The Mathematics of Paul Erdős I, R.L. Graham and J. Nešetřil editors, Algorithms and Combinatorics n°13, Springer Verlag, 228-240.

On Landau's Function $g(n)$

Jean-Louis Nicolas

Mathématiques, Bâtiment 101, Université Claude Bernard (LYON 1), F-69622 Villeurbanne cédex

1. Introduction

Let S_n be the symmetric group of n letters. Landau considered the function $g(n)$ defined as the maximal order of an element of S_n; Landau observed that (cf. [5])

$$g(n) = max\ lcm(m_1, \ldots, m_k) \tag{1.1}$$

where the maximum is taken on all the partitions $n = m_1 + m_2 + \ldots + m_k$ of n and proved that, when n tends to infinity

$$\log g(n) \sim \sqrt{n \log n}. \tag{1.2}$$

S.M. Shah in 1938 determined two more terms in the above asymptotic expansion (cf. [16]), and I do not know any other paper about $g(n)$ before the sixties. A nice survey paper was written by W. Miller in 1987 (cf. [9]). One may add two more recent references: [7] and [8].[1]

My very first mathematical paper (cf. [11]) was about Landau's function, and the main result was that $g(n)$, which is obviously non decreasing, is constant on arbitrarily long intervals. First time I met A. Schinzel in Paris in May 1967. He told me that he was interested in my results, but that P. Erdős would be more interested than himself. Then I wrote my first letter to Paul with a copy of my work. I received an answer dated of June 12 1967 saying "I sometimes thought about $g(n)$ but my results were very much less complete than yours". Afterwards, I met my advisor, the late Professor Pisot, who, in view of this letter, told me that my work was good for a thesis.

The main idea of my work about $g(n)$ was to use the tools introduced by S. Ramanujan to study highly composite numbers (cf. [14]). P. Erdős was very well aware of this paper of Ramanujan (cf. [1], [2] and [3]) as well as of the symmetric group and the order of its elements, (cf. [4]) and I think that he enjoyed the connection between these two areas of mathematics. Anyway, since these first letters, we had many occasions to discuss Landau's function.

Let us define $n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 5, n_6 = 7$, etc $\ldots, n_k$, such that

[1] One may add also the nice paper of J. Grantham. The largest prime dividing the maximal order of an element of S_n, to appear in Mathematics of Computation.

$$g(n_k) > g(n_k - 1). \tag{1.3}$$

The above mentioned result can be read:

$$\overline{\lim}\ n_{k+1} - n_k = +\infty. \tag{1.4}$$

Here, I shall prove the following result:

Theorem 1.1.

$$\underline{\lim}\ n_{k+1} - n_k < +\infty. \tag{1.5}$$

Let $p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_k$ the k-th prime. It is easy to deduce Theorem 1.1 from the twin prime conjecture (i.e. $\underline{\lim}\ p_{k+1} - p_k = 2$) or even from the weaker conjecture $\underline{\lim}\ p_{k+1} - p_k < +\infty$. (cf. § IV below). But I shall prove Theorem 1.1 independently of these deep conjectures. Moreover I shall explain below why it is reasonable to conjecture that the mean value of $n_{k+1} - n_k$ is 2; in other terms one may conjecture that

$$n_k \sim 2k \tag{1.6}$$

and that $n_{k+1} - n_k = 2$ has infinitely many solutions. Due to a parity phenomenon, $n_{k+1} - n_k$ seems to be much more often even than odd; nevertheless, I conjecture that:

$$\underline{\lim}\ n_{k+1} - n_k = 1. \tag{1.7}$$

The steps of the proof of Theorem 1.1 are first to construct the set G of values of $g(n)$ corresponding to the so called superior highly composite numbers introduced by S. Ramanujan, and then, when $g(n) \in G$, to build the table of $g(n+d)$ when d is small. This will be done in §4 and §5. Such values of $g(n+d)$ will be linked with the number of distinct differences of the form $P-Q$ where P and Q are primes satisfying $x - x^\alpha \le Q \le x < P \le x + x^\alpha$, where x goes to infinity and $0 < \alpha < 1$. Our guess is that these differences $P-Q$ represent almost all even numbers between 0 and $2x^\alpha$, but we shall only prove in §3 that the number of these differences is of the order of magnitude of x^α, under certain strong hypothesis on x and α, and for that a result due to Selberg about the primes between x and $x + x^\alpha$ will be needed (cf. §2).

