```
> ### Computation of the paper "Highly composite numbers and the
  Riemann hypothesis ###

> read "hcnHR.mpl": # contains the procedures used below
> read "hcnHR.m": # this file contains two lists : hcn2HR and
  lishc10p100
>    # hcn2HR contains the 7739 first shc numbers of type 2 [q,al,
  eps,p,(log d(N))/log 2,log N]
        # computed with "shcntype2(2*10^8,100)"
     # lishc10p100 contains the 1381 first hc numbers [M,d(M)]
        # computed with "highlycomp(10^100,0,0)
> Digits:=30: # precise computations are carried out with 30
  decimals, other ones with 10
> lisS:=constrlisteS(10000): # allows the computation of S(t) defined
  in (1.3)
> lisshc:=shc(300,0): # list of shc numbers N with P+(N) <= 300 [q,
  al,eps,p,d(N),N,xi]

> ######
> ##############################   Section 1   ##################
> ######

> betak(40,0); evalf(op([be[2],be[3],be[4],be[5]]),10);
     # determines be[j] = beta_j for 1 <= j <= 40, cf. (1.2)
```
0.5849625007, 0.4150374993, 0.3219280949, 0.2630344058 **(1)**

```
> log2:=log(2.): tautau:=evalf(2+gamma-log(4*Pi)): taup2:=2+tautau:
  "log2 = ",log2, "tautau = ",tautau,"taup2 = ",evalf(taup2,6); # cf
  (1.5)
```
"log2 = ", 0.693147180559945309417232121458, "tautau = ", **(2)**

0.04619141793224206762862049581, "taup2 = ", 2.04619

```
> ######
> ##############################   Section 2   ##################
> ######

> #################  Proof of Lemma 2.1, (2.6) ################

> comaxpi(376),comaxpi(2090);
```
"t0=", 376, "comaxpi=", 1.197678720

"t0=", 2090, "comaxpi=", 1.159620946 **(3)**

```
> #################  Proof of Lemma 2.2, (2.11) ################
```

```
> lempipip1(89693): # computes the champion primes for (p_{i+1}-p_i)
  /p_{i+1}
```

"x = ", 89693, "ecarmax = ", 0.0006742353

"p=", 89689, "pp=", 89753, "diff=", 64, "quo=", 0.0007130681

"p=", 79699, "pp=", 79757, "diff=", 58, "quo=", 0.0007272089

"p=", 60539, "pp=", 60589, "diff=", 50, "quo=", 0.0008252323

"p=", 59281, "pp=", 59333, "diff=", 52, "quo=", 0.0008764094

"p=", 58831, "pp=", 58889, "diff=", 58, "quo=", 0.0009849038

"p=", 48679, "pp=", 48731, "diff=", 52, "quo=", 0.001067083

"p=", 45893, "pp=", 45943, "diff=", 50, "quo=", 0.001088305

"p=", 44293, "pp=", 44351, "diff=", 58, "quo=", 0.001307750

"p=", 43331, "pp=", 43391, "diff=", 60, "quo=", 0.001382775

"p=", 35677, "pp=", 35729, "diff=", 52, "quo=", 0.001455400

"p=", 35617, "pp=", 35671, "diff=", 54, "quo=", 0.001513835

"p=", 34061, "pp=", 34123, "diff=", 62, "quo=", 0.001816956

"p=", 31397, "pp=", 31469, "diff=", 72, "quo=", 0.002287966

"p=", 19609, "pp=", 19661, "diff=", 52, "quo=", 0.002644830

"p=", 15683, "pp=", 15727, "diff=", 44, "quo=", 0.002797736

"p=", 11743, "pp=", 11777, "diff=", 34, "quo=", 0.002886983

"p=", 10799, "pp=", 10831, "diff=", 32, "quo=", 0.002954483

"p=", 9973, "pp=", 10007, "diff=", 34, "quo=", 0.003397622

"p=", 9551, "pp=", 9587, "diff=", 36, "quo=", 0.003755085

"p=", 8467, "pp=", 8501, "diff=", 34, "quo=", 0.003999529

"p=", 7253, "pp=", 7283, "diff=", 30, "quo=", 0.004119182

"p=", 6917, "pp=", 6947, "diff=", 30, "quo=", 0.004318411

"p=", 6491, "pp=", 6521, "diff=", 30, "quo=", 0.004600521

"p=", 5953, "pp=", 5981, "diff=", 28, "quo=", 0.004681491

"p=", 5749, "pp=", 5779, "diff=", 30, "quo=", 0.005191210

"p=", 5591, "pp=", 5623, "diff=", 32, "quo=", 0.005690912

"p=", 4831, "pp=", 4861, "diff=", 30, "quo=", 0.006171570

"p=", 4297, "pp=", 4327, "diff=", 30, "quo=", 0.006933210

"p=", 3271, "pp=", 3299, "diff=", 28, "quo=", 0.008487420

"p=", 2971, "pp=", 2999, "diff=", 28, "quo=", 0.009336445 "p=", 2477, "pp=", 2503, "diff=", 26, "quo=", 0.01038753

"p=", 2179, "pp=", 2203, "diff=", 24, "quo=", 0.01089424

"p=", 1951, "pp=", 1973, "diff=", 22, "quo=", 0.01115053

"p=", 1669, "pp=", 1693, "diff=", 24, "quo=", 0.01417602

"p=", 1327, "pp=", 1361, "diff=", 34, "quo=", 0.02498163

"p=", 523, "pp=", 541, "diff=", 18, "quo=", 0.03327172

"p=", 317, "pp=", 331, "diff=", 14, "quo=", 0.04229607

"p=", 293, "pp=", 307, "diff=", 14, "quo=", 0.04560261

"p=", 211, "pp=", 223, "diff=", 12, "quo=", 0.05381166

"p=", 199, "pp=", 211, "diff=", 12, "quo=", 0.05687204

"p=", 139, "pp=", 149, "diff=", 10, "quo=", 0.06711409

"p=", 113, "pp=", 127, "diff=", 14, "quo=", 0.1102362

"p=", 47, "pp=", 53, "diff=", 6, "quo=", 0.1132075

"p=", 31, "pp=", 37, "diff=", 6, "quo=", 0.1621622

"p=", 23, "pp=", 29, "diff=", 6, "quo=", 0.2068966

"p=", 13, "pp=", 17, "diff=", 4, "quo=", 0.2352941

"p=", 7, "pp=", 11, "diff=", 4, "quo=", 0.3636364

"p=", 3, "pp=", 5, "diff=", 2, "quo=", 0.4000000 **(4)**

```
> ############  Section 2.2, logarithmic integral, Table (2.14)
  ################
```

```
> "Li(0.5) = ",evalf(Li(0.5),10); # computation of the table (2.14)
  "Li(1.45136) = ", evalf(Li(1.45136),10),"Li(1.45137) = ", evalf(Li
  (1.45137),10);
  "Li(1.96904) = ", evalf(Li(1.96904),10),"Li(1.96905) = ", evalf(Li
  (1.96905),10);
  "Li(2) = ", evalf(Li(2.),10);
```

"Li(0.5) = ", -0.3786710431

"Li(1.45136) = ", -0.00002479135175, "Li(1.45137) = ", 0.000002053962201

"Li(1.96904) = ", 0.9999889466, "Li(1.96905) = ", 1.000003706

"Li(2) = ", 1.045163780 **(5)**

```
> ############  Section 2.3, function S #########################
```

```
> sig2:=1-evalf(Pi^2/8)+evalf(2*gamma(1)+gamma^2); # cf. Definition
  2.4
```

$sig2 := -0.0461543172958046027571079903 75$ **(6)**

```
> #################  Proof of Lemma 2.5, (2.20) ################
  assume(x > 1):f:=-int(log(1-1/t^2)/(2*t),t=1..x): series(f,x=
  infinity,14); f:='f':
```

$$\frac{1}{24}\pi^2 - \frac{1}{4\,x\sim^2} - \frac{1}{16\,x\sim^4} - \frac{1}{36\,x\sim^6} - \frac{1}{64\,x\sim^8} - \frac{1}{100\,x\sim^{10}} - \frac{1}{144\,x\sim^{12}} + O\left(\frac{1}{x\sim^{14}}\right) \tag{7}$$

```
> Snum(10000,0); # computes [S(t),S(t)/sqrt(t)] by using the list
  "lisS", cf. (2.19)
```

