Highly Composite Numbers and the Riemann hypothesis

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Abstract Let us denote by $d(n)$ the number of divisors of n, by $\mathrm{li}(t)$ the logarithmic integral of t, by β_2 the number $\frac{\log 3/2}{\log 2} = 0.584\ldots$ and by $R(t)$ the function $t \mapsto \frac{2\sqrt{t}+\sum_\rho t^\rho/\rho^2}{\log^2 t}$, where ρ runs over the non-trivial zeros of the Riemann ζ function. In his PHD thesis about highly composite numbers, Ramanujan proved, under the Riemann hypothesis, that

$$\frac{\log d(n)}{\log 2} \leqslant \mathrm{li}(\log n) + \beta_2 \,\mathrm{li}(\log^{\beta_2} n) - \frac{\log^{\beta_2} n}{\log\log n} - R(\log n) + \mathcal{O}\left(\frac{\sqrt{\log n}}{(\log\log n)^3}\right)$$

holds when n tends to infinity. The aim of this paper is to give an effective form to the above asymptotic result of Ramanujan.

1 Introduction

Let us denote by $d(n)$ the number of divisors of n and by $\mathrm{li}(t)$ the logarithmic integral of t (see, below, §2.2). In [24, (235)], under the Riemann hypothesis,

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Ramanujan proved that when $N \to \infty$, $D(N) = \max_{1 \leqslant n \leqslant N} d(n)$ satisfies

$$\frac{\log D(N)}{\log 2} = \mathrm{li}(\log N) + \beta_2 \,\mathrm{li}(\log^{\beta_2} N) - \frac{\log^{\beta_2} N}{\log\log N} - R(\log N) + \frac{\mathcal{O}(\sqrt{\log N})}{(\log\log N)^3}, \tag{1.1}$$

with, for $k \geqslant 1$,

$$\beta_k = \frac{\log(1+1/k)}{\log 2} \tag{1.2}$$

and, for $t > 1$,

$$R(t) = \frac{2\sqrt{t} + S(t)}{\log^2 t} \quad \text{with} \quad S(t) = \sum_\rho \frac{t^\rho}{\rho^2}, \tag{1.3}$$

where ρ runs over the non-trivial zeros of the Riemann ζ function. Moreover, in [24, (226)], Ramanujan writes under the Riemann hypothesis

$$\begin{aligned} |S(t)| = \left|\sum_\rho \frac{t^\rho}{\rho^2}\right| &\leqslant \sum_\rho \left|\frac{t^\rho}{\rho^2}\right| = \sqrt{t}\sum_\rho \frac{1}{\rho(1-\rho)} \\ &= \sqrt{t}\sum_\rho \left(\frac{1}{\rho} + \frac{1}{1-\rho}\right) = 2\sqrt{t}\sum_\rho \frac{1}{\rho} = \tau\sqrt{t} \end{aligned} \tag{1.4}$$

with

$$\tau = 2 + \gamma_0 - \log(4\pi) = 0.046\ 191\ 417\ 932\ 242\ 0\ldots \tag{1.5}$$

where γ_0 is the Euler constant. The value of $\tau = 2\sum_\rho 1/\rho$ can be found in several books, for instance [8, p. 67] or [5, p. 272]. (1.4) implies that $R(t)$ defined in (1.3) satisfies

$$(2-\tau)\frac{\sqrt{t}}{\log^2 x} \leqslant R(t) \leqslant (2+\tau)\frac{\sqrt{t}}{\log^2 t}. \tag{1.6}$$

It is convenient to use the following notation for $t > 1$

$$F(t) = \mathrm{li}(t) + \beta_2\,\mathrm{li}(t^{\beta_2}) - \frac{t^{\beta_2}}{\log t} \quad \text{with} \quad \beta_2 = \frac{\log 3/2}{\log 2} = 0.584\ldots \tag{1.7}$$

The aim of this paper is to give an effective form to the result (1.1) of Ramanujan and, more precisely, to prove

Theorem 1.1. *(i) Under the Riemann hypothesis, for* $n > 183783600$*,*

$$\frac{\log d(n)}{\log 2} \leqslant F(\log n) - R(\log n) - 5.12\frac{\sqrt{\log n}}{(\log\log n)^3} + \frac{1.52\log^{\beta_3} n}{\log\log n} \tag{1.8}$$

with $\beta_3 = (\log(4/3))/\log 2 = 0.415\ldots$

(ii) If the Riemann hypothesis is not true, there exists infinitely many n's for which (1.8) *does not hold. In other words, (i) is equivalent to the Riemann hypothesis.*

(iii) Independently of the Riemann hypothesis, there exists infinitely many n's such that

$$\frac{\log d(n)}{\log 2} \geqslant F(\log n) - R(\log n) - 25.3\frac{\sqrt{\log n}}{(\log\log n)^3} - \frac{1.45\log^{\beta_3} n}{\log\log n}. \tag{1.9}$$

Corollary 1.2. *Under the Riemann hypothesis, we have, for* $n > 122522400$*,*

$$\frac{\log d(n)}{\log 2} \leqslant F(\log n) - \frac{(2-\tau)\sqrt{\log n}}{(\log\log n)^2} - 5.12\frac{\sqrt{\log n}}{(\log\log n)^3} + \frac{1.52\log^{\beta_3} n}{\log\log n} \tag{1.10}$$

for $n \notin \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 18, 24\}$*,*

$$\frac{\log d(n)}{\log 2} \leqslant \mathrm{li}(\log n) + \beta_2\,\mathrm{li}(\log^{\beta_2} n), \tag{1.11}$$

for $4324320 < n \leqslant \exp(10^8)$ *or for* $n > \exp(1.56 \times 10^{17})$*,*

$$\frac{\log d(n)}{\log 2} \leqslant F(\log n) \tag{1.12}$$

and, for $N^{(1)} < n < \exp(10^8)$ *or for* $n > \exp(1.11 \times 10^{40})$

$$\frac{\log d(n)}{\log 2} \leqslant F(\log n) - R(\log n), \tag{1.13}$$

where

$$N^{(1)} = 2^{10}3^6 5^4 7^3 \prod_{p=11}^{19} p^2 \prod_{p=23}^{157} p = 1.245143\ldots \times 10^{75}. \tag{1.14}$$

Corollary 1.2 is easy to prove from Theorem 1.1 and some computation (see below Section 4.3). Under the Riemann hypothesis, (1.12) probably holds for all $n > 4324320$ and also (1.13) for all $n > N^{(1)}$ but we have not been able to prove it.

Let us recall some effective upper bounds for $\frac{\log d(n)}{\log 2}$, obtained without any hypothesis:

$$\frac{\log d(n)}{\log 2} \leqslant 1.5379398606\ldots\frac{\log n}{\log\log n},\ n \geqslant 3 \tag{1.15}$$

with equality for $n = 6983776800 = 2^5 3^3 5^2 \left(\prod_{p=7}^{19} p\right)$,

$$\frac{\log d(n)}{\log 2} \leqslant \frac{\log n}{\log\log n} + 1.9348509679\ldots\frac{\log n}{(\log\log n)^2},\ n \geqslant 2, \qquad (1.16)$$

with equality for $n = 2^8 3^5 5^3 7^2 11^2 13^2 \left(\prod_{p=17}^{83} p\right)$ and

$$\frac{\log d(n)}{\log 2} \leqslant \sum_{i=1}^{2} \frac{(i-1)!\log n}{(\log\log n)^i} + 4.7623501211\ldots\frac{\log n}{(\log\log n)^3},\ n \geqslant 2 \quad (1.17)$$

with equality for $n = 2^{11} 3^6 5^4 7^3 \left(\prod_{p=11}^{23} p^2\right)\left(\prod_{p=29}^{293} p\right)$.

Inequality (1.15) is proved in [18]. Inequalities (1.16) and (1.17) are proved in [28, pp. 41–49], cf. also [19, Section VII].

1.1 Notation

$d(n) = \sum_{m|n} 1$ is the divisor function.

$\pi(x) = \sum_{p\leqslant x} 1$ is the prime counting function. $\Pi(x) = \sum_{p^k \leqslant x} \frac{1}{k} = \sum_{k\geqslant 1} \frac{\pi(x^{1/k})}{k}$.

p_i denotes the i-th prime. $\mathcal{P} = \{2, 3, 5, 7, 11, \ldots\}$ is the set of primes.

$\theta(x) = \sum_{p\leqslant x} \log p$ and $\psi(x) = \sum_{p^k\leqslant x} \log p$ are the Chebychev functions.

$\mathrm{li}(x)$ denotes the logarithmic integral of x (cf. below Section 2.2).

F is defined in (1.7), $R(t)$ and $S(t)$ in (1.3).

G is defined in (2.32), G_1 in (2.33), G_2 in (4.10), G_3 in Section 4.3.2, H in (2.34) and H_0 in (4.9).

τ is defined in (1.5), β_k in (1.2), ξ in (3.12) and $\xi_k = \xi^{\beta_k}$ in (3.13).

$A(x) = \mathrm{li}(\theta(x)) - \pi(x)$, $A_1(x)$ and $A_2(x)$ are defined in (2.41)–(2.43).

σ_2 is defined in Definition 2.4.

$\xi^{(0)} = 10^8 + 7$ is defined in (3.21) and $\xi_k^{(0)}$ in (3.22).

The value of $N^{(0)}$ is given in (3.23) and the one of $\log N^{(0)}$ in (3.24). The value of $N^{(1)}$ and $N^{(2)}$ are given respectively in (1.14) and in (4.7).

Highly composite (hc) numbers are defined in Section 3.4. M_j denotes the j-th hc number.

As $\log N$ and $\log\log N$ occur many times in the article, they are often replaced by L and λ. Similarly, L_0 means $\log N^{(0)}$ and λ_0 means $\log\log N^{(0)}$.

We often implicitly use the following results : for u and v positive and w real, the function

$$t \mapsto \frac{(\log t - w)^u}{t^v} \quad \text{is decreasing for} \quad t > \exp(w + u/v) \tag{1.18}$$

and

$$\max_{t \geqslant e^w} \frac{(\log t - w)^u}{t^v} = \left(\frac{u}{v}\right)^u \exp(-u - vw). \tag{1.19}$$

1.2 Plan of the article

The proof of Theorem 1.1 follows the proof of (1.1) in [24]. We have replaced the asymptotic estimates in number theory used by Ramanujan by effective ones.

In Section 2, we recall and prove some results that we use in the sequel, first, in Section 2.1, about effective estimates of classical functions of prime number theory and later, in Section 2.2, about the logarithmic integral. In Section 2.3, the function S, defined in (1.3), is studied and, finally, in Section 2.4, several lemmas in calculus are proved.

Under the Riemann hypothesis, Ramanujan proved that, $A(x) = \mathrm{li}(\theta(x)) - \pi(x)$ is positive for x large enough, and it was an important argument for his proof of (1.1). In [20], it is proved that, under the Riemann hypothesis, $A(x) > 0$ holds for $x \geqslant 11$. In Section 2.5, some results of [20] about $A(x)$ are recalled to be used in the proof of Theorem 1.1.

Section 3 is devoted to the study of *superior highly composite* (shc) numbers. These numbers, introduced by Ramanujan to study the large values of the divisor function, play an important role in the proof of Theorem 1.1. In Section 3.2, the definition of shc numbers is recalled, and some properties and examples are given.

A shc number N is associated to a parameter ε and its largest prime factor is $\leqslant \xi = 2^{1/\varepsilon}$. In Section 3.7 (without any hypothesis) and in Section 3.8 (under the Riemann hypothesis) effective estimates of N in terms of ξ are given.

Let ε be a positive real number. The theorem of the six exponentials implies that there are at most four shc numbers associated to ε. However, no ε is known with more than two shc numbers associated to it. This question is discussed in Section 3.1 and in Proposition 3.7.

The definition of highly composite numbers (hc) introduced by Ramanujan is recalled in Section 3.4. These numbers are used to determine the largest number (i.e. 183783600) not satisfying (1.8), see Lemma 3.14 and Section 4.2. The notion of benefit, recalled in Section 3.5, is convenient to find, on some interval, the numbers with a large number of divisors.

In Section 3.6, an argument of convexity is given, which is used in Lemma 4.3 to shorten the computation in the proof of Theorem 1.1.

The proofs of Theorem 1.1 and Corollary 1.2 are given in Section 4.

The computations, both algebraic and numerical, have been carried out with Maple. On the website [32], one can find the code and a Maple sheet with the results.

2 Preliminary results

2.1 Effective estimates

Without any hypothesis, Büthe [4, Theorem 2] has shown by computation that

$$\theta(x) < x, \quad \text{for} \quad 1 < x \leqslant 10^{19} \tag{2.1}$$

while, Platt and Trudjan, [23, Theorem 1, Corollary 1] proved for $x > 0$,

$$\theta(x) < (1+\eta)x \quad \text{with} \quad \eta = 1 + 7.5 \cdot 10^{-7}. \tag{2.2}$$

Without any hypothesis, Dusart [7, Théorème 5.2] has proved that

$$|\theta(x) - x] < \frac{x}{\log^3 x} \text{ for } x \geqslant 89\ 967\ 803. \tag{2.3}$$

From (2.3) and the computation of $\theta(x)$ for $x < 89\ 967\ 803$, it is possible to show (cf. [6, Table 1, p. 114] or [33]) that we have $\theta(x) > bx$ for $x \geqslant a$ for each of the following pairs of values of a and b

a	127	367	1993	47491
b	0.8499	0.9134	0.9629	0.9927

(2.4)

We shall also use the inequality (cf. [29, (3.5)]):

$$\pi(x) \geqslant \frac{x}{\log x} \quad \text{for} \quad x \geqslant 17. \tag{2.5}$$

Lemma 2.1. *For each of the following pairs of values of a and b, we have $\pi(x) < bx/\log x$ for $x \geqslant a$.*

a	2	376	2090
b	1.25506	1.19768	1.15963

(2.6)

Proof : For $a = 2$, the result is quoted in [29, (3.6))]. For the two other values of a, we start from the inequality (cf. [7, Théorème 6.9]) valid for $x \geqslant 60184$

$$\pi(x) \leqslant \frac{x}{\log x - 1.1} \leqslant \frac{x}{(\log x)(1 - 1.1/\log 60184)} \leqslant 1.1114 \frac{x}{\log x}.$$

Furthermore, if p and p' are two consecutive primes, on the interval $[p, p')$, the function $f(t) = \pi(t)(\log t)/t$ is decreasing. For $a = 376$, we check that $f(376) < 1.19768$ holds and that for all prime p satisfying $376 < p < 60184$, we also have $f(p) < 1.19768$. A similar computation shows the result for $a = 2090$. □

Under the Riemann hypothesis, we shall use the upper bounds (cf. [30, (6.2) and (6.3)])

$$|\psi(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}\log^2 x, \quad \text{for} \quad x \geqslant 73.2 \tag{2.7}$$

and

$$|\theta(x) - x| \leqslant \frac{1}{8\pi}\sqrt{x}\log^2 x, \quad \text{for} \quad x \geqslant 599. \tag{2.8}$$

Let us introduce

$$\delta(t) = \begin{cases} 0 & \text{if } \ t \leqslant 10^{19} \\ 1 & \text{if } \ t > 10^{19}. \end{cases} \tag{2.9}$$

Then (2.1) and (2.8) imply

$$\theta(x) \leqslant x + \frac{\delta(x)}{8\pi}\sqrt{x}\log^2 x, \quad \text{for} \quad x > 0. \tag{2.10}$$

Lemma 2.2. *Let us denote by p_i the i-th prime. Then, for $p_i \geqslant 127$, we have*

$$\frac{p_{i+1} - p_i}{p_{i+1}} \leqslant \frac{149 - 139}{149} \leqslant 0.0672. \tag{2.11}$$

We order the prime powers p^m, with $m \geqslant 1$, in a sequence $(a_i)_{i \geqslant 1} = (2, 3, 4, 5, 7, 8, 9, 11, \ldots)$. Then, for $a_i \geqslant 127$,

$$\frac{a_{i+1} - a_i}{a_{i+1}} = 1 - \frac{a_i}{a_{i+1}} \leqslant 0.0672\ . \tag{2.12}$$

Proof : In [7, Proposition 5.4], it is proved that, for $x \geqslant 89693$, there exists a prime p satisfying $x < p \leqslant x(1 + 1/\log^3 x)$. This implies that for $p_i \geqslant 89693$, we have $p_{i+1} \leqslant p_i + p_i/\log^3 p_i$ and

$$\frac{p_{i+1} - p_i}{p_{i+1}} \leqslant \frac{p_{i+1} - p_i}{p_i} \leqslant \frac{1}{\log^3 p_i} \leqslant \frac{1}{\log^3 89693} = 0.00067423\ldots$$

For $2 \leqslant p_i < 89693$, the computation of $(p_{i+1} - p_i)/p_{i+1}$ completes the proof of (2.11).

