```
> read "gdenHR.mpl":
> read "TabT2.m": # [q,alpha,p,n,log g(n),rho,excadd,excmul]
> read "superhden.m": # [p,n=pi_1(p),log(h(n)=theta(p)]
> x0:=10.^10+19: logx0:=log(x0):sqrtx0:=sqrt(x0): c:=
  0.04611764442150902382709 08: # cf. (1.11)
  "x0=",x0,"logx0=",logx0,"sqrtx0=",sqrtx0,"c=",c;
```

"x0=", $1.0000000019\ 10^{10}$, "logx0=", 23.02585093184045683 8, "sqrtx0=", $1.0000000009499999995\ 10^{5}$, "c=", 0.046117644421509023827 0908 **(1)**

```
> "pi_1(x0) = ",pi1(x0);
```

"pi_1(x0) = ", 2220822442581729257 **(2)**

```
> nu0:=2220832950051364840: nu0f:=evalf(nu0): L0:=log(nu0f):lambda0:=
  log(L0):
  "nu0 =",nu0,"nu0f =",nu0f,"L0 =",L0,"lambda0 =",lambda0;
```

"nu0 =", 2220832950051364840, "nu0f =", $2.220832950051364840\ 10^{18}$, "L0 =", 42.24441400213190936 4, "lambda0 =", 3.743472132094985958 0 **(3)**

```
> f746(48757,70877,0.0746); # Proof of (2.4)
```

"for t satisfying pmin < t <= pmax with pmin = ", 48757, "pmax = ", 70877, "thetaminus(t) is > ", $t\left(1-\frac{0.0746}{\ln(t)}\right)$, " holds" **(4)**

```
> f746(48000,70877,0.0746); # Proof of (2.4)
```

"for t satisfying p < t <= p+ with p = ", 48751, "p+ = ", 48757, "theta(t) is > ", $t\left(1-\frac{0.0746}{\ln(t)}\right)$, " fails" **(5)**

```
> calculW(10^6): # Proof of Lemma 2.2, (2.8)
```

"list of primes <= y = ", 1000000, " s.t. W(p) > p :"

"p = ", 3, "W(p) = ", 3.034212794, "W(p)/p = ", 1.011404265

"p = ", 5, "W(p) = ", 5.046010185, "W(p)/p = ", 1.009202037

"p = ", 7, "W(p) = ", 7.316238692, "W(p)/p = ", 1.045176956

"y=", 1000000, "W(y) = ", $9.98497413976641\ 10^{5}$, "theta(y) = ", $9.98484175025634\ 10^{5}$, " a = ", 12.24046583, " b = ", 1.000013990 **(6)**

```
> a:=0.045: K:=2: alpha:=1/2: b:=(1-a/logx0^(K+1))*(a-2*alpha/logx0-
  a*alpha/logx0^(K+1)):
  "a=",a,"b=",b,"sqrt(x0)/logx0^3=",sqrtx0/logx0^3,"8.19 b=",8.19*b;
  a:='a':K:='K':alpha:='alpha': b:='b':a,b,K,alpha: # Proof of
  Corollary 2.4, (2.12)
```

"a=", 0.045, "b=", 0.0015687029879308942561, "sqrt(x0)/logx0^3=", 8.1913019292091958460, "8.19 b=", 0.012847677471154023957 **(7)**

```
> Lim1Newtonx0(1,0,6); # Computation of li^(-1)(x). To avoid the
  details, execute Lim1Newtonx0(1,0,3)    # cf. Section 2.2 and
  Section 8.1 where the result is used
```

"computation of Li^(-1)(x), x=", 1

"x=", 1., "yanc=", 1.4, "ynou=", 1.785257684292964378141217, "dif=",
0.385257684292964378141217
"x=", 1., "yanc=", 1.785257684292964378141217, "ynou=", 1.954849863902451855418587,
"dif=", 0.169592179609487477277369 8
"x=", 1., "yanc=", 1.954849863902451855418587, "ynou=", 1.968971308048332347947542,
"dif=", 0.014121444145880492528954 80
"x=", 1., "yanc=", 1.968971308048332347947542, "ynou=", 1.969047487049610247978378,
"dif=", 0.0000761790012779000308355 7618
"x=", 1., "yanc=", 1.969047487049610247978378, "ynou=", 1.969047489224750848374546,
"dif=", 2.175140600396167970986952 10^{-9}
"x=", 1., "yanc=", 1.969047489224750848374546, "ynou=", 1.969047489224750850147703,
"dif=", 1.773156943069393933440876 10^{-18}
"x=", 1., "yanc=", 1.969047489224750850147703, "ynou=", 1.969047489224750850147703,
"dif=", 0.

1.9690474892247508501 **(8)**

```
> #################### Study of pi_r(x), Section 2.3
  #########################
```

```
> lis:=[227325003281036, 13501147086873627946348,
  90660888060477797746621036 5146,
  6506440811740555953424106561636940128,
  4868694384660993648813605308147109183464084346]:
  # lis has been computed by lis:=sommepr(5,89967803); in about 450
  seconds
> calculC0(5,lis); # The values of C_0 and C_0h (defined by (2.15)
  and (2.18)) are used in the proofs                    of Corollary 2.7
  and Corollary 2.8. A[0] is the sum of positive terms, A[1] negative
```

"r=", 1, "A[0]=", 8.2082626525957611659 10^{14}, "A[1]=", 8.2083250195076881395 10^{14},
"C0=", -6.23669119269736 10^{9}
"r=", 1, "Ah[0]=", 8.4986765008763478598 10^{14}, "Ah[1]=", 8.4979737672803381019 10^{14},
"C0h=", 7.027335960097579 10^{10}
"r=", 2, "A[0]=", 1.2921973093720948207 10^{23}, "A[1]=", 1.2922077098998901986 10^{23},
"C0=", -1.04005277953779 10^{18}
"r=", 2, "Ah[0]=", 1.3154396844999538478 10^{23}, "Ah[1]=", 1.3153594634478156376 10^{23},
"C0h=", 8.02210521382102 10^{18}
"r=", 3, "A[0]=", 2.3829697485026415367 10^{31}, "A[1]=", 2.3829814063566907065 10^{31},
"C0=", -1.16578540491698 10^{26}
"r=", 3, "Ah[0]=", 2.4026013375556204250 10^{31}, "Ah[1]=", 2.4025222390607564728 10^{31},
"C0h=", 7.90984948639522 10^{26}

"r=", 4, "A[0]=", $4.115706563290281630110^{39}$, "A[1]=", $4.115718278376569206810^{39}$,
"C0=", $-1.17150862875767\ 10^{34}$

"r=", 4, "Ah[0]=", $4.132683487907013752410^{39}$, "Ah[1]=", $4.132608721906844023210^{39}$,
"C0h=", $7.47660001697292\ 10^{34}$

"r=", 5, "A[0]=", $6.475024936242360124010^{47}$, "A[1]=", $6.475036169828649961110^{47}$,
"C0=", $-1.12335862898371\ 10^{42}$

"r=", 5, "Ah[0]=", $6.489889673697204699710^{47}$, "Ah[1]=", $6.489820259233829280410^{47}$, **(9)**
"C0h=", $6.94144633754193\ 10^{42}$

```
> verifpi2max(60169,70000,0.001); # To prove (2.21) execute
  verifpi2max(60169,89967803,0.001)
```

"p=", 60169, "pi2(p)=", $6.838284919887\ 10^{12}$, "est=", $6.836996363522283026910^{12}$, "S/est=",
1.000189

"p=", 60223, "pi2(p)=", $6.849162940386\ 10^{12}$, "est=", $6.854837207182149068210^{12}$, "S/est=",
0.9991723

"p=", 60259, "pi2(p)=", $6.860055176517\ 10^{12}$, "est=", $6.866748075292289341510^{12}$, "S/est=",
0.9990253

60172.903253337664416 **(10)**

```
> verifpi2maxbis(60293,315011,0.0001); # Proof of (2.22)
```

"p=", 60293, "pi2(p)=", $6.870957779328\ 10^{12}$, "c=", $6.869770137142241757\ 10^{12}$, "S/c=",
1.000172

"p=", 60353, "pi2(p)=", $6.889160065205\ 10^{12}$, "c=", $6.889662982333562958310^{12}$, "S/c=",
0.9999270

"p=", 96017, "pi2(p)=", $2.6582175089330\ 10^{13}$, "c=", $2.658328951372453742610^{13}$, "S/c=",
0.9999582

60296.565793200319808 **(11)**

```
> verifpi2min(1091200,1100000,0.00001): # To prove (2.23) execute
  verifp2min(32300,89967803,0.0001)
```

"F=", $\frac{t^3}{3\ln(t)} + \frac{t^3}{9\ln(t)^2} + \frac{2t^3}{27\ln(t)^3} - \frac{1069\,t^3}{648\ln(t)^4}$, "F1=",
$\frac{t^2\left(648\ln(t)^4 - 3351\ln(t) + 4276\right)}{648\ln(t)^5}$

"FF1=", $648\,T^4 - 3351\,T + 4276$, "FF2=diff of FF1=", $2592\,T^3 - 3351$

"zero of FF2=(3351/2592)^(1/3)=", 1.089381, "the minimum of FF1=", 1538.113, "is positive"

"p=", 1091219, "p+=", 1091221, "pi2(p)=", $3.1879086471058513\ 10^{16}$, "F(p+)=",
$3.187942984093176963310^{16}$, "S/est=", 0.9999892291

"p=", 1091221, "p+=", 1091239, "pi2(p)=", $3.1880277234329354\ 10^{16}$, "F(p+)=",

$3.1880968855097752579\ 10^{16}$, "S/est=", 0.9999783059

"p=", 1091239, "p+=", 1091243, "pi2(p)=", $3.1881468036884475\ 10^{16}$, "F(p+)=",

$3.1881310864894094542\ 10^{16}$, "S/est=", 1.000004930

"p=", 1091243, "p+=", 1091257, "pi2(p)=", $3.1882658848169524\ 10^{16}$, "F(p+)=", **(12)**

$3.1882507918219931002\ 10^{16}$, "S/est=", 1.000004734

```
> verifpi2minbis(32300,120000,0.0001): # To prove (2.24) execute
  verifp2minbis(32300,1100000,0.0001)
```

$$\text{"F="}, \frac{t^3}{3\ln(t)} + \frac{t^3}{9\ln(t)^2} + \frac{2\,t^3}{27\ln(t)^3} - \frac{1069\,t^3}{648\ln(t)^4}, \text{"f="}, \frac{t^3}{3\ln(t)} + \frac{31\,t^3}{375\ln(t)^2}$$

$$\text{"F-f="}, \frac{t^3\left(2304\ln(t)^2 + 6000\ln(t) - 133625\right)}{81000\ln(t)^4}, \text{"trinom="}, 2304\,T^2 + 6000\,T - 133625$$

"zero of trinom=", -9.028167905, 6.424001239, "exp(6.425)=", 617.0808

"p=", 32303, "p+=", 32309, "p+=", 32309, "pi2(p)=", $1.108260898891\ 10^{12}$, "f(p+)=",

$1.1085961771097900465\ 10^{12}$, "S/est=", 0.9996975653

"p=", 32309, "p+=", 32321, "p+=", 32321, "pi2(p)=", $1.109304770372\ 10^{12}$, "f(p+)=",

$1.1097912620024024453\ 10^{12}$, "S/est=", 0.9995616365

"p=", 69959, "p+=", 69991, "p+=", 69991, "pi2(p)=", $1.0472883706424\ 10^{13}$, "f(p+)=", **(13)**

$1.0472266558975163065\ 10^{13}$, "S/est=", 1.000058932

```
> verifpi3max(600,70000,0.01);# To prove (2.25) execute verifpi3max
  (600,89967803,0.01)
```

"r=3, coef=", 0.25380853990085065148

"p=", 619, "pi3(p)=", $6.171715615\ 10^{9}$, "est=", $6.1894050708312291332\ 10^{9}$, "S3/est=",

0.9971421

"p=", 661, "pi3(p)=", $8.076458694\ 10^{9}$, "est=", $7.9667154778525718127\ 10^{9}$, "S3/est=",

1.013775

"p=", 677, "pi3(p)=", $8.691568644\ 10^{9}$, "est=", $8.7343685487645216716\ 10^{9}$, "S3/est=",

0.9950998

"p=", 683, "pi3(p)=", $9.010180631\ 10^{9}$, "est=", $9.0359143427976750482\ 10^{9}$, "S3/est=",

0.9971521

"p=", 1129, "pi3(p)=", $6.2106256796\ 10^{10}$, "est=", $6.2639185948171564573\ 10^{10}$, "S3/est=",

0.9914921

"p=", 1307, "pi3(p)=", $1.09476567065\ 10^{11}$, "est=", $1.1020976973386898407\ 10^{11}$, "S3/est=",

0.9933472

"p=", 1321, "pi3(p)=", $1.14076510985\ 10^{11}$, "est=", $1.1483773743956371603\ 10^{11}$, "S3/est=",

0.9933715

"p=", 1327, "pi3(p)=", 1.16413263768 10^{11}, "est=", 1.1686467608217762700 10^{11}, "S3/est=", 0.9961374

"p=", 1621, "pi3(p)=", 2.52427126454 10^{11}, "est=", 2.5316868568936880752 10^{11}, "S3/est=", 0.9970707

"p=", 1627, "pi3(p)=", 2.56734005337 10^{11}, "est=", 2.5680950525219821688 10^{11}, "S3/est=", 0.9997060

"p=", 1637, "pi3(p)=", 2.61120787190 10^{11}, "est=", 2.6296374104779834565 10^{11}, "S3/est=", 0.9929918