[-0.008396442417964910648160534190078, -0.00008396442417964910648160534190078] **(8)**

```
> psiproc(128,0); # for a_i <= 127 returns [a_i,co_i] where co_i is
  the parenthesis of 2.25
```

[[2., 0.17595576], [3., 0.10065818], [4., 0.13952546], [5., 0.15401257], [7., 0.016217800], [8., 0.048325811], [9., 0.12129620], [11., 0.071936988], [13., 0.085576317], [16., 0.098579414], [17., 0.044331082], [19., 0.082496117], [23., 0.030470842], [25., 0.042674772], [27., 0.070810865], [29., 0.022133708], [31., -0.054621072], [32., 0.069158253], [37., 0.071899259], [41., 0.028703965], [43., 0.031340985], [47., -0.0076967229], [49., 0.031640003], [53., 0.062823625], [59., 0.026705691], [61., 0.0080962657], [64., 0.042164226], [67., 0.036905610], [71., 0.0048988093], [73., 0.026166328], [79., -0.0037322449], [81., 0.0072177379], [83., 0.024496878], [89., 0.058676039], [97., 0.050661987], [101., 0.024288722], [103., 0.017448653], [107., -0.0073928858], [109., -0.013249339], [113., 0.014672990], [121., 0.027020275], [125., 0.029670043], [127., -0.00059446945]] **(9)**

```
> 0.0672+log(128.)^2/(8*evalf(Pi)*sqrt(128.)); # between (2.25) and
  (2.26)
```

0.1499946277003643824789800399981 **(10)**

```
> y0:=10.^8: 1/(8*evalf(Pi))+1/(sqrt(y0)*log(y0)^2)*(\log(2*evalf(Pi)
  )+1/2/(y0^2-1));
     # between (2.28) and (2.29)
```

0.03978927740596194986930204789660 **(11)**

```
> y0:=10.^8: 0.0924/log(y0)^3; # after (2.30)
```

0.00001478274018998555307466585646443 **(12)**

```
> ################# Section 2.4 : Four lemmas of calculus
  ###############
```

```
> #################### Proof of Lemma 2.8 #################
```

```
> Digits:=10: exp(2/(1/2-be[3])), fsolve(5.12*t^(1/2-be[3])/log(t)^2=
  1.52,t=2..10),
```

```
fsolve(5.12*t^(1/2-be[3])/log(t)^2=1.52,t=10^11..10^50);
```

$$1.671875165\ 10^{10},\ 7.373534649,\ 1.100260199\ 10^{40} \tag{13}$$

```
> #################### Proof of Lemma 2.10 ################
```

```
> a[1]:=0.352: a[2]:=0.9132: a[3]:=0.0143:
Phi:=a1*exp((b[3]-b[2]/2)*t)+a2*exp((b[4]-b[2]/2)*t)-a3*t^2;
```

$$\Phi := a1\, \mathrm{e}^{\left(b_3 - \frac{1}{2} b_2\right) t} + a2\, \mathrm{e}^{\left(b_4 - \frac{1}{2} b_2\right) t} - a3\, t^2 \tag{14}$$

```
> Phi1:=diff(Phi,t);Phi2:=diff(Phi1,t); Phi3:=diff(Phi2,t);
```

$$\Phi 1 := a1 \left(b_3 - \frac{1}{2} b_2\right) \mathrm{e}^{\left(b_3 - \frac{1}{2} b_2\right) t} + a2 \left(b_4 - \frac{1}{2} b_2\right) \mathrm{e}^{\left(b_4 - \frac{1}{2} b_2\right) t} - 2\, a3\, t$$

$$\Phi 2 := a1 \left(b_3 - \frac{1}{2} b_2\right)^2 \mathrm{e}^{\left(b_3 - \frac{1}{2} b_2\right) t} + a2 \left(b_4 - \frac{1}{2} b_2\right)^2 \mathrm{e}^{\left(b_4 - \frac{1}{2} b_2\right) t} - 2\, a3$$

$$\Phi 3 := a1 \left(b_3 - \frac{1}{2} b_2\right)^3 \mathrm{e}^{\left(b_3 - \frac{1}{2} b_2\right) t} + a2 \left(b_4 - \frac{1}{2} b_2\right)^3 \mathrm{e}^{\left(b_4 - \frac{1}{2} b_2\right) t}$$

```
> Phinum:=subs(a1=a[1],a2=a[2],a3=a[3],b[2]=be[2],b[3]=be[3],b[4]=be
[4],Phi);
```

$$Phinum := 0.352\, \mathrm{e}^{0.1225562489\, t} + 0.9132\, \mathrm{e}^{0.02944684445\, t} - 0.0143\, t^2 \tag{15}$$

```
> Phinum1:=subs(a1=a[1],a2=a[2],a3=a[3],b[2]=be[2],b[3]=be[3],b[4]=be
[4],Phi1);
```

$$Phinum1 := 0.04313979961\, \mathrm{e}^{0.1225562489\, t} + 0.02689085840\, \mathrm{e}^{0.02944684445\, t} - 0.0286\, t \tag{16}$$

```
> Phinum2:=subs(a1=a[1],a2=a[2],a3=a[3],b[2]=be[2],b[3]=be[3],b[4]=be
[4],Phi2);
```

$$Phinum2 := 0.005287052017\, \mathrm{e}^{0.1225562489\, t} + 0.0007918509257\, \mathrm{e}^{0.02944684445\, t} - 0.0286 \tag{17}$$

```
> Phinum3:=subs(a1=a[1],a2=a[2],a3=a[3],b[2]=be[2],b[3]=be[3],b[4]=be
[4],Phi3); # Phinum3 is positive
```

$$Phinum3 := 0.0006479612631\, \mathrm{e}^{0.1225562489\, t} + 0.00002331751108\, \mathrm{e}^{0.02944684445\, t} \tag{18}$$

```
> evalf(subs(t=0.,Phinum2)),evalf(subs(t=0.,Phinum1)),evalf(subs(t=0.
,Phinum));
t0:=fsolve(subs(Phinum2)=0,t=0...100);evalf(subs(t=t0,Phinum1)),
evalf(subs(t=t0,Phinum));
```

$$-0.02252109706,\ 0.07003065801,\ 1.2652$$

$$t0 := 13.43184612$$

$$-0.1204470482,\ 0.602151091 \tag{19}$$

```
> t1:=fsolve(Phinum1=0,t=0..t0);evalf(subs(t=t1,Phinum2)),evalf(subs
(t=t1,Phinum));
```