If p_j is the largest prime $\leqslant a_i$ then $a_{i+1} \leqslant p_{j+1}$ holds, so that, by (2.11), for $a_i \geqslant 127$,

$$\frac{a_{i+1}-a_i}{a_{i+1}} = 1 - \frac{a_i}{a_{i+1}} \leqslant 1 - \frac{p_j}{p_{j+1}} \leqslant 0.0672$$

holds, which proves (2.12). □

2.2 The logarithmic integral

For x real > 1, we define $\mathrm{li}(x)$ as (cf. [1, p. 228])

$$\mathrm{li}(x) = \fint_0^x \frac{dt}{\log t} = \lim_{\varepsilon \to 0^+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \frac{dt}{\log t} \right) = \int_2^x \frac{dt}{\log t} + \mathrm{li}(2). \tag{2.13}$$

For $0 < x < 1$, (cf. [22, p. 1–3]),

$$\mathrm{li}(x) = \gamma_0 + \log(-\log(x)) + \sum_{k=1}^{\infty} \frac{\log^k x}{k \cdot k!}$$

where γ_0 is the Euler constant. We have the following values:

x	0.5	1	1.45136...	1.96904...	2
$\mathrm{li}(x)$	$-0.37867\ldots$	$-\infty$	0	1	1.145163...

(2.14)

From the definition of $\mathrm{li}(x)$, it follows that

$$\frac{d}{dx}\,\mathrm{li}(x) = \frac{1}{\log x} \qquad \text{and} \qquad \frac{d^2}{dx^2}\,\mathrm{li}(x) = -\frac{1}{x \log^2 x}. \tag{2.15}$$

The function $x \mapsto \mathrm{li}(x)$ is increasing for $x > 1$. For $x > 1$ the second derivative of $\mathrm{li}(x)$ is negative and increasing. Therefore, from Taylor's formula, if a, c and h are three real numbers satisfying $a > 1$, $a+h > 1$ and $c = \min(a, a+h) > 1$, one has

$$\mathrm{li}(a) + \frac{h}{\log a} - \frac{h^2}{2c \log^2 c} \leqslant \mathrm{li}(a+h) \leqslant \mathrm{li}(a) + \frac{h}{\log a}. \tag{2.16}$$

We also have for $x \to \infty$

$$\mathrm{li}(x) = \sum_{k=1}^{N} \frac{(k-1)!\,x}{(\log x)^k} + \mathcal{O}\left(\frac{x}{(\log x)^{N+1}}\right). \tag{2.17}$$

2.3 Study of the function $S(t)$ defined by (1.3)

Lemma 2.3. *If a, b are fixed real numbers satisfying $1 \leqslant a < b < \infty$, and g any function with a continuous derivative on the interval $[a, b]$, then*

$$\sum_\rho \int_a^b \frac{g(t)t^\rho}{\rho} dt = \int_a^b g(t) \left[t - \psi(t) - \log(2\pi) - \frac{1}{2} \log \left(1 - \frac{1}{t^2}\right)\right] dt, \quad (2.18)$$

where ρ runs over the non-trivial zeros of the Riemann ζ function.

Proof : This is Théorème 5.8(b) of [10, p. 169] or Theorem 5.8(b) of [9, p. 162]. □

Definition 2.4. One defines σ_2 by

$$\sigma_2 = \sum_\rho \frac{1}{\rho^2} = 1 - \frac{\pi^2}{8} + 2\gamma_1 + \gamma_0^2 = -0.046\ 154\ 317\ 295\ 804...$$

where the coefficients γ_m are defined by the Laurent expansion of $\zeta(s)$ around 1 (see [5, p. 206 and 272])

$$\zeta(s) = \frac{1}{s-1} + \sum_{m=0}^{\infty} (-1)^m \frac{\gamma_m}{m!} (s-1)^m$$

Lemma 2.5. *For $x > 1$, one has*

$$S(x) = \int_1^x \frac{t - \psi(t)}{t} dt - (\log(2\pi)) \log x + \sigma_2 + \frac{\pi^2}{24} - \sum_{j=1}^{\infty} \frac{1}{4j^2 x^{2j}}. \quad (2.19)$$

Proof : Applying Lemma 2.3 with $g(t) = 1/t, a = 1, b = x$ yields

$$S(x) = \sum_\rho \frac{x^\rho}{\rho^2} = \int_1^x \left[t - \psi(t) - \log(2\pi) - \frac{1}{2} \log \left(1 - \frac{1}{t^2}\right)\right] \frac{dt}{t} + \sigma_2 \quad (2.20)$$

which, by expanding $\log(1 - 1/t^2)/(2t)$ in power series, implies (2.19). □

Lemma 2.6. *Let a be a real number $> 1/2$. The function $Y := t \mapsto -a/t - \log(1 - 1/t^2)/(2t)$ is increasing and negative for $t \geqslant 2$.*

Proof : We have $Y = -a/t + \sum_{j=1}^{\infty} 1/(2jt^{2j+1})$,

$$Y' = \frac{a}{t^2} - \sum_{j=1}^{\infty} \frac{2j+1}{2jt^{2j+2}} \geqslant \frac{a}{t^2} - \sum_{j=1}^{\infty} \frac{3}{2t^{2j+2}} \geqslant \frac{a}{t^2} - \sum_{j=1}^{\infty} \frac{3/2}{2^{2j} t^2} = \frac{a - 1/2}{t^2} > 0.$$

Therefore, for $t \geqslant 2$, Y in increasing and, as $\lim_{t\to\infty} Y = 0$, Y is negative. □

Lemma 2.7. *Let S be defined by (1.3) and x and y be real numbers. Then, under the Riemann hypothesis,*

$$S(x) - S(y) \leqslant 0.18(x-y) \quad \textit{for} \quad 2 \leqslant y \leqslant x, \tag{2.21}$$

$$S(x) - S(y) \leqslant 0.0398\,\frac{|x-y|\log^2 y}{\sqrt{y}} \quad \textit{for} \quad 10^8 \leqslant y \leqslant x \tag{2.22}$$

and

$$\left|\frac{S(x)}{\log^2 x} - \frac{S(y)}{\log^2 y}\right| \leqslant 0.04\,\frac{|x-y|}{\sqrt{y}} \quad \textit{for} \quad 10^8 \leqslant y \leqslant x. \tag{2.23}$$

Proof : In a first step, we assume that x and y satisfy $a_i \leqslant y < x \leqslant a_{i+1}$ for some $i \geqslant 1$, with a_i defined as in Lemma 2.2. Then, from (2.20), we get

$$S(x) - S(y) = \int_y^x \left[1 - \frac{\psi(a_i)}{t} - \frac{\log(2\pi)}{t} - \frac{\log(1-1/t^2)}{2t}\right] dt. \tag{2.24}$$

From Lemma 2.6, the above square bracket is increasing for $t \geqslant 2$ and (2.24) yields

$$\begin{aligned} S(x) - S(y &\leqslant \int_y^x \left[1 - \frac{\psi(a_i)}{a_{i+1}} - \frac{\log(2\pi)}{a_{i+1}} - \frac{\log(1-1/a_{i+1}^2)}{2a_{i+1}}\right] dt \\ &= (x-y)\left(1 - \frac{\psi(a_i)}{a_{i+1}} - \frac{\log(2\pi)}{a_{i+1}} - \frac{\log(1-1/a_{i+1}^2)}{2a_{i+1}}\right). \end{aligned} \tag{2.25}$$

For $a_1 = 2 \leqslant a_i \leqslant a_{43} = 127$, one computes the parenthesis of the right-hand side of (2.25). The maximum $0.1759\ldots$ is attained for $a_1 = 2$.

For $a_i \geqslant a_{44} = 128$, we use the inequality $\psi(a_i) \geqslant a_i - (\sqrt{a_i}\log^2 a_i)/(8\pi) > a_i - (\sqrt{a_{i+1}}\log^2 a_{i+1})/(8\pi)$ (cf. (2.7)), whence, from (2.25) and Lemma 2.6,

$$\frac{S(x)-S(y)}{x-y} \leqslant 1 - \frac{\psi(a_i)}{a_{i+1}} \leqslant 1 - \frac{a_i}{a_{i+1}} + \frac{\log^2 a_{i+1}}{8\pi\sqrt{a_{i+1}}}$$

and, from (2.12) and (1.18),

$$\frac{S(x)-S(y)}{x-y} \leqslant 0.0672 + \frac{\log^2 128}{8\pi\sqrt{128}} = 0.149994\ldots$$

In the general case, if $2 < y < x$ holds, we determine the two integers $i \geqslant 1$ and $j \geqslant 1$ such that $a_i \leqslant y \leqslant a_{i+1} \leqslant a_{i+j} \leqslant x \leqslant a_{i+j+1}$. We have

$$\begin{aligned}
&S(x) - S(y)\\
&\quad = S(a_{i+1}) - S(y) + \left(\sum_{k=1}^{j-1} S(a_{i+k+1}) - S(a_{i+k})\right) + S(x) - S(a_{i+j})\\
&\quad \leqslant 0.18\left(a_{i+1} - y + \left(\sum_{k=1}^{j-1} a_{i+k+1} - a_{i+k}\right) + x - a_{i+j}\right) = 0.18(x-y),
\end{aligned}$$

which completes the proof of (2.21).

From (2.7) and (1.18), for $73.2 \leqslant y \leqslant x$, one has

$$\left|\int_y^x \frac{t-\psi(t)}{t}dt\right| \leqslant \int_y^x \frac{\log^2 t}{8\pi\sqrt{t}}dt \leqslant \frac{|x-y|}{8\pi}\left(\frac{\log^2 y}{\sqrt{y}}\right), \tag{2.26}$$

$$\int_y^x \frac{dt}{t} = \log(x/y) \leqslant x/y - 1 = |x-y|/y \tag{2.27}$$

and

$$\begin{aligned}
\left|\int_y^x -\frac{\log(1-1/t^2)}{2t}dt\right| &= \int_y^x \frac{1}{2t}\sum_{j=1}^{\infty}\frac{1}{jt^{2j}}dt \leqslant \frac{|x-y|}{2y}\sum_{j=1}^{\infty}\frac{1}{jy^{2j}}\\
&\leqslant \frac{|x-y|}{2y}\sum_{j=1}^{\infty}\frac{1}{y^{2j}} = \frac{|x-y|}{2y(y^2-1)},
\end{aligned} \tag{2.28}$$

whence, from (2.20), (2.26), (2.27) and (2.28), with $y_0 = 10^8$,

$$\begin{aligned}
|S(x) - S(y)| &\leqslant |x-y|\left(\frac{\log^2 y}{8\pi\sqrt{y}} + \frac{\log(2\pi)}{y} + \frac{1}{2y(y^2-1)}\right)\\
&= \frac{|x-y|\log^2 y}{\sqrt{y}}\left(\frac{1}{8\pi} + \frac{1}{\sqrt{y}\log^2 y}\left(\log(2\pi) + \frac{1}{2(y^2-1)}\right)\right)\\
&\leqslant \frac{|x-y|\log^2 y}{\sqrt{y}}\left(\frac{1}{8\pi} + \frac{1}{\sqrt{y_0}\log^2 y_0}\left(\log(2\pi) + \frac{1}{2(y_0^2-1)}\right)\right)\\
&\leqslant 0.0397\ldots\frac{|x-y|\log^2 y}{\sqrt{y}}
\end{aligned}$$

which proves (2.22).

To prove (2.23), one writes

$$\left|\frac{S(x)}{\log^2 x} - \frac{S(y)}{\log^2 y}\right| \leqslant \frac{|S(x)-S(y)|}{\log^2 x} + \left|S(y)\left(\frac{1}{\log^2 x} - \frac{1}{\log^2 y}\right)\right|. \tag{2.29}$$

From (2.22), it follows

$$\frac{|S(x)-S(y)|}{\log^2 x} \leqslant \frac{|S(x)-S(y)|}{\log^2 y} \leqslant 0.0398\frac{|x-y|}{\sqrt{y}}. \tag{2.30}$$

(1.4) and (1.5) imply $|S(y)| \leqslant \tau\sqrt{y} \leqslant 0.0462\sqrt{y}$, whence, with $y_0 = 10^8$,

$$\left|S(y)\left(\frac{1}{\log^2 x} - \frac{1}{\log^2 y}\right)\right| = |S(y)| \int_y^x \frac{2dt}{t\,\log^3 t} \leqslant 2|S(y)|\frac{|x-y|}{y\,\log^3 y}$$
$$\leqslant 2\tau\,\frac{|x-y|}{\sqrt{y}\log^3 y} \leqslant \frac{0.0924}{\log^3 y_0}\frac{|x-y|}{\sqrt{y}} = 0.000014827\ldots\,\frac{|x-y|}{\sqrt{y}},$$

which, together with (2.29) and (2.30), proves (2.23). □

2.4 Four lemmas in calculus

Lemma 2.8. *The function*

$$f(t) = \frac{1.52\,t^{\beta_3}}{\log t} - \frac{5.12\,\sqrt{t}}{\log^3 t}$$

is positive for $7.38 < t < 1.1 \times 10^{40}$ *and negative for* $1 < t < 7.37$ *and* $t > 1.11 \times 10^{40}$.

Proof : Let us write $f(t) = (t^{\beta_3}/\log t)[1.52 - 5.12\,t^{1/2-\beta_3}/\log^2 t]$. From (1.18), the above square bracket is maximal for $t = t_0 = \exp(2/(1/2-\beta_3)) = 1.67\ldots 10^{10}$, is increasing for $t < t_0$, decreasing for $t > t_0$ and vanishes for $t = 7.3735\ldots$ and $1.10026\ldots 10^{40}$. □

Lemma 2.9. *Let a be a positive real number and, for $n \geqslant 0$, $\varphi_n = \log(n+1) - an$.*

(i) If, for some positive integer k, a is equal to $\log(1+1/k)$, (i.e. $k = 1/(e^a-1)$), then the sequence $(\varphi_n)_{n\geqslant 0}$ attains its maximum on the two points k and $k-1$. More precisely, for $0 \leqslant n \leqslant k-2$ or $n \geqslant k+1$, $\varphi_n < \varphi_{k-1} = \varphi_k$ holds.

(ii) If $\log(1+1/(k+1)) < a < \log(1+1/k)$ holds ($\log(1+1/0) = \infty$ is assumed), then the maximum of φ_n is attained on only one point $k = \lfloor 1/(e^a-1)\rfloor$. More precisely, for $0 \leqslant n \leqslant k-1$ or $n \geqslant k+1$, $\varphi_n < \varphi_k$ holds.

Proof : For $n \geqslant 1$, we have $\Delta_n = \varphi_n - \varphi_{n-1} = \log(1+1/n) - a$.

If $a = \log(1+1/k)$ holds, then Δ_n is positive for $n < k$, vanishes for $n = k$ and is negative for $n > k$, which implies (i).

If $\log(1+1/(k+1)) < a < \log(1+1/k)$ holds, then Δ_n is positive for $n \leqslant k$, and is negative for $n > k$, which implies (ii). Note that $k < 1/(e^a-1) < k+1$ holds and thus, $k = \lfloor 1/(e^a-1)\rfloor$. □

Lemma 2.10. *Let us set* $\beta_k = (\log(1+1/k))/\log 2$ *as in* (1.2) *above. The function*

$$\Phi(t) = 0.352\ \exp((\beta_3 - \beta_2/2)t) + 0.9132\ \exp((\beta_4 - \beta_2/2)t) - 0.0143\, t^2 \quad (2.31)$$

is positive for $t \geqslant 0$.

Proof : Let $a = \beta_3 - \beta_2/2 = 0.1225\dots$ and $b = \beta_4 - \beta_2/2 = 0.02944\dots$. One has $\Phi' = 0.352\, a \exp(a\,t) + 0.9132\, b\ \exp(b\,t) - 0.0286\, t$ and $\Phi'' = 0.352\, a^2 \exp(a\,t) + 0.9132\, b^2\ \exp(b\,t) - 0.0286$, The third derivative is positive and thus, the second derivative is increasing. The variation of Φ' and Φ is displayed in the array below (cf. [32]).

t	0		3.294		13.43		20.62		∞
Φ''	-0.02	$-$	-0.019	$-$	0	$+$	0.039	$+$	∞
Φ'	0.07	↘	0	↘	-0.12	↗	0	↗	∞
Φ	1.26	↗	1.37	↘	0.6	↘	0.002	↗	∞

The minimum of $\Phi(t)$ for $t \geqslant 0$ is > 0.0019, which proves lemma 2.10. □

Lemma 2.11. *Let* a, b, c *be three real numbers satisfying* $0 \leqslant a \leqslant 3$, $0 \leqslant b \leqslant 30$ *and* $c \geqslant 0$. $F(t)$ *is defined by* (1.7), $S(t)$ *by* (1.3) *and* β_k *by* (1.2).

(i) The function G *defined by*

$$G = G(a,b,c,t) = F(t) - \frac{a\sqrt{t}}{\log^2 t} - \frac{b\sqrt{t}}{\log^3 t} + \frac{c\, t^{\beta_3}}{\log t} \quad (2.32)$$

is increasing for $t \geqslant 12$.

(ii) The function

$$G_1(t) = G(2+\tau, 0, 0, t) = F(t) - \frac{(2+\tau)\sqrt{t}}{\log^2 t} \quad (2.33)$$

is increasing and concave for $t > 1$.

(iii) The function H *defined by*

$$H = H(a,b,c,t) = F(t) - \frac{a\sqrt{t}}{\log^2 t} - \frac{S(t)}{\log^2 t} - \frac{b\sqrt{t}}{\log^3 t} + \frac{c\, t^{\beta_3}}{\log t} \quad (2.34)$$

is continuous for $t > 1$ *and increasing for* $t \geqslant 12$.