"solution of pi_3(t) log(t) = 0.271 t^4 is", 663.3555 **(14)**

```
> verifpi4max(150,300000,0.01); # To prove (2.26) execute verifpi4max
  (150,89967803,0.01)
```

"r=4, coef=", 0.20254144050183350040

"p=", 199, "pi4(p)=", 1.4185215405 10^{10}, "est=", 1.3972908775495022172 10^{10}, "S4/est=", 1.015194

"p=", 283, "pi4(p)=", 7.5911179980 10^{10}, "est=", 7.6204777072473669256 10^{10}, "S4/est=", 0.9961472

"solution of pi_4(t) log(t) = 0.237 t^5 is", 199.6247 **(15)**

```
> verifpi5max(31,100000,0.01); # To prove (2.27) execute verifpi3max
  (31,89967803,0.01)
```

"r=5, coef=", 0.16852606262213529074

"p=", 31, "pi5(p)=", 6.0025150 10^{7}, "c=", 5.8409021339400157698 10^{7}, "S5/c=", 1.027669

"p=", 43, "pi5(p)=", 3.92233751 10^{8}, "c=", 3.7983303337492375910 10^{8}, "S5/c=", 1.032648

"p=", 109, "pi5(p)=", 8.0528815750 10^{10}, "c=", 8.0792265797600247385 10^{10}, "S5/c=", 0.9967392

"p=", 113, "pi5(p)=", 9.8953167543 10^{10}, "c=", 9.9530885582769023493 10^{10}, "S5/c=", 0.9941956

"solution of pi_5(t) log(t)=0.226 t^6 is", 43.24158 **(16)**

```
> verifpi5maxbis(5,43);# Compute the max of pi_5(p)/(p^6/log(p)) for
  pmin <= p <= pmax
                         (proof of (2.28))
```

[0.35021, 0.33422, 0.24534, 0.29362, 0.23152, 0.27842, 0.23055, 0.17773, 0.23225, 0.18207, 0.19171, 0.23338]

"pmaxi=", 5, "maxi=", 0.35021369, "log(3.)/3=", 0.36620410 **(17)**

```
>
> ################ Definition of Superchampions, Section 3.1
  ################
>
>  n:=nu0: SupChn(n);
   # determines rho, xi, n', n'', N', N'' (as defined in Definition
  3.6) and g(n)/N'
```

" rho = pquo/log(pquo) = ", 4.342944827 10^8, " xi = pquo = ", 10000000019
" n = ", 2220832950051364840, " n' =ell(N') = ", 2220832950051364840, " n" = ell(N") = ",
2220832960051364859
" N' = ", [2, 29], [3, 18], [5, 12], [7, 10], [11, 8], [13, 8], [17, 7], [19, 7], [23, 6], ... , [31,
6], [37, 5], ... , [71, 5], [73, 4], ... , [211, 4], [223, 3], ... , [1459, 3], [1471, 2], ... ,
[69557, 2], 69593, ... , 9999999967
" N"/N' = pquo = ", 10000000019, " g(n)/N' = ", 1, " = ", 1., " time = ", 170.371 **(18)**

```
> calculdexij(10^10+19); # from xi_1, this proc determines xi[j] as
  described in Definition 3.6,
  p1[j] and p2[j] are the primes surrounding xi[j] cf. also (3.19).
  "expo" is the exponent of p in N_rho = N' (cf. (3.6) and (3.18)). J
  is defined in (3.9)
```

[["j=", 2, "p1=", 69557, "xi[j]=", 69588.85912, "p2=", 69593], ["j=", 3, "p1=", 1459, "xi[j]=", 1468.859920, "p2=", 1471], ["j=", 4, "p1=", 211, "xi[j]=", 220.2592091, "p2=", 223], ["j=", 5, "p1=", 71, "xi[j]=", 71.59540467, "p2=", 73], ["j=", 6, "p1=", 31, "xi[j]=", 34.12559548, "p2=", 37], ["j=", 7, "p1=", 19, "xi[j]=", 20.20220124, "p2=", 23], ["j=", 8, "p1=", 13, "xi[j]=", 13.67872852, "p2=", 17], ["j=", 9, "p1=", 7, "xi[j]=", 10.12216225, "p2=", 11], ["j=", 10, "p1=", 7, "xi[j]=", 7.967318437, "p2=", 11], ["j=", 11, "p1=", 5, "xi[j]=", 6.557245414, "p2=", 7], ["j=", 12, "p1=", 5, "xi[j]=", 5.579189860, "p2=", 7], ["j=", 13, "p1=", 3, "xi[j]=", 4.869318441, "p2=", 5], ["j=", 14, "p1=", 3, "xi[j]=", 4.335130355, "p2=", 5], ["j=", 15, "p1=", 3, "xi[j]=", 3.921159360, "p2=", 5], ["j=", 16, "p1=", 3, "xi[j]=", 3.592464449, "p2=", 5], ["j=", 17, "p1=", 3, "xi[j]=", 3.326110823, "p2=", 5], ["j=", 18, "p1=", 3, "xi[j]=", 3.106512206, "p2=", 5], ["j=", 19, "p1=", 2, "xi[j]=", 2.922756826, "p2=", 3], ["j=", 20, "p1=", 2, "xi[j]=", 2.767006733, "p2=", 3], ["j=", 21, "p1=", 2, "xi[j]=", 2.633504368, "p2=", 3], ["j=", 22, "p1=", 2, "xi[j]=", 2.517936236, "p2=", 3], ["j=", 23, "p1=", 2, "xi[j]=", 2.417013857, "p2=", 3], ["j=", 24, "p1=", 2, "xi[j]=", 2.328190911, "p2=", 3], ["j=", 25, "p1=", 2, "xi[j]=", 2.249468069, "p2=", 3], ["j=", 26, "p1=", 2, "xi[j]=", 2.179255654, "p2=", 3], ["j=", 27, "p1=", 2, "xi[j]=", 2.116275247, "p2=", 3], ["j=", 28, "p1=", 2, "xi[j]=", 2.059488059, "p2=", 3], ["j=", 29, "p1=", 2, "xi[j]=", 2.008042006, "p2=", 3], ["j=", 30, "p1=", 1, "xi[j]=", 1.961232078, "p2=", 2]]

[["expo=", 0, [10000000019, "... ,infinity"]], ["expo=", 1, [69593, "...", 9999999967]], ["expo=", 2, [1471, "...", 69557]], ["expo=", 3, [223, "...", 1459]], ["expo=", 4, [73, "...", 211]], ["expo=", 5, [37, "...", 71]], ["expo=", 6, [23, "...", 31]], ["expo=", 7, [17, "...", 19]], ["expo=", 8, [11, "...", 13]]]
[["p=", 7, "expo=", 10], ["p=", 5, "expo=", 12], ["p=", 3, "expo=", 18], ["p=", 2, "expo=", 29]]
"J=", 29.165332010963717010 **(19)**

```
> calculch(1,6): # Computation of table of Fig. 1 and enumeration of superchampion numbers
```

"i= ", "ell(N)= ", " N= ", " rho= ", " xi= "

1, " ", 7, " ", [2, 2], 3, " ", 3.1067, " ", 5

1, " ", 12, " ", [2, 2], 3, 5, " ", 3.5973, " ", 7

1, " ", 19, " ", [2, 2], 3, ... , 7, " ", 4.5874, " ", 11

1, " ", 30, " ", [2, 2], 3, ... , 11, " ", 5.0683, " ", 13

1, " ", 43, " ", [2, 2], 3, ... , 13, " ", 5.4614, " ", 14.667

2, " ", 49, " ", [2, 2], [3, 2], 5, ... , 13, " ", 5.7708, " ", 16.000

3, " ", 53, " ", [2, 3], [3, 2], 5, ... , 13, " ", 6.0003, " ", 17

3, " ", 70, " ", [2, 3], [3, 2], 5, ... , 17, " ", 6.4528, " ", 19

3, " ", 89, " ", [2, 3], [3, 2], 5, ... , 19, " ", 7.3354, " ", 23

3, " ", 112, " ", [2, 3], [3, 2], 5, ... , 23, " ", 8.6123, " ", 29

3, " ", 141, " ", [2, 3], [3, 2], 5, ... , 29, " ", 9.0274, " ", 31

3, " ", 172, " ", [2, 3], [3, 2], 5, ... , 31, " ", 10.247, " ", 37

3, " ", 209, " ", [2, 3], [3, 2], 5, ... , 37, " ", 11.041, " ", 41

3, " ", 250, " ", [2, 3], [3, 2], 5, ... , 41, " ", 11.433, " ", 43

3, " ", 293, " ", [2, 3], [3, 2], 5, ... , 43, " ", 11.542, " ", 43.559

4, " ", 301, " ", [2, 4], [3, 2], 5, ... , 43, " ", 12.207, " ", 47

4, " ", 348, " ", [2, 4], [3, 2], 5, ... , 47, " ", 12.427, " ", 48.143

5, " ", 368, " ", [2, 4], [3, 2], [5, 2], 7, ... , 47, " ", 13.349, " ", 53

5, " ", 421, " ", [2, 4], [3, 2], [5, 2], 7, ... , 53, " ", 14.470, " ", 59

5, " ", 480, " ", [2, 4], [3, 2], [5, 2], 7, ... , 59, " ", 14.839, " ", 61

5, " ", 541, " ", [2, 4], [3, 2], [5, 2], 7, ... , 61, " ", 15.935, " ", 67

5, " ", 608, " ", [2, 4], [3, 2], [5, 2], 7, ... , 67, " ", 16.384, " ", 69.488

6, " ", 626, " ", [2, 4], [3, 3], [5, 2], 7, ... , 67, " ", 16.656, " ", 71

6, " ", 697, " ", [2, 4], [3, 3], [5, 2], 7, ... , 71, " ", 17.014, " ", 73

6, " ", 770, " ", [2, 4], [3, 3], [5, 2], 7, ... , 73, " ", 18.080, " ", 79

6, " ", 849, " ", [2, 4], [3, 3], [5, 2], 7, ... , 79, " ", 18.783, " ", 83

6, " ", 932, " ", [2, 4], [3, 3], [5, 2], 7, ... , 83, " ", 19.828, " ", 89

6, " ", 1021, " ", [2, 4], [3, 3], [5, 2], 7, ... , 89, " ", 21.204, " ", 97

6, " ", 1118, " ", [2, 4], [3, 3], [5, 2], 7, ... , 97, " ", 21.584, " ", 99.230 **(20)**

```
> lemma3_8ii(9); # Proof of Lemma 3.8 (ii) and computation of lambda [j]
```

"j=", 2, "e^j=", 7.3890561, "lambda[j]=", 79.877781309766419843, 80

"j=", 3, "e^j=", 20.085537, "lambda[j]=", 585.45826414894707700, 586

"j=", 4, "e^j=", 54.598150, "lambda[j]=", 6380.4599469708695867, 6381

"j=", 5, "e^j=", 148.41316, "lambda[j]=", 89016.683602195247707, 89017

$$"j=", 6, "e^j=", 403.42879, "lambda[j]=", 1.4997496877890318902\ 10^{6}, 1499750$$

$$"j=", 7, "e^j=", 1096.6332, "lambda[j]=", 2.9511243343943428453\ 10^{7}, 29511244$$

$$"j=", 8, "e^j=", 2980.9580, "lambda[j]=", 6.6318407453999995851\ 10^{8}, 663184075$$

$$"j=", 9, "e^j=", 8103.0839, "lambda[j]=", 1.6743263688753398443\ 10^{10}, 16743263689 \quad \mathbf{(21)}$$

```
> lemma3_11(t); # Convexity : Proof of Lemma 3.11, cf. (3.25)
```

$$"Phi =", \sqrt{t \ln(t)} \left(1 + \frac{1}{2} \frac{\ln(\ln(t)) - 1}{\ln(t)} - \frac{u \ln(\ln(t))^2}{\ln(t)^2}\right)$$

$$"Phi\_1 = d\ Phi/dt=", \frac{1}{4} \frac{-2 L \lambda^2 u + 2 L^3 + L^2 \lambda + 6 \lambda^2 u + L^2 - L \lambda - 8 \lambda u + 3 L}{L^2 \sqrt{t L}}$$

"Phi_1 as written =",

$$\frac{1}{4} \frac{2 L \left(-\lambda^2 u + L^2\right) + L \left(L - \lambda + 3\right) + \lambda \left(L^2 - 2 u\right) + 6 u \lambda \left(\lambda - 1\right)}{L^2 \sqrt{t L}}$$

"Phi1(L,lambda) - Phi1 as written =", 0

"Phi_2 = d Phi1/dt=",

$$-\frac{1}{8} \frac{-2 L^2 \lambda^2 u + 2 L^4 + L^3 \lambda - L^3 + 30 \lambda^2 u + 2 L^2 - 3 L \lambda - 64 \lambda u + 11 L + 16 u}{L^2 (t L)^{3/2}}$$

$$"Phi\_2 as written =", -\frac{1}{8} \frac{1}{L^2 (t L)^{3/2}} \left(L^2 \left(-2 \lambda^2 u + L^2\right) + L^3 \left(\lambda - 1\right) + L \left(2 L - 3 \lambda\right) + L^4 + 11 L - 22 u \lambda + 2 u \left(15 \lambda^2 - 21 \lambda + 8\right)\right)$$