$$t1 := 3.294863590$$

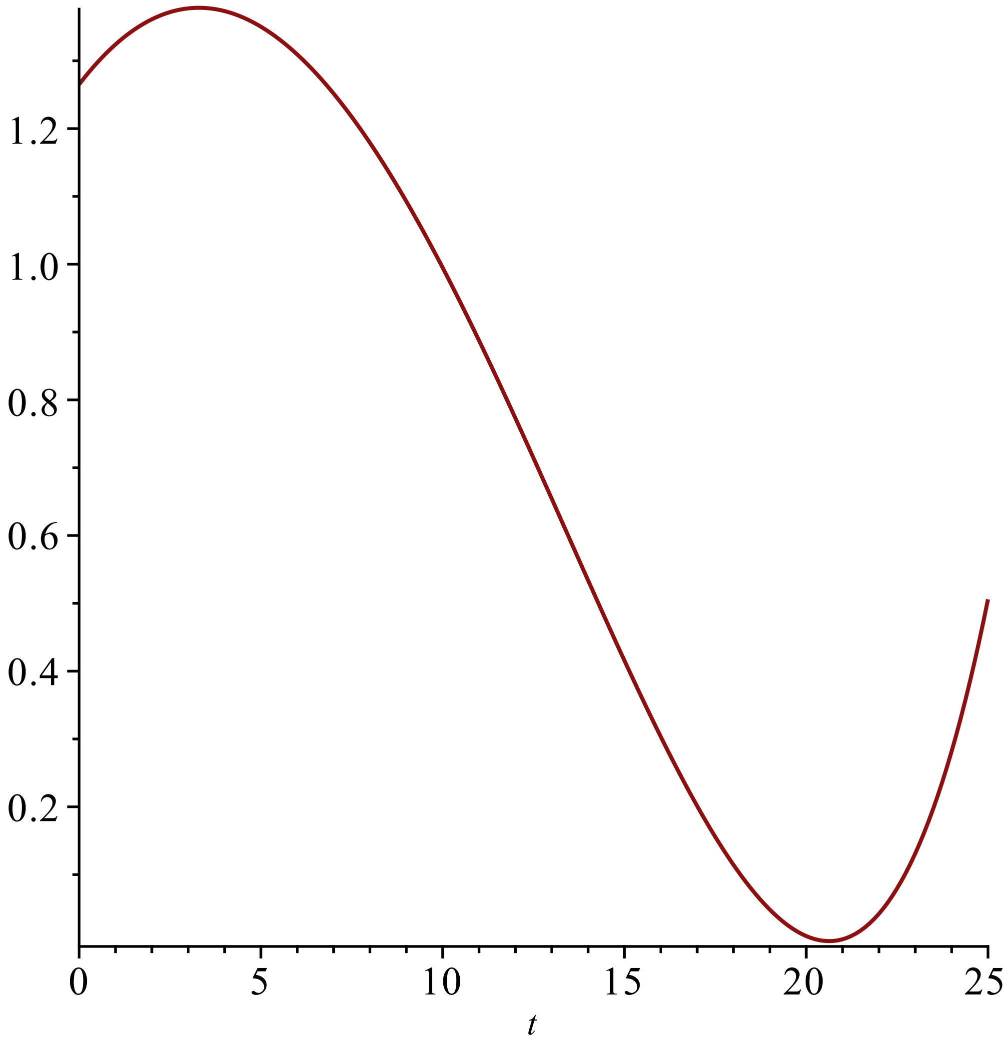
$$-0.01981003827, 1.378124554 \tag{20}$$

```
> t2:=fsolve(Phinum1=0,t=t0..100);evalf(subs(t=t2,Phinum2)),evalf
  (subs(t=t2,Phinum));
```

$$t2 := 20.62990647$$

$$0.03911357968, 0.001915134 \tag{21}$$

```
> plot(Phinum, t=0..25);
```




```
> #################### Proof of Lemma 2.11 ###################
```

```
> g1:=Li(t): g11:=normal(diff(g1,t)): g12:=normal(diff(g11,t)):
  "g1 = ",g1,"  g11 = ",g11, "  g12 = ",g12;
```

$$\text{"g1 = "}, \mathrm{Li}(t), \text{" g11 = "}, \frac{1}{\ln(t)}, \text{" g12 = "}, -\frac{1}{\ln(t)^2\, t} \tag{22}$$

```
> g2:=b2*Li(t^b2)-t^b2/log(t): g21:=simplify(diff(g2,t),assume=
```

```
positive):
g22:=simplify(diff(g21,t),assume=positive):  "g2 = ",g2,"  g21 = ",
g21, "  g22 = ",g22;
```

$$"g2 = ", b2\,\mathrm{Li}\left(t^{b2}\right) - \frac{t^{b2}}{\ln(t)}, "\ \ g21 = ", \frac{t^{b2-1}}{\ln(t)^2}, "\ \ g22 = ", \frac{(-2+(b2-1)\ln(t))\,t^{b2-2}}{\ln(t)^3} \tag{23}$$

```
> a:='a':g3:=Li(t)/6-a*sqrt(t)/log(t)^2: g31:=normal(diff(g3,t)):
g32:=normal(simplify(diff(g31,t),assume=positive)): " g3 = ",g3,"
g31 = ",g31,  "g32 = ",g32;
```

$$"\ g3 = ", \frac{1}{6}\,\mathrm{Li}(t) - \frac{a\sqrt{t}}{\ln(t)^2}, "\ g31 = ", \frac{1}{6}\,\frac{\sqrt{t}\,\ln(t)^2 - 3\,a\ln(t) + 12\,a}{\sqrt{t}\,\ln(t)^3}, "g32 = ", \tag{24}$$

$$-\frac{1}{12}\,\frac{-3\ln(t)^2 a + 2\sqrt{t}\,\ln(t)^2 + 72\,a}{t^{3/2}\ln(t)^4}$$

```
> b:='b': g4:=Li(t)/6-b*sqrt(t)/log(t)^3: g41:=normal(diff(g4,t)): "
g4 = ",g4," g41 = ",g41;
```

$$"\ g4 = ", \frac{1}{6}\,\mathrm{Li}(t) - \frac{b\sqrt{t}}{\ln(t)^3}, "\ g41 = ", \frac{1}{6}\,\frac{\sqrt{t}\,\ln(t)^3 - 3\,b\ln(t) + 18\,b}{\sqrt{t}\,\ln(t)^4} \tag{25}$$

```
> g5:=c*t^b3/log(t): g51:=normal(diff(g5,t)): "g5 = ",g5,"g51 = ",
g51,"exp(1/be[3])= ",evalf(exp(1/be[3]),8);
```

$$"g5 = ", \frac{c\,t^{b3}}{\ln(t)}, "g51 = ", \frac{c\,t^{b3}\,(\ln(t)\,b3 - 1)}{\ln(t)^2\,t}, "exp(1/be[3])= ", 11.127514 \tag{26}$$

```
> 1/2-0.18/log(2.)-2*tautau/sqrt(2.)/log(2.)^2; ############ end of
Section 2.4
```

$$0.10435044281317747287762680199 7 \tag{27}$$

```
> ##########  Section 2.5, Proposition 2.11 : Estimates of A(x)
###########
```

```
> x:=10.^8;-(2/300+0.0009*log(x)^5/sqrt(x)),2/300+log(2.)*log(x)
^3/sqrt(x);
   # proof of (2.51), cf. (2.46) and (2.47)
```

$$x := 1.00000000\ 10^8$$

$$-0.1975512646,\ 0.4399205738 \tag{28}$$

```
> 8-log(x)^3/x^(1/4)/8/evalf(Pi)-0.0009*log(x)^5/sqrt(x); # lower
bound of (2.52), cf. (2.49)
```

$$5.322107490 \tag{29}$$

```
> F1til(95),L(1,95)/(95./log(95.)^2), F2til(381),L(2,381)/(381./log
(381.)^3); # cf.(2.50)
```

$$1.785,\ 1.784109620,\ 4.05,\ 4.040414604 \tag{30}$$

```
> Q(5,10^8): # upper bound of (2.52) cf. (2.49) with kappa_1 = 5
```
"COMPUTATION of an upper bound of the bracket of (2.50)"

"ka1 = ", 5, "x = ", 100000000, "log x = ", 18.42068074, "sqrt(x) = ", 10000.0000

"T1 = ", 13.28822641

"[k,T2[k] = ", [[3, 4.250187540], [4, 1.314395987], [5, 0.6545064041]]

"T2sum = ", 6.219089931

"T3 = ", 3.597893553, "T4 = ", 1.468875169, "T5 = ", 0.002770234301, "T6 = ", 0.1908845979

"bracket <= ", 24.76773989 **(31)**

```
> ######
> ###################### Section 3, shc numbers ############
> ######
```

```
> setE(23); # computation of the set E, see (3.1) and (3.5)
```
"set E = ", [[1.0000, 2., 1.], [0.63093, 3., 1.], [0.58496, 2., 2.], [0.43068, 5., 1.], [0.41504, 2., 3.], [0.36907, 3., 2.], [0.35621, 7., 1.], [0.32193, 2., 4.], [0.28906, 11., 1.], [0.27024, 13., 1.], [0.26303, 2., 5.], [0.26186, 3., 3.], [0.25193, 5., 2.], [0.24465, 17., 1.], [0.23541, 19., 1.], [0.22239, 2., 6.], [0.22106, 23., 1.]] **(32)**

```
> shc(19,1): # List of shc numbers N with P+(N) <= 19, cf Table (3.20)
    # the list "lisshc" contains the first 89 shc numbers
```
"i=", 1, [1, 0], "eps = ", ∞, "xi = ", 1, "N = ", 1, "d(N) = ", 1

"i=", 2, [2, 1], "eps = ", 1.0000, "xi = ", 2, "N = ", 2, [2, 1], "d(N) = ", 2, (2)

"i=", 3, [3, 1], "eps = ", 0.63093, "xi = ", 3, "N = ", 6, [2, 1], [3, 1], "d(N) = ", 4, $(2)^2$