Proof : Let $T = \log t$ and assume $t > 1$. It is convenient to define

$$g_1 = \mathrm{li}(t), \quad g_1' = 1/T, \quad g_1'' = -1/(t\,T^2), \tag{2.35}$$

$$g_2 = \beta_2\,\mathrm{li}(t^{\beta_2}) - \frac{t^{\beta_2}}{\log t}, \quad g_2' = \frac{1}{t^{1-\beta_2}T^2}, \quad g_2'' = -\frac{(1-\beta_2)T+2}{t^{2-\beta_2}T^3}, \tag{2.36}$$

$$g_3 = \frac{\mathrm{li}(t)}{6} - a\frac{\sqrt{t}}{\log^2 t}, \quad g_3' = \frac{\sqrt{t}\,T^2 - 3\,a\,T + 12\,a}{6\sqrt{t}\,T^3},$$
$$g_3'' = -\frac{2\sqrt{t}\,T^2 - 3a\,T^2 + 72\,a}{12\,t^{3/2}\,T^4}, \tag{2.37}$$

$$g_4 = \frac{\mathrm{li}(t)}{6} - b\frac{\sqrt{t}}{\log^3 t}, \quad g_4' = \frac{\sqrt{t}\,T^3 - 3\,b\,T + 18\,b}{6\,\sqrt{t}\,T^4}, \tag{2.38}$$

$$g_5 = c\frac{t^{\beta_3}}{\log t}, \quad g_5' = c\frac{\beta_3(T - 1/\beta_3)}{t^{1-\beta_3}\,T^2}. \tag{2.39}$$

From (2.35) and (2.36), it is clear that g_1 and g_2 are increasing and concave for $t > 1$.

If $T \leqslant 4$, $-3aT + 12a \geqslant 0$ holds while, if $T > 4$, then $\sqrt{t}\,T^2 > 16e^2 > 9 \geqslant 3a$ so that, from (2.37), g_3 is increasing for $t > 1$.

If $T \leqslant 4$ then $-3aT^2 + 72a \geqslant 0$ while, if $T > 4$, $2\sqrt{t} \geqslant 2e^2 > 9 \geqslant 3a$ so that g_3 is concave for $t > 1$.

If $T \leqslant 6$, one has $3bT \leqslant 18b$. If $T > 6$, $\sqrt{t}\,T^3 > 36e^3 T > 90T \geqslant 3bT$ so that, from (2.38), g_4 is increasing for $t > 1$.

From (2.39), g_5 is increasing for $t \geqslant \exp(1/\beta_3) = 11.12\ldots$

In conclusion, $G = 2g_1/3 + g_2 + \ldots + g_5$ is increasing for $t \geqslant 12$, which proves (i) and $G_1 = 5g_1/6 + g_2 + g_3$ (with $a = 2 + \tau$) is increasing and concave for $t > 1$, which proves (ii).

Proof of (iii). From (2.19), $S(t)$ is continuous for $t > 1$, and consequently also $H(a, b, c, t)$. Moreover, we introduce

$$g_6 = \frac{2\,\mathrm{li}(t)}{3} - \frac{S(t)}{\log^2 t}. \tag{2.40}$$

so that, $H = g_2 + \ldots + g_6$. We shall prove the increasingness of g_6 for $t \geqslant 2$. For that, we consider two real numbers satisfying $2 \leqslant y < x < y^{4/3}$. From (2.40), we get

$$g_6(x) - g_6(y) = \frac{2}{3}\int_y^x \frac{dt}{\log t} - \left(\frac{S(x)}{\log^2 x} - \frac{S(y)}{\log^2 y}\right)$$
$$= \frac{2}{3}\int_y^x \frac{dt}{\log t} - \frac{S(x) - S(y)}{\log^2 x} - S(y)\int_y^x \frac{2}{t\log^3 t}dt$$

and, from (2.21) and (1.4),

$$\begin{aligned} g_6(x) - g_6(y) &\geqslant \frac{2(x-y)}{3\log(y^{4/3})} - \frac{0.18(x-y)}{\log^2 y} - \frac{2\tau\sqrt{y}(x-y)}{y\log^3 y} \\ &= \frac{x-y}{\log y}\left(\frac{1}{2} - \frac{0.18}{\log y} - \frac{2\tau}{\sqrt{y}\log^2 y}\right) \\ &\geqslant \frac{x-y}{\log y}\left(\frac{1}{2} - \frac{0.18}{\log 2} - \frac{2\tau}{\sqrt{2}\log^2 2}\right) = 0.104\ldots\frac{x-y}{\log y} > 0, \end{aligned}$$

which proves (iii) and ends the proof of Lemma 2.11. □

2.5 Study of $A(x) = \mathrm{li}(\theta(x)) - \pi(x)$

Let us set

$$A(x) = \mathrm{li}(\theta(x)) - \pi(x) = A_1(x) + A_2(x), \tag{2.41}$$

$$A_1(x) = \mathrm{li}(\psi(x)) - \Pi(x), \tag{2.42}$$

$$A_2(x) = \mathrm{li}(\theta(x)) - \mathrm{li}(\psi(x)) + \Pi(x) - \pi(x) \tag{2.43}$$

with

$$\psi(x) = \sum_{p^m \leqslant x} \log p, \quad \kappa = \left\lfloor \frac{\log x}{\log 2} \right\rfloor \quad \text{and} \quad \Pi(x) = \sum_{k=1}^{\kappa} \frac{\pi(x^{1/k})}{k}.$$

In [20], under the Riemann hypothesis, the following results are given.

For $x \geqslant 11$ (cf. [20, Theorem 1.1, (1.8)]),

$$A(x) > 0, \tag{2.44}$$

for $x \geqslant 2$ (cf. [20, Theorem 1.1, (1.10)]),

$$A(x) \leqslant 5.07\sqrt{x}/\log^2 x, \tag{2.45}$$

for $x \geqslant 599$, (cf. [20, Proposition 3.3] and (1.3)),

$$A_1(x) \geqslant \frac{S(x)}{\log^2 x} - \frac{\sqrt{x}}{\log^3 x}\left[\frac{2}{300} + \frac{0.0009\log^5 x}{\sqrt{x}}\right] \tag{2.46}$$

and

$$A_1(x) \leqslant \frac{S(x)}{\log^2 x} + \frac{\sqrt{x}}{\log^3 x}\left[\frac{2}{300} + \frac{(\log 2)\log^3 x}{\sqrt{x}}.\right], \tag{2.47}$$

for $x \geqslant 941$, (cf. [20, p. 604, l. 8]),

$$A_2(x) \geqslant \frac{2\sqrt{x}}{\log^2 x} + \frac{\sqrt{x}}{\log^3 x}\left[8 - \frac{\log^3 x}{8\pi x^{1/4}} - \frac{9\log^5 x}{10000\sqrt{x}}\right], \tag{2.48}$$

and, for $x > 10^8$, (cf. [20, Proposition 3.5 and Lemma 3.6]),

$$A_2(x) \leqslant \frac{2\sqrt{x}}{\log^2 x} + \frac{\sqrt{x}}{\log^3 x}\left[4\widetilde{F_2}(\sqrt{x}) + \sum_{k=3}^{\kappa_1} \frac{k\widetilde{F_1}(x^{1/k})\log x}{x^{1/2-1/k}} + \frac{7.23\,\kappa_1^3}{x^{1/2-1/\kappa_1}} + 2.35\frac{\log^3 x}{\sqrt{x}} + \frac{0.94}{\log^2 x} + \frac{9\log^5 x}{10000\sqrt{x}}\right], \tag{2.49}$$

where $\kappa_1 = 5$ and $\widetilde{F_1}$ and $\widetilde{F_2}$ are the non-increasing functions defined by (cf. [20, Lemma 2.1 and (3.16)])

$$\widetilde{F_1}(t) = \begin{cases} 1.785 & \text{if } t \leqslant 95 \\ \frac{\mathrm{li}(t) - \frac{t}{\log t}}{t/\log^2 t} & \text{if } t > 95 \end{cases} \quad \text{and} \quad \widetilde{F_2}(t) = \begin{cases} 4.05 & \text{if } t \leqslant 381 \\ \frac{\mathrm{li}(t) - \frac{t}{\log t} - \frac{t}{\log^2 t}}{t/\log^3 t} & \text{if } t > 381. \end{cases} \tag{2.50}$$

Proposition 2.12. *For $x \geqslant 10^8$,*

$$\frac{S(x)}{\log^2(x)} - 0.198\frac{\sqrt{x}}{\log^3(x)} \leqslant A_1(x) \leqslant \frac{S(x)}{\log^2(x)} + 0.44\frac{\sqrt{x}}{\log^3(x)}, \tag{2.51}$$

$$\frac{2\sqrt{x}}{\log^2(x)} + 5.32\frac{\sqrt{x}}{\log^3(x)} \leqslant A_2(x) \leqslant \frac{2\sqrt{x}}{\log^2(x)} + 24.77\frac{\sqrt{x}}{\log^3(x)} \tag{2.52}$$

and

$$R(x) + 5.12\frac{\sqrt{x}}{\log^3(x)} \leqslant A(x) \leqslant R(x) + 25.3\frac{\sqrt{x}}{\log^3(x)}. \tag{2.53}$$

Proof : For $x \geqslant 10^8$, the terms in the square brackets of (2.46) and (2.47) are decreasing in x, so, these two brackets are maximal for $x = 10^8$, which proves (2.51). For $x \geqslant 10^8$, the bracket of (2.48) is increasing and minimal for $x = 10^8$ while the one of (2.49) is non-increasing and maximal for $x = 10^8$, which, by choosing $\kappa_1 = 5$, proves (2.52) (cf. [32]). Finally, from (2.41) and (1.3), (2.53) results from the addition of (2.51) and (2.52). □

For $k \geqslant 2$, let us set $S_k(x) = \sum_\rho x^\rho/\rho^k$. Note that $S_2 = S$ and, as $|\rho| \geqslant 14$ holds, from (1.4), one has $|S_k(x)| \leqslant \tau\sqrt{x}/(14)^{k-2}$. From the formulas [20, p. 598, l. -5] and [20, p. 601, l. -7], with the prime number theorem, one gets, for $x \to \infty$,

$$A(x) = \sum_\rho \int_0^\infty \frac{x^{\rho-t}}{\rho - t}dt - \frac{1}{\log x}\sum_\rho \frac{x^\rho}{\rho} + \frac{\mathrm{li}(\sqrt{x})}{2} - \frac{\sqrt{x}}{\log x} + \mathcal{O}\left(\frac{x^{1/3}}{\log x}\right).$$

By partial integration and (2.17), one can deduce, for $x \to \infty$,

$$A(x) = R(x) + \frac{8\sqrt{x} + 2S_3(x)}{\log^3 x} + \dots + \frac{(k-1)!(2^{k-1}\sqrt{x} + S_k(x))}{\log^k x} + \mathcal{O}\left(\frac{\sqrt{x}}{\log^{k+1} x}\right).$$

In particular, as $|S_3(x)/\sqrt{x}| \leqslant \tau/14 \leqslant 0.0033$ holds, for x large enough,

$$7.99\sqrt{x}/\log^3 x \leqslant A(x) - R(x) \leqslant 8.01\sqrt{x}/\log^3 x. \tag{2.54}$$

Therefore, the coefficient 5.12 in (2.53) and (1.8) cannot be replaced by a number exceeding 8.01 (see below (4.22)) .

3 Superior Highly Composite (shc) Numbers

3.1 The Theorem of the 6 exponentials

In the next section (Section 3.2) introducing the shc numbers, we need some diophantine properties of the set

$$\mathcal{E} = \left\{ \frac{\log(1+1/k)}{\log p}, \quad k \geqslant 1, \quad p \text{ prime} \right\}. \tag{3.1}$$

Let us recall first two main results (cf. for instance [31, Theorem 1.12, p. 13 and Theorem 1.4, p. 3] or [14, Theorem 1, chap. 2]).

Lemma 3.1. (Six Exponentials Theorem.) *Let x_1, x_2 be two complex numbers linearly independent over $\mathbb{Q}$ and y_1, y_2, y_3 three complex numbers linearly independent over $\mathbb{Q}$. Then one of the six numbers $\exp(x_i y_j)$ is transcendantal.*

Lemma 3.2. (Gelfond-Schneider Theorem.) *If α is algebraic and different of 0 and 1 an if β is algebraic and irrationnal, then α^β is transcendental.*

First, let us observe that the elements of $\mathcal{E}$ but $\log(1+1/1)/\log 2 = 1$ are irrationnal. Indeed, assume that $\varepsilon = \frac{\log(1+1/k)}{\log p} = a/b \neq 1$ with a and b positive integers. One would have

$$(1+1/k)^b = p^a$$

which is impossible, because, for $k > 1$, the left-hand side is a fraction while the right-hand side is an integer and $k = 1$ implies $p = 2$, $a = b$ and $\varepsilon = 1$.

Lemma 3.3. *Three elements of $\mathcal{E}$ are always distinct. In other words, there do not exist three distinct primes q_1, q_2, q_3 and three positive integers k_1, k_2, k_3 such that*

$$\frac{\log(1+1/k_1)}{\log q_1} = \frac{\log(1+1/k_2)}{\log q_2} = \frac{\log(1+1/k_3)}{\log q_3}.$$

Proof : Ab absurdum, let us assume that $\frac{\log(1+1/k_1)}{\log q_1} = \frac{\log(1+1/k_2)}{\log q_2} = \frac{\log(1+1/k_3)}{\log q_3} = \varepsilon$ holds. In the Six Exponentials Theorem (cf. Lemma 3.1), we choose $x_1 = 1$ and $x_2 = \varepsilon$. As ε is irrationnal, x_1 and x_2 are linearly independent over $\mathbb{Q}$. Afterwards, we choose $y_j = \log q_j$ that are also linearly independent over $\mathbb{Q}$. The six exponentials $\exp(x_i y_j)$ are $q_1, q_2, q_3, 1+1/k_1, 1+1/k_2, 1+1/k_3$. They are all rationnal, which contradicts the theorem of the six exponentials. □

It was known by Siegel (cf. [2, p. 455] or [31, p. 14]) or [14, Historical Note, p. 19–20] that for real t and three different primes q_1, q_2, q_3, the numbers q_1^t, q_2^t, q_3^t cannot be all rational, except when t is an integer.

Remark 3.4. It has not been proved that it cannot exist two distinct primes q, q' and two positive integers k, k' such that

$$\frac{\log(1+1/k)}{\log q} = \frac{\log(1+1/k')}{\log q'}. \tag{3.2}$$

No example is known. In particular, there is no example with $q, q' < 10^8$.

It is possible to show that $\varepsilon = \frac{\log(1+1/k)}{\log q} \in \mathcal{E} \setminus \{1\}$ is transcendental. Indeed, by the Gelfond-Scneider Theorem (cf. Lemma 3.2), as ε is irrational, if ε were algebraic, q^ε should be transcendental. But $q^\varepsilon = 1 + 1/k$ is rational.

3.2 Definition of shc numbers

Definition 3.5. A number N is said *superior highly composite* (shc) if there exists $\varepsilon > 0$ such that (cf. [24, Section 32] and Remark 3.8 below)

$$\frac{d(M)}{M^\varepsilon} \leqslant \frac{d(N)}{N^\varepsilon} \tag{3.3}$$

holds for all positive integer M. The number ε is called a *parameter* of the shc number N.

From the definition (3.3), note that two shc numbers N of parameter ε and N' of parameter ε' satisfy the following implication

$$N < N' \quad \Longrightarrow \quad \epsilon \geqslant \epsilon'. \tag{3.4}$$

It is convenient to order the elements of $\mathcal{E} \cup \{\infty\}$ defined in (3.1) in the decreasing sequence

$$\varepsilon_1 = \infty > \varepsilon_2 = 1 > \varepsilon_3 = \frac{\log 2}{\log 3} > \varepsilon_4 = \frac{\log 3/2}{\log 2} > \ldots > \varepsilon_i > \ldots \tag{3.5}$$

From Remark 3.4, there could exist two equal elements in the set $\mathcal{E}$ defined by (3.1), but not three. We call ε_i *extraordinary* if

$$\varepsilon_i = \frac{\log(1+1/k_i)}{\log q_i} = \frac{\log(1+1/k_i')}{\log q_i'} \tag{3.6}$$

with $k_i > k_i' \geqslant 1$ and $q_i < q_i'$. If ε_i is not extraordinary, it is said *ordinary* and satisfies in only one way

$$\varepsilon_i = \frac{\log(1+1/k_i)}{\log q_i}. \tag{3.7}$$

For $\varepsilon > 0$, let us introduce

$$N_\varepsilon = \prod_{p\in\mathcal{P}} p^{\lfloor 1/(p^\varepsilon - 1)\rfloor} \tag{3.8}$$

which will be proved shc of parameter ε. We observe that N_ε is a non-increasing function of ε. More precisely, if $\varepsilon \leqslant \varepsilon'$ then $N_{\varepsilon'}$ divides N_ε.