"Phi2(L,lambda) - Phi2 as written =", 0 **(22)**

```
> subs(a=log(2),devx2(3,0)); # Computation of (3.30) with t = log(xi)
```

$$1 - \frac{1}{2} \frac{\ln(2)}{t} + \frac{-\frac{1}{2} \ln(2) - \frac{1}{8} \ln(2)^2}{t^2} + \frac{-\frac{1}{2} \ln(2) - \frac{1}{2} \ln(2)^2 - \frac{1}{16} \ln(2)^3}{t^3} \quad \mathbf{(23)}$$

```
> prop3_14(); # Proof of Proposition 3.14 : effective estimate of
  xi_2, a satisfies 0 < a <= 0.4
```

$$"phi = ", \frac{1}{2} \sqrt{2} \sqrt{x} \left(1 - \frac{a}{\ln(x)}\right)$$

"Wx = (phi^2-phi)*log(x)/x-log(phi) = ",

$$\frac{\left(\frac{1}{2} x \left(1 - \frac{a}{\ln(x)}\right)^2 - \frac{1}{2} \sqrt{2} \sqrt{x} \left(1 - \frac{a}{\ln(x)}\right)\right) \ln(x)}{x} - \ln\left(\frac{1}{2} \sqrt{2} \sqrt{x} \left(1 - \frac{a}{\ln(x)}\right)\right)$$

$$"Wt = ", \frac{3}{2} \ln(2) + \ln(t) - \ln(2 t - a) + \frac{1}{4} \frac{a^2}{t} - a + \frac{\frac{1}{2} \sqrt{2}\, a - t \sqrt{2}}{e^t},$$

"change of variable x = e^(2t), t = log(x)/2"

"W as written = ", $-a+\frac{1}{4}\frac{a^2}{t}+\frac{3}{2}\ln(2)+\ln(t)-\ln(2t-a)-\frac{1}{2}e^{-t}\sqrt{2}(2t-a)$

"W as written - Wt = ", 0

"W1 = dW/dt = ", $-\frac{1}{4}\frac{1}{t^2(-2t+a)}\left(2e^{-t}\sqrt{2}a^2t^2-8e^{-t}\sqrt{2}at^3+8e^{-t}\sqrt{2}t^4\right.$
$\left.+4e^{-t}\sqrt{2}t^2a-8e^{-t}\sqrt{2}t^3+a^3-2a^2t-4ta\right)$

"U=", $\left(8\sqrt{2}t^4+\left(-8\sqrt{2}a-8\sqrt{2}\right)t^3+\left(2\sqrt{2}a^2+4\sqrt{2}a\right)t^2\right)e^{-t}+\left(-2a^2-4a\right)t$
$+a^3$

"UU1 = dU/dt = ", $\left(-8\sqrt{2}t^4+\left(8\sqrt{2}a+40\sqrt{2}\right)t^3+\left(-2\sqrt{2}a^2-28\sqrt{2}a\right.\right.$
$\left.\left.-24\sqrt{2}\right)t^2+\left(4\sqrt{2}a^2+8\sqrt{2}a\right)t\right)e^{-t}-2a^2-4a$

"U1 as writte n =", $-2a(a+2)+2\sqrt{2}e^{-t}\left(-4t^4+4(a+5)t^3-\left(a^2+14a+12\right)t^2\right.$
$\left.+2a(a+2)t\right)$

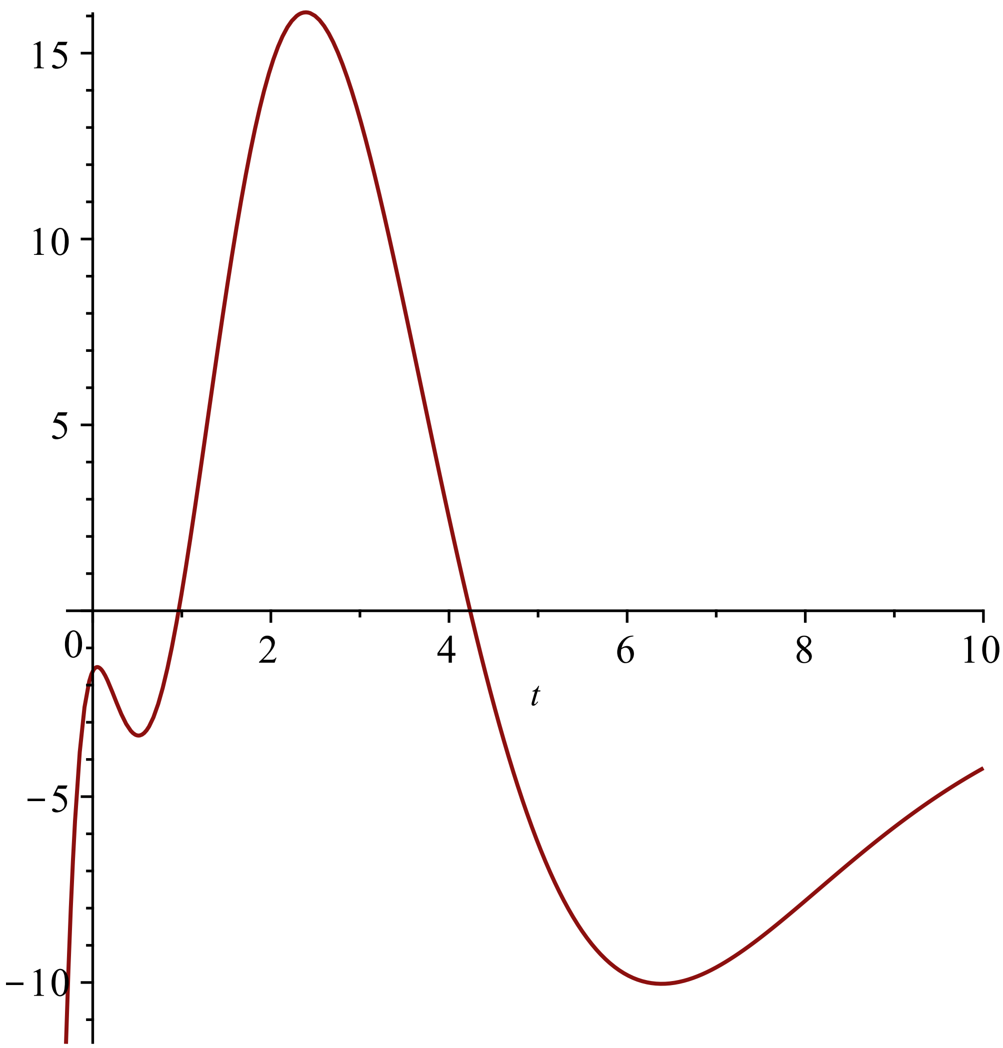
"U1 as written - UU1 = ", 0

"U2 = dU1/dt = ", $\left(8\sqrt{2}t^4+\left(-8\sqrt{2}a-72\sqrt{2}\right)t^3+\left(2\sqrt{2}a^2+52\sqrt{2}a\right.\right.$
$\left.\left.+144\sqrt{2}\right)t^2+\left(-8\sqrt{2}a^2-64\sqrt{2}a-48\sqrt{2}\right)t+4\sqrt{2}a^2+8\sqrt{2}a\right)e^{-t}$

"V = U2* exp(t)/(2*sqrt(2)) = ", $4t^4+(-4a-36)t^3+\left(a^2+26a+72\right)t^2+\left(-4a^2\right.$
$\left.-32a-24\right)t+2a^2+4a$

"******** UPPER BOUND a = log(2)/2 = ", 0.34657359, " **********"

"real roots of Va = ", 0.051671859, 0.51498620, 2.3922609, 6.3876546

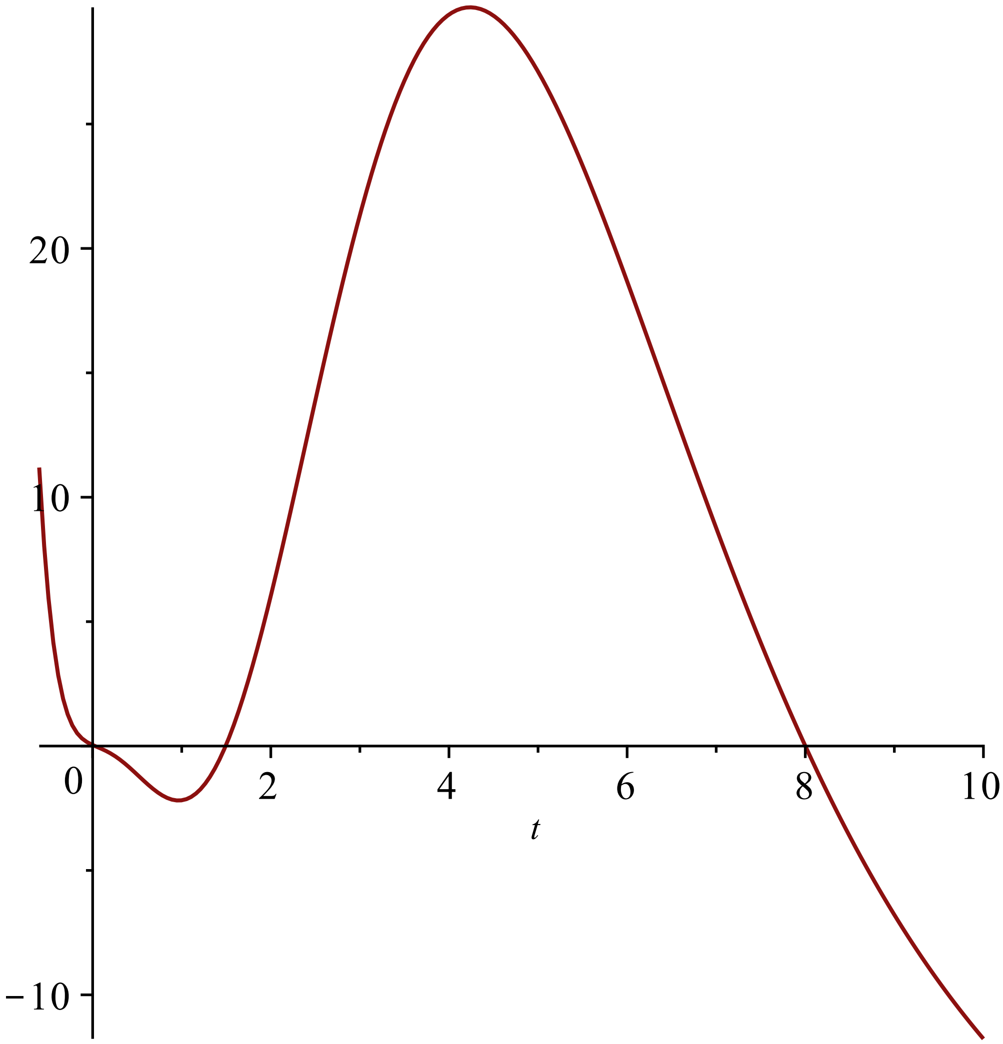


"U1a = subs(a = log(2)/2,U1),   t = -0.3..10)"

"list of extrema of U1a = ", [−1.5146439, −3.3570388, 16.098644, −10.036034]

"roots of U1a = ", 0.96460820, 4.2370082

"lim U1a_{t --> -inf} = ", −∞, "U1a(log 2) = ", −2.7975979, "lim U1a_{t --> +inf} = ", −1.6265209

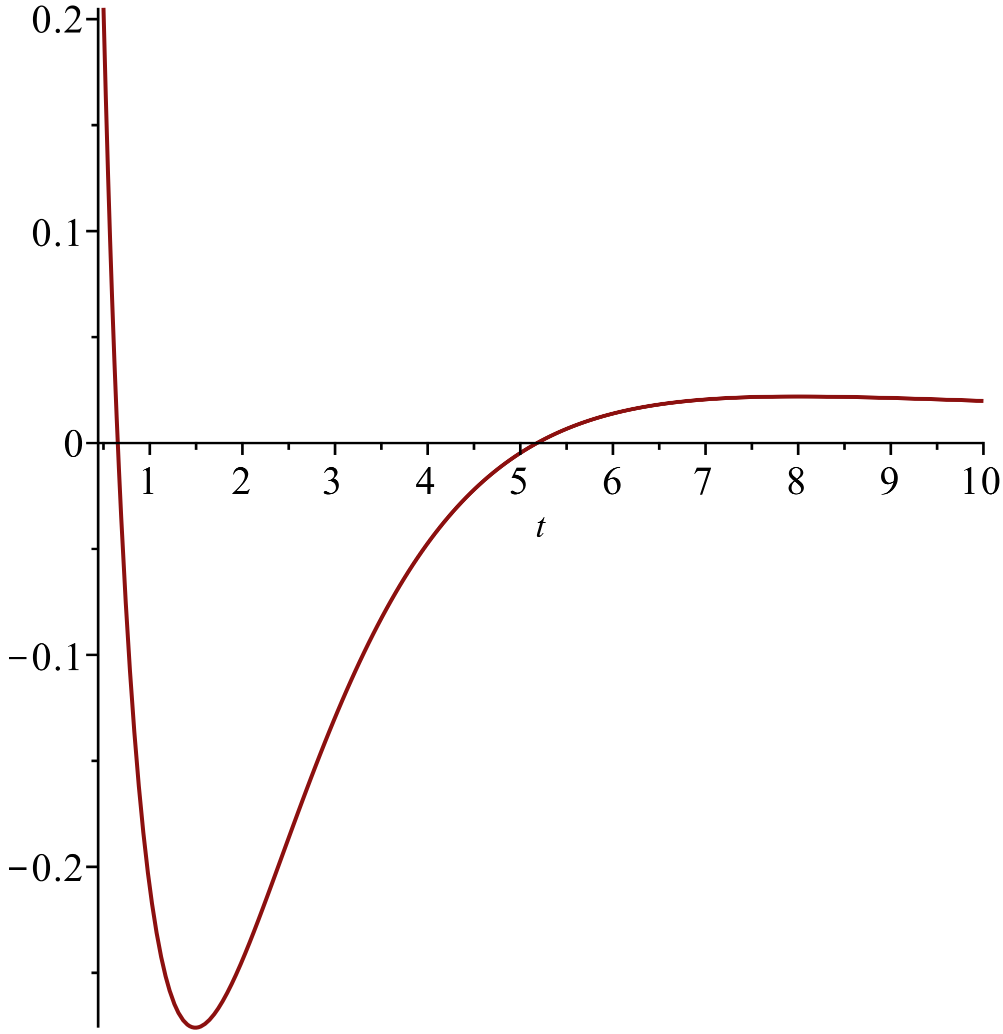


"Ua = subs(a = log(2)/2,U), t = -1..10)"