To support conjecture (1.6), I think that what has been done here with $g(n) \in G$ can also be done for many more values of $g(n)$, but, unfortunately, even assuming strong hypotheses, I do not see for the moment how to manage it.

I thank very much E. Fouvry who gave me the proof of Proposition 3.1.

Notation 1.1. p will denote a generic prime, p_k the k-th prime; P, Q, P_i, Q_j will also denote primes. As usual $\pi(x) = \sum_{p \le x} 1$ is the number of primes up to x.

$| S |$ will denote the number of elements of the set S. The sequence n_k is defined by (1.3).

2. About the distribution of primes

Proposition 2.1. *Let us define $\pi(x)=\sum_{p\le x}1$, and let α be such that $\frac{1}{6}<\alpha<1$, and $\varepsilon>0$. When ξ goes to infinity, and $\xi'=\xi+\xi/\log\xi$, then for all x in the interval $[\xi,\xi']$ but a subset of measure $O((\xi'-\xi)/\log^3\xi)$ we have:*

$$|\pi(x+x^\alpha)-\pi(x)-\frac{x^\alpha}{\log x}|\le\varepsilon\frac{x^\alpha}{\log x} \tag{2.1}$$

$$|\pi(x)-\pi(x-x^\alpha)-\frac{x^\alpha}{\log x}|\le\varepsilon\frac{x^\alpha}{\log x} \tag{2.2}$$

$$|\frac{x}{\log x}-\frac{Q^k-Q^{k-1}}{\log Q}|\ge\frac{\sqrt{x}}{\log^4 x}\ \textit{for all primes } Q,\textit{ and } k\ge 2. \tag{2.3}$$

Proof. This proposition is an easy extension of a result of Selberg (cf. [15]) who proved that (2.1) holds for most x in (ξ,ξ'). In [13], I gave a first extension of Selberg's result by proving that (2.1) and (2.2) hold simultaneously for all x in (ξ,ξ') but for a subset of measure $O((\xi'-\xi)/\log^3\xi)$. So, it suffices to prove that the measure of the set of values of x in (ξ,ξ') for which (2.3) does not hold is $O((\xi'-\xi)/\log^3\xi)$.

We first count the number of primes Q such that for one k we have:

$$\frac{\xi}{\log\xi}\le\frac{Q^k-Q^{k-1}}{\log Q}\le\frac{\xi'}{\log\xi'}. \tag{2.4}$$

If Q satisfies (2.4), then $k\le\frac{\log\xi'}{\log 2}$, for ξ' large enough. Further, for k fixed, (2.4) implies that $Q\le(\xi')^{1/k}$, and the total number of solutions of (2.4) is

$$\le\sum_{k=2}^{\log\xi'/\log 2}(\xi')^{1/k}=O(\sqrt{\xi'})=O(\sqrt{\xi}).$$

With more careful estimations, this upper bound could be divided by $\log^2\xi$, but this crude result is enough for our purpose. Now, for all values of $y=\frac{Q^k-Q^{k-1}}{\log Q}$ satisfying (2.4), we cross out the interval $\left(y-\frac{\sqrt{\xi'}}{\log^4\xi'},y+\frac{\sqrt{\xi'}}{\log^4\xi'}\right)$. We also cross out this interval whenever $y=\frac{\xi}{\log\xi}$ and $y=\frac{\xi'}{\log\xi'}$. The total sum of the lengths of the crossed out intervals is $O\left(\frac{\xi}{\log^4\xi}\right)$, which is smaller than the length of the interval $\left(\frac{\xi}{\log\xi},\frac{\xi'}{\log\xi'}\right)$, and if $\frac{x}{\log x}$ does not fall into one of these forbidden intervals, (2.3) will certainly hold. Since the derivative of the function $\varphi(x)=x/\log x$ is $\varphi'(x)=\frac{1}{\log x}-\frac{1}{\log^2 x}$ and satisfies $\varphi'(x)\sim\frac{1}{\log\xi}$ for all $x\in(\xi,\xi')$, the measure of the set of values of $x\in(\xi,\xi')$ such that $\varphi(x)$ falls into one of the above forbidden intervals is, by the mean value theorem $O\left(\frac{\xi}{\log^3\xi}\right)$, and the proof of Proposition 2.1 is completed.