Lemma 3.6. *Let ε_{i-1} and ε_i be two consecutive elements of the sequence* (3.5) *and ε a number satisfting $\varepsilon_{i-1} \geqslant \varepsilon > \varepsilon_i$. Then, with the notation* (3.8), *we have*

$$N_\varepsilon = N_{\varepsilon_{i-1}}. \tag{3.9}$$

Proof : From (3.8), we have $N_\varepsilon = \prod_p p^{\lfloor 1/(p^\varepsilon - 1)\rfloor}$. As $\varepsilon \leqslant \varepsilon_{i-1}$ is assumed, it follows that $N_\varepsilon \geqslant N_{\varepsilon_{i-1}}$. Assume that $N_\varepsilon > N_{\varepsilon_i - 1}$. Then there exists a prime p such that $\lfloor 1/(p^\varepsilon - 1)\rfloor > \lfloor 1/(p^{\varepsilon_{i-1}} - 1)\rfloor$ thus, there exists an integer k such that $1/(p^\varepsilon - 1) \geqslant k > 1/(p^{\epsilon_{i-1}} - 1)$. We can write $k = 1/(p^\eta - 1)$, i.e. $\eta = \log(1+1/k)/\log p$, with $\varepsilon_{i-1} > \eta \geqslant \varepsilon > \varepsilon_i$, which is impossible since $\eta \in \mathcal{E}$. □

Proposition 3.7. *If ε_i (with $i \geqslant 2$) belongs to the sequence* (3.5) *and ε satisfies $\varepsilon_{i-1} > \varepsilon > \varepsilon_i$, there is only one shc number of parameter ε, namely $N_\varepsilon = N_{\varepsilon_{i-1}}$ defined by* (3.8).

If ε_i is ordinary and satisfies (3.7), *there are two shc numbers of parameter ε_i, namely N_{ε_i} and $N_{\varepsilon_{i-1}}$ satisfying*

$$N_{\varepsilon_i} = q_i N_{\varepsilon_{i-1}} \tag{3.10}$$

with q_i defined by (3.7).

If ε_i in (3.5) *is extraordinary of the form* (3.6), *there are four shc numbers of parameter ε_i, namely $N_{\varepsilon_{i-1}}$, $q_i N_{\varepsilon_{i-1}}$, $q_i' N_{\varepsilon_{i-1}}$, $N_{\varepsilon_i} = q_i q_i' N_{\varepsilon_{i-1}}$.*

In conclusion, if there is no extraordinary ε_i, any shc number is of the form N_{ε_i} (with $i \geqslant 1$). If it exists extraordinary ε_i's, for each of them, there *are two extra shc numbers* N_{ε_i}/q'_i *and* N_{ε_i}/q_i *which have only one parameter* ε_i*. In both cases, the set of parameters of* N_{ε_i} *is* $[\varepsilon_{i+1}, \varepsilon_i]$ *and two consecutive shc numbers* $N_{\varepsilon_{i-1}}, N_{\varepsilon_i}$ *have one and only one common parameter* ε_i*.*

Proof : For $\varepsilon > 0$, what is the maximum of $d(n)/n^\varepsilon$? Writing n under the form $\prod_p p^{\alpha_p}$ implies $d(n)/n^\varepsilon = \prod_p (\alpha_p + 1)/p^{\alpha_p \varepsilon}$ and, for each prime p, we have to maximize $(t+1)/p^{t\varepsilon}$ for t integer, i.e. to maximize $\varphi(t) = \log(t+1) - \varepsilon\, t \log p$.
• If $\varepsilon \notin \mathcal{E}$, then, for each prime p, $\varepsilon \log p \neq \log(1 + 1/k)$ for all integer k and, from Lemma 2.9 (ii), $\varphi(t)$ attains its maximum in only one point $\alpha_p = \left\lfloor 1/(e^{\varepsilon \log p} - 1) \right\rfloor = \lfloor 1/(p^\varepsilon - 1) \rfloor$ so that the maximum of $d(n)/n^\varepsilon$ is attained in N_ε defined by (3.8). Moreover, if $\varepsilon_{i-1} > \varepsilon > \varepsilon_i$ is assumed, then, by Lemma 3.6, N_ε is constant and equal to $N_{\varepsilon_{i-1}}$.
• If $\varepsilon = \varepsilon_i$ with ε_i an ordinary element (3.7) of the sequence (3.5) then, for $p \neq q_i$, $\varphi(t)$ attains its maximum in one integer $\alpha_p = \lfloor 1/(p^{\varepsilon_i} - 1) \rfloor$ while, if $p = q_i$, Lemma 2.9 (i) claims that $\varphi(t)$ attains its maximum in two integers $k_i = 1/(q_i^{\varepsilon_i} - 1)$ and $k_i - 1$. Therefore, $d(n)/n^{\varepsilon_i}$ attains its maximum in two numbers N_{ε_i} and$N_{\varepsilon_i}/q_i = N_{\varepsilon_{i-1}}$.
• If $\varepsilon = \varepsilon_i$ with ε_i an extraordinary element (3.6) of the sequence (3.5) then, for $p \neq q_i, q'_i$, $\varphi(t)$ attains its maximum in one integer $t = \lfloor 1/(p^{\varepsilon_i} - 1) \rfloor$ while, if $p = q_i$ or $p = q'_i$, Lemma 2.9 (i) claims that $\varphi(t)$ attains its maximum in two integers k_i, $k_i - 1$ or k'_i, $k'_i - 1$. Therefore, $d(n)/n^{\varepsilon_i}$ attains its maximum in four numbers N_{ε_i}, N_{ε_i}/q_i, N_{ε_i}/q'_i and $N_{\varepsilon_i}/(q_i q'_i) = N_{\varepsilon_{i-1}}$. □

Remark 3.8. Our definition 3.5 of shc numbers is slightly different of the definition given by Ramanujan in [24, Section 32]. Ramanujan calls shc of parameter ε a number N such that

$$\text{for } M < N, \quad \frac{d(M)}{M^\varepsilon} \leqslant \frac{d(N)}{N^\varepsilon} \quad \text{and for } M > N, \quad \frac{d(M)}{M^\varepsilon} < \frac{d(N)}{N^\varepsilon}. \quad (3.11)$$

Clearly, if N satisfies (3.11), it also satisfies definition 3.5.

If $\varepsilon \notin \mathcal{E}$, we have seen in the proof of Proposition 3.7 that the mapping $n \mapsto d(n)/n^\varepsilon$ has a unique maximum on say N, and thus, for $M \neq N$, $d(M)/M^\varepsilon < d(N)/N^\varepsilon$ so that N satisfies (3.11).

If ε_i is an ordinary number, the mapping $n \mapsto d(n)/n^\varepsilon$ attains its maximum on two numbers $N_{\varepsilon_{i-1}}$ and N_{ε_i}. Only N_{ε_i} satisfies (3.11) with $\varepsilon = \varepsilon_i$ and the set of parameters for which $N = N_{\varepsilon_i}$ satisfies (3.11) is $[\varepsilon_i, \varepsilon_{i-1})$.

If ε_i is an extraordinary number, from Proposition 3.7 with the same notation, there are four numbers maximizing $d(n)/n^{\varepsilon_i}$ and only the largest one $N = N_{\varepsilon_i}$ satisfies (3.11) with $\varepsilon = \varepsilon_i$. The three other ones $N_{\varepsilon_i}/(q_i q'_i), N_{\varepsilon_i}/q_i$ and N_{ε_i}/q'_i do not satisfies (3.11) with $\varepsilon = \varepsilon_i$. Since N_{ε_i}/q_i and N_{ε_i}/q'_i have only one parameter ε_i, they are not considered as shc by (3.11). But, as the existence of extraordinary numbers is highly unprobable, the difference between the two definitions of shc numbers does not matter so much.

Definition 3.9. Let N be a shc number satisfying $N_{\varepsilon_{i-1}} < N \leqslant N_{\varepsilon_i}$ (where ε_i is an element of the sequence (3.5) and N_ε is defined by (3.8)). From Proposition 3.7, N is either equal to N_{ε_i} or ε_i is extraordinary. In both cases, the largest parameter of N is ε_i. We define $\xi = \xi(N)$ by

$$\xi = \xi(N) = 2^{1/\varepsilon_i} \quad \text{i.e.} \quad \varepsilon_i = \frac{\log 2}{\log \xi}\ , \tag{3.12}$$

for $k \geqslant 1$, the numbers

$$\xi_k = \xi^{\beta_k} = 2^{\beta_k/\varepsilon_i}, \tag{3.13}$$

with β_k defined in (1.2), and

$$K = K(N) = \left\lfloor \frac{1}{2^{\varepsilon_i}-1} \right\rfloor < \frac{1}{\varepsilon_i \log 2} = \frac{\log \xi}{(\log 2)^2}. \tag{3.14}$$

We observe that $K+1 > 1/(2_i^{\varepsilon}-1)$ holds, so that, for $k \geqslant K+1$, $\log(1+1/k) \leqslant \log(1+1/(K+1)) < \varepsilon_i \log 2$. Therefore, for $k > K$, $\xi_k = \xi^{\beta_k} < \xi_i^{\varepsilon} = 2$.

Proposition 3.10. *Let N be a shc number and ε_i the element of (3.5) such that $N_{\varepsilon_{i-1}} < N \leqslant N_{\varepsilon_i}$. The numbers ξ, ξ_k and K are defined by Definition 3.9. Then*

$$\log N_{\varepsilon_i} - \log \xi = \sum_{k=1}^{K} \theta(\xi_k) - \log \xi \leqslant \log N \leqslant \log N_{\varepsilon_i} = \sum_{k=1}^{K} \theta(\xi_k) \tag{3.15}$$

and

$$\frac{\log d(N_{\varepsilon_i})}{\log 2} - 1 \leqslant \frac{\log d(N)}{\log 2} \leqslant \frac{\log d(N_{\varepsilon_i})}{\log 2} = \sum_{k=1}^{K} \beta_k \pi(\xi_k) \tag{3.16}$$

with ξ_k defined by (3.13), K by (3.14) and β_k by (1.2).

Proof : By observing that, for $k \geqslant 1$, $1/(p^{\varepsilon_i}-1) = k$ is equivalent to $\varepsilon_i = (\log(1+1/k))/\log p$ and also to $p = \xi_k$, from (3.8), it follows that

$$\log N_{\varepsilon_i} = \sum_{1 \leqslant k \leqslant K} \theta(\xi_k) \tag{3.17}$$

and that

$$\log d(N_{\varepsilon_i}) = (\log 2) \sum_{1 \leqslant k \leqslant K} \beta_k \pi(\xi_k). \tag{3.18}$$

• If ε_i is ordinary, from Proposition 3.7, $N_{\varepsilon_{i-1}}$ and N_{ε_i} are two consecutive shc numbers so that $N_{\varepsilon_{i-1}} < N \leqslant N_{\varepsilon_i}$ implies $N = N_{\varepsilon_i}$, so that (3.17) and (3.18) prove (3.15) and (3.16).

• If ε_i is extraordinary and given by (3.6), from Proposition 3.7, N is equal to N_{ε_i}, N_{ε_i}/q_i, or N_{ε_i}/q_i'. But q_i and q_i' divide N_{ε_i} and so are both $\leqslant \xi$, which proves (3.15). We also observe that $d(N_{\varepsilon_i})/d(N_{\varepsilon_i}/q_i) = (k_i+1)/k_i \leqslant 2$ and $d(N_{\varepsilon_i})/d(N_{\varepsilon_i}/q'_i) = (k'_i+1)/k'_i \leqslant 2$ which, from (3.18), proves (3.16). □

Proposition 3.11. *Let $n \geqslant 2$ be an integer. There exists two consecutive shc numbers $N' < N$ such that*

$$N/\xi \leqslant N' < n \leqslant N \quad \text{and} \quad d(n) \leqslant d(N) \leqslant 2d(N') \tag{3.19}$$

where $\xi = \xi(N)$ is defined in Definition 3.9.

Proof : First, we determine the element ε_i of the sequence (3.5) such that $N_{\varepsilon_i - 1} < n \leqslant N_{\varepsilon_i}$.

• If ε_i is ordinary and given by (3.7), from Proposition 3.7, we choose $N = N_{\varepsilon_i}$, $N' = N_{\varepsilon_{i-1}} = N_{\varepsilon_i}/q_i \geqslant N_{\varepsilon_i}/\xi$ and, from (3.3), $d(n) \leqslant d(N)(n/N)^{\varepsilon_i} \leqslant d(N)$ follows. We also have $d(N) = d(N'q_i) \leqslant 2d(N')$.

• If ε_i is extraordinary and given by (3.6), from Proposition 3.7, there are four consecutive shc numbers of parameter ε_i. We determine $(N', N]$ containing n among $(N_{\varepsilon_i}/(q_i q'_i), N_{\varepsilon_i}/q_i]$, $(N_{\varepsilon_i}/q_i, N_{\varepsilon_i}/q'_i]$, $(N_{\varepsilon_i}/q'_i, N_{\varepsilon_i}]$. As q_i and q'_i divide N_{ε_i} they are $\leqslant \xi$ and, from Defition 3.9, $\xi(N_{\varepsilon_i}/q_i) = \xi(N_{\varepsilon_i}/q'_i) = \xi(N_{\varepsilon_i}) = 2^{1/\varepsilon_i}$, it is easy to see that (3.19) is still satisfied. □

The first shc numbers are (for a longer table cf. [24, Section 37] or [32]):

i	ε_i	$N = N_{\varepsilon_i}$	d(N)	parameter	$\xi = \xi(N)$
1	∞	1	1	$[\varepsilon_2, \varepsilon_1)$	1
2	1	2	2	$[\varepsilon_3, \varepsilon_2]$	2
3	$\frac{\log 2}{\log 3} = 0.63$	$6 = 2 \cdot 3$	4	$[\varepsilon_4, \varepsilon_3]$	3
4	$\frac{\log 3/2}{\log 2} = 0.58$	$12 = 2^2 \cdot 3$	6	$[\varepsilon_5, \varepsilon_4]$	3.27
5	$\frac{\log 2}{\log 5} = 0.43$	$60 = 2^2 \cdot 3 \cdot 5$	12	$[\varepsilon_6, \varepsilon_5]$	5
6	$\frac{\log 4/3}{\log 2} = 0.41$	$120 = 2^3 \cdot 3 \cdot 5$	16	$[\varepsilon_7, \varepsilon_6]$	5.31
7	$\frac{\log 3/2}{\log 3} = 0.36$	$360 = 2^3 \cdot 3^2 \cdot 5$	24	$[\varepsilon_8, \varepsilon_7]$	6.54
8	$\frac{\log 2}{\log 7} = 0.35$	$2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$	48	$[\varepsilon_9, \varepsilon_8]$	7

(3.20)

3.3 Computation

How to compute shc numbers? For a short table, one determines the sequence ε_i (cf. (3.5)) and, if ε_i satisfies (3.7), then $N_{\varepsilon_i} = q_i N_{\varepsilon_{i-1}}$. In the proof of Lemma 4.3, we have to compute the shc numbers up to $N^{(0)}$. Let us say that N_{ε_i} is a shc number of type 2 if ε_i satisfies (3.7) with $k_i \geqslant 2$. We have precomputed the table of shc numbers of type 2 with $\xi_i = 2^{1/\varepsilon_i} < 2 \times 10^8$. If N is of type 2 with its largest prime factor equal to the r-th prime p_r, then the following shc numbers are Np_{r+1}, $Np_{r+1}p_{r+2}$, etc. up to the next shc number of type 2 (cf. [32]).

Note that we have not found any exceptionnal case (cf. Section 3.2). The smallest difference $\varepsilon_{i-1}-\varepsilon_i = 1.65\ldots\times 10^{-13}$ has been obtained with $\varepsilon_{i-1} = \log(3/2)/\log(62129)$ and $\varepsilon_i = \log(2)/\log(156383467)$.