"list of extrema of Ua = ", [−2.1898697, 29.691640]

"roots of Ua = ", 0.026388103, 1.4925111, 8.0010041

"lim Ua_{t--> -inf} = ", ∞, "Ua(log 2) = ", −1.7641814, "lim Ua_{t--> +inf} = ", −∞



"Wa = subs(a = log(2)/2,W), t = 0.5..10 (Wa is defined for t > a/2)"

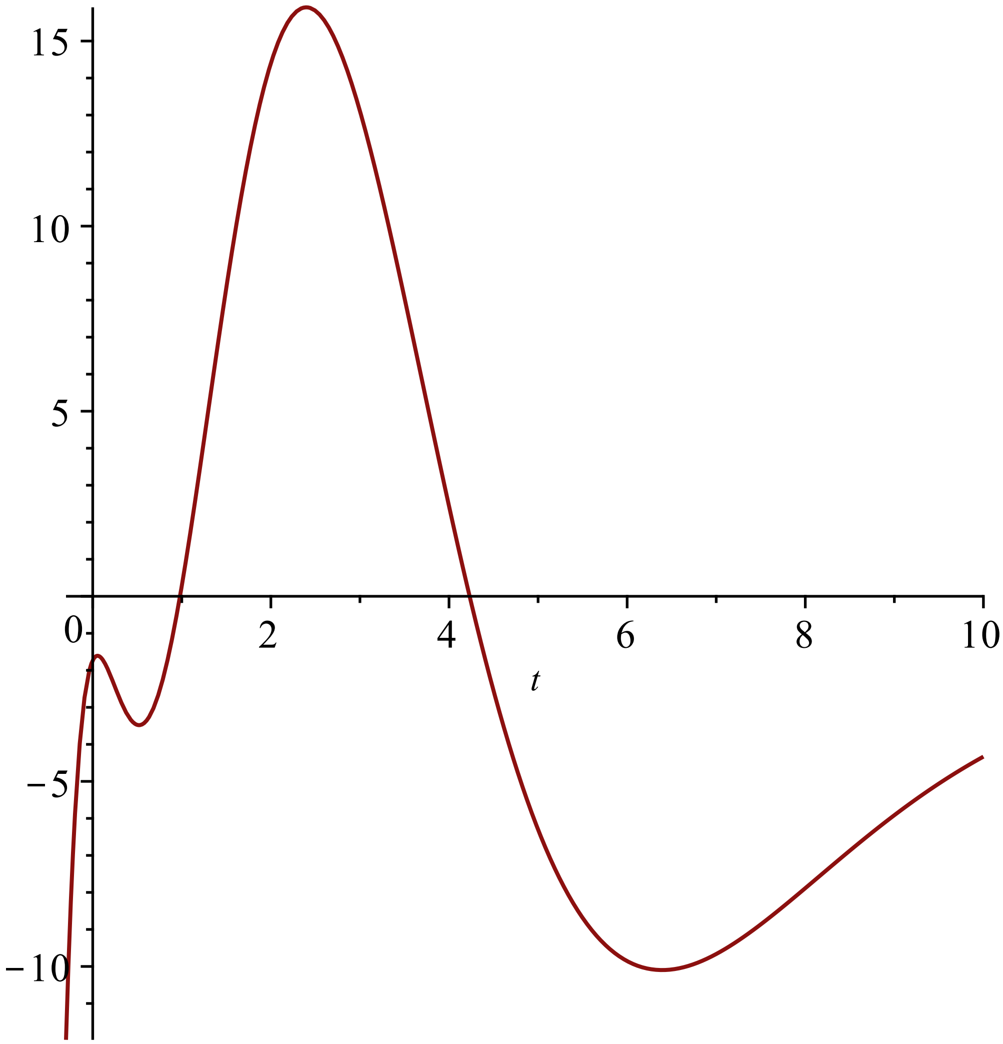
"list of extrema of Wa = ", [−0.27587993, 0.021939313]

"roots of Wa: w1 = ", 0.65426442, "w2 = ", 5.1811243, "exp(2*w1)=", 3.7007252, "exp(2*w2)=", 31642.265

"lim Wa_{t --> a/2,right} = ", ∞, "Wa(log 2) = ", −0.03659308, "lim Wa_{t --> +inf} = ", 0

"*************** LOWER BOUND a = aa = 0.366 *****************"

"real roots of Va = ", 0.054243563, 0.52068533, 2.3978548, 6.3932163

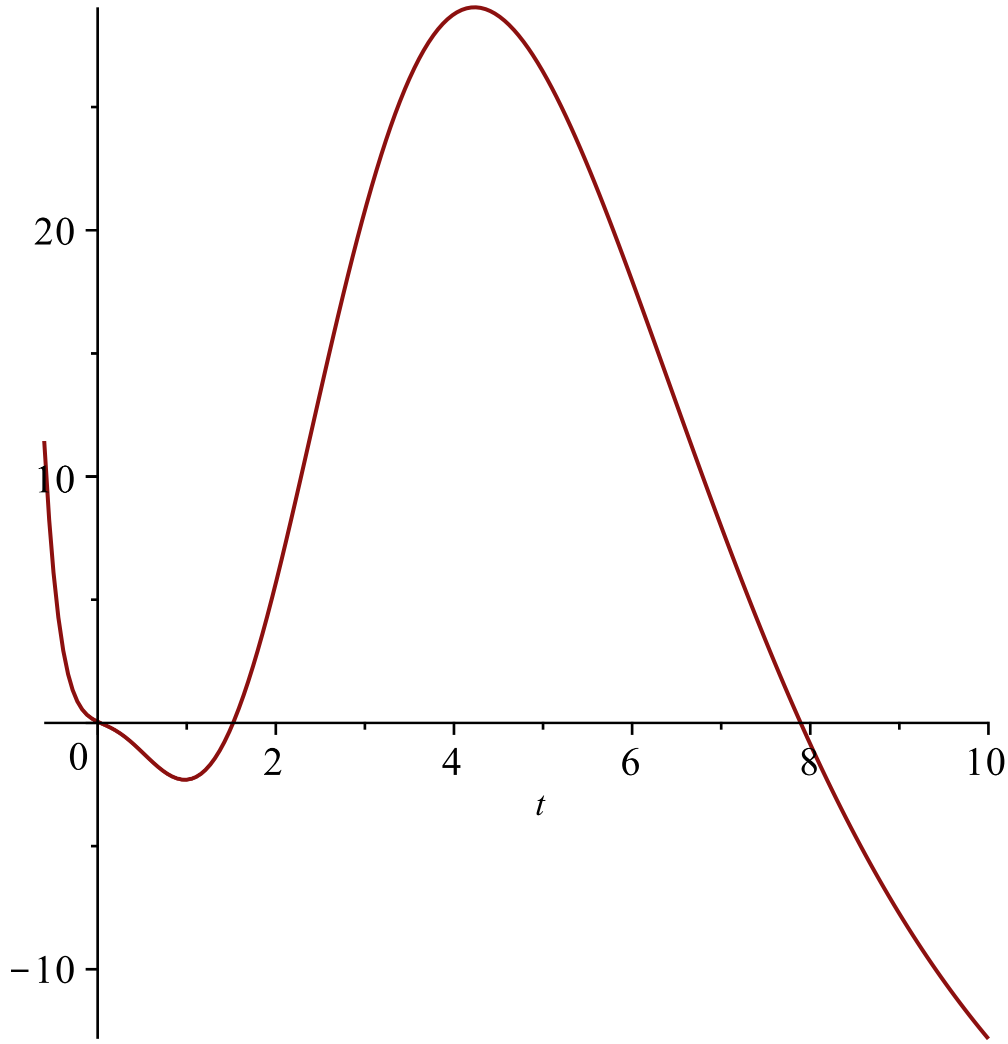


"U1a = subs(a = log(2)/2,U1),   t = -0.3..10)"

"list of extrema of U1a = ", [−1.6071816, −3.4814326, 15.910864, −10.097391]

"roots of U1a = ", 0.97973211, 4.2315297

"lim U1a_{t -->-inf} = ", $-\infty$, "U1a(log 2) = ", −2.9572804, "lim U1a_{t --> +inf} = ", −1.7319120

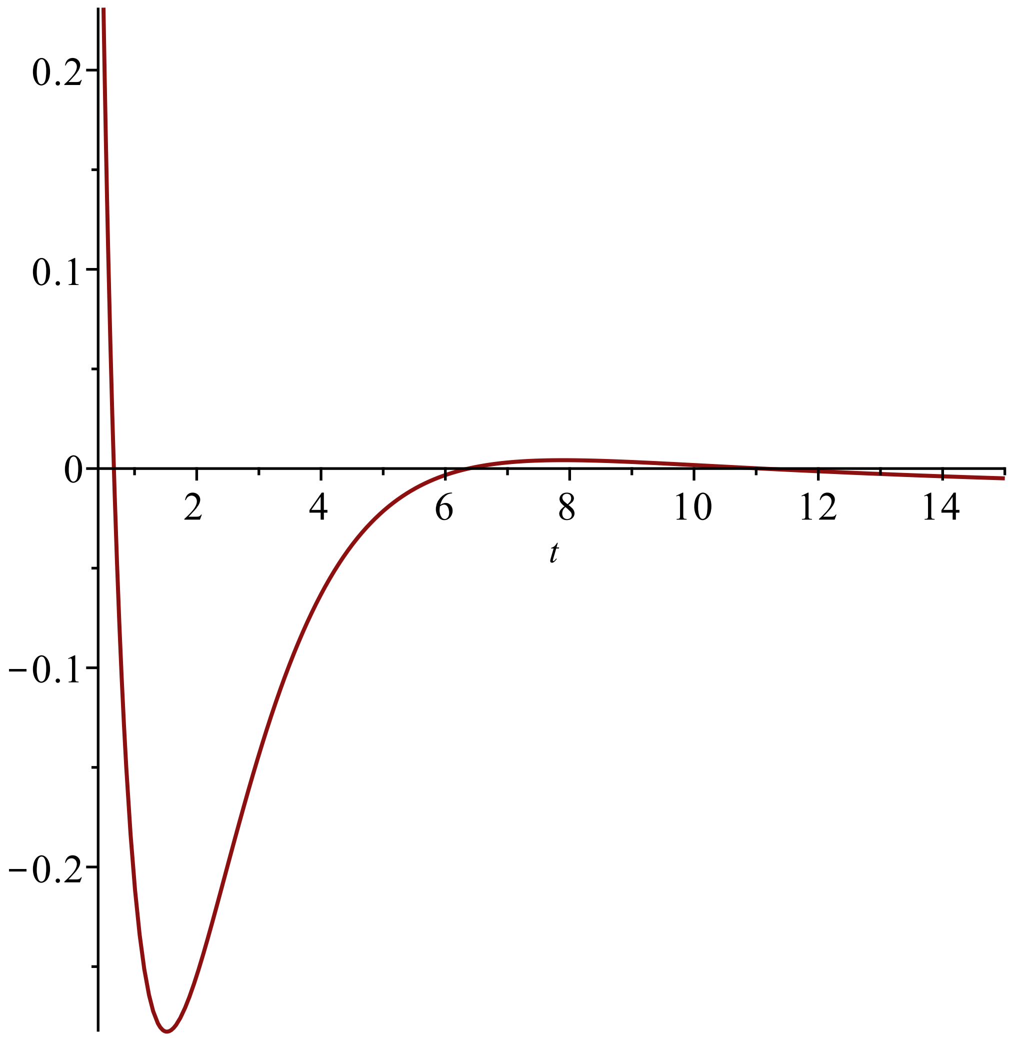


"Ua = subs(a = log(2)/2,U), t = -1..10)"

"list of extrema of Ua = ", [-2.3080321, 29.049736]

"roots of Ua = ", 0.029272690, 1.5208774, 7.8921186

"lim Ua_{t--> -inf} = ", ∞, "Ua(log 2) = ", -1.8306254, "lim Ua_{t--> +inf} = ", -∞



"Wa = subs(a = 0.366,W), t = 0.5..15 (Wa is defined for t > a/2)"

"list of extrema of Wa = ", [−0.28265348, 0.0042037371]