"i=", 4, [2, 2], "eps = ", 0.58496, "xi = ", 3.2705, "N = ", 12, [2, 2], [3, 1], "d(N) = ", 6, $(2)\,(3)$

"i=", 5, [5, 1], "eps = ", 0.43068, "xi = ", 5, "N = ", 60, [2, 2], [3, 1], [5, 1], "d(N) = ", 12, $(2)^2\,(3)$

"i=", 6, [2, 3], "eps = ", 0.41504, "xi = ", 5.3126, "N = ", 120, [2, 3], [3, 1], [5, 1], "d(N) = ", 16, $(2)^4$

"i=", 7, [3, 2], "eps = ", 0.36907, "xi = ", 6.5410, "N = ", 360, [2, 3], [3, 2], [5, 1], "d(N) = ", 24, $(2)^3\,(3)$

"i=", 8, [7, 1], "eps = ", 0.35621, "xi = ", 7, "N = ", 2520, [2, 3], [3, 2], [5, 1], [7, 1], "d(N) = ", 48, $(2)^4\,(3)$

"i=", 9, [2, 4], "eps = ", 0.32193, "xi = ", 8.6116, "N = ", 5040, [2, 4], [3, 2], [5, 1], [7, 1],
"d(N) = ", 60, $(2)^2\ (3)\ (5)$

"i=", 10, [11, 1], "eps = ", 0.28906, "xi = ", 11, "N = ", 55440, [2, 4], [3, 2], [5, 1], [7, 1],
[11, 1], "d(N) = ", 120, $(2)^3\ (3)\ (5)$

"i=", 11, [13, 1], "eps = ", 0.27024, "xi = ", 13, "N = ", 720720, [2, 4], [3, 2], 5, ... , 13,
"d(N) = ", 240, $(2)^4\ (3)\ (5)$

"i=", 12, [2, 5], "eps = ", 0.26303, "xi = ", 13.946, "N = ", 1441440, [2, 5], [3, 2], 5, ... , 13,
"d(N) = ", 288, $(2)^5\ (3)^2$

"i=", 13, [3, 3], "eps = ", 0.26186, "xi = ", 14.112, "N = ", 4324320, [2, 5], [3, 3], 5, ... , 13,
"d(N) = ", 384, $(2)^7\ (3)$

"i=", 14, [5, 2], "eps = ", 0.25193, "xi = ", 15.664, "N = ", 21621600, [2, 5], [3, 3], [5, 2], 7,
... , 13, "d(N) = ", 576, $(2)^6\ (3)^2$

"i=", 15, [17, 1], "eps = ", 0.24465, "xi = ", 17, "N = ", 367567200, [2, 5], [3, 3], [5, 2], 7,
... , 17, "d(N) = ", 1152, $(2)^7\ (3)^2$

"i=", 16, [19, 1], "eps = ", 0.23541, "xi = ", 19, "N = ", 6983776800, [2, 5], [3, 3], [5, 2], 7, **(33)**
... , 19, "d(N) = ", 2304, $(2)^8\ (3)^2$

```
> shcntype2(50,1): # compute the shc numbers of type 2 with largest
    # prime <= 50 (cf. between Lemma 3.12 and Lemma 3.13)
    # To compute the list of shc numbers N of type 2 with p < 2*
  10^8, execute
    # "hcn2HR:=shcntype2(2*10^8,100):",with Digits = 70, the running
  time is about one hour
```

"kmaxre=", 7.6525953, "kmax=", 7

" [q, al, eps, p, (log d(N))/log 2, log N, pi(p), theta(p)] "

"i=", 1, "T[i] = ", [2, 2, 0.5849625007, 3, 2.584962501, 2.484906651, 2, 1.791759470],
"time = ", 0.003

"i=", 2, "T[i] = ", [2, 3, 0.4150374989, 5, 4.000000000, 4.787491743, 3, 3.401197382],
"time = ", 0.003

"i=", 3, "T[i] = ", [3, 2, 0.3690702463, 5, 4.584962500, 5.886104032, 3, 3.401197382],
"time = ", 0.004

"i=", 4, "T[i] = ", [2, 4, 0.3219280948, 7, 5.906890595, 8.525161362, 4, 5.347107531], "time = ", 0.004

"i=", 5, "T[i] = ", [2, 5, 0.2630344058, 13, 8.169925001, 14.18115317, 6, 10.30995216],

"time = ", 0.004

"i=", 6, "T[i] = ", [3, 3, 0.2618595068, 13, 8.584962500, 15.27976546, 6, 10.30995216],

"time = ", 0.004

"i=", 7, "T[i] = ", [5, 2, 0.2519296365, 13, 9.169925001, 16.88920337, 6, 10.30995216],

"time = ", 0.004

"i=", 8, "T[i] = ", [2, 6, 0.2223924217, 19, 11.39231742, 23.36000287, 8, 16.08760448],

"time = ", 0.004

"i=", 9, "T[i] = ", [7, 2, 0.2083678469, 23, 12.97727992, 28.44140724, 9, 19.22309870],

"time = ", 0.004

"i=", 10, "T[i] = ", [3, 4, 0.2031140135, 29, 14.29920802, 32.90731536, 10, 22.59039453],

"time = ", 0.004

"i=", 11, "T[i] = ", [2, 7, 0.1926450780, 31, 15.49185310, 37.03444974, 11, 26.02438173],

"time = ", 0.004

"i=", 12, "T[i] = ", [5, 3, 0.1787469216, 47, 19.90689060, 53.57972535, 15, 40.96021943],

"time = ", 0.005

"total time = ", 0.005 **(34)**

```
> ######### small differences between two epsilons ################
```

```
> Digits:=30: mineps22(1,7738,hcn2HR):log(1.2)/log(101.)-log(1.5)/log
  (28669.);
     # smallest difference between two epsilons of type 2
```

"mineps22=", $5.78795504825783435071 1\ 10^{-10}$, "epsmax = ", [5, 101], "epsmin = ", [2, 28669]

$5.78795504825783435071 4\ 10^{-10}$ **(35)**

```
> mineps21(1,7739,hcn2HR);log(1.5)/log(62129.)-log(2.)/log(156383467.
  );
     # smallest difference between an epsilons of type 2 and an
  epsilon of type 1
```

"mineps21=", $1.658180660611488482\ 10^{-13}$, "epsmin = ", [2, 62129], [1, 156383467]

$1.658180660611488482\ 10^{-13}$ **(36)**

```
> pmax:=hcn2HR[7739][4]: pf:=evalf(pmax): mineps11:=2*log(2.)/(pf*log
  (pf)^2);
     # the difference between two epsilons of type 1 is bigger than
```

```
mineps11:
```

$$mineps11 := 1.89747174628194616821538836321\ 10^{-11} \quad (37)$$

```
> listN0:=calcullogN(10^8+7,1): logN0:=listN0[1]; llN0:=log(logN0);
    # Computation of xi_k^{(0)}, log(N_0) and log log(N_0) cf.
  (3.22), (3.23) and (3.24)
```

"kmaxre=", 37.842407

"k=", 1, "beta=", 1.000000000, "xi0=", $1.00000007\ 10^{8}$, "pp = ", 100000007, "np = ", 100000037

"k=", 2, "beta=", 0.5849625007, "xi0=", 47829.96065, "pp = ", 47819, "np = ", 47837

"k=", 3, "beta=", 0.4150374993, "xi0=", 2090.739897, "pp = ", 2089, "np = ", 2099