By (3.8), we define $N^{(0)} = N_{\varepsilon_{(0)}}$ with $\varepsilon^{(0)} = \frac{\log 2}{\log \xi^{(0)}} \in \mathcal{E}$ and

$$\xi^{(0)} = 2^{1/\varepsilon^{(0)}} = \xi(N^{(0)}) = 10^8 + 7, \tag{3.21}$$

the smallest prime exceeding 10^8. For $k \geqslant 1$, we set $\xi_k^{(0)} = (\xi^{(0)})^{\beta_k}$. From (1.2), one has

$$\xi_2^{(0)} = 47829.9,\ \ \xi_3^{(0)} = 2090.7,\ \ \xi_4^{(0)} = 376.2,\ \ \xi_5^{(0)} = 127.1, \ldots,\ \ \xi_{37}^{(0)} = 2.03, \tag{3.22}$$

from (3.8),

$$N^{(0)} = N_{\varepsilon^{(0)}} = 2^{37}3^{23}5^{16}7^{13}11^{10}13^{9}17^{8}19^{8}\prod_{p=23}^{31}p^7\prod_{p=37}^{59}p^6\prod_{p=61}^{127}p^5 \prod_{p=131}^{373}p^4\prod_{p=379}^{2089}p^3\prod_{p=2099}^{47819}p^2\prod_{p=47837}^{100000007}p \tag{3.23}$$

from (3.14), $K = K(N^{(0)}) = 37$, $\log d(N^{(0)}) = 3995657.8341\ldots$

$$\log N^{(0)} = 100037943.8694\ldots \ \text{ and } \ \log\log N^{(0)} = 18.421060\ldots \tag{3.24}$$

Lemma 3.12. *Let N be a shc number $> N^{(0)} = N_{\varepsilon^{(0)}}$ and $\varepsilon_i, \xi, \xi_k$ and K defined in Definition 3.9. Then $\varepsilon_i \leqslant \varepsilon^{(0)}$, $\xi \geqslant \xi^{(0)}$, $\xi_k \geqslant \xi_k^{(0)}$ for $k \geqslant 1$, and $K = K(N) \geqslant 37$.*

Proof : By (3.15), $N \leqslant N_{\varepsilon_i}$ holds. Since $N^{(0)} < N$ is assumed, this implies $N^{(0)} < N_{\varepsilon_i}$ and, from (3.4), $\varepsilon_i \leqslant \varepsilon^{(0)}$. Moreover, one has $\xi = \xi(N) = 2^{1/\varepsilon_i} \geqslant 2^{1/\varepsilon^{(0)}} = \xi^{(0)} = 10^8 + 7$, $\xi_k = \xi^{\beta_k} \geqslant (\xi^{(0)})^{\beta_k} = \xi_k^{(0)}$ and $K = \lfloor 1/(2^{\varepsilon_i}-1)\rfloor \geqslant \left\lfloor 1/(2^{\varepsilon^{(0)}}-1)\right\rfloor = 37$. □

Lemma 3.13. *Let N be a shc number satisfying $N \geqslant N^{(0)}$ defined by (3.23); $\xi = \xi(N)$, ξ_k and K are defined by Definition 3.9, so that, from Lemma 3.12, $\xi \geqslant \xi^{(0)}$ and $K \geqslant 37$ hold. Then*

$$T_5 = \sum_{k=5}^{K}\frac{\xi_k}{\xi_3} \leqslant 0.1815 \quad \text{and} \quad T_3 = \sum_{k=3}^{K}\frac{\xi_k}{\xi_3} \leqslant 1.3615. \tag{3.25}$$

Proof : Let us fix $k_0 = 34$. As $\xi_k/\xi_3 = 1/\xi^{\beta_3-\beta_k}$ is decreasing in ξ for $k > 3$, one may write from (3.14)

$$T_5 \leqslant \sum_{k=5}^{k_0-1} \frac{\xi_k}{\xi_3} + \frac{K-k_0+1}{\xi^{\beta_3-\beta_{k_0}}} \leqslant \sum_{k=5}^{k_0-1} \frac{1}{\xi^{\beta_3-\beta_k}} + \frac{\log\xi - (k_0-1)\log^2 2}{(\log^2 2)\xi^{\beta_3-\beta_{k_0}}}.$$

By applying (1.18) with $u = 1$, $v = \beta_3 - \beta_{k_0} = 0.3732\ldots$ and $w = (k_0 - 1)\log^2 2 = 15.85\ldots$, we have

$$T_5 \leqslant \sum_{k=5}^{k_0-1} \frac{1}{\left(\xi^{(0)}\right)^{\beta_3-\beta_k}} + \frac{(u/v)^u \exp(-u-vw)}{\log^2 2} = 0.181497\ldots$$

which proves the upper bound for T_5. Furthermore, one has

$$T_3 = 1 + \frac{\xi_4}{\xi_3} + T_5 = 1 + \frac{1}{\xi^{\beta_3-\beta_4}} + T_5 \leqslant 1 + \frac{1}{\left(\xi^{(0)}\right)^{\beta_3-\beta_4}} + T_5 \leqslant 1.3615,$$

which completes the proof of Lemma 3.13. □

3.4 Highly composite numbers

A positive integer M is said *highly composite*, (for short *hc*) if $n < M$ implies $d(n) < d(M)$. This notion was introduced by Ramanujan in [24] and studied in [2, 11, 12, 15, 19, 21, 26].

Lep p_k denote the k-th prime and $\mathcal{N}_k$ the subset of $\mathbb{N}$ made of the numbers whose prime factors are $\leqslant p_k$. Let us call k-hc number (cf. [3]) an integer $M \in \mathcal{N}_k$ such that $n < M$ and $n \in \mathcal{N}_k$ imply $d(n) < d(M)$. A k-hc number (and, as well, a hc number) has the property that the p-adic valuation $v_p(M)$ is a non-increasing function of p. The 1-hc numbers are the powers of 2 and, by induction on k, it is easy to compute the set of k-hc numbers $\leqslant n_0$. If the product $p_1p_2\ldots p_k$ exceeds n_0, then this set is the set of hc numbers $\leqslant n_0$. We have computed the 1381 hc numbers $< 10^{100}$ and we shall refer to them as M_j, $1 \leqslant j \leqslant 1381$.

Lemma 3.14. *Let M_J and M_{J+1} be two consecutive hc numbers and f an increasing function on $[\log M_j, \log M_{j+1}]$ such that $(\log d(M_j))/\log 2 \leqslant f(\log M_j)$. Then, for $M_j \leqslant n < M_{j+1}$, $(\log d(n))/\log 2 \leqslant f(\log n)$ holds.*

Proof. From the defintion of hc numbers, $M_j \leqslant n < M_{j+1}$ implies $d(n) \leqslant d(M_j)$ so that $(\log d(n))/\log 2 \leqslant (\log d(M_j))/\log 2 \leqslant f(\log M_j) \leqslant f(\log n)$ holds, since f is assumed to be increasing.

3.5 Benefit

Definition 3.15. Let ε be a positive real number and N a shc number of parameter ε. For a positive integer n, we introduce the *benefit* of n

$$\mathrm{ben}_\varepsilon(n) = \log\left(\frac{d(N)}{d(n)}\right) + \varepsilon \log\left(\frac{n}{N}\right). \tag{3.26}$$

Note that that this notion depends only of ε but not on N. Indeed, if $\widetilde{N}$ is another shc number of parameter ε, (3.3) yields $d(N)/N^\varepsilon \leqslant d(\widetilde{N})/\widetilde{N}^\varepsilon$ and $d(\widetilde{N})/\widetilde{N}^\varepsilon \leqslant d(N)/N^\varepsilon$, so that $d(\widetilde{N})/\widetilde{N}^\varepsilon = d(N)/N^\varepsilon$, which implies $\log d(\widetilde{N}) - \varepsilon \log \widetilde{N} = \log d(N) - \varepsilon \log N$.

From (3.3), it follows that, for any n,

$$\mathrm{ben}_\varepsilon(n) \geqslant 0 \tag{3.27}$$

holds. Let us write $N = \prod_{p\in\mathcal{P}} p^{a_p}$ and $n = \prod_{p\in\mathcal{P}} p^{b_p}$. We define

$$\mathrm{Ben}_{p,\varepsilon}(n) = \log\left(\frac{a_p+1}{b_p+1}\right) + \varepsilon(b_p - a_p)\log p = \mathrm{ben}_\varepsilon(Np^{b_p-a_p}) \geqslant 0 \tag{3.28}$$

so that, from (3.26),

$$\mathrm{ben}_\varepsilon(n) = \sum_{p\in\mathcal{P}} \mathrm{Ben}_{p,\varepsilon}(n). \tag{3.29}$$

This notion of benefit has been used in [15, 16, 19, 21, 17, 12] for theoretical results on numbers having many divisors.

For $t \geqslant 0$, the mapping $t \mapsto \log((a_p+1)/(t+1)) + \varepsilon(t-a_p)\log p$ is convex, vanishes for $t = a_p$, and, from (3.28), is non-negative for t integer. Therefore, $Ben_{p,\varepsilon}$ defined in (3.28), is non-increasing in b_p for $0 \leqslant b_p \leqslant a_p$ and, for $b_p \geqslant a_p$, is non-decreasing and tends to infinity with b_p. Consequently, formulas (3.28) and (3.29) yield an algorithm to compute all integers n such that $\mathrm{ben}_\varepsilon(n) \leqslant B$ for a B not too large. With B close to $\varepsilon/3$, this algorithm has been used in [26] to compute the hc numbers between two consecutive shc numbers of common parameter ε.

Lemma 3.16. *Let N be a shc number of parameter ε, M_j and M_{j+1} two consecutive hc numbers and $f : [\log M_J, \log M_{j+1}] \to \mathbb{R}$ a continuous increasing function such that $f(\log M_j) < (\log d(M_j))/\log 2 < f(\log M_{j+1})$. Let us denote by $\mu \in (M_j, M_{j+1})$ the number satisfying $f(\log\mu) = (\log d(M_j))/\log 2$. If $n \in (M_j, M_{j+1})$ is any integer such that $f(n) \leqslant (\log d(n))/\log 2$, then one has*

$$\mathrm{ben}_\varepsilon(n) \leqslant \log d(N) - (\log 2) f(\log M_j) + \varepsilon(\log\mu - \log N). \tag{3.30}$$

Proof : From the defnition 3.15 of hc numbers, $M_j < n < M_{j+1}$ implies $d(n) \leqslant d(M_j)$ so that, as f is assumed to be increasing, $f(\log M_j) < f(\log n) \leqslant (\log d(n))/\log 2 \leqslant (\log d(M_j))/\log 2 = f(\log \mu)$ implies $n \leqslant \mu$ and $\log d(n) \geqslant (\log 2) f(\log M_j)$. Therefore, (3.30) follows from (3.26). □

3.6 Convexity

Lemma 3.17. *Let N' and N be two consecutive shc numbers and f a concave function on the interval $[\log N, \log N']$ such that*

$$\log d(N) \leqslant f(\log N) \quad \textit{and} \quad \log d(N') \leqslant f(\log N'). \tag{3.31}$$

Let n be an integer satisfying $N' \leqslant n \leqslant N$. Then we have

$$\log d(n) \leqslant f(\log n).$$

Proof : From Proposition 3.7, N and N' share a common parameter, say ε. From the definition (3.3) of shc numbers, one deduces that $\log d(N) - \varepsilon \log N = \log d(N') - \varepsilon \log N'$. For $n \in (N', N)$, from (3.3), one has

$$\log d(n) - \varepsilon \log n \leqslant \log d(N) - \varepsilon \log N = \log d(N') - \varepsilon \log N'. \tag{3.32}$$

In view of using a convexity argument, one writes

$$\log n = \lambda \log N + \mu \log N' \quad \text{with} \quad 0 \leqslant \lambda \leqslant 1 \quad \text{and} \quad \mu = 1 - \lambda.$$

From (3.32), it follows

$$\begin{aligned} \log d(n) &\leqslant \varepsilon \log n + \lambda(\log d(N) - \varepsilon \log N) + \mu(\log d(N') - \varepsilon \log N') \\ &= \varepsilon(\lambda \log N + \mu \log N') + \lambda \log d(N) + \mu \log d(N') \\ &\qquad - \lambda\varepsilon \log N - \mu\varepsilon \log N' = \lambda \log d(N) + \mu \log d(N'). \end{aligned}$$

From (3.31) and from the concavity of f, the above result implies

$$\log d(n) \leqslant \lambda f(\log N) + \mu f(\log N') \leqslant f(\lambda \log N + \mu \log N') = f(\log n),$$

which completes the proof of Lemma 3.17. □

3.7 Estimates of shc numbers without any hypothesis

To get shorter formulas, we use the notation $L = \log N$, $\lambda = \log\log N$, $L_0 = \log N^{(0)}$, and $\lambda_0 = \log\log N^{(0)}$ (cf. (3.24)).

Lemma 3.18. *Let N be a shc number $> N^{(0)} = N_{\varepsilon^{(0)}}$ (cf. (3.23)) with ε_i, ξ, ξ_k and K defined in Definition 3.9. Then*

$$\sum_{1\leqslant k\leqslant 4} \theta(\xi_k) \leqslant L = \log N \leqslant \theta(\xi) + \theta(\xi_2) + 1.362\ \xi_3, \tag{3.33}$$

$$0.99949\ L \leqslant \xi \leqslant 1.00017\ L, \tag{3.34}$$

$$0.99997\ \lambda \leqslant \log \xi \leqslant 1.00001\ \lambda, \tag{3.35}$$

$$\pi(\xi)+\beta_2\pi(\xi_2)+\frac{0.9997\ L^{\beta_3}}{\lambda} \leqslant \frac{\log d(N)}{\log 2} \leqslant \pi(\xi)+\beta_2\pi(\xi_2)+\frac{1.604\ L^{\beta_3}}{\lambda}, \tag{3.36}$$

$$\frac{\sqrt{\xi}\log^2 \xi}{8\pi} \leqslant 0.0398\sqrt{L}\,\lambda^2 \tag{3.37}$$

and

$$\frac{\sqrt{\xi_2}\log^2 \xi_2}{8\pi} \leqslant 0.0137\, L^{\beta_2/2}\lambda^2, \tag{3.38}$$

Proof : Since $N > N^{(0)} = N_{\varepsilon^{(0)}}$ is assumed, Lemma 3.12 implies $K = K(N) \geqslant 37$, $\varepsilon_i \leqslant \varepsilon^{(0)}$, $\xi \geqslant \xi^{(0)} = 10^8 + 7$, and, from (3.22), $\xi_5 \geqslant \xi_5^{(0)} > 127$, $\xi_4 \geqslant \xi_4^{(0)} > 376$ and $\xi_3 \geqslant \xi_3^{(0)} > 2090$.

From (3.15), we have

$$L = \log N \geqslant \sum_{k=1}^{K} \theta(\xi_k) - \log \xi \geqslant \sum_{k=1}^{4} \theta(\xi_k) + (\theta(\xi_5) - \log \xi). \tag{3.39}$$

From (2.4), $\theta(\xi_5) \geqslant 0.8499\,\xi_5$ holds. As $\log \xi = (\log \xi_5)/\beta_5 = 3.80\ldots \times \log \xi_5$, it follows that $\theta(\xi_5) - \log \xi \geqslant 0.8499\,\xi_5 - 3.81 \log \xi_5$. But the function $t \mapsto 0.8499\,t - 3.81 \log t$ is increasing for $t \geqslant 4.49$ and positive for $t > 11$. Therefore, the lower bound of (3.33) follows from (3.39).

From (3.15) we also have

$$L \leqslant \log N_{\varepsilon_i} = \sum_{k=1}^{K} \theta(\xi_k) = \theta(\xi) + \theta(\xi_2) + \Sigma \quad \text{with} \quad \Sigma = \sum_{k=3}^{K} \theta(\xi_k).$$

From (2.2) and (3.25), one gets

$$\Sigma \leqslant (1+\eta) \sum_{k=3}^{K} \xi_k = (1+\eta)\, T_3\, \xi_3 \leqslant 1.00000075 \times 1.3615\, \xi_3 \leqslant 1.362\, \xi_3,$$

which proves the upper bound of (3.33).

To prove the upper bound of (3.34), from (3.33) and (2.3), one writes

$$L \geqslant \theta(\xi) \geqslant \xi \left(1 - \frac{1}{\log^3 \xi}\right) \geqslant \xi \left(1 - \frac{1}{\log^3 \xi^{(0)}}\right) = \frac{\xi}{1.00016001\ldots}$$

From (3.33) and (2.2), it follows that

$$\begin{aligned} L &\leqslant (1+\eta)(\xi+\xi_2)+1.362\,\xi_3 = \xi\left((1+\eta)(1+\xi^{\beta_2-1})+1.362\,\xi^{\beta_3-1}\right)\\ &\leqslant \xi\left((1+\eta)(1+(\xi_2^{(0)})^{\beta_2-1})+1.362\,(\xi^{(0)})^{\beta_3-1}\right) = \xi/0.99949273\ldots, \end{aligned}$$

which proves the lower bound of (3.34).

From (3.34), we deduce $\log\xi \leqslant \lambda(1+0.00017/\lambda) \leqslant \lambda(1+0.00017/\lambda_0) = 1.0000092\ldots\,\lambda$ which proves the upper bound of (3.35). Similarly, we have $\log\xi \geqslant \lambda(1+\log(0.99949)/\lambda_0) = 0.9999723\ldots\,\lambda$ which completes the proof of (3.35).

From (3.16) and (3.18), one has

$$\frac{\log d(N)}{\log 2} \geqslant \frac{\log d(N_{\varepsilon_i})}{\log 2} - 1 \geqslant \sum_{k=1}^{4}\beta_k\pi(\xi_k) - 1.$$

But, from (3.22), one gets

$$\beta_4\pi(\xi_4) \geqslant \beta_4\pi(\xi_4^{(0)}) \geqslant \beta_4\pi(376) = 74\,\beta_4 > 1$$

which implies

$$\frac{\log d(N)}{\log 2} \geqslant \sum_{k=1}^{3}\beta_k\pi(\xi_k).$$

Then, as $\xi_3 \geqslant \xi_3^{(0)} > 2090$ holds, we may apply (2.5) to get, with (3.34) and (3.35), $\beta_3\pi(\xi_3) \geqslant \beta_3\xi_3/\log\xi_3 = \xi_3/\log\xi \geqslant (0.99949\,L)^{\beta_3}/(1.00001\,\lambda) \geqslant 0.9997\,L^{\beta_3}/\lambda$ which proves the lower bound of (3.36).