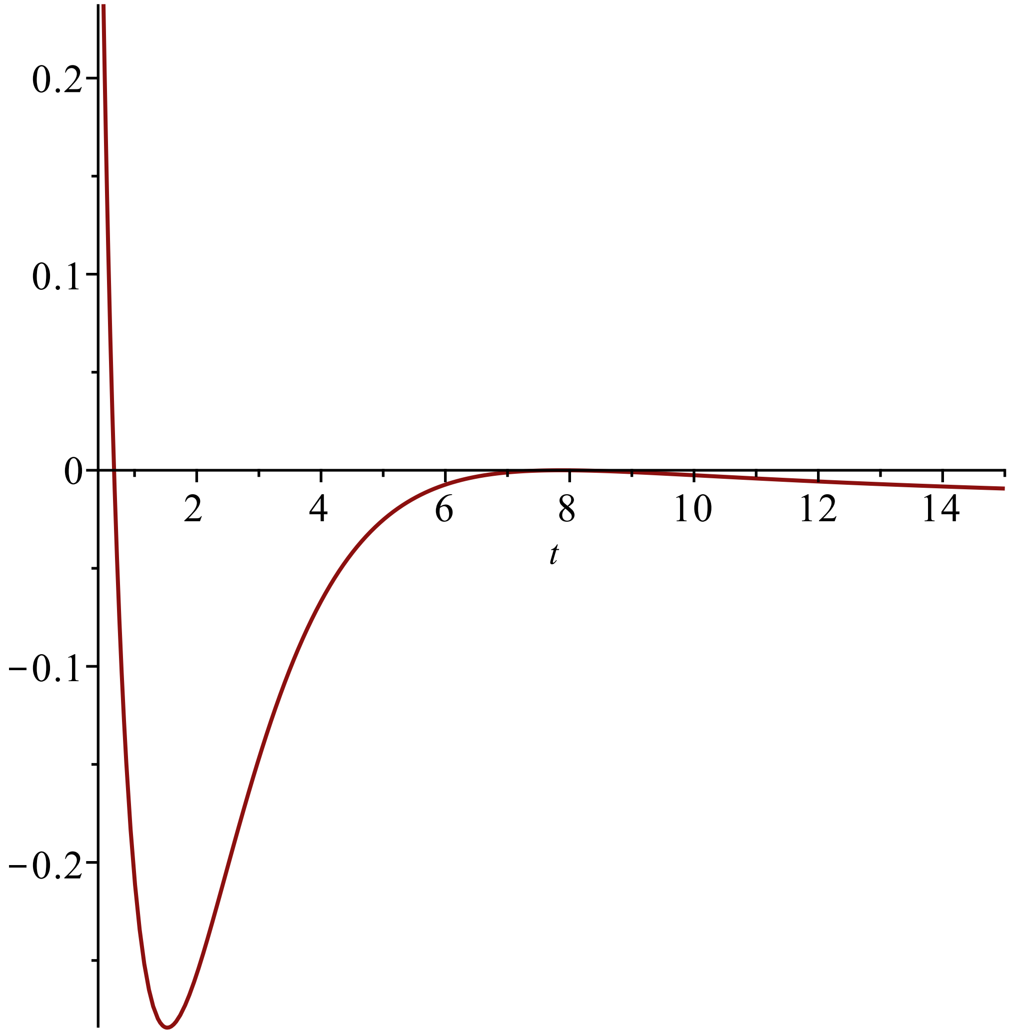
"root of Wa: w1 = ", 0.66717791, " w2 = ", 6.3701275, "w3 = ", 11.088031

"exp(2*w1) = ", 3.7975488, "exp(2*w2) = ", $3.4121055\ 10^5$, "exp(2*w3) = ", $4.2750516\ 10^9$

"lim Wa_{t --> a/2,right} = ", ∞, "Wa(log 2) = ", −0.02529747, "lim Wa_{t --> +inf} = ", −0.01942641

"*************** REMARK a = 0.370612465... ******************"

"roots of {U=0,W=0} = ", $a = 0.37061246510421873686$, $t = 7.8668240753167659431$



"subs(a = 0.3706..,W),   t = 0.5..15"

"lim Wa_{t --> aa/2,right} = ", ∞, "Wa(log 2) = ", -0.02252275, "lim Wa_{t --> +inf} = ", -0.02403888 **(24)**

```
> -log(2.)/2+log(1-0.366/logx0),2/(1-0.726/logx0),0.726/(1
  -0.726/logx0), 1+0.75/logx0;
  # Proof of Corollary 3.16, (3.36)
```

-0.36259645149756300309, 2.0651125428792345374, 0.74963585306516213708, 1.0325720861400561704 **(25)**

```
> 0.0746*2.07,0.000079*2.07,log(2.)/2-0.000164; # Proof of Corollary
  3.17, (3.37)
```

0.154422, 0.00016353, 0.34640959027997265471 **(26)**

```
> 0.13-0.496*0.366/logx0,log(2.)/2-sqrt(2.)*(1+2/sqrtx0)*
  logx0^2/sqrtx0,2.07*0.385,0.797-0.339;          #  Proof of
  Corollary 3.18, (3.38)
```

0.12211599169397168407, 0.33907542411137549851, 0.79695, 0.458 **(27)**

```
> addexc(x0); # Computation of the additive excess, Section 3.5.1,
  (3.40)
```

10517469635602 **(28)**

```
> 0.122-0.057*3*sqrt(2.)*logx0^2/sqrt(x0); # Proof of Proposition
```

```
3.19, (3.42), the lower bound
```
0.12071783922838529422 **(29)**

```
> majexcadd(8,3); # Proof of Proposition 3.19, (3.42), the upper
  bound
```
"j=", 3, "tot=", 1.9173213501338885334, "be=", 3., "ga=", 0.19732, "de=", 0.4152847, "et=", 1.459321, "xi0=", 1468.859920

"j=", 4, "tot=", 1.0917301233710986024, "be=", 4., "ga=", 0.17832, "de=", 0.05508744, "et=", 0.2184455, "xi0=", 220.2592091

"j=", 5, "tot=", 0.99917712229214889068, "be=", 5., "ga=", 0.17611, "de=", 0.01720423, "et=", 0.07080502, "xi0=", 71.59540467

"j=", 6, "tot=", 0.97921454582973396434, "be=", 6., "ga=", 0.32045, "de=", 0.01453040, "et=", 0.03363820, "xi0=", 34.12559548

"j=", 7, "tot=", 0.97995402905538685099, "be=", 7., "ga=", 0.32066, "de=", 0.008403627, "et=", 0.01984729, "xi0=", 20.20220124

"j=", 8, "tot=", 0.98190626528105069885, "be=", 8., "ga=", 0.32253, "de=", 0.005602969, "et=", 0.01339590, "xi0=", 13.67872852

"total minimum =", 0.97921454582973396434, "j0 =", 6 **(30)**

```
> remark3_20(x0); # Remark 3.20
```
"x = ", $1.0000000019\ 10^{10}$, "additive excess = ", 10517469635602, "coe = ", 0.63219636253854929280 **(31)**

```
> mulexc(x0); # Computation of the multiplicative excess, Section
  3.5.2, (3.49)
```
70954.464472086132180 **(32)**

```
> remark3_22(x0); # Remark 3.22
```
"x = ", $1.0000000019\ 10^{10}$, "multiplicative excess = ", 70954.464472086132180, "coe = ", 0.079385275754404565538 **(33)**

```
> majexcmul(29,3); # Proof of Proposition 3.21 the upper bound,
  (3.55)
```
"j0=", 3, "total[j0] = ", 15.812988, "be = ", 3., "S1 = ", 0., "S2 = ", 16.15898, "xi0 = ", 1468.8599

"j0=", 4, "total[j0] = ", 2.5592690, "be = ", 4., "S1 = ", 0.4864336, "S2 = ", 2.418833, "xi0 = ", 220.25921

"j0=", 5, "total[j0] = ", 0.99726837, "be = ", 5., "S1 = ", 0.5592478, "S2 = ", 0.7840196, "xi0 = ", 71.595405

"j0=", 6, "total[j0] = ", 0.60932361, "be = ", 6., "S1 = ", 0.5828491, "S2 = ", 0.3724738, "xi0 = ", 34.125595

"j0=", 7, "total[j0] = ", 0.46783010, "be = ", 7., "S1 = ", 0.5940617, "S2 = ", 0.2197678, "xi0 = ", 20.202201

"j0=", 8, "total[j0] = ", 0.40300986, "be = ", 8., "S1 = ", 0.6006774, "S2 = ", 0.1483319, "xi0 = ", 13.678729

```
"j0=", 9, "total[j0] = ", 0.36733275, "be = ", 9.9866, "S1 = ", 0.6051426, "S2 = ", 0.1081896,
    "xi0 = ", 10.122162
"j0=", 10, "total[j0] = ", 0.34718051, "be = ", 11.435, "S1 = ", 0.6083994, "S2 = ", 0.08478056,
    "xi0 = ", 7.9673184
"j0=", 11, "total[j0] = ", 0.33445562, "be = ", 12.979, "S1 = ", 0.6109516, "S2 = ", 0.06950353,
    "xi0 = ", 6.5572454
"j0=", 12, "total[j0] = ", 0.32597944, "be = ", 14.621, "S1 = ", 0.6130438, "S2 = ", 0.05893509,
    "xi0 = ", 5.5791899
"j0=", 13, "total[j0] = ", 0.32010274, "be = ", 16.360, "S1 = ", 0.6148180, "S2 = ", 0.05128427,
    "xi0 = ", 4.8693184
"j0=", 14, "total[j0] = ", 0.31590357, "be = ", 18.198, "S1 = ", 0.6163618, "S2 = ", 0.04554130,
    "xi0 = ", 4.3351304
"j0=", 15, "total[j0] = ", 0.31283469, "be = ", 20.135, "S1 = ", 0.6177327, "S2 = ", 0.04110149,
    "xi0 = ", 3.9211594
"j0=", 16, "total[j0] = ", 0.31055495, "be = ", 22.172, "S1 = ", 0.6189700, "S2 = ", 0.03758447,
    "xi0 = ", 3.5924644
"j0=", 17, "total[j0] = ", 0.30884279, "be = ", 24.308, "S1 = ", 0.6201014, "S2 = ", 0.03474091,
    "xi0 = ", 3.3261108
"j0=", 18, "total[j0] = ", 0.30754928, "be = ", 26.545, "S1 = ", 0.6211472, "S2 = ", 0.03240159,
    "xi0 = ", 3.1065122
"j0=", 19, "total[j0] = ", 0.30657129, "be = ", 28.882, "S1 = ", 0.6221226, "S2 = ", 0.03044822,
    "xi0 = ", 2.9227568
"j0=", 20, "total[j0] = ", 0.30583556, "be = ", 31.319, "S1 = ", 0.6230392, "S2 = ", 0.02879591,
    "xi0 = ", 2.7670067
"j0=", 21, "total[j0] = ", 0.30528891, "be = ", 33.856, "S1 = ", 0.6239060, "S2 = ", 0.02738241,
    "xi0 = ", 2.6335044
"j0=", 22, "total[j0] = ", 0.30489193, "be = ", 36.493, "S1 = ", 0.6247303, "S2 = ", 0.02616114,
    "xi0 = ", 2.5179362
"j0=", 23, "total[j0] = ", 0.30461493, "be = ", 39.231, "S1 = ", 0.6255178, "S2 = ", 0.02509661,
    "xi0 = ", 2.4170139
"j0=", 24, "total[j0] = ", 0.30443521, "be = ", 42.070, "S1 = ", 0.6262733, "S2 = ", 0.02416141,
    "xi0 = ", 2.3281909
"j0=", 25, "total[j0] = ", 0.30433513, "be = ", 45.009, "S1 = ", 0.6270006, "S2 = ", 0.02333400,
    "xi0 = ", 2.2494681
"j0=", 26, "total[j0] = ", 0.30430085, "be = ", 48.048, "S1 = ", 0.6277031, "S2 = ", 0.02259730,
    "xi0 = ", 2.1792557
"j0=", 27, "total[j0] = ", 0.30432139, "be = ", 51.188, "S1 = ", 0.6283833, "S2 = ", 0.02193759,
    "xi0 = ", 2.1162752
```

"j0=", 28, "total[j0] = ", 0.30438791, "be = ", 54.428, "S1 = ", 0.6290437, "S2 = ", 0.02134372, "xi0 = ", 2.0594881

"j0=", 29, "total[j0] = ", 0.30449327, "be = ", 57.769, "S1 = ", 0.6296862, "S2 = ", 0.02080657, "xi0 = ", 2.0080420

"total minimum =", 0.30430085464403772470, "j0min =", 26 **(34)**

```
> 0.98+3*sqrt(2.)*logx0^2/sqrt(x0),1.01+3*sqrt(2.)*logx0^2/sqrt(x0),
  1/3/sqrt(2.)/logx0*(1+1.033/logx0); #(3.60)--(3.62)
```

1.0024940486248193996, 1.0324940486248193996, 0.010695651685592331819 **(35)**

```
> 0.045/logx0,0.12-0.002,0.118-0.002*0.12/logx0,0.1179-3*sqrt(2.)*
  logx0^2/sqrt(x0);
  # (3.63)--(3.64)
```

0.0019543251684033702237, 0.118, 0.11798957693243518203, 0.095405951375180600404 **(36)**

```
> 0.002/logx0,(1.0331+0.0001033/logx0)/3, 0.095-3*sqrt(2.)*
  logx0^2/sqrt(x0);
  # (3.65)--(3.67)
```

0.000086858896373483121052, 0.34436816208733256347, 0.072505951375180600404 **(37)**

```
> ############### Section 5 : Proof of Theorem 1.7  ###############
```

```
> logx0^4/x0+logx0^3/3/sqrt(2*x0)*(1+0.98/logx0),107/160+0.0301; #
  (5.7)--(5.8)
```

0.030027486686917837222, 0.69885000000000000000 **(38)**

```
> Digits:=8: P:=expand((t/2+t^2/4+t^3/4+7*t^4/10)*(2/t-1-0.584*t));
  fsolve(P-1=0);1/logx0; Digits:=20: # between (5.8) and (5.9)
```

$P := 1 - 0.04200000\, t^2 + 1.0040000\, t^3 - 0.84600000\, t^4 - 0.40880000\, t^5$