3. About the differences between primes

Proposition 3.1. *Suppose that there exists $\alpha, 0<\alpha<1$, and x large enough such that the inequalities*

$$\pi(x+x^\alpha)-\pi(x)\ge(1-\varepsilon)x^\alpha/\log x \tag{3.1}$$

$$\pi(x)-\pi(x-x^\alpha)\ge(1-\varepsilon)x^\alpha/\log x \tag{3.2}$$

hold. Then the set

$$E=E(x,\alpha)=\{P-Q;P,Q\ \textit{primes},\ x-x^\alpha<Q\le x<P\le x+x^\alpha\}$$

satisfies:

$$|E|\ge C_2x^\alpha$$

where $C_2=C_1\alpha^4(1-\varepsilon)^4$ and C_1 is an absolute constant. ($C_1=0.00164$ works).

Proof. The proof is a classical application of the sieve method that Paul Erdős enjoys very much. Let us set, for $d\le 2x^\alpha$,

$$r(d)=|\{(P,Q);x-x^\alpha<Q\le x<P\le x+x^\alpha,P-Q=d\}|.$$

Clearly we have

$$|E|=\sum_{\substack{0<d\le 2x^\alpha\\ r(d)\ne 0}}1 \tag{3.3}$$

and

$$\sum_{0<d\le 2x^\alpha}r(d)=(\pi(x+x^\alpha)-\pi(x))(\pi(x)-\pi(x-x^\alpha))\ge(1-\varepsilon)^2x^{2\alpha}/\log^2 x. \tag{3.4}$$

Now to get an upper bound for $r(d)$, we sift the set

$$A=\{n;x-x^\alpha<n\le x\}$$

with the primes $p\le z$. If p divides d, we cross out the n's satisfying $n\equiv 0\ mod\ p$, and if p does not divide d, the n's satisfying

$$n\equiv 0\ mod\ p\ or\ n\equiv -d\ mod\ p$$

so that we set for $p\le z$:

$$\omega(p)=1\ if\ p\ divides\ d$$
$$\omega(p)=2\ if\ p\ does\ not\ divide\ d.$$

By applying the large sieve (cf. [10]), we have

$$r(d) \le \frac{|A|}{L(z)}$$

with

$$L(z) = \sum_{n \le z} \left(1 + \frac{3}{2} n \,|A|^{-1} z\right)^{-1} \mu(n)^2 \left(\prod_{p|n} \frac{\omega(p)}{p - \omega(p)}\right)$$

(μ is the Möbius function), and with the choice $z = (\frac{2}{3}\,|A|)^{1/2}$, it is proved in [17] that

$$\frac{|A|}{L(z)} \le 16 \prod_{p \ge 3} \left(1 - \frac{1}{(p-1)^2}\right) \frac{|A|}{\log^2(|A|)} \prod_{\substack{p|d \\ p>2}} \frac{p-1}{p-2}.$$

The value of the above infinite product is $0.6602\ldots < 2/3$. We set $f(d) = \prod_{\substack{p|d \\ p>2}} \frac{p-1}{p-2}$, and we observe that $|A| \ge x^\alpha - 1$, so that for x large enough

$$r(d) \le \frac{32}{3\alpha^2} \frac{|A|}{\log^2 x} f(d). \qquad (3.5)$$

Now, we shall need for the next step an upper bound for $\sum_{n \le x} f^2(n)$. By using the convolution method and defining

$$h(n) = \sum_{a|n} \mu(a) f^2(n/a)$$

one gets $h(p) = \frac{2p-3}{(p-2)^2}, h(p^2) = h(p^3) = \ldots = 0,\ for\ p \ge 3,\ h(2) = 0$, and

$$\begin{aligned} \textstyle\sum_{n \le x} f^2(n) &= \textstyle\sum_{n \le x} \sum_{a|n} h(a) = \sum_{a \le x} h(a)[\frac{x}{a}] \\ &\le \textstyle x \sum_{a=1}^{\infty} \frac{h(a)}{a} = x \prod_{p \ge 3} \left(1 + \frac{2p-3}{p(p-2)^2}\right) \\ &= 2.63985\ldots x \le \tfrac{8}{3} x. \end{aligned} \qquad (3.6)$$

From (3.4) and (3.5), one can deduce

$$\frac{(1-\varepsilon)^2 x^{2\alpha}}{\log^2 x} \le \sum_{\substack{0<d \le 2x^\alpha \\ r(d) \ne 0}} r(d) \le \frac{32}{3\alpha^2} \frac{|A|}{\log^2 x} \sum_{\substack{0<d \le 2x^\alpha \\ r(d) \ne 0}} f(d).$$

and:

$$\sum_{\substack{0<d \le 2x^\alpha \\ r(d) \ne 0}} f(d) \ge \frac{3\alpha^2 x^{2\alpha}(1-\varepsilon)^2}{32\,|A|}.$$