From (3.16), one has

$$\frac{\log d(N)}{\log 2} \leqslant \pi(\xi)+\beta_2\pi(\xi_2)+\Sigma' \quad \text{with} \quad \Sigma' = \sum_{k=3}^{K}\beta_k\pi(\xi_k). \tag{3.40}$$

From (2.6), (3.22), (3.25), (3.34) and (3.35), one gets

$$\begin{aligned} \Sigma' &\leqslant 1.15963\,\beta_3\,\frac{\xi_3}{\log\xi_3} + 1.19768\,\beta_4\,\frac{\xi_4}{\log\xi_4} + 1.25506\sum_{k=5}^{K}\beta_k\frac{\xi_k}{\log\xi_k}\\ &= \frac{\xi_3}{\log\xi}\left(1.15963+1.19768\,\frac{\xi_4}{\xi_3}+1.25506\,T_5\right)\\ &\leqslant \frac{1.00017^{\beta_3}}{0.99997}\left(1.15963+1.19768\,\frac{\xi_4^{(0)}}{\xi_3^{(0)}}+125506\times 0.1815\right)\frac{L^{be_3}}{\lambda}\\ &= 1.60309\ldots\frac{L^{be_3}}{\lambda}, \end{aligned}$$

which, with (3.40), proves the upper bound of (3.36).

From (3.34) and (3.35), one has

$$\frac{\sqrt{\xi}\log^2\xi}{8\pi} \leqslant \frac{\sqrt{1.00017}\,(1.00001)^2}{8\pi}\sqrt{L}\,\lambda^2 = 0.039792\ldots\sqrt{L}\,\lambda^2$$

which proves (3.37).

From (3.34) and (3.35), one may write

$$\frac{\sqrt{\xi_2}\log^2\xi_2}{8\pi} \leqslant \frac{(1.00017\,L)^{\beta_2/2}}{8\pi}(1.00001\,\beta_2\lambda)^2 = 0.0136159\ldots L^{\beta_2/2}\lambda^2$$

which proves (3.38) and completes the proof of Lemma 3.18. □

Lemma 3.19. *Let N be a shc number tending to infinity. There is a positive number α such that*

$$\frac{\log d(N)}{\log 2} = \mathrm{li}(\log N) + \mathcal{O}((\log N)e^{-\alpha\sqrt{\log\log N}}). \qquad (3.41)$$

Let n be a number tending to infinity. One has

$$\frac{\log d(n)}{\log 2} \leqslant \mathrm{li}(\log n) + \mathcal{O}((\log n)e^{-\alpha\sqrt{\log\log n}}). \qquad (3.42)$$

Proof : These results are due to Ramanujan, cf. [24, Section 1 and Section 39]. Equation (3.41) follows from (3.36), (3.33), (2.16) and from the prime number theorem under the forms $\theta(x) = x + \mathcal{O}(xe^{-\alpha\sqrt{\log x}})$ and $\pi(x) = \mathrm{li}(x) + \mathcal{O}(xe^{-\alpha\sqrt{\log x}})$.

By observing that the function $t \mapsto \mathrm{li}(t) + Ate^{-\alpha\sqrt{\log t}}$ is concave for t large enough, one deduces (3.42) from (3.41) and from Lemma 3.17. □

3.8 Estimates of shc numbers under the Riemann hypothesis

Lemma 3.20. *Let N be a shc number $> N^{(0)}$ (defined by (3.23)) with ε_i, ξ and ξ_k defined in Definition 3.9. Then, under the Riemann hypothesis, with β_2 and β_3 defined by (1.2), we have*

$$-2.92\,L^{\beta_2} \leqslant \xi - L \leqslant 0.0266\,\lambda^2\sqrt{L}, \qquad (3.43)$$

$$-1.71\,L^{2\beta_2-1} \leqslant \xi_2 - L^{\beta_2} \leqslant 0.0156\,L^{\beta_2-1/2}\,\lambda^2, \qquad (3.44)$$

$$-0.0143\,L^{\beta_2/2}\lambda^2 \leqslant \theta(\xi_2) - L^{\beta_2} \leqslant 0.0141\,L^{\beta_2/2}\,\lambda^2, \qquad (3.45)$$

$$-0.0245\, L^{\beta_2/2}\, \lambda \leqslant \mathrm{li}(\theta(\xi_2)) - \mathrm{li}(L^{\beta_2}) \leqslant 0.0242\, L^{\beta_2/2}\, \lambda, \tag{3.46}$$

$$\textit{for } 2 \leqslant k \leqslant 3, \quad \frac{\sqrt{L}}{\lambda^k} - 1.461 \frac{L^{\beta_2 - 1/2}}{\lambda^k} \leqslant \frac{\sqrt{\xi}}{\log^k \xi} \leqslant \frac{\sqrt{L}}{\lambda^k} + 0.02, \tag{3.47}$$

$$\theta(\xi) + L^{\beta_2} + 0.61\, L^{\beta_3} \leqslant L \leqslant \theta(\xi) + L^{\beta_2} + 1.872\, L^{\beta_3}, \tag{3.48}$$

$$\mathrm{li}(L) - \frac{L^{\beta_2}}{\lambda} - 1.873 \frac{L^{\beta_3}}{\lambda} \leqslant \mathrm{li}(\theta(\xi)) \leqslant \mathrm{li}(L) - \frac{L^{\beta_2}}{\lambda} - \frac{0.61\, L^{\beta_3}}{\lambda}. \tag{3.49}$$

and

$$\left| \frac{S(\xi)}{\log^2 \xi} - \frac{S(L)}{\lambda^2} \right| \leqslant 0.12\, L^{\beta_2 - 1/2}. \tag{3.50}$$

Proof : As β_2 exceeds $1/2$, from (3.34) and (2.8), it follows that, for N large enough, $L = \log N > \xi(N)$ holds. But, ξ^{β_2} is smaller than $\sqrt{\xi}\log^2 \xi/(8\pi)$ for $\xi < 10^{24}$ and we have not been able to replace the upper bound of (3.43) by 0.

From Lemma 3.12, $\xi \geqslant \xi^{(0)}$ holds and, from (3.21), (3.33), (2.8) and (3.37), one has

$$\begin{aligned} L \geqslant \theta(\xi) + \theta(\xi_2) \geqslant \xi - \frac{\sqrt{\xi}\log^2 \xi}{8\pi} + \theta(\xi_2) \\ \geqslant \xi - 0.0398\sqrt{L}\lambda^2 + \theta(\xi_2). \end{aligned} \tag{3.51}$$

But $\xi_2 \geqslant \xi_2^{(0)}$ holds, and from (3.22), (2.4), and (3.34), we get

$$\begin{aligned} \theta(\xi_2) \geqslant 0.9927\, \xi_2 \geqslant 0.9927 \times (0.99949\, L)^{\beta_2} \geqslant 0.9924\, L^{\beta_2} \\ = 0.9924\sqrt{L}\lambda^2 \left(\frac{L^{\beta_2 - 1/2}}{\lambda^2} \right). \end{aligned} \tag{3.52}$$

From (1.19), the above parenthesis is $\geqslant (\beta_2 - 1/2)^2 e^2/4 \geqslant 0.013334$, which, from (3.51), shows that

$$L - \xi \geqslant (-0.0398 + 0.9924 \times 0.013334)\sqrt{L}\lambda^2 = -0.0265673\ldots \sqrt{L}\lambda^2$$

and yields the upper bound of (3.43).

Furthermore, one writes

$$\begin{aligned}
L &\leqslant \theta(\xi) + \theta(\xi_2) + 1.362\, \xi_3 && \text{from (3.33)} \\
&\leqslant \xi + \delta(\xi)\sqrt{\xi}(\log^2 \xi)/(8\pi) && \text{from (2.10)} \\
&\quad + \xi_2(1 + \eta + 1.362\, \xi_3/\xi_2) && \text{from (2.2)} \\
&\leqslant \xi + 0.0398\, \delta(\xi)\sqrt{L}\, \lambda^2 && \text{from (3.37)} \\
&\quad + (1.00017\, L)^{\beta_2}(1 + \eta + 1.362\, \xi_3^{(0)}/\xi_2^{(0)}) && \text{from (3.34)} \\
&= \xi + \big(0.0398\, \delta(\xi)\lambda^2/L^{\beta_2 - 1/2} + 1.0596\ldots\big)\, L^{\beta_2}.
\end{aligned}$$

If $\xi \leqslant 10^{19}$ then (2.9) yields $\delta(\xi) = 0$ so that

$$L \leqslant \xi + 1.06\, L^{\beta_2}. \tag{3.53}$$

If $\xi > 10^{19}$ then (2.9) implies $\delta(\xi) \leqslant 1$ and, from (3.34), $L \geqslant 10^{19}/1.00017$. Thus, as the function $t \mapsto (\log^2 t)/t^{\beta_2 - 1/2}$ is decreasing for $t \geqslant 2 \times 10^{10}$,

$$L - \xi \leqslant L^{\beta_2}\left(1.06 + \frac{0.0398\, \log^2(10^{19}/1.00017)}{(10^{19}/1.00017)^{\beta_2 - 1/2}}\right) = 2.911556\ldots L^{\beta_2}$$

which, together with (3.53), proves the lower bound of (3.43).

For $t \geqslant 0$, from the concavity of $t \mapsto (1+t)^{\beta_2}$ for $0 \leqslant t \leqslant 1$, one has $(1+t)^{\beta_2} \leqslant 1 + \beta_2 t$, which, with (3.13) and (3.43), yields

$$\xi_2 = \xi^{\beta_2} \leqslant L^{\beta_2}\left(1 + \frac{0.0266\,\lambda^2}{\sqrt{L}}\right)^{\beta_2} \leqslant L^{\beta_2}\left(1 + \frac{0.0266\,\beta_2\lambda^2}{\sqrt{L}}\right),$$

which proves the upper bound of (3.44) since $0.0266\,\beta_2 = 0.015560002\ldots$

Let

$$h = \frac{2.92}{L^{1-\beta_2}} \leqslant h_0 = \frac{2.92}{L_0^{1-\beta_2}} = 0.00139\ldots \tag{3.54}$$

and

$$b = \frac{1 - (1 - h_0)^{\beta_2}}{h_0} \leqslant 0.5852.$$

From (3.43), it follows that $\xi \geqslant L(1-h)$ holds. From the concavity of $t \mapsto (1-t)^{\beta_2}$ for $0 \leqslant t \leqslant 1$, one has

$$\begin{aligned} \xi_2 &\geqslant L^{\beta_2}(1-h)^{\beta_2} \geqslant L^{\beta_2}(1 - bh) \\ &\geqslant L^{\beta_2} - 0.5852 \times 2.92\, L^{2\beta_2 - 1} = L^{\beta_2} - 1.708784\, L^{2\beta_2 - 1}, \end{aligned}$$

which proves the lower bound of (3.44).

From (2.8) and (3.38), one deduces

$$-0.0137 L^{\beta_2/2}\lambda^2 \leqslant \theta(\xi_2) - \xi_2 \leqslant 0.0137\, L^{\beta_2/2}\lambda^2. \tag{3.55}$$

With the lower bound of (3.44), one gets

$$\begin{aligned} \theta(\xi_2) - L^{\beta_2} &\geqslant -1.71\, L^{2\beta_2 - 1} - 0.0137\, L^{\beta_2/2}\lambda^2 \\ &= -L^{\beta_2/2}\lambda^2\left(0.0137 + \frac{1.71}{L^{1-3\beta_2/2}\,\lambda^2}\right) \\ &\geqslant -L^{\beta_2/2}\,\lambda^2\left(0.0137 + \frac{1.71}{L_0^{1-3\beta_2/2}\,\lambda_0^2}\right) = -0.0142271\ldots\ L^{\beta_2/2}\,\lambda^2 \end{aligned}$$

which proves the lower bound of (3.45).

To find an upper bound of (3.45), from (3.55) and (3.44), one gets

$$\theta(\xi_2) - L^{\beta_2} \leqslant L^{\beta_2/2}\lambda^2\, U \tag{3.56}$$

with

$$U = 0.0137 + \frac{0.0156}{L^{1/2-\beta_2/2}} \leqslant 0.0137 + \frac{0.0156}{L_0^{1/2-\beta_2/2}} = 0.0140411\ldots,$$

which, from (3.56), proves the upper bound of (3.45).

From the upper bound of (3.45) and (2.16), one gets

$$\mathrm{li}(\theta(\xi_2)) \leqslant \mathrm{li}(L^{\beta_2} + 0.0141\, L^{\beta_2}\lambda^2) \leqslant \mathrm{li}(L^{\beta_2}) + \frac{0.0141\, L^{\beta_2/2}\lambda^2}{\beta_2\lambda}$$

which proves the upper bound of (3.46) since $0.0141/\beta_2 = 0.0241041\ldots$

From (3.52), one has $\theta(\xi_2) \geqslant 0.9924\, L^{\beta_2} \geqslant L^{\beta_2}/1.008$ and (3.24) yields

$$\log\theta(\xi_2) \geqslant (\beta_2 - \log 1.008/\lambda)\lambda \geqslant (\beta_2 - \log 1.008/\lambda_0)\lambda \geqslant 0.5845\,\lambda. \tag{3.57}$$

If $\theta(\xi_2) > L^{\beta_2}$ then $L(\theta(\xi_2)) - \mathrm{li}(L^{\beta_2})$ is positive and the lower bound of (3.46) clearly holds. If $\theta(\xi_2) \leqslant L^{\beta_2}$ then (2.13), (3.57) and (3.45) give

$$\mathrm{li}(L^{\beta_2}) - \mathrm{li}(\theta(\xi_2)) = \int_{\theta(\xi_2)}^{L^{\beta_2}} \frac{dt}{\log t} \leqslant \frac{L^{\beta_2} - \theta(\xi_2)}{\log\theta(\xi_2)} \leqslant \frac{0.0143\, L^{\beta_2/2}\,\lambda^2}{0.5845\,\lambda}$$

which, from (3.45), proves the lower bound of (3.46) since $0.0143/0.5845 = 0.0244653\ldots$.

From (3.43), one has $\xi \leqslant L(1+0.0266\,\lambda^2/\sqrt{L})$ and, from the increasingness of the function $t \mapsto \sqrt{t}/\log^k t$ for $t \geqslant 10^8$ and $2 \leqslant k \leqslant 3$, one gets

$$\frac{\sqrt{\xi}}{\log^k\xi} \leqslant \frac{\sqrt{L}\left(1 + 0.0266\,\lambda^2/\sqrt{L}\right)^{1/2}}{\left(\lambda + \log\left(1 + 0.0266\,\lambda^2/\sqrt{L}\right)\right)^k} \leqslant \frac{\sqrt{L}}{\lambda^k}\left(1 + \frac{0.0266\,\lambda^2}{\sqrt{L}}\right)^{1/2}.$$

By using the inequality $\sqrt{1+t} \leqslant 1 + t/2$ valid for $t \geqslant -1$, one has

$$\frac{\sqrt{\xi}}{\log^k\xi} \leqslant \frac{\sqrt{L}}{\lambda^k}\left(1 + \frac{0.0133\,\lambda^2}{\sqrt{L}}\right) = \frac{\sqrt{L}}{\lambda^k} + \frac{0.0133}{\lambda^{k-2}} \leqslant \frac{\sqrt{L}}{\lambda^k} + 0.0133,$$

which proves the upper bound of (3.47).

To prove the lower bound, from (3.43) and (3.54), we write $\xi \geqslant L(1-h)$ so that we have

$$\frac{\sqrt{\xi}}{\log^k\xi} \geqslant \frac{\sqrt{L}\sqrt{1-h}}{(\lambda + \log(1-h))^k} \geqslant \frac{\sqrt{L}\sqrt{1-h}}{\lambda^k}.$$

From (3.54), $h \leqslant h_0$ holds and, by setting $b' = (1-(1-h_0)^{1/2})h_0 \leqslant 0.5002$, the concavity of $t \mapsto \sqrt{1-t}$ yields

$$\frac{\sqrt{\xi}}{\log^k \xi} \geqslant \frac{\sqrt{L}}{\lambda^k}(1-b'h) \geqslant \frac{\sqrt{L}}{\lambda^k} - (0.5002 \times 2.92)\,\frac{L^{\beta_2-1/2}}{\lambda^k} = \frac{\sqrt{L}}{\lambda^k} - 1.460584\frac{L^{\beta_2-1/2}}{\lambda^k},$$

which completes the proof of (3.47).

From (3.22), one has $\xi_3 \geqslant \xi_3^{(0)} > 2090$, $\xi_4 \geqslant \xi_4^{(0)} > 376$, from (2.4), $\theta(\xi_3) \geqslant 0.9629\,\xi_3$, $\theta(\xi_4) \geqslant 0.9134\,\xi_4$ and, from (3.13) and (3.34),

$$\theta(\xi_3) \geqslant 0.9629 \times (0.99949\,L)^{\beta_3} \geqslant 0.9626\,L^{\beta_3}$$

and

$$\theta(\xi_4) \geqslant 0.9134 \times (0.99949\,L)^{\beta_4} \geqslant 0.9132\,L^{\beta_4}.$$

Therefore, from (3.33) and (3.45), one gets

$$L \geqslant \sum_{i=1}^{4} \theta(\xi_i) \geqslant \theta(\xi) + L^{\beta_2} - 0.0143\,(L^{\beta_2/2})\lambda^2 + 0.9626\,L^{\beta_3} + 0.9132\,L^{\beta_4} = \theta(\xi) + L^{\beta_2} + 0.6106\,L^{\beta_3} + L^{\beta_2/2}\,\Phi(\lambda)$$

with Φ defined in (2.31). From Lemma 2.10, $\Phi(\lambda)$ is positive, which proves the lower bound of (3.48).