-2.9220110, 0., 0., 0.043457433, 0.80908191

0.043429448 **(39)**

```
> 1+1.168/L0; # (5.11)
```

1.0276486259210757577 **(40)**

```
> (lambda0-1)/2/L0; # between (5.14) and (5.15)
```

0.032471418966263015433 **(41)**

```
> 2*(1+1/2/L0),1/lambda0,2.03*lambda0/L0,(-2+0.27+0.18)/8; # (5.17)
```

2.0236717687680443131, 0.26713168008555813164, 0.17988765160215779833, -0.19375000000000000000 **(42)**

```
> (0.584+0.48*1.03)/lambda0; # (5.18)
```

0.28807480380426588916 **(43)**

```
> 3/20-logx0^4/x0; # between (5.18) and (5.19)
```

0.14997188987647023669 **(44)**

```
> Digits:=8: Q:=expand((t/2+t^2/4+t^3/4+0.149*t^4)*(2/t-1-0.492*t));
  fsolve(Q-1=0);1/logx0; Digits:=20: # between (5.19) and ((5.20)
```

$Q := 1 + 0.00400000\, t^2 - 0.07500000\, t^3 - 0.27200000\, t^4 - 0.073308\, t^5$

-3.4052224, -0.35082465, 0., 0., 0.045674419

0.043429448 **(45)**

```
> 2-log(2*logx0)/logx0,1.8336*0.492; # before (5.20)
```

1.8336754315737963672, 0.9021312 **(46)**

```
> 1+(lambda0-1)/2/L0,normal(expand((1+(\lambda-1)/(2*L))^2-(1+
  (lambda-1-b/L)/L+(lambda-1)/L^2))),evalf([fsolve(0.989*lambda^2-6*
  lambda+8.608)],8),0.011/(4*2.066); # between (5.21)and (5.22)
```

$$1.0324714189662630154, \frac{1}{4}\,\frac{\lambda^2+4\,b-6\,\lambda+5}{L^2}, [2.3279775, 3.7387566],$$
$$0.001331074540174249758 \tag{47}$$

```
> y:=1+(t^2/4-0.76*t)*exp(-t): y1:=normal(diff(y,t)):y,evalf(y1,3);
  A:=[solve(exp(t)*y1,t)]:A,evalf(subs(t=A[1],y));# Proof of
  Corollary 5.3
```

$$1+\left(\frac{1}{4}\,t^2-0.76\,t\right)\mathrm{e}^{-t}, 1.26\,\mathrm{e}^{-1.\,t}\,t-0.76\,\mathrm{e}^{-1.\,t}-0.250\,\mathrm{e}^{-1.\,t}\,t^2$$
$$[4.3394504664870654291, 0.70054953351293457094], 1.0183880248221891576 \tag{48}$$

```
> np0:=nu0-addexc(x0): # np0 is equal to pi_1(x0)
  Lp0:=log(evalf(np0)):lambdap0:=log(Lp0):np0,Lp0,lambdap0;
  1/2+logx0^4/x0,8*0.500029,0.38-0.026*(1+lambdap0/2/Lp0+0.38*
  lambdap0/Lp0^2);
  # Section 5.2, (5.24)--(5.25)
```

2220822432581729238, 42.244409266298653707, 3.7434720199894299222

0.50002811012352976331, 4.000232, 0.35282728508916522658 **(49)**

```
> minslice1_7(380000000,400000000,1): # Proof of Theorem 1.7, (1.13)
  cf. Section 5.3
  # To prove (1.13), execute minslice1_7(400000000,
  2220822442581729257,100):
```

"i=", 74, "wmin = ", -0.0066219950, "ppmin = ", 88813, "nppmin = ", 361092676, "temps = ", 0.142

"Conclusion"

"smallest difference = ", $2.67300676360905\ 10^{-8}$, " for the interval = ", [361181493, 361270312], "time = ", 0.142

"For ", 398898277, " <= n <= ", 411137805, "log h(n) > Phi_{1/8}(n) holds"

"For p = ", 93337, "pp = ", 93371, "np = ", 398804906, "npp =", 398898277, "k1 = pi(pp) = ", 9018, " log h(np) = ", 93084.9779186447, " < Phi_{1/8}(npp) = ", 93084.9901194923, "holds", " w = ", -0.000018031046 **(50)**

```
> okrec1_7(2,398898277,500); # completes the proof of (1.13), see
  Section 5.3
```

"n1=", 2, "n2=", 398898277, "test log h(n) > phi1s8(n)"

"comp=", 500, [392202900, 392227247], "log h(n1)=", 92267.2008144, "phi1s8(n2)=", 92265.6222929, "time=", 19.958

"comp=", 1000, [386651825, 386663998], "log h(n1)=", 91577.3325886, "phi1s8(n2)=", 91577.2084028, "time=", 40.350

"comp=", 1500, [382147664, 382245051], "log h(n1)=", 91016.7948888, "phi1s8(n2)=", 91027.0126491, "time=", 59.176

"comp=", 2000, [378690415, 378714762], "log h(n1)=", 90583.2159550, "phi1s8(n2)=", 90585.2735537, "time=", 78.558

"comp=", 2500, [376067045, 376073132], "log h(n1)=", 90253.0781314, "phi1s8(n2)=", 90253.4417908, "time=", 95.753

"comp=", 3000, [374161908, 374356683], "log h(n1)=", 90012.8133791, "phi1s8(n2)=", 90037.2302934, "time=", 112.710

"comp=", 3500, [373623233, 373624755], "log h(n1)=", 89944.6953125, "phi1s8(n2)=", 89944.8893040, "time=", 129.366

"Conclusion"

"number of intervals=", 3577, "mini=", 0.000010316671, "lismini=", [373714534, 373714724], "time=", 132.184

"For n s.t. ", 373623863, " <= n <= ", 398898277, " log h(n) > phi1s8(n) holds"

"For n =", 373623862, "log h(n) =", 89944.7747662060, "phi1s8(n) =", 89944.7765891595, "log h(n) >= phi1s8(n) is false" **(51)**

$Digits := 20$ **(52)**

```
> 0.72+logx0/sqrt(x0),0.73*(1+(lambda0-1)/4/L0),0.75*L0^(7/4)/evalf
  (n0)^(1/4)/lambda0^2;
  #(5.28)--(5.29)
```

0.72023025850909965898, 0.74185206792268600063, 0.00097046379802116620280 **(53)**

```
> nu1:=10.^36:L1:=log(nu1):lambda1:=log(L1):"L1 = ",L1,"lambda1 = ",
  lambda1;
  evalf(1+(lambda1-1)/2/L1,8),evalf(1.2*1.021,5),evalf
  (1.23/L1/lambda1^2,8),
  evalf(0.75*L1^(7/4)/nu1^(1/4)/lambda1^2,8);# (5.28)--(5.29)
```

"L1 = ", 82.893063347785644625, "lambda1 = ", 4.4175513837040658014

1.0206142, 1.2252, 0.00076036781, $8.7519510\ 10^{-8}$ **(54)**

```
> calculzn(19); # Proof of (5.32) for n <= 19 [Computation of z_n
  defined by (3.30)]
```

"n=", 2, "g(n)=", 2., "zn=", -2.054536

"n=", 3, "g(n)=", 3., "zn=", -2.383142

"n=", 4, "g(n)=", 4., "zn=", 3.033933

"n=", 5, "g(n)=", 6., "zn=", 2.351073

"n=", 6, "g(n)=", 6., "zn=", 3.183046

"n=", 7, "g(n)=", 12., "zn=", 2.057540

"n=", 8, "g(n)=", 15., "zn=", 2.191439

"n=", 9, "g(n)=", 20., "zn=", 2.165139

"n=", 10, "g(n)=", 30., "zn=", 1.944815

"n=", 11, "g(n)=", 30., "zn=", 2.342347

"n=", 12, "g(n)=", 60., "zn=", 1.730138

"n=", 13, "g(n)=", 60., "zn=", 2.073518

"n=", 14, "g(n)=", 84., "zn=", 1.963177

"n=", 15, "g(n)=", 105., "zn=", 1.988366

"n=", 16, "g(n)=", 140., "zn=", 1.933928

"n=", 17, "g(n)=", 210., "zn=", 1.752946

"n=", 18, "g(n)=", 210., "zn=", 1.997081

"n=", 19, "g(n)=", 420., "zn=", 1.531456 **(55)**

```
> Digits:=30; majslice1_7(6466*10^12,6482*10^12,1,3): Digits:=20:
  # To prove Theorem 1.7 (1.14) execute majslice1_7(19,nu0,50,3), cf.
  Section 5.5
```

$$Digits := 30$$

"i=", 1999, "minz=", 0.00545504803632000559309654456114, "nminz=", 6473549497145122, " pminz = ", 502748593, "time = ", 12.436

"Conclusion"

"mini = ", 3.05746569703860669 10^{-15}, "nmini = ", 6472496283558071, " pmini = ", 502706629, "time = ", 12.436

"For ", 6465984410320754, " <= n <= ", 6482105188612314

"minimum of z_n = ", 0.00545504803632000559309654456114, " for n = ", 6473549497145122, " and p = ", 502748593 **(56)**

```
> ##################### Section 6 : Study of g(n)/h(n) for large n's
  ##################
```

```
> (lambda0-1.019)^2/8/L0,-11.433-10.392*(lambda0-1.041)/4/L0,3/2*
  (0.305-0.0724/3),2.3704+3.3704*(lambda0-1)/4/L0; # Proof of
  Proposition 6.1, (6.5)--(6.6)
```

0.021963697964266949319, -11.599200023483068084, 0.42130000000000000000, 2.4251208352419464336 **(57)**

```
> ######### Section 7.1 : Proof of Theorem 1.5(i), Asymptotic
  expansion of log(g(n)/h(n)) ########
```

```
> devx2(4,0); # Asymptotic expansion of xi_2/sqrt(xi/2) with t=log
  (xi), a=log 2 (cf. (3.31))
```

$$1+\frac{-\frac{1}{2}a}{t}+\frac{-\frac{1}{2}a-\frac{1}{8}a^2}{t^2}+\frac{-\frac{1}{2}a-\frac{1}{2}a^2-\frac{1}{16}a^3}{t^3}+\frac{-\frac{1}{2}a-\frac{9}{8}a^2-\frac{23}{48}a^3-\frac{5}{128}a^4}{t^4} \quad \mathbf{(58)}$$

```
> devx2log(4,0); # Asymptotic expansion of log(xi_2)/(log(xi)/2))
  with t=log(xi), a=log 2
```

(59)

$$1-\frac{a}{t}+\frac{-a}{t^2}+\frac{-a-\frac{1}{2}a^2}{t^3}+\frac{-a-\frac{3}{2}a^2-\frac{1}{3}a^3}{t^4} \quad (59)$$

```
> loggmoinsloghx(4,0): # log(g(n)/h(n)) = sqrt(2 xi)/3*F(1/log xi),
  a=log 2 cf. (7.1)--(7.2)
```

$$\text{"F(t) =", } 1+\frac{-\frac{1}{3}-\frac{1}{2}a}{t}+\frac{-\frac{4}{9}-\frac{2}{3}a-\frac{1}{8}a^2}{t^2}+\frac{-\frac{8}{9}-\frac{4}{3}a-\frac{5}{8}a^2-\frac{1}{16}a^3}{t^3}$$
$$+\frac{-\frac{64}{27}-\frac{32}{9}a-\frac{55}{24}a^2-\frac{7}{12}a^3-\frac{5}{128}a^4}{t^4} \quad (60)$$

```
> Limoinsunasy(4,logn,0); # Expansion of Li^(-1)(n) with logn=log(n),
  L=log(logn))
                          # cf. Salvy [30, p. 235, B2]
```

$$1+\frac{L-1}{logn}+\frac{-2+L}{logn^2}+\frac{-\frac{11}{2}-\frac{1}{2}L^2+3L}{logn^3}+\frac{-\frac{131}{6}+\frac{1}{3}L^3-\frac{7}{2}L^2+14L}{logn^4} \quad (61)$$

```
> loggdenasy(4,t,0); # Asymptotic expansion of sqrt(Li^(-1)(n))/sqrt
  (n*log(n))
     # with logn=log(n), L=log(logn) : cf. (1.7) and Salvy [30, p.
  235, B3]
```

$$1+\frac{\frac{1}{2}L-\frac{1}{2}}{logn}+\frac{-\frac{9}{8}+\frac{3}{4}L-\frac{1}{8}L^2}{logn^2}+\frac{-\frac{11}{16}L^2+\frac{39}{16}L-\frac{53}{16}+\frac{1}{16}L^3}{logn^3}$$
$$+\frac{-\frac{5071}{384}+\frac{61}{96}L^3-\frac{239}{64}L^2+\frac{343}{32}L-\frac{5}{128}L^4}{logn^4} \quad (62)$$

```
> Loggmoinsloghn(3,0); # log(g(n)/h(n)) with logn=log(n), L=log(logn)
  ) cf. Thm. 1.5 (i)
```

$$1+\frac{-\frac{11}{12}-a+\frac{1}{4}L}{logn}+\frac{-\frac{635}{288}-\frac{29}{12}a-\frac{1}{2}a^2+\left(\frac{15}{16}+\frac{3}{4}a\right)L-\frac{3}{32}L^2}{logn^2}$$
$$+\frac{1}{logn^3}\left(-\frac{1119}{128}-\frac{333}{32}a-\frac{39}{8}a^2-\frac{1}{2}a^3+\left(\frac{5525}{1152}+\frac{239}{48}a+\frac{7}{8}a^2\right)L+\left(-\frac{117}{128}-\frac{21}{32}a\right)L^2+\frac{7}{128}L^3\right) \quad (63)$$

```
> #################### The method of B. Salvy : cf. [30]
  ###########################
>
> E:= convert([seq(i!*n^i, i=0..10)],`+`); G:=expand(subs(a=log(2),t=
  1/2/n,loggmoinsloghx(5,0)));
```