By Cauchy-Schwarz's inequality, one has

$$\left(\sum_{\substack{0<d \le 2x^\alpha \\ r(d) \ne 0}} 1\right) \left(\sum_{\substack{0<d \le 2x^\alpha \\ r(d) \ne 0}} f^2(d)\right) \ge \frac{9\alpha^4 x^{4\alpha}(1-\varepsilon)^4}{1024\,|A|^2}$$

and, by (3.3) and (3.6)

$$|E| \ge \frac{9\alpha^4 \cdot 3 \cdot x^{4\alpha}(1-\varepsilon)^4}{1024 \cdot 8 \cdot 2\,|A|^2\, x^\alpha}.$$

Since $|A| \le x^\alpha + 1$, and x has been supposed large enough, proposition 2 is proved.

4. Some properties of $g(n)$.

Here, we recall some known properties of $g(n)$ which can be found for instance in [12]. Let us define the arithmetic function l in the following way: l is additive, and, if p is a prime and $k \ge 1$, then $l(p^k) = p^k$. It is not difficult to deduce from (1.1) (cf. [9] or [12]) that

$$g(n) = max_{l(M) \le n}\ M\,. \qquad (4.1)$$

Now the relation (cf. [12], p. 139)

$$M \in g(\mathbb{N}) \Leftrightarrow (M' > M \Rightarrow l(M') > l(M)) \qquad (4.2)$$

easily follows from (4.1), and shows that the values of the Landau function g are the "champions" for the small values of l. So the methods introduced by Ramanujan (cf. [14]) to study highly composite numbers can also be used for $g(n)$. Indeed M is highly composite, if it is a "champion" for the divisor function d, that is to say if

$$M' < M \Rightarrow d(M') < d(M).$$

Corresponding to the so called superior highly composite numbers, one introduces the set $G : N \in G$ if there exists $\rho > 0$ such that

$$\forall M \ge 1, l(M) - \rho \log M \ge l(N) - \rho \log N. \qquad (4.3)$$

(4.2) and (4.3) easily imply that $G \subset g(\mathbb{N})$. Moreover, if $\rho > 2/\log 2$, let us define $x > 4$ such that $\rho = x/\log x$ and

$$N_\rho = \prod_{p \le x} p^{\alpha_p} = \prod_p p^{\alpha_p} \qquad (4.4)$$

with

$$\begin{aligned} \alpha_p &= 0 \quad if \quad p > x, \\ \alpha_p &= 1 \quad if \quad \tfrac{p}{\log p} \le \rho < \tfrac{p^2 - p}{\log p} \end{aligned}$$

and

$$\alpha_p \quad = \quad k \geq 2 \quad if \quad \tfrac{p^k - p^{k-1}}{\log p} \quad \leq \quad \rho < \tfrac{p^{k+1}-p^k}{\log p}$$

then $N_\rho \in G$. It is not difficult to show that, with the above definition,

$$p^{\alpha_p} \leq x \tag{4.5}$$

holds for $p \leq x$, whence, N_ρ is a divisor of the l.c.m. of the integers $n \leq x$. Here we can prove

Proposition 4.1. *For every prime p, there exists n such that the largest prime factor of $g(n)$ is equal to p.*

Proof. We have $g(2) = 2, g(3) = 3$. If $p \geq 5$, let us choose $\rho = p/\log p > 2/\log 2$, and N_ρ defined by (4.4) belongs to $G \subset g(\mathbb{N})$, and its largest prime factor is p.

From Proposition 4.1, it is easy to deduce a proof of Theorem 1.1,,under the twin prime conjecture. Let $P = p + 2$ be two twin primes, and n such that the largest prime factor of $g(n)$ is p. The sequence n_k being defined by (1.3), we define k in terms of n by $n_k \leq n < n_{k+1}$, so that $g(n_k) = g(n)$ has its largest prime factor equal to p. Now, from (4.1) and (4.2),

$$l(g(n_k)) = n_k$$

and $g(n_k + 2) > g(n_k)$ since M satisfies $M = \frac{P}{p} g(n_k) > g(n_k)$ and $l(M) = n_k + 2$. So $n_{k+1} \leq n_k + 2$, and Theorem 1.1 is proved under this strong hypothesis.