"k=", 4, "beta=", 0.3219280949, "xi0=", 376.2051781, "pp = ", 373, "np = ", 379

"k=", 5, "beta=", 0.2630344058, "xi0=", 127.1379647, "pp = ", 127, "np = ", 131

"k=", 6, "beta=", 0.2223924213, "xi0=", 60.13667022, "pp = ", 59, "np = ", 61

"k=", 7, "beta=", 0.1926450779, "xi0=", 34.76647259, "pp = ", 31, "np = ", 37

"k=", 8, "beta=", 0.1699250014, "xi0=", 22.87704975, "pp = ", 19, "np = ", 23

"k=", 9, "beta=", 0.1520030934, "xi0=", 16.44465445, "pp = ", 13, "np = ", 17

"k=", 10, "beta=", 0.1375035237, "xi0=", 12.59007143, "pp = ", 11, "np = ", 13

"k=", 11, "beta=", 0.1255308821, "xi0=", 10.09827191, "pp = ", 7, "np = ", 11

"k=", 12, "beta=", 0.1154772174, "xi0=", 8.391077705, "pp = ", 7, "np = ", 11

"k=", 13, "beta=", 0.1069152039, "xi0=", 7.166739760, "pp = ", 7, "np = ", 11

"k=", 14, "beta=", 0.09953567355, "xi0=", 6.255836526, "pp = ", 5, "np = ", 7

"k=", 15, "beta=", 0.09310940439, "xi0=", 5.557445827, "pp = ", 5, "np = ", 7

"k=", 16, "beta=", 0.08746284125, "xi0=", 5.008442967, "pp = ", 5, "np = ", 7

"k=", 17, "beta=", 0.08246216019, "xi0=", 4.567696968, "pp = ", 3, "np = ", 5

"k=", 18, "beta=", 0.07800251200, "xi0=", 4.207460993, "pp = ", 3, "np = ", 5

"k=", 19, "beta=", 0.07400058144, "xi0=", 3.908450840, "pp = ", 3, "np = ", 5

"k=", 20, "beta=", 0.07038932789, "xi0=", 3.656912974, "pp = ", 3, "np = ", 5

"k=", 21, "beta=", 0.06711419586, "xi0=", 3.442814068, "pp = ", 3, "np = ", 5

"k=", 22, "beta=", 0.06413033742, "xi0=", 3.258687399, "pp = ", 3, "np = ", 5

"k=", 23, "beta=", 0.06140054466, "xi0=", 3.098877146, "pp = ", 3, "np = ", 5

"k=", 24, "beta=", 0.05889368905, "xi0=", 2.959030994, "pp = ", 2, "np = ", 3

"k=", 25, "beta=", 0.05658352837, "xi0=", 2.835751880, "pp = ", 2, "np = ", 3

"k=", 26, "beta=", 0.05444778402, "xi0=", 2.726354177, "pp = ", 2, "np = ", 3 "k=", 27, "beta=", 0.05246741989, "xi0=", 2.628689926, "pp = ", 2, "np = ", 3

"k=", 28, "beta=", 0.05062607307, "xi0=", 2.541022942, "pp = ", 2, "np = ", 3

"k=", 29, "beta=", 0.04890960048, "xi0=", 2.461936263, "pp = ", 2, "np = ", 3

"k=", 30, "beta=", 0.04730571478, "xi0=", 2.390263169, "pp = ", 2, "np = ", 3

"k=", 31, "beta=", 0.04580368961, "xi0=", 2.325035125, "pp = ", 2, "np = ", 3

"k=", 32, "beta=", 0.04439411936, "xi0=", 2.265442044, "pp = ", 2, "np = ", 3

"k=", 33, "beta=", 0.04306872189, "xi0=", 2.210801632, "pp = ", 2, "np = ", 3

"k=", 34, "beta=", 0.04182017569, "xi0=", 2.160535501, "pp = ", 2, "np = ", 3

"k=", 35, "beta=", 0.04064198450, "xi0=", 2.114150388, "pp = ", 2, "np = ", 3

"k=", 36, "beta=", 0.03952836419, "xi0=", 2.071223260, "pp = ", 2, "np = ", 3

"k=", 37, "beta=", 0.03847414781, "xi0=", 2.031389408, "pp = ", 2, "np = ", 3

"k=", 38, "beta=", 0.03747470542, "xi0=", 1.994332857, "pp = ", 1, "np = ", 2

"xi=", 100000007, "log N=", $1.00037943869466\ 10^{8}$, "log d(N)/log 2=", $5.76451574240844\ 10^{6}$, "log d(N) = ", $3.99565783414383\ 10^{6}$, "log log N = ", 18.4210601106784

"list of exponents of N = ", [2, 37], [3, 23], [5, 16], [7, 13], [11, 10], [13, 9], [19, 8], [31, 7], [59, 6], [127, 5], [373, 4], [2089, 3], [47819, 2], [100000007, 1]

$$logN0 := 1.00037943869465743905928343761\ 10^{8}$$

$$llN0 := 18.4210601106783659628373398078 \tag{38}$$

```
> ############### proof of Lemma 3.13 ###################
```

```
> T3T5(10.^8+7,34): # cf. (3.25)
```

"k0 = ", 34, "T5 <= ", 0.1814976520, "T3 <= ", 1.361436428 **(39)**

```
> ############### Section 3.4 : highly composite numbers
  ##############
```

```
> highlycomp(1000,0,1); # computes the list [M,d(M)] of hc numbers M
  up to 1000
     # the list lishc10p100 contains the 1381 hc numbers M < 10^100
```

"p=", 3, "nops(lord) = ", 26, "nops(lis) =", 15

"p=", 5, "nops(lord) = ", 26, "nops(lis) =", 14

"p=", 7, "nops(lord) = ", 24, "nops(lis) =", 15

[[1, 1], [2, 2], [4, 3], [6, 4], [12, 6], [24, 8], [36, 9], [48, 10], [60, 12], [120, 16], [180, **(40)** 18], [240, 20], [360, 24], [720, 30], [840, 32]]

```
> ############### Section 3.5 : benefit ##############
```

```
> Digits:=10: N:=367567200; eps:=log(2.)/log(17.): print("eps = ",
  eps);
  bensmall(N,eps,0.13,N,2*N,0); # cf. Section 3.5
     # computes the list of [n,d(n),ben(n)] of numbers n, N <= n <=
  2*N with ben(n) <= 0.13
```

$$N := 367567200$$

"eps = ", 0.2446505421

"nops(liso)=", 286, "nops(lisocourte) = ", 23

[[367567200, 1152, 0.], [410810400, 1152, 0.0272114119], [428828400, 1080, 0.1022515681], [441080640, 1120, 0.0727759449], [454053600, 1080, 0.1162354056], [465585120, 1152, 0.0578326424], [479278800, 1080, 0.1294629800], [490089600, 1152, 0.0703815746], [492972480, 1120, 0.0999873568], [497296800, 1152, 0.0739531792], [514594080, 1152, 0.0823181152], [537213600, 1152, 0.0928423418], [547747200, 1152, 0.0975929865], [551350800, 1200, 0.0583752641], [563603040, 1152, 0.1045744097], [575134560, 1152, 0.1095295271], [581981400, 1152, 0.1124248330], [588107520, 1152, 0.1149866429], [605404800, 1152, 0.1220784593], [612612000, 1152, 0.1249737652], [616215600, 1200, 0.0855866760], [698377680, 1280, 0.0516693856], [735134400, 1344, 0.0154281536]] **(41)**

```
> ################ Section 3.7, estimates without RH, proof of Lemma
  3.18 ############
```

```
> evalf(1/be[5]); f:=0.8499*t-3.81*log(t);f1:=diff(f,t); t0:=solve
  (f1=0.,t);evalf(subs(t=11,f));
     # lower bound of (3.33), cf. (3.39)
```

3.801784017

$$f := 0.8499\, t - 3.81 \ln(t)$$

$$f1 := 0.8499 - \frac{3.81}{t}$$

$$t0 := 4.482880339$$

0.212919010 **(42)**

```
> eta:=7.5*10^(-7):(1+eta)*1.3615; # upper bound of (3.33)
```

1.361501021 **(43)**

```
> ub334:=1/(1-1/log(10.^8+7)^3); # upper bound of (3.34)
```

$ub334 := 1.000160012$ **(44)**

```
> A:=(1+eta)*(1+(10.^8+7)^(be[2]-1))+1.362*(10.^8+7)^(be[3]-1):
  lb334:=1/A; # lower bound of (3.34)
```

$lb334 := 0.9994927315$ **(45)**

```
> lb335:=1+0.00017/llN0: ub335:=1+log(0.99949)/llN0:
     "lb335 = ",evalf(lb335,10),"ub335 = ",evalf(ub335,10); # proof
  of (3.35)
```

"lb335 = ", 1.000009229, "ub335 = ", 0.9999723072 **(46)**

```
> numtheory[pi](376),74*be[4],(0.99949)^be[3]/1.00001; # lower bound
  of (3.36)
```