$$E := 3628800\,n^{10}+362880\,n^9+40320\,n^8+5040\,n^7+720\,n^6+120\,n^5+24\,n^4+6\,n^3$$

$$+ 2 n^{2} + n + 1$$

$$G := 1 - \frac{2}{3} n - n \ln(2) - \frac{16}{9} n^{2} - \frac{8}{3} n^{2} \ln(2) - \frac{1}{2} n^{2} \ln(2)^{2} - \frac{64}{9} n^{3} - \frac{32}{3} n^{3} \ln(2) - 5 n^{3} \ln(2)^{2} - \frac{1}{2} n^{3} \ln(2)^{3} - \frac{1024}{27} n^{4} - \frac{512}{9} n^{4} \ln(2) - \frac{110}{3} n^{4} \ln(2)^{2} - \frac{28}{3} n^{4} \ln(2)^{3} - \frac{5}{8} n^{4} \ln(2)^{4} - \frac{20480}{81} n^{5} - \frac{10240}{27} n^{5} \ln(2) - \frac{2452}{9} n^{5} \ln(2)^{2} - \frac{938}{9} n^{5} \ln(2)^{3} - \frac{211}{12} n^{5} \ln(2)^{4} - \frac{7}{8} n^{5} \ln(2)^{5} \quad \textbf{(64)}$$

```
> sqrt(2)/3*theorem2_part2(1,1/4,0,E,subs(t=1/(2*n),G),n,3); # Salvy method for Thm. 1.5 (i)
```

$$\frac{1}{3} \sqrt{2}\, n^{1/4} \ln(n)^{1/4} \left(1 + \frac{\frac{1}{4} \ln(\ln(n)) - \frac{11}{12} - \ln(2)}{\ln(n)} + \frac{1}{\ln(n)^{2}} \left(-\frac{3}{32} \ln(\ln(n))^{2} + \frac{1}{4} \ln(\ln(n)) - \frac{3}{4} \left(-\frac{11}{12} - \ln(2)\right) \ln(\ln(n)) - \frac{635}{288} - \frac{29}{12} \ln(2) - \frac{1}{2} \ln(2)^{2}\right) + \frac{1}{\ln(n)^{3}} \left(\frac{7}{128} \ln(\ln(n))^{3} - \frac{5}{16} \ln(\ln(n))^{2} + \frac{21}{32} \left(-\frac{11}{12} - \ln(2)\right) \ln(\ln(n))^{2} + \frac{1}{4} \ln(\ln(n)) - \frac{3}{4} \left(-\frac{11}{12} - \ln(2)\right) \ln(\ln(n)) - \frac{7}{4} \left(-\frac{635}{288} - \frac{29}{12} \ln(2) - \frac{1}{2} \ln(2)^{2}\right) \ln(\ln(n)) - \frac{1119}{128} - \frac{333}{32} \ln(2) - \frac{39}{8} \ln(2)^{2} - \frac{1}{2} \ln(2)^{3}\right) + O\left(\frac{\ln(\ln(n))^{4}}{\ln(n)^{4}}\right)\right) \quad \textbf{(65)}$$

```
> thm2(1,1/4,0,E,subs(t=1/(2*n),G),n,6): # C is defined by Salvy [30, p.229, (8)]
```

$$\text{"C=", } -n - \frac{5}{2} n^{2} - \frac{47}{6} n^{3} - \frac{379}{12} n^{4} - \frac{9337}{60} n^{5} - \frac{109139}{120} n^{6} + O(n^{7})$$

$$\text{"generating series of constant coeff = ", } 1 + \left(-\frac{11}{12} - \ln(2)\right) n + \left(-\frac{635}{288} - \frac{29}{12} \ln(2) - \frac{1}{2} \ln(2)^{2}\right) n^{2} + \left(-\frac{1119}{128} - \frac{333}{32} \ln(2) - \frac{39}{8} \ln(2)^{2} - \frac{1}{2} \ln(2)^{3}\right) n^{3} + \left(-\frac{2553311}{55296} - \frac{67297}{1152} \ln(2) - \frac{6935}{192} \ln(2)^{2} - \frac{221}{24} \ln(2)^{3} - \frac{5}{8} \ln(2)^{4}\right) n^{4} + \left(-\frac{199949215}{663552} - \frac{22023041}{55296} \ln(2) - \frac{637417}{2304} \ln(2)^{2} - \frac{59381}{576} \ln(2)^{3} - \frac{1673}{96} \ln(2)^{4} - \frac{7}{8} \ln(2)^{5}\right) n^{5} + \left(-\frac{8635843619}{26542080} - \frac{37818307}{221184} \ln(2) - \frac{3170699}{36864} \ln(2)^{2} \right. \quad \textbf{(66)}$$

$$- \frac{18913}{2304} \ln(2)^3 + \frac{1741}{768} \ln(2)^4 + \frac{7}{32} \ln(2)^5 \Big) n^6 + O\left(n^7\right)$$

```
> ################  Section 7.2 : Proof of Theorem 1.5 (ii)
  #################
```

```
> majslice1_5_2(22*10^12,24*10^12,2.43,1,5):
  # To prove Theorem 1.5 (ii) execute majslice1_5_2(22*10^12,nu0,
  2.43,50,3):
```

"i=", 603, "betamaj > a=", 2.43, 2.4300064058024262524, "grnp=", 21962830729180,
"grnpp=", 21962857728827

"et2 fin i=", 603, [22255802707198, 22255815714040], "pg=", 27183829, "temps=", 10.611

"i=", 604, "betamaj > a=", 2.43, 2.4300064058024262524, "grnp=", 21962830729180,
"grnpp=", 21962857728827

"et2 fin i=", 604, [22397215968791, 22397229018947], "pg=", 27274099, "temps=", 15.074

"i=", 605, "betamaj > a=", 2.43, 2.4300140812407573615, "grnp=", 22488775657598,
"grnpp=", 22488802989495

"et2 fin i=", 605, [22493094303400, 22493107382472], "pg=", 27334357, "temps=", 18.340

"i=", 606, "betamaj > a=", 2.43, 2.4300140812407573615, "grnp=", 22488775657598,
"grnpp=", 22488802989495

"et2 fin i=", 606, [22637672158340, 22637685280846], "pg=", 27424871, "temps=", 23.013

"i=", 607, "betamaj > a=", 2.43, 2.4300140812407573615, "grnp=", 22488775657598,
"grnpp=", 22488802989495

"et2 fin i=", 607, [22832033620345, 22832046800875], "pg=", 27545821, "temps=", 29.126

"i=", 608, "betamaj > a=", 2.43, 2.4300140812407573615, "grnp=", 22488775657598,
"grnpp=", 22488802989495

"et2 fin i=", 608, [22980073934412, 22980087158544], "pg=", 27636683, "temps=", 33.965

"i=", 609, "betamaj > a=", 2.43, 2.4300143514977784168, "grnp=", 22998745945409,
"grnpp=", 22998773594040

"et2 fin i=", 609, [23125585762674, 23125599030480], "pg=", 27727709, "temps=", 38.536

"i=", 610, "betamaj > a=", 2.43, 2.4300080809917574916, "grnp=", 23270400847898,
"grnpp=", 23270428664755

"et2 fin i=", 610, [23522231323698, 23522244708320], "pg=", 27971147, "temps=", 51.235

"i=", 611, "betamaj > a=", 2.43, 2.4300133958176945950, "grnp=", 23542024585503,
"grnpp=", 23542052569006

"et2 fin i=", 611, [23821956516556, 23821969989126], "pg=", 28154449, "temps=", 60.541

"i=", 612, "betamaj > a=", 2.43, 2.4300133958176945950, "grnp=", 23542024585503,
"grnpp=", 23542052569006

"et2 fin i=", 612, [23871407733729, 23871421220985], "pg=", 28185019, "temps=", 62.062

"i=", 613, "betamaj > a=", 2.43, 2.4300133958176945950, "grnp=", 23542024585503, "grnpp=", 23542052569006

"et2 fin i=", 613, [23973111551325, 23973125067977], "pg=", 28246289, "temps=", 65.184

"i=", 614, "betamaj > a=", 2.43, 2.4300133958176945950, "grnp=", 23542024585503, "grnpp=", 23542052569006

"et2 fin i=", 614, [24328169348333, 24328182968123], "pg=", 28461241, "temps=", 76.342

"Conclusion"

"mini = ", 2.35154099257 10^{-8}, "lismini = ", [22436166605748, 22436193904301], " time = ", 76.342

"For ", 23542052569006, " <= n <= ", 24328182968123, "beta_n < a = ", 2.43, "holds"

"For [n1,n2] = ", [23542024585503, 23542052569006], "betamaj(n1,n2) = ", 2.4300133958176945950, " < a fails" **(67)**

```
> okrec1_5_2(3997000000000,3997100000000,2.43,50); # cf. Section 7.2
  # To complete the proof of Theorem 1.5(ii) execute okrec1_5_2(3997*
  10 ^9,24*10^12,2.43,10000);
```

"n1=", 3997000000000, "n2=", 3997100000000, "a=", 2.43, "test beta_n = beta(n,n) < a"

"comp=", 50, [3997087500000, 3997088281250], "beta(n1,n2) = ", 2.39351707394114, "time=", 23.478

"comp=", 100, [3997078027343, 3997078076171], "beta(n1,n2) = ", 2.39727296570309, "time=", 48.813

"comp=", 150, [3997057031250, 3997057812500], "beta(n1,n2) = ", 1.61529486876097, "time=", 75.550

"comp=", 200, [3997043750000, 3997045312500], "beta(n1,n2) = ", 2.53868961373140, "time=", 99.137

"comp=", 250, [3997032812500, 3997033593750], "beta(n1,n2) = ", 2.46854345234483, "time=", 125.551

"comp=", 300, [3997022460937, 3997022656250], "beta(n1,n2) = ", 2.43154734638921, "time=", 147.300

"conclusion"

"number of intervals=", 336, "mini=", 5.2129424 10^{-7}, "lismini=", [3997089332579, 3997089332960], "time=", 163.924

"For n s.t. ", 3997022083663, " <= n <= ", 3997100000000, " beta_n = beta(n,n) < ", 2.43, " holds"

"For n =", 3997022083662, " beta_n = ", 2.43000186560077, " < a = ", 2.43, " is wrong" **(68)**

```
> ############### Section 7.2 Proof of Theorem 1.5(iii)
  ##########################
```

```
> minslice1_5_3(10^6,2*10^6,-11.6,1,5):
  # To prove Theorem 1.5(iii) execute minslice1_5_3(10^6,nu0,-11.6,
  50,3): cf. Section 7.2
```

"i=", 27, "betamij < a=", -11.6, -14.259827401957466163, "grnp=", 1013843, "grnpp=", 1017810

"et2 fin i=", 27, [1017810, 1019616], "pg=", 3967, "betamin = ", -14.259827401957466163, "temps=", 0.039

"i=", 28, "betamij < a=", -11.6, -14.259827401957466163, "grnp=", 1013843, "grnpp=", 1017810

"et2 fin i=", 28, [1129985, 1131195], "pg=", 4211, "betamin = ", -9.6175507206549377901, "temps=", 0.061

"i=", 29, "betamij < a=", -11.6, -14.259827401957466163, "grnp=", 1013843, "grnpp=", 1017810

"et2 fin i=", 29, [1417703, 1419865], "pg=", 4751, "betamin = ", -8.9517383512064067588, "temps=", 0.120

"i=", 30, "betamij < a=", -11.6, -14.259827401957466163, "grnp=", 1013843, "grnpp=", 1017810

"et2 fin i=", 30, [2218331, 2221087], "pg=", 6043, "betamin = ", -9.5553174164055259285, "temps=", 0.209

"Conclusion"

"mini = ", 0.0062114090788275400, "lismini = ", [1448575, 1453376], " time = ", 0.209

"For ", 1017810, " <= n <= ", 2221087, "beta_n > a = ", -11.6, "holds"

"For [n1,n2] = ", [1013843, 1017810], "betamij(n1,n2) = ", -14.259827401957466163, " > a fails" **(69)**

```
> okrec1_5_3(2,1017810,-11.6,200); # completes the proof of Theorem
  1.5(iii), cf. Section 7.2
```

"n1=", 2, "n2=", 1017810, "a=", -11.6, "test beta_n > a"

"comp=", 200, [747454, 763358], "beta116(n1,n2) = ", -69.4051239908777, "time=", 6.641

"comp=", 400, [548663, 550651], "beta116(n1,n2) = ", -8.51372480376293, "time=", 12.959

"comp=", 600, [373728, 381680], "beta116(n1,n2) = ", -50.7659876545890, "time=", 18.759

"comp=", 800, [239544, 240538], "beta116(n1,n2) = ", -9.77690435726574, "time=", 24.383

"comp=", 1000, [139155, 141143], "beta116(n1,n2) = ", -20.3678436990832, "time=", 29.918

"comp=", 1200, [73554, 75542], "beta116(n1,n2) = ", -38.4158042644535, "time=", 34.133

"comp=", 1400, [36777, 37771], "beta116(n1,n2) = ", -34.4180583442533, "time=", 36.478

"comp=", 1600, [17395, 17892], "beta116(n1,n2) = ", -26.7926306456169, "time=", 38.485