Let us introduce now the so called benefit method. For a fixed $\rho > 2/\log 2, N = N_\rho$ is defined by (4.4), and for any integer M,

$$M = \prod_p p^{\beta_p}$$

one defines the benefit of M:

$$benM = l(M) - l(N) - \rho \log M/N. \tag{4.6}$$

Clearly, from (4.3), $benM \geq 0$ holds, and from the additivity of l one has:

$$benM = \sum_p \left(l(p^{\beta_p}) - l(p^{\alpha_p}) - \rho(\beta_p - \alpha_p) \log p \right). \tag{4.7}$$

In the above formula, let us observe that $l(p^\beta) = p^\beta$ if $\beta \geq 1$, but that $l(p^\beta) = 0 \neq p^\beta = 1$ if $\beta = 0$, and, due to the choice of α_p in (4.4), that all the terms in the sum are non negative: for all p, and $\beta \geq 0$, we have

$$l(p^\beta) - l(p^{\alpha_p}) - \rho(\beta - \alpha_p) \log p \geq 0 \tag{4.8}$$

Indeed, let us consider the set of points $(0, 0)$ and for β integer ≥ 1, $(\beta, p^\beta/\log p)$. For all p, the piecewise linear curve going through these points

is convex, and for a given ρ, α_p is chosen so that the straight line of slope ρ going through $\left(\alpha_p, \frac{p^{\alpha_p}}{\log p}\right)$ does not cut that curve. The right-hand side of (4.8), (which is $ben(Np^{\beta - \alpha_p})$) can be seen to be the product of $\log p$ by the vertical distance of the point $\left(\beta, \frac{p^\beta}{\log p}\right)$ to the above straight line, and because of convexity, we shall have for all p,

$$ben(Np^t) \geq t\ ben(Np), t \geq 1 \tag{4.9}$$

and for $p \leq x$,

$$ben(Np^{-t}) \geq t\ ben(Np^{-1}), 1 \leq t \leq \alpha_p. \tag{4.10}$$

5. Proof of Theorem 1.1

First the following Proposition will be proved:

Proposition 5.1. *Let $\alpha < 1/2$, and x large enough such that (2.3) holds. Let us denote the primes surrounding x by:*

$$\ldots < Q_s < \ldots < Q_2 < Q_1 \leq x < P_1 < P_2 < \ldots < P_r < \ldots$$

Let us define $\rho = x/\log x, N = N_\rho$ by (4.4), $n = l(N)$. Then for $n \leq m \leq n + 2x^\alpha, g(m)$ can be written

$$g(m) = N \frac{P_{i_1} P_{i_2} \ldots P_{i_r}}{Q_{j_1} Q_{j_2} \ldots Q_{j_r}} \tag{5.1}$$

with $i_1 < i_2 < \ldots < i_r, j_1 < j_2 < \ldots < j_r, P_{i_r} \leq x + 4x^\alpha, Q_{j_r} \geq x - 4x^\alpha$.

Proof. First, one has from (4.1) $l(g(m)) \leq m$, and from (4.6)

$$ben(g(m)) = l(g(m)) - l(N) - \rho \log \frac{g(m)}{N} \leq m - n \leq 2x^\alpha \tag{5.2}$$

for $n \leq m \leq 2x^\alpha$.

Now let $Q \leq x$ be a prime, and $k = \alpha_Q \geq 1$ the exponent of Q in the standard factorization of N. Let us suppose that for a fixed m, Q divides $g(m)$ with the exponent $\beta_Q = k + t, t > 0$. Then, from (4.7), (4.8), and (4.9), one gets

$$beng(m) \geq benNQ^t \geq benNQ \tag{5.3}$$

and

$$\begin{aligned} ben(NQ) &= Q^{k+1} - Q^k - \rho \log Q \\ &= \log Q \left(\tfrac{Q^{k+1} - Q^k}{\log Q} - \rho \right). \end{aligned}$$

From (4.4), the above lower bound is non negative, and from (2.3), one gets:

$$benNQ \geq \log 2 \frac{\sqrt{x}}{\log^4 x}. \tag{5.4}$$

For x large enough, there is a contradiction between (5.2), (5.3) and (5.4), and so, $\beta_Q \leq \alpha_Q$.

Similarly, let $Q \leq x, k = \alpha_Q \geq 2$, and suppose that $\beta_Q = k - t, 1 \leq t \leq k$. One has, from (4.7), (4.8) and (4.10),

$$beng(m) \geq benNQ^{-t} \geq benNQ^{-1}$$

and

$$\begin{aligned} benNQ^{-1} &= Q^{k-1} - Q^k + \rho \log Q \\ &= \log Q \left(\rho - \tfrac{Q^k - Q^{k-1}}{\log Q}\right) \\ &\geq \log 2 \tfrac{\sqrt{x}}{\log^4 x} \end{aligned}$$

which contradicts (5.2), and so, for these $Q's, \beta_Q = \alpha_Q$.