From successively (3.33), (3.45) and (3.34), one has

$$L - \theta(\xi) \leqslant \theta(\xi_2) + 1.362\,\xi_3 \leqslant L^{\beta_2} + 0.0141\,L^{\beta_2/2}\lambda^2 + 1.362 \times 1.00017^{\beta_3} L^{\beta_3} \leqslant L^{\beta_2} + L^{\beta_3}\left(1.363 + \frac{0.0141\,\lambda^2}{L^{\beta_3-\beta_2/2}}\right).$$

But, from (1.19), $\lambda^2/L^{\beta_3-\beta_2/2} \leqslant 4e^{-2}/(\beta_3-\beta_2/2)^2 \leqslant 36.05$ and $1.363 + 0.0141 \times 36.05 = 1.871305$, which completes the proof of (3.48).

From (3.48), we have $\theta(\xi) \leqslant L - L^{\beta_2} - 0.61\,L^{\beta_3}$ and the upper bound of (3.49) follows from (2.16).

To prove the lower bound of (3.49), we set $h = L^{\beta_2} + 1.872\,L^{\beta_3}$. One has

$$h = L^{\beta_2}\left(1 + \frac{1.872}{L^{\beta_2-\beta_3}}\right) \leqslant L^{\beta_2}\left(1 + \frac{1.872}{L_0^{\beta_2-\beta_3}}\right) \leqslant 1.082\,L^{\beta_2} \tag{3.58}$$

and

$$L-h = L\left(1-\frac{1}{L^{1-\beta_2}}-\frac{1.872}{L^{1-\beta_3}}\right) \geqslant L\left(1-\frac{1}{L_0^{1-\beta_2}}-\frac{1.872}{L_0^{1-\beta_3}}\right) = \frac{L}{1.000517\ldots}.$$

Let us set $c = L/1.0006$ so that $L-h \geqslant c$ holds. We have

$$c\log^2 c = \frac{L\lambda^2}{1.0006}\left(1-\frac{\log 1.0006}{\lambda}\right)^2 \geqslant \frac{L\lambda^2}{1.0006}\left(1-\frac{0.0006}{\lambda_0}\right)^2 \geqslant \frac{L\lambda^2}{1.0007}$$

and, from (3.58),

$$\begin{aligned}\frac{h^2}{2c\log^2 c} &\leqslant \frac{1.0007\times(1.082)^2 L^{2\beta_2}}{2\,L\,\lambda^2}. \leqslant \frac{0.5858\,L^{2\beta_2}}{L\,\lambda^2}\\ &= \left(\frac{0.5858}{\lambda\,L^{\beta_3-2\beta_2+1}}\right)\frac{L^{\beta_3}}{\lambda} \leqslant \left(\frac{0.5858}{\lambda_0\,L_0^{\beta_3-2\beta_2+1}}\right)\frac{L^{\beta_3}}{\lambda} \leqslant \frac{0.00035\,L^{\beta_3}}{\lambda}.\end{aligned}$$

Now, from (3.48) and (2.16), we deduce $\theta(\xi) \geqslant L-h$ and

$$\begin{aligned}\mathrm{li}(\theta(\xi)) &\geqslant \mathrm{li}(L-h) \geqslant \mathrm{li}(L) - \frac{h}{\lambda} - \frac{h^2}{2c\log^2 c}\\ &\geqslant \mathrm{li}(L) - \frac{h}{\lambda} - \frac{0.00035\,L^{\beta_3}}{\lambda} = \mathrm{li}(L) - L^{\beta_2} - \frac{1.87235\,L^{\beta_3}}{\lambda},\end{aligned}$$

which proves the lower bound of (3.49).

Lemma 3.12 implies $\xi \geqslant \xi^{(0)}$. As $N > N^{(0)}$ is assumed, $L > L^{(0)}$ follows, so that, from (3.21) and (3.24), $\min(\xi, L) \geqslant \min(\xi^{(0)}, L_0) = \xi^{(0)} = 10^8+7$. Moreover, (3.34) yields $\min(\xi, L) \geqslant 0.99949\,L$. Applying (2.23) gives

$$\left|\frac{S(\xi)}{\log^2\xi} - \frac{S(L)}{\lambda^2}\right| \leqslant 0.04\frac{|\xi-L|}{\sqrt{0.99949\,L}} \leqslant 0.041\frac{|\xi-L|}{\sqrt{L}}. \tag{3.59}$$

By (1.19), $\lambda^2/L^{\beta_2-1/2} \leqslant 4e^{-2}/(\beta_2-1/2)^2 \leqslant 75$ holds, which from (3.43) implies

$$-2.92\,L^{\beta_2} \leqslant \xi - L \leqslant 0.0266\,\lambda^2\sqrt{L} \leqslant 0.0266\times 75\,L^{\beta_2} = 1.995\,L^{\beta_2}$$

so that $|\xi - L| \leqslant 2.92\ L^{\beta_2}$ holds. As $0.041\times 2.92 = 0.11972$, with (3.59), this completes the proof of (3.50) and of Lemma 3.20. □

4 Proof of Theorem 1.1

4.1 N is shc

As in Sections 3.7 and 3.8, we use the notation $L = \log N$, $\lambda = \log\log N$, $L_0 = \log N^{(0)}$, and $\lambda_0 = \log\log N^{(0)}$ (cf. (3.24)).

Proposition 4.1. *Let N be a shc number, F be defined by* (1.7)*, R by* (1.3) *and β_k by* (1.2)*. Then, under the Riemann hypothesis, for $N \geqslant N^{(0)}$ defined by* (3.23)*, we have*

$$\frac{\log d(N)}{\log 2} \leqslant F(L) - R(L) - 5.12\frac{\sqrt{L}}{\lambda^3} + 1.51\,\frac{L^{\beta_3}}{\lambda} \tag{4.1}$$

and

$$\frac{\log d(N)}{\log 2} \geqslant F(L) - R(L) - \frac{25.3\sqrt{L}}{\lambda^3} - \frac{1.45\,L^{\beta_3}}{\lambda}. \tag{4.2}$$

Proof : Defining $\xi = \xi(N)$ and $\xi_k = \xi_k(N)$ by Definition 3.9, from (3.36), one has

$$\frac{\log d(N)}{\log 2} \leqslant \pi(\xi) + \beta_2\pi(\xi_2) + 1.604\,\frac{L^{\beta_3}}{\lambda}.$$

From Lemma 3.12 and (3.21), $\xi \geqslant \xi^{(0)} > 10^8$ holds, so that (2.41) and (2.53) imply $\pi(\xi) = \mathrm{li}(\theta(\xi)) - A(\xi) \leqslant \mathrm{li}(\theta(\xi)) - R(\xi) - 5.12\sqrt{\xi}/\log^3\xi$. From (2.44), $A(\xi_2)$ is positive, which, from (2.41), implies $\pi(\xi_2) = \mathrm{li}(\theta(\xi_2)) - A(\xi_2) \leqslant \mathrm{li}(\theta(\xi_2))$. Therefore, from (1.3), one gets

$$\frac{\log d(N)}{\log 2} \leqslant \mathrm{li}(\theta(\xi)) - \frac{2\sqrt{\xi}}{\log^2\xi} - \frac{S(\xi)}{\log^2\xi} - \frac{5.12\sqrt{\xi}}{\log^3\xi} + \beta_2\,\mathrm{li}(\theta(\xi_2)) + 1.604\,\frac{L^{\beta_3}}{\lambda}. \tag{4.3}$$

From (3.49), (3.46), (3.47) and (3.50) we deduce

$$\frac{\log d(N)}{\log 2} \leqslant \mathrm{li}(L) - \frac{L^{\beta_2}}{\lambda} + \beta_2\,\mathrm{li}(L^{\beta_2}) - \frac{2\sqrt{L}}{\lambda^2} - \frac{S(L)}{\lambda^2} - \frac{5.12\,\sqrt{L}}{\lambda^3} + B$$

with

$$\begin{aligned}
B &= -\frac{0.61L^{\beta_3}}{\lambda} + 0.0242\,\beta_2 L^{\beta_2/2}\lambda + \frac{2\times 1.461\,L^{\beta_2-1/2}}{\lambda^2} + 0.12\,L^{\beta_2-1/2}\\
&\qquad + \frac{5.12\times 1.461\,L^{\beta_2-1/2}}{\lambda^3} + 1.604\,\frac{L^{\beta_3}}{\lambda}\\
&= \left(0.994 + \frac{0.0242\,\beta_2\,\lambda^2}{L^{\beta_3-\beta_2/2}} + \frac{2.922/\lambda + 0.12\,\lambda + 7.48032/\lambda^2}{L^{\beta_3-\beta_2+1/2}}\right)\frac{L^{\beta_3}}{\lambda}\\
&\leqslant \left(0.994 + \frac{0.0242\,\beta_2\,\lambda_0^2}{L_0^{\beta_3-\beta_2/2}} + \frac{2.922/\lambda_0 + 0.12\,\lambda_0 + 7.48032/\lambda_0^2}{L_0^{\beta_3-\beta_2+1/2}}\right)\frac{L^{\beta_3}}{\lambda}\\
&= 1.50193\ldots\frac{L^{\beta_3}}{\lambda},
\end{aligned}$$

which, together with (1.3) and (1.7), proves (4.1).

To prove the lower bound (4.2), from (3.36) and (2.41), one gets

$$\frac{\log d(N)}{\log 2} \geqslant \mathrm{li}(\theta(\xi)) - A(\xi) + \beta_2\,\mathrm{li}(\theta(\xi_2)) - \beta_2 A(\xi_2) + 0.9997\frac{L^{\beta_3}}{\lambda}. \qquad (4.4)$$

First, from (2.53), (1.3), (3.47) and (3.50), one has

$$\begin{aligned}
A(\xi) &\leqslant \frac{2\sqrt{\xi}}{\log^2\xi} + \frac{S(\xi)}{\log^2\xi} + \frac{25.3\sqrt{\xi}}{\log^3\xi}\\
&\leqslant \frac{2\sqrt{L}}{\lambda^2} + 0.04 + \frac{S(L)}{\lambda^2} + 0.12\,L^{\beta_2-1/2} + \frac{25.3\sqrt{L}}{\lambda^3} + 0.506\\
&= R(L) + \frac{25.3\sqrt{L}}{\lambda^3} + \left(\frac{0.546\,\lambda}{L^{\beta_3}} + \frac{0.12\,\lambda}{L^{\beta_3-\beta_2+1/2}}\right)\frac{L^{\beta_3}}{\lambda}\\
&\leqslant R(L) + \frac{25.3\sqrt{L}}{\lambda^3} + \left(\frac{0.546\,\lambda_0}{L_0^{\beta_3}} + \frac{0.12\,\lambda_0}{L_0^{\beta_3-\beta_2+1/2}}\right)\frac{L^{\beta_3}}{\lambda}\\
&= R(L) + \frac{25.3\sqrt{L}}{\lambda^3} + 0.00986633\ldots\frac{L^{\beta_3}}{\lambda} \qquad (4.5)
\end{aligned}$$

After, from (2.45), (3.13), (3.34), (3.35) and (3.24), we have successively

$$\begin{aligned}
\beta_2\,A(\xi_2) &\leqslant 5.07\,\beta_2\frac{\sqrt{\xi_2}}{\log^2\xi_2} \leqslant \left(\frac{5.07\,\beta_2\times 1.00017^{\beta_2/2}}{(0.9997\,\beta_2)^2}\right)\frac{L^{\beta_2/2}}{\lambda^2}\\
&\leqslant 8.67\,\frac{L^{\beta_3}}{\lambda^2 L^{\beta_3-\beta_2/2}} \leqslant 8.67\,\frac{L^{\beta_3}}{\lambda\lambda_0 L_0^{\beta_3-\beta_2/2}} \leqslant 0.05\,\frac{L^{\beta_3}}{\lambda}. \qquad (4.6)
\end{aligned}$$

Finally, (4.4), (3.49), (4.5), (3.46) and (4.6) yield

$$\frac{\log d(N)}{\log 2} \geqslant \mathrm{li}(L) - \frac{L^{\beta_2}}{\lambda} + \beta_2\,\mathrm{li}(L^{\beta_2}) - R(L) - \frac{25.3\sqrt{L}}{\lambda^3} + B'$$

with

$$\begin{aligned} B' &= (-1.873 - 0.0099 - 0.05 + 0.9997)\frac{L^{\beta_3}}{\lambda} - 0.0245\,\beta_2 L^{\beta_2/2}\lambda \\ &= -\left(0.9332 + 0.0245\beta_2\frac{\lambda^2}{L^{\beta_3-\beta_2/2}}\right)\frac{L^{\beta_3}}{\lambda} \\ &\geqslant -\left(0.9332 + 0.0245\beta_2\frac{\lambda_0^2}{L_0^{\beta_3-\beta_2/2}}\right)\frac{L^{\beta_3}}{\lambda} = -1.44188\ldots\frac{L^{\beta_3}}{\lambda} \end{aligned}$$

which, with (1.3) and (1.7), completes the proof of Proposition 4.1. □

Corollary 4.2. *Let n be an integer $\geqslant N^{(0)}$ defined by* (3.23)*.Then, under the Riemann hypothesis,* (1.8) *holds.*

Proof : From Proposition 3.11, there exists two consecutive shc numbers N' and N satisfying $N^{(0)} \leqslant N' \leqslant n < N$ with $d(n) \leqslant d(N) \leqslant 2d(N')$. From (2.34), let us set

$$H_1(t) = H(2, 5.12, 1.51, t) = F(t) - R(t) - 5.12\,\sqrt{t}/\log^3 t + 1.51\,t^{\beta_3}/\log t.$$

Note that, from Lemma 2.11 (iii), $H_1(t)$ is increasing for $t \geqslant 12$. From (4.1), one deduces

$$\frac{\log d(n)}{\log 2} \leqslant \frac{\log d(N)}{\log 2} \leqslant 1 + \frac{\log d(N')}{\log 2} \leqslant 1 + H_1(\log N') \leqslant 1 + H_1(\log n).$$

But

$$\begin{aligned} 1 + \frac{1.51\log^{\beta_3} n}{\log\log n} &= \frac{\log^{\beta_3} n}{\log\log n}\left(1.51 + \frac{\log\log n}{\log^{\beta_3} n}\right) \\ &\leqslant \frac{\log^{\beta_3} n}{\log\log n}\left(1.51 + \frac{\log\log N^{(0)}}{\log^{\beta_3} N^{(0)}}\right) = 1.5188\ldots\frac{\log^{\beta_3} n}{\log\log n}, \end{aligned}$$

which ends the proof of Corollary 4.2. □

It remains to consider the numbers n satisfying $n \leqslant N^{(0)}$. For that, we start by proving the following lemma.

Lemma 4.3. *Let us introduce the shc number*

$$N^{(2)} = N_{(\log 2)/\log 179} = 2^{10}3^6 5^4 7^3 \prod_{p=11}^{19} p^2 \prod_{p=23}^{179} p = 1.049597\ldots 10^{84}. \quad (4.7)$$

For $N^{(2)} < n \leqslant N^{(0)}$, the function G_1 defined in (2.33) *satisfies*

$$\frac{\log d(n)}{\log 2} \leqslant G_1(\log n) = F(\log n) - \frac{(2+\tau)\sqrt{\log n}}{(\log(\log n))^2.} \quad (4.8)$$

Proof : Note that, from Lemma 2.11 (ii), the maping $t \mapsto G_1(t)$ is increasing and concave for $t > 1$. So, if N and $\widetilde{N}$ are two consecutive shc numbers such that $(\log d(N))/\log 2 \leqslant G_1(\log N)$ and $(\log d(\widetilde{N}))/\log 2 \leqslant G_1(\log \widetilde{N})$, we may apply Lemmma 3.17 to prove (4.8) for $N \leqslant n \leqslant \widetilde{N}$.

Therefore, one computes the difference $G_1(\log N) - (\log d(N))/\log 2$ for all shc number N satisfying $N^{(2)} \leqslant N \leqslant N^{(0)}$. It turns out that this difference is negative for $N = N^{(2)}$ but positive for all shc number N satisfying $N^{(3)} \leqslant N \leqslant N^{(0)}$ where $N^{(3)} = 181 N^{(2)}$ is the shc number following $N^{(2)}$ and, consequently, for all n's satisfying $N^{(3)} \leqslant n \leqslant N^{(0)}$.

Now, we have to look at the n's satisfying $N^{(2)} < n < N^{(3)}$. For that, we consider the hc numbers M_j between $N^{(2)}$ and $N^{(3)}$. The 1125-th hc number is $M_{1125} = N^{(2)}$ while $M_{1162} = N^{(3)}$. For j satisfying $1126 \leqslant j \leqslant 1161$, we check that the difference $G_1(\log M_j) - (\log d(M_j)/\log 2$ is positive, which, from Lemma 3.14, proves (4.8) for $M_{1126} \leqslant n < N^{(3)}$.