"comp=", 1800, [7673, 7704], "beta116(n1,n2) = ", -12.7975973642547, "time=", 39.792

"conclusion"

"number of intervals=", 1948, "mini=", 0.0092603867, "lismini=", [616252, 618240], "time=", 40.807

"For n s.t. ", 4230, " <= n <= ", 1017810, " beta_n > ", -11.6, " holds"

"For n =", 4229, " beta_n = ", -12.3466353525204, " > a = ", -11.6, " is wrong" **(70)**

```
> ################## Section 7.3 : Proof of Theorem 1.5(iv)
  ##################
```

```
> proc4230(4230): # cf. Section 7.3
```

"list of n's s.t. g(n)=h(n)", [1, 2, 3, 5, 6, 8, 10, 11, 15, 17, 18, 28, 41, 58, 77] **(71)**

```
> ################## Section 7.4 : Proof of Theorem 1.5(v)
  ##################
```

```
> majslice1_5_5(4*10^7,7*10^7,0.62,1,5): # see Section 7.4
  # To prove Theorem 1.5 (v), execute majslices1_5_5(7,4*10^12,0.62,
  50,3)
```

"i=", 53, "Mn1n2 = ", 0.62285689, " > a = ", 0.62, "for n1 = ", 47242215, "n2 = ", 47272712

"fin i=", 53, [47272712, 47274760], "pg=", 30497, "max = ", 0.62685834, "temps=", 0.150

"i=", 54, "Mn1n2 = ", 0.62002071, " > a = ", 0.62, "for n1 = ", 49435860, "n2 = ", 49467083

"fin i=", 54, [60123385, 60139387], "pg=", 34513, "max = ", 0.62499533, "temps=", 0.344

"i=", 55, "Mn1n2 = ", 0.62002071, " > a = ", 0.62, "for n1 = ", 49435860, "n2 = ", 49467083

"fin i=", 55, [67443934, 67460964], "pg=", 36713, "max = ", 0.61358790, "temps=", 0.455

"i=", 56, "Mn1n2 = ", 0.62002071, " > a = ", 0.62, "for n1 = ", 49435860, "n2 = ", 49467083

"fin i=", 56, [76655164, 76666802], "pg=", 39251, "max = ", 0.61469502, "temps=", 0.748

"Conclusion"

"For ", 49467083, " <= n <= ", 76666802, "log(g(n)/h(n))/(n*log n)^(1/4) < ", 0.62, "holds"

"For [n1,n2] = ", [49435860, 49467083], "M(n1,n2) = ", 0.62002070729155826870 **(72)**

```
> okrec1_5_5(2,5*10^7,0.62059,3000); # see Section 7.4
```

"n1=", 2, "n2=", 50000000, "a=", 0.62059, "test log g(n)/h(n) < a(n log n)^(1/4)"

"comp=", 3000, [19604493, 19628907], "log(g(n2)/h(n1)) = ", 83.061683, "a(n log n)^(1/4) = ", 83.591970, "time=", 201.974

"comp=", 6000, [5957032, 5963135], "log(g(n2)/h(n1)) = ", 60.874494, "a(n log n)^(1/4) = ", 60.931797, "time=", 363.801

"comp=", 9000, [1149749, 1150512], "log(g(n2)/h(n1)) = ", 32.997979, "a(n log n)^(1/4) = ", 39.277010, "time=", 488.763

"comp=", 12000, [128175, 128938], "log(g(n2)/h(n1)) = ", 21.285434, "a(n log n)^(1/4) = ", 21.745433, "time=", 577.182

"Conclusion"

"number of intervals=", 14289, "mini=", 0.000032758901, "lismini=", [2244], "time=", 619.307

"For n s.t. ", 2244, " <= n <= ", 50000000, " log(g(n)/h(n)) <= ", 0.62059,
" (n log n)^(1/4) holds"
"For n =", 2243, "log g(n)/h(n) <= ", 0.62059, " (n log n)^(1/4) is false" **(73)**

```
> proc2243(120000,0.62); # computes the n's up to 120000 with d_n >
  0.62, cf. (7.12)
```

"list of n's s.t. log(g(n)/h(n))/(n*log(n))^(1/4) > ", 0.62, [2243, 2244, 118869]
"n=", 2243, "log(g[n]/h[n])/(nf*log(nf))^(1/4)=", 0.62066526568375620112
"n=", 2244, "log(g[n]/h[n])/(nf*log(nf))^(1/4)=", 0.62058714421610282114
"n=", 118869, "log(g[n]/h[n])/(nf*log(nf))^(1/4)=", 0.62014609683999469464
[2243, 2244, 118869] **(74)**

```
> ###################### Section 8.2 : Proof of Theorem 1.1(ii)
  #######################
```

```
> 0.23+sqrt(2.)/12+2.43*sqrt(2.)/lambda0/12; # Section 8.2
```

0.42435183229335061604 **(75)**

```
> Digits:=30: minslice1_1_2(6466*10^12,6482*10^12,1,3): Digits:=20: #
  cf. Lemma 8.1
  # To prove Lemma 8.1, execute minslice1_1_2(43,nu0,50,3), cf.
  Section 8.2
```

"i=", 1999, "minan=", 0.193938608602741816619480762052, "nmin=", 6473580667603736,
"pmin=", 502750063, "time=", 229.446

"Conclusion"

"mini = ", 2.327448795937405289 10^{-12}, "nmini = ", 6473581170353835, " pmini = ",
502750099, " time = ", 229.446
"For ", 6465984410320754, " <= n <= ", 6482105188612314
"minimum of a_n = ", 0.193938608602741816619480762052, " for n = ", 6473580667603736, **(76)**
" and p = ", 502750063

```
> functLandau(6473580667603736,2): # computation of g
  (6473580667603736) cf. Lemma 8.1, Section 8.2
```

N=, [2, 25], [3, 15], [5, 11], [7, 9], [11, 7], [13, 7], [17, 6], [19, 6], [23, 5], ... , [37, 5],
[41, 4], ... , [103, 4], [107, 3], ... , [523, 3], [541, 2], ... , [15559, 2], 15569, ... ,
502750063
n=, 6473580667603736, *ell(N)*=, 6473580667603736, *n-ell(N)*=, 0
rho=, 2.5092835 10^7, *prime number corresponding to rho*=, 502750099, "xi = ", 5.0275010 10^8
n=, 6473580667603736, *g(n)/N*=, 1, *n-ell(g(n))*=, 0, *total time*=, 0.909 **(77)**

```
> ###################### Section 8.3 : Proof of Theorem 1.1(iii)
  #######################
```

```
> 0.77-sqrt(2.)/12+11.6*sqrt(2.)/lambda0/12; # Section 8.3
```

1.0173374065548656454 **(78)**

```
> majslice1_1_3(5000000,7000000,1.02,1);
  # To prove Theorem 1.1(iii), execute majslice1_1_3(5000000,nu0,
  1.02,50) cf. Section 8.3
```

"et2 fin i=", 35, [5389332, 5393754], "Rmax = ", 1.0815115344417261072, "temps=", 0.960

"et2 fin i=", 36, [5451806, 5453864], "Rmax = ", 1.0242496353855823670, "temps=", 0.992

"et2 fin i=", 37, [6673601, 6678571], "Rmax = ", 1.0164861795871539357, "temps=", 1.557

"et2 fin i=", 38, [7402436, 7407692], "Rmax = ", 0.85062122654949382314, "temps=", 1.954

" Conclusion"

"mini=", 0.0000029920292, "lismini=", [4116861, 4125250], "time=", 1.954

"For n in ", [5432420, 7000000], "a_n < (2-sqrt(2))/3 + c + ", 1.02, "log(log(n))/log n holds"

"R(", 5422741, 5432420, ") is equal to", 1.02036341295143, " > ", 1.02

[[35, 1.0815115344417261072, 4442781], [36, 1.0242496353855823670, 5393754], [37, 1.0164861795871539357, 5453864], [38, 0.85062122654949382314, 6678571]] **(79)**

```
> okrec1_1_3(2,5500000,1.02,1000);
  # proves the second step of the proof of Theorem 1.1(iii), cf.
  Section 8.3
```

"n1=", 2, "n2=", 5500000, "a=", 1.02,

"test sqrt(Li^(-1)(n) < (n1*log n1)^(1/4)(((2-sqrt(2))/3 + c) + a*log(log n)/log n)"

"comp=", 1000, [2814453, 2825195], " sqrt(Li^(-1)(n2) = ", 6846.8354, " fa102(n1,a) = ", 6840.4235, "time=", 28.933

"comp=", 2000, [1326660, 1329346], " sqrt(Li^(-1)(n2) = ", 4580.8164, " fa102(n1,a) = ", 4582.6575, "time=", 54.184

"comp=", 3000, [520997, 523682], " sqrt(Li^(-1)(n2) = ", 2782.2470, " fa102(n1,a) = ", 2779.2417, "time=", 75.068

"comp=", 4000, [162475, 163818], " sqrt(Li^(-1)(n2) = ", 1488.2739, " fa102(n1,a) = ", 1484.7226, "time=", 93.122

"comp=", 5000, [43221, 43305], " sqrt(Li^(-1)(n2) = ", 722.72295, " fa102(n1,a) = ", 723.06874, "time=", 108.046

"Conclusion"

"fa102(n,a) = log g(n) + (n*log n)^(1/4)(deux3pc + a*log(log n)/log n)", " with a = ", 1.02

"number of intervals=", 5993, "mini=", 0.000025672391, "lismini=", [43473, 43641], "time=", 120.586

"For n s.t. ", 19425, " <= n <= ", 5500000, "and a= ", 1.02, "sqrt(Li^(-1)(n) < fa102(n,a) holds"

"For n =", 19424, " sqrt(Li^(-1)(n) = ", 465.831202938283, " < fa102(n,a) = ", 465.809244112473, " fails" **(80)**

(81)

```
> ###################### Section 8.4 : Proof of Theorem 1.1(iv)
  #######################
```

```
> "c=",c, (2-sqrt(2.))/3+c+1.02*log(log(19425.))/log(19425.); # cf.
  Section 8.4
```

"c=", 0.046117644421509023827090 8, 0.47792637844227154606 **(82)**

```
> test19424(19424,0.477927): # upper bound of a_n : cf. Section 8.4
```

"com=", 1, " For n >= ", 19425, " a_n < ", 0.47793
"com=", 2, " For n >= ", 18962, " a_n < ", 0.47898
"com=", 3, " For n >= ", 18501, " a_n < ", 0.48205
"com=", 4, " For n >= ", 18500, " a_n < ", 0.48293
"com=", 5, " For n >= ", 18043, " a_n < ", 0.48314
"com=", 6, " For n >= ", 17595, " a_n < ", 0.48584
"com=", 7, " For n >= ", 17152, " a_n < ", 0.48710
"com=", 8, " For n >= ", 7517, " a_n < ", 0.48771
"com=", 9, " For n >= ", 7499, " a_n < ", 0.48855
"com=", 10, " For n >= ", 7228, " a_n < ", 0.49027
"com=", 11, " For n >= ", 6959, " a_n < ", 0.49656
"com=", 12, " For n >= ", 6696, " a_n < ", 0.49944
"com=", 13, " For n >= ", 6436, " a_n < ", 0.50280
"com=", 14, " For n >= ", 5662, " a_n < ", 0.50292
"com=", 15, " For n >= ", 5429, " a_n < ", 0.50654
"com=", 16, " For n >= ", 5200, " a_n < ", 0.51198
"com=", 17, " For n >= ", 4086, " a_n < ", 0.51480
"com=", 18, " For n >= ", 3896, " a_n < ", 0.51542
"com=", 19, " For n >= ", 3705, " a_n < ", 0.52049
"com=", 20, " For n >= ", 3521, " a_n < ", 0.52315
"com=", 21, " For n >= ", 3340, " a_n < ", 0.52904
"com=", 22, " For n >= ", 2970, " a_n < ", 0.53044
"com=", 23, " For n >= ", 2786, " a_n < ", 0.53046
"com=", 24, " For n >= ", 2629, " a_n < ", 0.53196
"com=", 25, " For n >= ", 2477, " a_n < ", 0.53559
"com=", 26, " For n >= ", 1409, " a_n < ", 0.53934
"com=", 27, " For n >= ", 1304, " a_n < ", 0.54145
"com=", 28, " For n >= ", 1302, " a_n < ", 0.54286
"com=", 29, " For n >= ", 1199, " a_n < ", 0.54621
"com=", 30, " For n >= ", 906, " a_n < ", 0.54840
"com=", 31, " For n >= ", 883, " a_n < ", 0.55563
"com=", 32, " For n >= ", 823, " a_n < ", 0.55740
"com=", 33, " For n >= ", 810, " a_n < ", 0.56014

"com=", 34, " For n >= ", 731, " a_n < ", 0.57408
"com=", 35, " For n >= ", 659, " a_n < ", 0.59835
"com=", 36, " For n >= ", 287, " a_n < ", 0.61324
"com=", 37, " For n >= ", 200, " a_n < ", 0.62046
"com=", 38, " For n >= ", 198, " a_n < ", 0.62710
"com=", 39, " For n >= ", 166, " a_n < ", 0.62923
"com=", 40, " For n >= ", 161, " a_n < ", 0.64447
"com=", 41, " For n >= ", 23, " a_n < ", 0.65227
"com=", 42, " For n >= ", 7, " a_n < ", 0.68836
"com=", 43, " For n >= ", 3, " a_n < ", 0.72355
"com=", 44, " For n >= ", 2, " a_n < ", 0.91024 **(83)**

```
> #####################   The END   ################################
```