Suppose now that $Q \leq x, \alpha_Q = 1$, and $\beta_Q = 0$ for some $m, n \leq m \leq n + 2x^\alpha$. Then

$$beng(m) \geq ben(NQ^{-1}) = -Q + \rho \log Q = \log Q \left(\frac{x}{\log x} - \frac{Q}{\log Q}\right)$$

$$\geq \log Q(x - Q)\left(\frac{1}{\log Q} - \frac{1}{\log^2 Q}\right) = (x - Q)\left(1 - \frac{1}{\log^2 Q}\right).$$

From (4.4), if x is large enough, as $\alpha_Q = 1$, Q will be large, and so,

$$beng(m) \geq \frac{1}{2}(x - Q)$$

which, from (5.2) yields

$$x - Q \leq 4x^\alpha$$

for x large enough. In conclusion, the only prime factors allowed in the denominator of $\frac{g(m)}{N}$ are the Q's, with $x - 4x^\alpha \leq Q \leq x$, and $\alpha_Q = 1$.

What about the numerator ? Let $P > x$ be a prime number and suppose that P^t divides $g(m)$ with $t \geq 2$. Then, from (4.9) and (4.6),

$$benNP^t \geq benNP^2 = P^2 - 2\rho \log P.$$

But the function $t \mapsto t^2 - 2\rho \log t$ is increasing for $t \geq \sqrt{\rho}$, so that,

$$ben(NP^t) \geq x^2 - 2x > 2x^\alpha$$

for x large enough, which contradicts (5.2). The only possibility is that P divides $g(m)$ with exponent 1. In that case,

$$\begin{aligned} ben(g(m)) &\geq ben(NP) = P - \rho \log P = \log P \left(\tfrac{P}{\log P} - \tfrac{x}{\log x}\right) \\ &\geq \log x (P - x)\left(\tfrac{1}{\log x} - \tfrac{1}{\log^2 x}\right) \geq \tfrac{1}{2}(P - x) \end{aligned}$$

for x large enough, and this relation implies with (5.2)

$$P - x \leq 4x^\alpha.$$

Up to now, we have shown that

$$g(m) = N \frac{P_{i_1} \dots P_{i_r}}{Q_{j_1} \dots Q_{j_s}}$$

with $P_{i_r} \leq x + 4x^\alpha, Q_{j_s} \geq x - 4x^\alpha$. It remains to show that $r = s$. First, since $n \leq m \leq n + 2x^\alpha$, and N belongs to G, we have from (4.1) and (4.2)

$$n \leq l(g(m)) \leq n + 2x^\alpha. \tag{5.5}$$

Further,

$$l(g(m)) - n = \sum_{t=1}^{r} P_{i_t} - \sum_{t=1}^{s} Q_{j_t}$$

and since $r \leq 4x^\alpha$, and $s \leq 4x^\alpha$,

$$\begin{aligned} l(g(m)) - n &\leq r(x + 4x^\alpha) - s(x - 4x^\alpha) \\ &\leq (r - s)x + 32x^{2\alpha}. \end{aligned}$$

As $\alpha < 1/2$, this implies with (5.2) that $r \geq s$. Similarly,

$$l(g(m)) - n \geq (r - s)x,$$

so $(r - s)x$ must be smaller than $2x^\alpha$, and for x large enough, this implies $r \leq s$, and finally $r = s$, and the proof of Proposition 5.1 is completed.

Lemma 5.1. *Let x be a positive real number, $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k$ be real number such that*

$$b_k \leq b_{k-1} \leq \dots \leq b_1 \leq x < a_1 \leq a_2 \leq \dots \leq a_k$$

and Δ be defined by $\Delta = \sum_{i=1}^{k}(a_i - b_i)$. Then the following inequalities

$$\frac{x + \Delta}{x} \leq \prod_{i=1}^{k} \frac{a_i}{b_i} \leq exp\left(\frac{\Delta}{x}\right)$$

hold.

Proof. It is easy, and can be found in [12], p.159.