It remains to prove (4.8) for $M_{1125} = N^{(2)} < n < M_{1126} = 4\, M_{1125}/3$. We apply Lemma 3.16 with $N = N^{(2)}$, $\varepsilon = (\log 2)/\log 179$, $j = 1125$, $f = G_1$. We have $G_1(\log M_{1125}) = 49.928530\ldots < \log d(M_{1125})/\log 2 = 49.928564\ldots < G_1(\log M_{1126}) = 49.94718\ldots$ The number μ such that $G_1(\log \mu) = (\log d(M_{1125})/\log 2$ is equal to $\exp(193.46573\ldots)$ and, if $n \in (M_{1125}, M_{1126})$ is a number satisfing $(\log d(n))/\log 2 \geqslant G_1(\log n)$, from (3.30), its benefit would satisfy

$$\mathrm{ben}_\varepsilon(n) \leqslant \log d(N) - (\log 2) G_1(\log M_{1125}) + \varepsilon(\log \mu - \log N) = 0.0000471\ldots$$

Such a number n does not exist, since the number $\nu \in (M_{1225}, M_{1126})$ with the smallest benefit is $\nu = (181/179) N^{(2)}$ and $\mathrm{ben}_\varepsilon(\nu) = 0.0148\ldots$. □

4.2 Proof of Theorem 1.1 (i)

It is convenient to introduce, from (2.34),

$$H_0(t) = H(2, 5.12, 1.52, t) = F(t) - R(t) - 5.12\,\frac{\sqrt{t}}{\log^3 t} + \frac{1.52\, t^{\beta_3}}{\log t}. \tag{4.9}$$

Note that, from Lemma 2.11 (iii), $H_0(t)$ is continuous for $t > 1$ and increasing for $t \geqslant 12$.

For $n \geqslant N^{(0)}$, (1.8) has been proved in Corollary 4.2. For $N^{(2)} < n < N^{(0)}$, from (4.7) and (3.24), we have $193 < \log n < 1.1 \times 10^8$ so that, from (1.6) and Lemma 2.8, H_0 defined in (4.9) and G_1 defined in (2.33) satisfy $H_0(\log n) > G_1(\log n)$, which, from Lemma 4.3, proves (1.8) for $N^{(2)} < n < N^{(0)}$.

Now, we consider the hc numbers M_j for $60 \leqslant j \leqslant 1125$. We have $M_{60} = 183783600$, $M_{61} = 245044800$, $M_{1125} = N^{(2)}$. For each j, we compute the difference $H_0(\log M_j) - (\log d(M_j))/\log 2$. This difference is negative for $j =$

60 but positive for $61 \leqslant j \leqslant 1125$, which, by Lemma 3.14, proves that (1.8) holds for $M_{61} \leqslant n \leqslant N^{(2)}$.

If n is a number satisfying $M_{60} < n < M_{61} = 4M_{60}/3$ and $(\log d(n))/\log 2 > H_0(\log n)$, we have

$$H_0(\log M_{60}) < (\log d(M_{60}))/\log 2 < H_0(\log M_{61})$$

so that we may apply Lemma 3.16 with $N = 367567200$, $\varepsilon = (\log 2)/\log 17$, $j = 60$, $f = H_0$. The number μ such that $H_0(\log \mu) = (\log d(M_{60}))/\log 2$ is equal to $\exp(19.0876\ldots)$ and, from (3.30), the benefit of such an n would satisfy

$$\text{ben}_\varepsilon(n) \leqslant \log d(N) - (\log 2)H_0(\log M_{60}) + \varepsilon(\log \mu - \log N) = 0.0442\ldots$$

There is only one number between M_{60} and M_{61} with a benefit $\leqslant 0.045$, namely $\nu = 205405200 = 19\, M_{60}/17$. But, $(\log d(\nu))/\log 2 = 9.9068\ldots < H_0(\log \nu) = 9.9293\ldots$ which completes the proof that $M_{60} = 183783600$ is the largest number that does not satisfy (1.8).

4.3 Proof of Corollary 1.2

4.3.1 Proof of (1.10)

Let

$$G_2(t) = F(t) - \frac{(2-\tau)\sqrt{t}}{\log^2 t} - \frac{5.12\sqrt{t}}{\log^3 t} + \frac{1.52\, t^{\beta_3}}{\log t}. \tag{4.10}$$

From (1.6), the inequality $G_2(t) \geqslant H_0(t)$ (defined in (4.9)) holds for $t > 1$. Therefore, (1.10) follows from (1.8) for $n > M_{60} = 183783600$.

Note that, from Lemma 2.11, $G_2(t) = G(2-\tau, 5.12, 1.52, t)$ is increasing for $t > 12$. We have $M_{58} = 122522400$, $M_{59} = 147026880 = 6\, M_{58}/5$. The difference $G_2(\log M_j) - (\log d(M_j))/\log 2$ is positive for $j \in \{59, 60\}$ but negative for $j = 58$ so that, from Lemma 3.14, (1.10) holds for $n \geqslant M_{59}$.

Assume that n satisfies $M_{58} < n < M_{59}$ and $G_2(\log n) \leqslant (\log d(n))/\log 2$. We should apply Lemma 3.16 with $N = 367567200$, $\varepsilon = (\log 2)/\log 17$, $j = 58$, $f = G_2$ and get $\log \mu = 18.653\ldots$ and

$$\text{ben}_\varepsilon(n) \leqslant \log d(N) - (\log 2)G_2(\log M_{58}) + \varepsilon(\log \mu - \log N) < 0.0347\ldots$$

But there is no number between M_{58} and M_{59} with a benefit < 0.04, which completes the proof that $M_{58} = 122522400$ is the largest number not satisfying (1.10).

4.3.2 Proof of (1.11)

Let $G_3(t) = \mathrm{li}(t) + \beta_2 \, \mathrm{li}(t^{\beta_2})$. For $t > 1$, this function is increasing and, from (1.6), $R(t) > 0$ holds, so one has

$$H_0(t) = G_3(t) - R(t) - \frac{t^{\beta_2}}{\log t} - \frac{5.12\sqrt{t}}{\log^3 t} + \frac{1.52\, t^{\beta_3}}{\log t} \leqslant G_3(t) + \frac{t^{\beta_3}}{\log t}(1.52 - t^{\beta_2-\beta_3})$$

which, for $t \geqslant (1.52)^{1/(\beta_2-\beta_3)} = 11.75\ldots$, yields $H_0(t) \leqslant G_3(t)$. Therefore, from (1.8), (1.11) holds for $n > 183783600$.

Then, one computes the difference $G_3(\log M_j) - (\log d(M_j))/\log 2$ for the hc numbers M_j satisfying $M_6 = 24 \leqslant M_j \leqslant M_{60} = 183783600$. This difference is negative for $j = 6$ but positive for $7 \leqslant j \leqslant 60$, which, from Lemma 3.14, proves (1.11) for $n \geqslant M_7 = 36$. We could apply Lemma 3.16 to check that 24 is the largest exception to (1.8) but it is easier to compute $G_3(\log n) - (\log d(n))/\log 2$ for $2 \leqslant n \leqslant 35$ in order to find the integers not satisfying (1.11).

4.3.3 Proof of (1.12)

From (4.10) and (1.7), we get

$$G_2(t) - F(t) = \frac{t^{\beta_3}}{\log t}\left(1.52 - \left[\frac{(2-\tau)t^{1/2-\beta_3}}{\log t} + \frac{5.12\, t^{1/2-\beta_3}}{\log^2 t}\right]\right).$$

From (1.18), the above square bracket is increasing in t for $t > 10^{11} > \exp(2/(1/2-\beta_3))$ and exceeds 1.52 for $t \geqslant 1.56 \times 10^{17}$. Therefore, from (1.10), (1.12) holds for $n \geqslant \exp(1.56 \times 10^{17})$.

From Lemma 4.3, for $N^{(2)} < n \leqslant N^{(0)}$, we also have $G_1(\log n) < F(\log n)$, which, from (4.8) implies (1.12) for $N^{(2)} < n \leqslant N^{(0)}$.

Next, we compute the difference $F(\log M_j) - (\log d(M_j))/\log 2$ for the hc numbers M_j satisfying $M_{44} = 4324320 \leqslant M_j \leqslant M_{1125} = N^{(2)}$. This difference is positive for $45 \leqslant j \leqslant 1125$, which, from Lemma 3.14, proves (1.12) for $M_{45} \leqslant n \leqslant N^{(0)}$ and is negative for $j = 44$.

If n is an integer such that $M_{44} < n < M_{45} = 6486480 = 3\,M_{44}/2$ and not satisfying (1.12), we apply Lemma 3.16 with $N = 4324320$, $\varepsilon = (\log(4/3))/\log 3$, $j = 44$ and $f = F$. We find $\log\mu = 15.364\ldots$ and

$$\mathrm{ben}_\varepsilon(n) < \log d(N) - (\log 2)F(\log M_{44}) + \varepsilon(\log\mu - \log N) = 0.0463\ldots$$

But there is no number between M_{44} and M_{45} with a benefit < 0.05, which completes the proof of (1.12).

4.3.4 Proof of (1.13)

From Lemma 2.8 and (4.9), for $t > 1.11 \times 10^{40}$, we have

$$F(t) - R(t) = H_0(t) - \left(\frac{1.52\, t^{\beta_3}}{\log t} - \frac{5.12\sqrt{t}}{\log^3 t}\right) > H_0(t)$$

which, from (1.8), proves (1.13) for $n > \exp(1.11 \times 10^{40})$.

From (1.6) and (2.33), for $t > 1$, we also have $F(t) - R(t) \geqslant G_1(t)$, which, from Lemma 4.3, proves (1.13) for $N^{(2)} < n \leqslant N^{(0)}$.

For $t > 12$, from Lemma 2.11 (iii), the mapping $t \mapsto F(t) - R(t) = H(2,0,0,t)$ is increasing and continuous. So, we may apply Lemma 3.14 to the hc numbers M_j satisfying $M_{975} = N^{(1)} \leqslant M_j \leqslant M_{1125} = N^{(2)}$. We compute the difference $F(M_j) - R(M_j) - (\log d(M_j))/\log 2$. This difference is positive for $976 \leqslant j \leqslant 1125$ and negative for $j = 975$, which proves (1.13) for $M_{976} = 4N^{(1)}/3 \leqslant n \leqslant N^{(0)}$.

if n were a number between M_{975} and M_{976} not satisfying (1.13), Lemma 3.16 with $N = N^{(1)}$, $\varepsilon = (\log 2)/\log 157$, $j = 975$ and $f = F - R$. would yield $\log\mu = 172.915\ldots$ and $\mathrm{ben}_\varepsilon(n) < \log d(N) - (\log 2)(F(\log M_{44}) - R(\log M_{44})) + \varepsilon(\log\mu - \log N) = 0.000742\ldots$ But, there is no number between M_{975} and M_{976} with a benefit < 0.005, which completes the proof of (1.13) and of Corollary 1.2.

4.4 Proof of Theorem 1.1 (ii)

In this section, we assume that the Riemann hypothesis does not hold, i.e. that $\Theta = \limsup \Re(\rho)$ when ρ runs over the non-trivial zeros of the Riemann ζ function satisfies $1/2 < \Theta \leqslant 1$. We shall use the following upper bounds (cf. [13, Theorem 30] or [10, Théorème. 5.10] or [9, Theorem 5.10]:

$$\pi(x) = \mathrm{li}(x) + \mathcal{O}(x^{\Theta}\log x) \tag{4.11}$$

and

$$\theta(x) = \psi(x) + \mathcal{O}(\sqrt{x}) = x + \mathcal{O}(x^{\Theta}\log^2 x). \tag{4.12}$$

For $A(x)$ defined in (2.41), we shall use the result of [27, Theorem 2]

$$\forall\, \omega < \Theta, \quad A(x) = \Omega_{\pm}(x^{\omega}). \tag{4.13}$$

As $A(x) = \mathrm{li}(\theta(x)) - \pi(x)$ is constant between two consecutive primes, note that (4.13) implies the existence of a sequence of primes $(p_{r_j})_{j\geqslant 1}$ tending to infinity such that, for $\omega < \Theta$,

$$\lim_{j\to\infty} A(p_{r_j})/p_{r_j}^{\omega} = -\infty. \tag{4.14}$$

Let us consider, for j fixed, the shc number N_ε with $\varepsilon = (\log 2)/\log p_{r_j}$. From Definition 3.9, we have $\xi = \xi(N) = 2^{1/\varepsilon} = p_{r_j}$ and $\xi_k = \xi^{\beta_k}$. Using the notation $L = \log N$, from (3.34), one has $\xi \asymp L$ and, for k fixed, $\xi_k \asymp L^{\beta_k}$. As $\beta_3 < 1/2$ holds, (3.33) implies

$$\theta(\xi) = L - \theta(\xi_2) + \mathcal{O}(\sqrt{L}). \tag{4.15}$$

We distinguish two cases : $\Theta > \beta_2$ and $1/2 < \Theta \leqslant \beta_2$:

First case : $\Theta > \beta_2$. We have $\theta(\xi_2) \asymp L^{\beta_2}$ and (4.15) yields $\theta(\xi) = L + \mathcal{O}(L^{\beta_2})$, which from (2.16) implies

$$\mathrm{li}(\theta(\xi)) = \mathrm{li}(L) + \mathcal{O}(L^{\beta_2}). \tag{4.16}$$

On the other hand, from (3.36), (2.41) and (4.16), we get

$$\begin{aligned}(\log d(N))/\log 2 = \pi(\xi) + \mathcal{O}(L^{\beta_2}) &= \mathrm{li}(\theta(\xi)) - A(\xi) + \mathcal{O}(L^{\beta_2})\\ &= \mathrm{li}(L) - A(\xi) + \mathcal{O}(L^{\beta_2}).\end{aligned} \tag{4.17}$$

Choosing ω such that $\beta_2 < \omega < \Theta$, as $\xi = \xi(N) = p_{r_j}$, (4.17) and (4.14) contradict (1.8) with $n = N$ for j large enough.

Second case : $1/2 < \Theta \leqslant \beta_2$. As $\Theta < 3/5$ holds, from (4.12), we have $\theta(\xi) = \xi + \mathcal{O}(\xi^{3/5})$, which, from (4.15), yields $\xi = L + \mathcal{O}(L^{3/5})$ and

$$\xi_2 = \xi^{\beta_2} = L^{\beta_2}\left(1 + \mathcal{O}\left(1/L^{2/5}\right)\right)^{\beta_2} = L^{\beta_2} + \mathcal{O}(L^{\beta_2 - 2/5}). \tag{4.18}$$

From (4.12) and (4.18), one gets

$$\theta(\xi_2) = \xi_2 + \mathcal{O}(\xi_2^{3/5}) = L^{\beta_2} + \mathcal{O}(\sqrt{L})$$

so that (4.15) gives $\theta(\xi) = L - L^{\beta_2} + \mathcal{O}(\sqrt{L})$ and (2.16) implies

$$\mathrm{li}(\theta(\xi)) = \mathrm{li}(L) - L^{\beta_2}/\log L + \mathcal{O}(\sqrt{L}). \tag{4.19}$$

Then, from (4.11), (4.18) and (2.16), one gets

$$\pi(\xi_2) = \mathrm{li}(\xi_2) + \mathcal{O}(\xi_2^{3/5}) = \mathrm{li}(L^{\beta_2}) + \mathcal{O}(\sqrt{L}). \tag{4.20}$$

Finally, from (3.36), (2.41) and (4.19),

$$\begin{aligned}(\log d(N))/\log 2 &= \pi(\xi) + \beta_2\pi_2(\xi_2) + \mathcal{O}(\sqrt{L})\\ &= \mathrm{li}(\theta(\xi)) - A(\xi) + \beta_2\pi_2(\xi_2) + \mathcal{O}(\sqrt{L})\\ &= \mathrm{li}(L) - L^{\beta_2}/\log L - A(\xi) + \beta_2\,\mathrm{li}(L^{\beta_2}) + \mathcal{O}(\sqrt{L})\\ &= F(L) - A(\xi) + \mathcal{O}(\sqrt{L}).\end{aligned} \tag{4.21}$$

Let us choose ω between $1/2$ and Θ. As ξ is equal to p_{r_j} and $\xi \asymp L$, (4.14) and (4.21) contradict (1.9) with $n = N$ for infinitely many values of j.

4.5 Proof of Theorem 1.1 (iii)

If the Riemann hypothesis is true, then, from Proposition 4.1 (4.2), then (1.9) holds for all shc numbers $N > N^{(0)}$.

Note that, by revisiting the proof of Proposition 4.1 with the upper bound (2.54) instead of (2.53), it is possible to prove

$$-8.01\frac{\sqrt{L}}{\lambda^3} \leqslant \frac{\log d(N)}{\log 2} - F(L) - R(L) \leqslant -7.99\frac{\sqrt{L}}{\lambda^3} \tag{4.22}$$

for N shc large enough.

If the Riemann hypothesis is not true, then, in the first case of the proof of Theorem 1.1 (ii), (4.17) and (4.13) imply, for $\beta_2 < \omega < \Theta$,

$$(\log d(N))/\log 2 = \mathrm{li}(L) + \Omega_+(L^\omega). \tag{4.23}$$

Similarly, in the second case, with $1/2 < \omega < \Theta$, (4.21) and (4.13) imply

$$(\log d(N))/\log 2 = F(L) + \Omega_+(L^\omega) \tag{4.24}$$

for $1/2 < \omega < \Theta$. In both cases, (4.23) or (4.24) proves (1.9) for infinitely many shc numbers.

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