq \hat{h}(m) + 4U(x + x^{2\alpha}, 2\alpha, 1) + \frac{41}{24} + o(1)$$

$$= \hat{h}(m) + 4(1 - 2\alpha) + \frac{41}{24} + o(1).$$

With (5), we obtain:

$$\hat{h}(n') \geq \hat{h}(n) + \frac{17}{24} - 28\alpha + o(1) \geq \hat{h}(n)$$

for x large enough. And since

$$n' \le n \frac{(x+x^{2\alpha})^4}{(x-x^{\alpha})^5} < n$$

n cannot be a champion number for $\hat{h}$.
Let $x = 41$, $n = \prod_{p \le 41} p$. J. Selfridge has observed that

$$\hat{h}\left(\frac{43n}{37}\right) > \hat{h}(n).$$

But it seems much more difficult to find the smallest x such that $\prod_{p \le x} p$ is not a champion for $\hat{h}$.

We shall end this paper with some remarks and problems. It is well known that the maximal order of $\omega(n)$ is $\frac{\log n}{\log\log n}(1+o(1))$. In [Erd 1], it is proved that

$$\text{Card}\left\{n \le x;\ \omega(n) \ge \frac{c \log x}{\log\log x}\right\} = x^{1-c+o(1)}$$

for $0 < c < 1$. In [Erd 2], it is proved that the maximal order of $F(n)$ is $(1 + o(1))\sqrt{\log n}$. It is interesting to study:

$$\Phi_c(x) = \text{Card}\{n \le x:\ F(n) \ge c\sqrt{\log x}\}$$

for $0 < c < 1$. For small c, it is easy to get a lower bound for $\Phi_c(x)$. We define k as the largest integer such that

$$2^{k(k+1)/2} \le x$$

and for $\sqrt{k} \le i \le k$, we consider a random prime p_i belonging to $[\alpha 2^i, 2^i]$, where α is a fixed real number, $\frac{1}{2} < \alpha < 1$. We set $n = \prod_{\sqrt{k} \le i \le k} p_i$. Clearly $n \le x$ and

$$F(n) > (k - \sqrt{k} - 2)\left(1 - \frac{1}{2\alpha}\right) \ge \sqrt{\frac{2}{\log 2}}\left(1 - \frac{1}{2\alpha} + o(1)\right)\sqrt{\log x}.$$

How many such n's do we have?

$$\prod_{\sqrt{k}\le i\le k} (\pi(2^i)-\pi(\alpha 2^i)) \ge \prod_{\sqrt{k}\le i\le k} \gamma(\frac{1-\alpha}{\log 2}) \frac{2^i}{i}$$

where γ is a fixed constant. An estimation of this last product shows that for $c < \sqrt{\frac{2}{\log 2}}\,(1 - \frac{1}{2\alpha})$, we have

$$\Phi_c(x) \ge x \exp(- \frac{1 + o(1)}{\sqrt{2\log 2}} \sqrt{\log x} \text{ loglog } x).$$

It is possible to improve the above reasonning, and for instance to get a lower bound for $\Phi_c(x)$ for all c, $0 < c < 1$, by using the technics of [Erd 2].

As observed by G. Tenenbaum, an upper bound of the same form, but with a different constant, can be obtained: Since $F(n) \le \omega(n)$, we have:

$$\Phi_c(x) \le \text{card}\{n \le x;\ \omega(n) \ge c\sqrt{\log x}\ \}$$
$$\le z^{-c\sqrt{\log x}} \ (\ \sum_{n\le x} z^{\omega(n)})$$

for all $z \ge 1$. The above sum can be evalued by convolution method, and we get

$$\Phi_c(x) \ll z^{-c\sqrt{\log x}}\, x(\log x)^{z-1}.$$

Choosing $z = (c\sqrt{\log x})/\text{loglog } x$ gives:

$$\Phi_c(x) \le x \exp(-(c/2 + o(1))\sqrt{\log x} \text{ loglog } x). \qquad (6)$$

It is possible to improve slightly the constant $c/2$ in the above expression. Using optimization results of [Erd 2] show that if $\omega(n) \le c\sqrt{\log n}$, with $0 < c < 2$, then $F(n) \le \lambda(c)c\sqrt{\log n}\,(1 + o(1))$, where

$$\lambda(c) = 1 - \frac{1}{2} \exp - (\frac{2(1 - c^2/4)}{c^2}) < 1.$$

So, (6) is valid with $\Phi_{c\lambda(c)}$ instead of Φ_c on the left hand side.

Let us denote by $\tau(n)$ the number of divisors of n, we write the divisors

$$d_1 = 1,\ d_2\ \ldots\qquad d_{\tau(n)} = n$$

and we define

$$g(n) = \sum_{i=1}^{\tau(n)-1} d_i/d_{i+1}\quad ;\quad G(n) = \sum_{i=1}^{\tau(n)-1} (1-d_i/d_{i+1})$$

$$H(n) = \sum_{i=1}^{\tau(n)-1} \frac{1}{d_{i+1}-d_i}\quad ;\quad \hat{H}(n) = \sum_{1\leq i<j\leq\tau(n)} \frac{1}{d_j-d_i}\ .$$

From the obvious inequality

$$1 - d_i/d_{i+1} \leq \log\ (d_{i+1}/d_i)$$

we easily deduce

$$(7)\qquad \tau(n) - 1 - \log n \leq g(n) \leq \tau(n) - 1.$$

In [Nic], $(\tau+f)$-champion numbers were considered when f is a slowly increasing function. By the same method, it is not difficult to prove that a τ-champion number large enough is a g-champion, and that if n is a g-champion, it is largely composite (i.e. $m \leq n \Rightarrow \tau(m) \leq \tau(n)$).

In fact, the calculation of τ-champions and g-champions shows that they exactly coincide from the very beginning up to 6 millions. We do not see how to prove that they coincide up to infinity.

The calculation of G-champions up to 6 millions shows that all τ-champions are G champions, and that largely composite numbers look like G-champions with a few exceptions. For instance 672 is a G-champion and is not largely composite, and 630 and 660 are largely composite but not G-champions. We do not see at all how to prove something about that. In fact, (7) tells us that $G(n) = \tau(n) - 1 - g(n) \leq \log n$, which is very small comparatively to high values of $\tau(n)$.

Computing $\hat{H}(n)$ gives 14 values of n, the largest of which is 5040, for which $\hat{H}(n) > \tau(n)$. We conjecture that for $n > 5040$, we have $\hat{H}(n) < \tau(n)$.

More information about these functions can be found in [Bal], [Erd 5], [Ten], [Vose].

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