Now it is time to prove Theorem 1.1. With the notation and hypotheses of Proposition 5.1, let us denote by B the set of integers M of the form

$$M = N\frac{P_{i_1}P_{i_2}\dots P_{i_r}}{Q_{j_1}Q_{j_2}\dots Q_{j_r}}$$

satisfying

$$l(M) - l(N) = \sum_{t=1}^{r}(P_{i_t} - Q_{j_t}) \leq 2x^\alpha.$$

From Proposition 5.1, for $n \leq m \leq 2x^\alpha, g(m) \in B$, and thus, from (4.1),

$$g(m) = max_{\substack{l(M)\leq m\\ M\in B}} M. \tag{5.6}$$

Further, for $0 \leq d \leq 2x^\alpha$, define

$$B_d = \{M \in B; l(M) - l(N) = d\}.$$

I claim that, if $d < d'$ (which implies $d \leq d' - 2$), any element of B_d is smaller than any element of $B_{d'}$. Indeed, let $M \in B_d$, and $M' \in B_{d'}$. From the above lemma, one has

$$\frac{M}{N} \leq exp(\frac{d}{x}) \ and \ \frac{M'}{N} \geq \frac{x+d'}{x} \geq \frac{x+d+2}{x}.$$

Since $d \leq 2x^\alpha < x$, and $e^t \leq \frac{1}{1-t}$ for $t < 1$, one gets

$$\frac{M}{N} \leq \frac{1}{1-d/x} = \frac{x}{x-d};$$

This last quantity is smaller than $\frac{x+d+2}{x}$ if $(d+1)^2 < 2x+1$, which is true, because $d \leq 2x^\alpha$ and $\alpha < 1/2$.

From the preceding claim, and from (5.6), it follows that, if B_d is non empty, then

$$g(n+d) = maxB_d.$$

Further, since $N \in G$, we know that $n = l(N)$ belongs to the sequence (n_k) where g is increasing, and so, $n = n_{k_0}$. If $0 < d_1 < d_2 < \dots < d_s \leq 2x^\alpha$ denote the values of d for which B_d is non empty, then one has

$$n_{k_0+i} = n + d_i, 1 \leq i \leq s. \tag{5.7}$$

Suppose now that $\alpha < 1/2$ and x have been chosen in such a way that (3.1) and (3.2) hold. With the notation of Proposition 3.1, the set $E(x,\alpha)$ is certainly included in the set $\{d_1, d_2, \dots, d_s\}$, and from Proposition 3.1,

$$s \geq C_2 x^\alpha \tag{5.8}$$

which implies that for at least one i, $d_{i+1} - d_i \leq \frac{2}{C_2}$, and thus

$$n_{k_0+i+1} - n_{k_0+i} \leq \frac{2}{C_2}.$$

Finally, for $\frac{1}{6} < \alpha < \frac{1}{2}$, Proposition 2.1 allows us to choose x as wished, and thus, the proof of Theorem 1.1 is completed. With ε very small, and α close to $1/2$, the values of C_1 and C_2 given in Proposition 3.1 yield that for infinitely many k's,

$$n_{k+1} - n_k \leq 20000.$$

We can precise how many such small differences we get:

Proposition 5.2. *Let $\gamma(n) = Card\{m \leq n; g(m) > g(m-1)\}$ (so, with the notation (1.3), $n_{\gamma(n)} = n$). Then $\gamma(n) \geq n^{3/4-\varepsilon}$ for all $\varepsilon > 0$, and n large enough.*

Proof. In [12], p. 162, it is proved that

$$n^{1-\tau/2} \leq \gamma(n) \leq n - c\frac{n^{3/4}}{\sqrt{\log n}}$$

where τ is such that the sequence of consecutive primes satisfy $p_{i+1} - p_i \leq p_i^\tau$. Without any hypothesis, the best known τ is $> 1/2$.

With the definition of $\gamma(n)$, (5.7) and (5.8) give

$$\gamma(n + 2x^\alpha) - \gamma(n) \geq s \gg x^\alpha \tag{5.9}$$

whenever $n = l(N), N = N_\rho, \rho = x/\log x$, and x satisfies Proposition 2.1. But, from (4.4), two close enough distinct values of x can yield the same N.

I now claim that, with the notation of Proposition 2.1, the number of primes p_i between ξ and ξ' such that there is at least one $x \in [p_i, p_{i+1}[$ satisfying (2.1), (2.2) and (2.3) is bigger than $\frac{1}{2}(\pi(\xi') - \pi(\xi))$. Indeed, for each i for which $[p_i, p_{i+1}[$ does not contain any such x, we get a measure $p_{i+1} - p_i \geq 2$, and if there are more than $\frac{1}{2}(\pi(\xi') - \pi(\xi))$ such i's, the total measure will be greater than $\pi(\xi') - \pi(\xi) \sim \xi/\log^2 \xi$, which contradicts Proposition 2.1.