74, 23.82267902, 0.9997783015 **(47)**

```
> 1.00017^be[3]/0.99997*(1.15963+1.19768*(10.^8+7)^(be[4]-be[3])
  +1.25506*0.1815);
     # upper bound of (3.36)
```

1.603093648 **(48)**

```
> sqrt(1.00017)*1.00001^2/8/evalf(Pi);  # proof of (3.37)
```

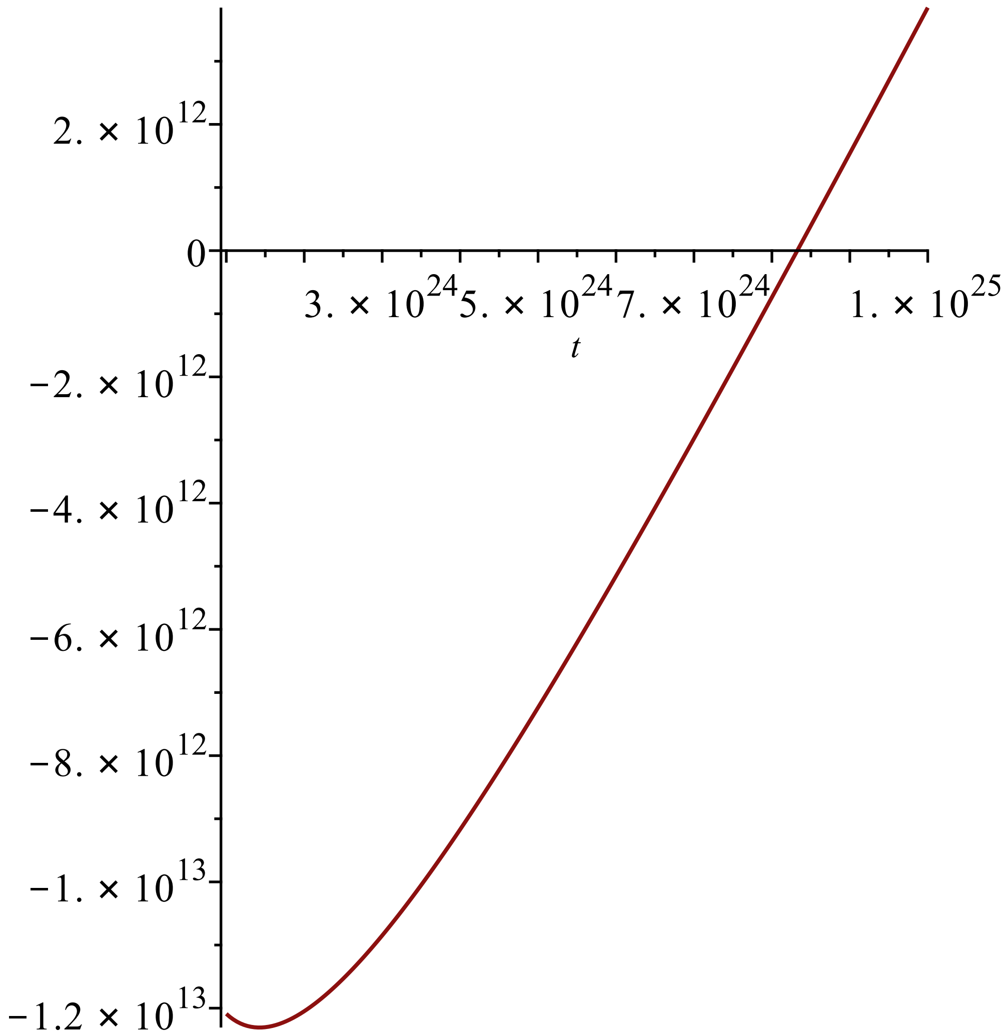
0.03979291349 **(49)**

```
> 1.00017^(be[2]/2)*(1.00001*be[2])^2/8/evalf(Pi); # proof of (3.38)
```

0.01361590369 **(50)**

```
> ######### Section 3.8, estimates under RH, proof of Lemma 3.20
  #########
```

```
> plot(t^be[2]-sqrt(t)*log(t)^2/8/evalf(Pi),t=10^24..10^25); #
  beginning of the proof of Lemma 3.20
```




```
> 0.9927*0.99949^be[2],(be[2]-1/2)^2*exp(2.)/4,-0.0398+0.9924*
  0.013334;
     # upper bound of (3.43), cf. (3.52)
```

0.9924038155, 0.01333470909, −0.0265673384 **(51)**

```
> eta:=7.5*10^(-7): 1.00017^be[2]*(1+eta+1.362*xi0[3]/xi0[2]);
     # lower bound of (3.43), cf. (3.53)
  exp(2/(be[2]-1/2)),1.06+(0.0398*log(10.^19/1.00017)^2)/(10.
  ^19/1.00017)^(be[2]-1/2);
```

1.059641759

$1.671875165\ 10^{10}$, 2.911556783 **(52)**

```
> 0.0266*be[2]; # upper bound of (3.44)
```

0.01556000252 **(53)**

```
> h0:=2.92/logN0^(1-be[2]):b:=(1-(1-h0)^be[2])/h0: # lower bound of
  (3.44), cf. (3.54)
  " h0 = ",h0, " b = ",b,0.5852*2.92;
```

" h0 = ", 0.001396414908, " b = ", 0.5851321089, 1.708784 **(54)**

```
> 0.0137+1.71/logN0^(1-3*be[2]/2)/llN0^2; # lower bound of (3.45)
                          0.01422710446                                    (55)
> 0.0137+0.0156/logN0^(1/2-be[2]/2);  # upper bound of (3.45), cf.
  (3.56)
                          0.01404114610                                    (56)
> 0.0141/be[2]; # upper bound of (3.46)
                          0.02410410920                                    (57)
> 1/1.008, be[2]-log(1.008)/llN0, 0.0143/0.5845; # lower bound of
  (3.46), cf. (3.57)
             0.9920634921, 0.5845299431, 0.02446535500                     (58)
> h0:=2.92/logN0^(1-be[2]):bb:=(1-(1-h0)^(1/2))/h0:
  " h0 = ",h0, " bb = ",bb,0.5002*2.92; # lower bound of (3.47)
     " h0 = ", 0.001396414908, " bb = ", 0.5001746945, 1.460584            (59)
> 0.9629*0.99949^be[3], 0.9134*0.99949^be[4]; # lower bound of (3.48)
                   0.9626961534, 0.9132500090                              (60)
> 1.362*1.00017^be[3], 4*exp(-2.)/(be[3]-be[2]/2)^2, 1.363+0.0141*
  36.05; # upper bound of (3.48)
               1.362096093, 36.04127180, 1.871305                          (61)
> 1+1.872/logN0^(be[2]-be[3]), 1/(1-1/logN0^(1-be[2])-1.872/logN0^(1-
  be[3]));
  A:=(1-0.0006/llN0)^2/1.0006: 1/A;
  0.5858/logN0^(be[3]-2*be[2]+1)/llN0; # lower bound of (3.49), cf.
  (3.58)
                   1.081823459, 1.000517622

                          1.000665185

                          0.0003479320029                                  (62)
> 0.04/sqrt(0.99949), 4*exp(-2.)/(be[2]-1/2)^2,0.0266*75,2.92*0.041;
  # proof of (3.50), cf. (3.59)
            0.04001020392, 74.99226216, 1.9950, 0.11972                    (63)
> ######
> ###################### Section 4, proofs of Theorem 1.1
  ############
> ######

> #########################  Proof of Proposition 4.1
  ###################

> 0.994 +0.0242*be[2]*llN0^2/logN0^(be[3]-be[2]/2)+(2.922/llN0+0.12*
  llN0+(7.48032)/llN0^2)/logN0^(be[3]+1/2-be[2]); # upper bound of B
                          1.501930765                                      (64)
```

```
> 0.546*llN0/logN0^be[3]+0.12*llN0/logN0^(be[3]-be[2]+1/2); # upper
  bound of A(xi), cf. (4.6)
```
0.009866332298 **(65)**

```
> 5.07*be[2]*1.00017^(be[2]/2)/(0.99997*be[2])^2,8.67/llN0/logN0^(be
  [3]-be[2]/2);
     # upper bound of be[2]*A(xi_2), cf. (4.7)
```
8.668173255, 0.04923050544 **(66)**

```
> -1.873-0.0099-0.05+0.9997, -(0.9332+0.0245*be[2]*llN0^2/logN0^(be
  [3]-be[2]/2));
     # lower bound of B'
```
-0.9332, -1.441889961 **(67)**

```
> 1.51+llN0/logN0^be[3]; # proof of Corollary 4.2
```
1.518809398 **(68)**

```
> ##################### Section 4, proof of Theorem 1.1 ###########
```

```
> Digits:=30: N3:=lishc10p100[1162][1]: print("N3 = M1162",impfact
  (N3),evalf(N3,8)):
  thm1_1(fonG1,N3,12,0); # Application of Lemma 3.16 with f = G_1 to
  end the proof of Lemma 4.3
```
"N3 = M1162", [2, 10], [3, 6], [5, 4], [7, 3], [11, 2], ... , [19, 2], 23, ... , 181,