From the above claim, there will be at least $\frac{1}{2}(\pi(\xi') - \pi(\xi))$ distinct N's, with $N = N_\rho, \rho = x/\log x$, and $\xi \leq x \leq \xi'$. Moreover, for two such distinct N, say $N' < N$", we have from (4.4), $l(N") - l(N') \geq \xi$.

Let $N^{(1)}$ and $N^{(0)}$ the biggest and the smallest of these N's, and $n^{(1)} = l(N^{(1)}), n^{(0)} = l(N^{(0)})$, then from (5.9),

$$\gamma(n^{(1)}) \geq \gamma(n^{(1)}) - \gamma(n^{(0)}) \gg \frac{\xi^{1+\alpha}}{\log^2 \xi}. \tag{5.10}$$

But from (4.4) and (4.5), $x \sim \log N_\rho$, and from (1.2),

$$x \sim \log N_\rho \sim \sqrt{n \log n} \quad with \ n = l(N_\rho)$$

so

$$\xi \sim \sqrt{n^{(1)} \log n^{(1)}}$$

and since α can be choosen in (5.10) as close as wished of $1/2$, this completes the proof of Proposition 5.2.

References

[1] L. Alaoglu and P. Erdős, "On highly composite and similar numbers", Trans. Amer. Math. Soc. 56, 1944, 448-469.

[2] P. Erdős, "On highly composited numbers", J. London Math. Soc., 19, 1944, 130-133.

[3] P. Erdős and J. L. Nicolas, "Répartition des nombres superabondants", Bull. Soc. Math. France, 103, 1975, 65-90.

[4] P. Erdős and P. Turan, "On some problems of a statistical group theory", I to VII, Zeitschr. fur Wahrschenlichkeitstheorie und verw. Gebiete, 4, 1965, 175-186; Acta Math. Hung., 18, 1967, 151-163; Acta Math. Hung. 18, 1967, 309-320; Acta Math. Hung. 19, 1968, 413-435;Periodica Math. Hung. 1, 1971, 5-13; J. Indian Math. Soc. 34, 1970, 175-192; Periodica Math. Hung. 2, 1972, 149-163.

[5] E. Landau, "Uber die Maximalordung der Permutation gegebenen Grades", Handbuch der Lehre von der Verteilung der Primzahlen, vol. 1, 2nd edition, Chelsea, New-York 1953, 222-229.

[6] J. P. Massias,"Majoration explicite de l'ordre maximum d'un élément du groupe symétrique", Ann. Fac. Sci. Toulouse Math. 6, 1984, 269-280.

[7] J. P. Massias, J. L. Nicolas, G. Robin, "Evaluation asymptotique de l'ordre maximum d'un élément du groupe symétrique", Acta Arithmetica, 50, 1988, 221-242.

[8] J. P. Massias, J. L. Nicolas, G. Robin, "Effective bounds for the Maximal Order of an Element in the Symmetric Group", Math. of Comp. 53, 1989, 665-678.

[9] W. Miller, "The Maximum Order of an Element of a Finite Symmetric Group", Amer. Math. Monthly, 94, 1987, 497-506.

[10] H. L. Montgomery and R. C. Vaughan, "The large sieve", Mathematika 20, 1973, 119-134.

[11] J. L. Nicolas, "Sur l'ordre maximum d'un élément dans le groupe S_n des permutations", Acta Arithmetica, 14, 1968, 315-332.

[12] J. L. Nicolas, "Ordre maximum d'un élément du groupe de permutations et highly composite numbers", Bull. Soc. Math. France, 97, 1969, 129-191.

[13] J. L. Nicolas, "Répartition des nombres largement composés" Acta Arithmetica, 34, 1979, 379-390.

[14] S. Ramanujan, "Highly composite numbers", Proc. London Math. Soc., Series 2, 14, 1915, 347-400; and "Collected papers" Cambridge at the University Press, 1927, 78-128.

[15] A. Selberg, "On the normal density of primes in small intervals and the difference between consecutive primes", Arch. Math. Naturvid, 47, 1943, 87-105.

[16] S.Shah, "An Inequality for the Arithmetical Function $g(x)$", J. Indian Math. Soc. 3, 1939, 316-318.

[17] H. Siebert, "Montgomery's weighted sieve for dimension two", Monatsch. Math. 82, 1976, 327-336.