$1.8997706\ 10^{86}$

"j=", 1125, "Mj=", [2, 10], [3, 6], [5, 4], [7, 3], [11, 2], ... , [19, 2], 23, ... , 179, "d(Mj)=",

1071508441006080

"f(logMj)-logdMj/log2 = ", -0.00003409330033

"jp1 = ", 1126, "Mjp1/Mj = ", $\frac{4}{3}$, "dMjp1/dMj = ", $\frac{78}{77}$, "f(logMjp1)-logdMjp1/log2 = ",

0.03696489

" N is the smallest shc number >= Mj"

"N/Mj = ", 1, "eps = ", 0.133621675, "logN = ", 193.4655541, "dN = ", 1071508441006080,

"logdN = ", 34.60784381

"logMj=", 193.465554127103, "f(logMj) = ", 49.9285305451666

"logmu =", 193.465730457522, "f(logmu)=", 49.928564638466888437

"logdMj/log2 = ", 49.928564638466888437, "logdMjp1/log2 = ", 49.9471803166342

"logMjp1=", 193.753236199555, "f(logMjp1) = ", 49.9841452138125

"log2*f(logMj) = ", 34.607820, "benmaxn=", 0.000047193241

"nops(liso)=", 34, "nops(lisocourte) = ", 4

"list of [n/Mj, d(n)/d(Mj), ben_eps(n)] with Mj <= n <= Mjp1 and ben_eps(n) <=
benmaxn + 0.01 = "

$$\left[[1, 1, 0.], \left[\frac{181}{179}, 1, 0.0014847005\right], \left[\frac{181}{173}, 1, 0.0060404262\right], \left[\frac{191}{179}, 1, 0.0086703927\right]\right]$$

"n/Mj = ", 1, "f(log n)-(log d(n))/log 2 = ", -0.00003409330033

"n/Mj = ", $\frac{181}{179}$, "f(log n)-(log d(n))/log 2 = ", 0.002114238120

"n/Mj = ", $\frac{181}{173}$, "f(log n)-(log d(n))/log 2 = ", 0.008706124391

"n/Mj = ", $\frac{191}{179}$, "f(log n)-(log d(n))/log 2 = ", 0.01251143548

"j=", 1125, "nmaxi/Mj=", 1, "d(nmaxi)/dMj=", 1, "bennmaxi=", 0., **(69)**
"f(nmaxi)-logdnmaxi/log2 = ", -0.00003409330033

```
> M1125:=lishc10p100[1125][1]: print("N2=M1125 = ",impfact(M1125),
  evalf(M1125,8)):
  thm1_1(fonH0,M1125,12,0);
     # Application of Lemma 3.16 with f=H_0 to end the proof of
  (1.8), Section 4.2 cf. (4.9)
```

"N2=M1125 = ", [2, 10], [3, 6], [5, 4], [7, 3], [11, 2], ... , [19, 2], 23, ... , 179,
$1.0495970\ 10^{84}$

"j=", 60, "Mj=", [2, 4], [3, 3], [5, 2], 7, ... , 17, "d(Mj)=", 960

"f(logMj)-logdMj/log2 = ", -0.02487234930

"jp1 = ", 61, "Mjp1/Mj = ", $\frac{4}{3}$, "dMjp1/dMj = ", $\frac{21}{20}$, "f(logMjp1)-logdMjp1/log2 = ",
0.026598307

" N is the smallest shc number >= Mj"

"N/Mj = ", 2, "eps = ", 0.244650542, "logN = ", 19.72241672, "dN = ", 1152, "logdN = ",
7.049254841

"logMj=", 19.0292695364837, "f(logMj) = ", 9.88201824630488

"logmu =", 19.0876861823733, "f(logmu)=", 9.9068905956085185293

"logdMj/log2 = ", 9.9068905956085185293, "logdMjp1/log2 = ", 9.97727992349992

"logMjp1=", 19.3169516089355, "f(logMjp1) = ", 10.0038782280108

"log2*f(logMj) = ", 6.8496931, "benmaxn=", 0.044274586

"nops(liso)=", 40, "nops(lisocourte) = ", 3

"list of [n/Mj, d(n)/d(Mj), ben_eps(n)] with Mj <= n <= Mjp1 and ben_eps(n) <=

benmaxn + 0.01 = "

$$\left[[1, 1, 0.0127427233], \left[\frac{19}{17}, 1, 0.0399541352\right], \left[\frac{4}{3}, \frac{21}{20}, 0.0343341337\right]\right]$$

"n/Mj = ", 1, "f(log n)-(log d(n))/log 2 = ", -0.02487234930

"n/Mj = ", $\frac{19}{17}$, "f(log n)-(log d(n))/log 2 = ", 0.02242833543

"n/Mj = ", $\frac{4}{3}$, "f(log n)-(log d(n))/log 2 = ", 0.02659830451

"j=", 60, "nmaxi/Mj=", 1, "d(nmaxi)/dMj=", 1, "bennmaxi=", 0.012742723, **(70)**

"f(nmaxi)-logdnmaxi/log2 = ", -0.02487234930

```
> M60:=183783600: print("M60 = ",M60,impfact(M60)); thm1_1(fonG2,M60,
  12,0);
    # Application of Lemma 3.16 with f=G_2 to end the proof of
  (1.10), Section 4.3.1, cf. (4.10)
```

"M60 = ", 183783600, [2, 4], [3, 3], [5, 2], 7, ... , 17

"j=", 58, "Mj=", [2, 5], [3, 2], [5, 2], 7, ... , 17, "d(Mj)=", 864

"f(logMj)-logdMj/log2 = ", -0.01235488515

"jp1 = ", 59, "Mjp1/Mj = ", $\frac{6}{5}$, "dMjp1/dMj = ", $\frac{28}{27}$, "f(logMjp1)-logdMjp1/log2 = ",

0.011226569

" N is the smallest shc number >= Mj"

"N/Mj = ", 3, "eps = ", 0.244650542, "logN = ", 19.72241672, "dN = ", 1152, "logdN = ",

7.049254841

"logMj=", 18.6238044283756, "f(logMj) = ", 9.74253261701293

"logmu =", 18.6533583495846, "f(logmu)=", 9.75488750216346854444

"logdMj/log2 = ", 9.75488750216346854444, "logdMjp1/log2 = ", 9.80735492205760

"logMjp1=", 18.8061259851695, "f(logMjp1) = ", 9.81858149152794

"log2*f(logMj) = ", 6.7530090, "benmaxn=", 0.034700117

"nops(liso)=", 30, "nops(lisocourte) = ", 2

"list of [n/Mj, d(n)/d(Mj), ben_eps(n)] with Mj <= n <= Mjp1 and ben_eps(n) <=

benmaxn + 0.01 = "

$$\left[[1, 1, 0.0189059801], \left[\frac{6}{5}, \frac{28}{27}, 0.0271434041\right]\right]$$

"n/Mj = ", 1, "f(log n)-(log d(n))/log 2 = ", -0.01235488515

$$"n/Mj = ", \frac{6}{5}, "f(log n)-(log d(n))/log 2 = ", 0.01122656947$$

"j=", 58, "nmaxi/Mj=", 1, "d(nmaxi)/dMj=", 1, "bennmaxi=", 0.018905980, **(71)**

"f(nmaxi)-logdnmaxi/log2 = ", -0.01235488515

```
> evalf(1.52^(1/(be[2]-be[3])),10);
  M60:=183783600: print("M60 = ",M60,impfact(M60)); thm1_1(fonG3,
  lishc10p100[60][1],1,0);
     # Application of Lemma 3.16 with f=G_3 to end the proof of
  (1.11), Section 4.3.2
```

11.75277106

"M60 = ", 183783600, [2, 4], [3, 3], [5, 2], 7, ... , 17

"j=", 6, "Mj=", [2, 3], [3, 1], "d(Mj)=", 8

"f(logMj)-logdMj/log2 = ", -0.09552541021

$$"jp1 = ", 7, "Mjp1/Mj = ", \frac{3}{2}, "dMjp1/dMj = ", \frac{9}{8}, "f(logMjp1)-logdMjp1/log2 = ",$$

0.185536359

" N is the smallest shc number >= Mj"

$$"N/Mj = ", \frac{5}{2}, "eps = ", 0.430676558, "logN = ", 4.094344562, "dN = ", 12, "logdN = ",$$

2.484906650

"logMj=", 3.17805383034795, "f(logMj) = ", 2.90447458979213

"logmu =", 3.26016009742919, "f(logmu)=", 3.0000000000000000000

"logdMj/log2 = ", 3.0000000000000000000, "logdMjp1/log2 = ", 3.16992500144231

"logMjp1=", 3.58351893845611, "f(logMjp1) = ", 3.35546135993709

"log2*f(logMj) = ", 2.0132284, "benmaxn=", 0.11241458

"nops(liso)=", 12, "nops(lisocourte) = ", 3

"list of [n/Mj, d(n)/d(Mj), ben_eps(n)] with Mj <= n <= Mjp1 and ben_eps(n) <=

benmaxn + 0.01 = "

$$\left[[1, 1, 0.0108401697], \left[\frac{5}{4}, 1, 0.1069428661\right], \left[\frac{3}{2}, \frac{9}{8}, 0.0676814514\right]\right]$$

"n/Mj = ", 1, "f(log n)-(log d(n))/log 2 = ", -0.09552541021

$$"n/Mj = ", \frac{5}{4}, "f(log n)-(log d(n))/log 2 = ", 0.1588672278$$

$$"n/Mj = ", \frac{3}{2}, "f(log n)-(log d(n))/log 2 = ", 0.1855363585$$

(72)

"j=", 6, "nmaxi/Mj=", 1, "d(nmaxi)/dMj=", 1, "bennmaxi=", 0.010840170, **(72)**

"f(nmaxi)-logdnmaxi/log2 = ", -0.09552541021

```
> lisG3n:=[1]: # list of exceptions <= 35 for (1.11), cf. Section
  4.3.2
  for n from 2 to 35 do
     logn:=log(evalf(n)): dn:=numtheory[tau](n): ldnsl2:=log(evalf
  (dn))/log2;
     dif:=fonG3(logn)-ldnsl2:
     if dif < 0 then lisG3n:=[op(lisG3n),n]: fi:
  od: lisG3n;
```

[1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 18, 24] **(73)**

```
> evalf(exp(2/(1/2-be[3])),10); # beginning of the proof of (1.12),
  Section 4.3.3
  evalf(subs(t=1.56*10^17,t^(1/2-be[3])/log(t)*(2-tautau+5.12/log(t))
  ),10);
```

$1.671875165\ 10^{10}$

1.520262334 **(74)**

```
> M1125:=lishc10p100[1125][1]: print("N2=M1125 = ",impfact(M1125),
  evalf(M1125,8)):
  thm1_1(fonF,M1125,12,0);
     # Application of Lemma 3.16 with f=F to end the proof of (1.12),
  Section 4.3.3
```

"N2=M1125 = ", [2, 10], [3, 6], [5, 4], [7, 3], [11, 2], ... , [19, 2], 23, ... , 179,

$1.0495970\ 10^{84}$

"j=", 44, "Mj=", [2, 5], [3, 3], 5, ... , 13, "d(Mj)=", 384

"f(logMj)-logdMj/log2 = ", -0.03479614862

"jp1 = ", 45, "Mjp1/Mj = ", $\frac{3}{2}$, "dMjp1/dMj = ", $\frac{25}{24}$, "f(logMjp1)-logdMjp1/log2 = ",

0.071636484

" N is the smallest shc number >= Mj"

"N/Mj = ", 1, "eps = ", 0.261859507, "logN = ", 15.27976546, "dN = ", 384, "logdN = ",

5.950642553

"logMj=", 15.2797654605534, "f(logMj) = ", 8.55016635210030

"logmu =", 15.3647087540624, "f(logmu)=", 8.5849625007211561815

"logdMj/log2 = ", 8.5849625007211561815, "logdMjp1/log2 = ", 8.64385618977472

"logMjp1=", 15.6852305686615, "f(logMjp1) = ", 8.71549267821705

"log2*f(logMj) = ", 5.9265237, "benmaxn=", 0.046362061 "nops(liso)=", 28, "nops(lisocourte) = ", 1

"list of [n/Mj, d(n)/d(Mj), ben_eps(n)] with Mj <= n <= Mjp1 and ben_eps(n) <= benmaxn + 0.01 = "

$$[[1, 1, 0.]]$$

"n/Mj = ", 1, "f(log n)-(log d(n))/log 2 = ", -0.03479614862

"j=", 44, "nmaxi/Mj=", 1, "d(nmaxi)/dMj=", 1, "bennmaxi=", 0., "f(nmaxi)-logdnmaxi/log2 = ", -0.03479614862 **(75)**

```
> M1125:=lishc10p100[1125][1]: print("N2=M1125 = ",impfact(M1125),
  evalf(M1125,8)):
  thm1_1(fonFmR,M1125,12,0);
     # Application of Lemma 3.16 with f=F-R to end the proof of
  (1.13), Section 4.3.4
```

"N2=M1125 = ", [2, 10], [3, 6], [5, 4], [7, 3], [11, 2], ... , [19, 2], 23, ... , 179, $1.0495970\ 10^{84}$

"j=", 975, "Mj=", [2, 10], [3, 6], [5, 4], [7, 3], [11, 2], ... , [19, 2], 23, ... , 157, "d(Mj)=", 66969277562880

"f(logMj)-logdMj/log2 = ", -0.0005356266061

"jp1 = ", 976, "Mjp1/Mj = ", $\frac{4}{3}$, "dMjp1/dMj = ", $\frac{78}{77}$, "f(logMjp1)-logdMjp1/log2 = ", 0.03788281

" N is the smallest shc number >= Mj"

"N/Mj = ", 1, "eps = ", 0.137087319, "logN = ", 172.9131327, "dN = ", 66969277562880, "logdN = ", 31.83525509

"logMj=", 172.913132713541, "f(logMj) = ", 45.9280290118607

"logmu =", 172.915844250895, "f(logmu)=", 45.928564638466888437

"logdMj/log2 = ", 45.928564638466888437, "logdMjp1/log2 = ", 45.9471803166342

"logMjp1=", 173.200814785993, "f(logMjp1) = ", 45.9850631281111

"log2*f(logMj) = ", 31.834884, "benmaxn=", 0.00074298546

"nops(liso)=", 108, "nops(lisocourte) = ", 4

"list of [n/Mj, d(n)/d(Mj), ben_eps(n)] with Mj <= n <= Mjp1 and ben_eps(n) <= benmaxn + 0.01 = "

$$\left[[1, 1, 0.], \left[\frac{163}{157}, 1, 0.0051413771\right], \left[\frac{167}{157}, 1, 0.0084648687\right], \left[\frac{163}{151}, 1, 0.0104831172\right]\right]$$

"n/Mj = ", 1, "f(log n)-(log d(n))/log 2 = ", -0.0005356266061 "n/Mj = ", $\frac{163}{157}$, "f(log n)-(log d(n))/log 2 = ", 0.006872551797

"n/Mj = ", $\frac{167}{157}$, "f(log n)-(log d(n))/log 2 = ", 0.01166101228

"n/Mj = ", $\frac{163}{151}$, "f(log n)-(log d(n))/log 2 = ", 0.01456875904

"j=", 975, "nmaxi/Mj=", 1, "d(nmaxi)/dMj=", 1, "bennmaxi=", 0., "f(nmaxi)-logdnmaxi/log2 = ", -0.0005356266061 **(76)**

```
>     #################### THE END ####################
```