APPENDIX: On Composite Numbers

By

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Highly composite numbers n are positive integers satisfying

$$d(n) > d(m) \text{ for all } m < n, \tag{1}$$

where d is the divisor function. Srinivasa Ramanujan studied highly composite numbers in great detail and his long paper [3] is quite famous. But there was much work on highly composite numbers and related topics that Ramanujan did not publish. During his centennial in December 1987, the first published copy [2] of his Lost Notebook and other unpublished papers was released and in this impressive volume a manuscript of Ramanujan on highly composite numbers (previously unpublished) is included (pages 280-308). It is to be noted, however, that at the top of page 295 of [2] the words - "Middle of another paper" is not handwritten by Ramanujan. A short analysis of this manuscript on highly composite numbers is given in [1] p. 238-239.

The table on page 280 of [2] is not a list of highly composite numbers. This table almost coincides with the list of largely composite numbers n which satisfy the weaker inequality

$$d(n) \geq d(m) \text{ for all } m \leq n. \tag{2}$$

Note the slight difference between (1) and (2). There are only four largely composite numbers which were omitted by Ramanujan in this table, namely, 4200, 151200, 415800, 491400. Also, as J. P. Massias has pointed out, the number 15080 in this table is not largely composite.

In this unpublished manuscript Ramanujan also has some very interesting results on $\sigma(n)$, the sum of the divisors of n. In this context we point out a result due to Robin [4] that $\sigma(n) < e^{\gamma}$ nloglogn for $n > 5041$. Here γ is Euler's constant. More precisely he showed that

$$\frac{\sigma(N)}{N\log\log N} < e^{\gamma}\exp\left\{\frac{2(1-\sqrt{2})+c}{\sqrt{x}\log x} + O\left(\frac{1}{\sqrt{x}\log^2 x}\right)\right\}, \tag{3}$$

where

$$c = \gamma + 2 - \log 4\pi.$$

In (3), N is a collossaly abundant number of parameter x and for such n we have

$$\log N = \sum_{\substack{p<x\\ p=\text{prime}}} \log p + O(\sqrt{x}) = x + O(\sqrt{x}\log^2 x) \tag{4}$$

under the assumption of the Riemann Hypothesis. Using (4) we may rewrite (3) as

$$\frac{\sigma(N)}{N\log\log N} < e^{\gamma}\left\{1 + \frac{2(1-\sqrt{2})+c}{\sqrt{\log N}\,\log\log N} + O\left(\frac{1}{\sqrt{\log N}\,(\log\log N)^2}\right)\right\}. \tag{5}$$

Ramanujan wrote down a similar formula about seventy years earlier with the notation $\Sigma_{-1}(N)$ for the maximal order of $\frac{\sigma(N)}{N}$ (see [2], p. 303):

$$\overline{\lim}\,(\Sigma_{-1}(N) - e^{\gamma}\log\log N)\sqrt{\log N} < e^{\gamma}(2\sqrt{2} + \gamma - \log 4\pi). \tag{6}$$

Unfortunately (5) and (6) do not agree; it seems that in formula (382) of Ramanujan ([2], p. 303) the sign of the term $2(\sqrt{2}-1)/\sqrt{\log N}$ is wrong and so the right hand side of (6) should read

$$e^{\gamma}(\gamma - \log 4\pi + 2(2-\sqrt{2})).$$

The wrong sign seems to come from Ramanujan's analysis of his formula (377) of [2]. As Ramanujan explains at the beginning of §71, p. 302 of [2], the term $(\log N)^{1/2 - s}/\log\log N$ arises from four terms of formula (377) and in formula (379) the coefficient of this term has the wrong sign!

In the same manuscript Ramanujan has a very nice estimation of the maximal order of $\sigma(n)/n^s$ for all s, which is not at all easy to obtain. This result of Ramanujan on the maximal order of $\sigma(n)/n^s$ for $s \neq 1$ under the assumption of the Riemann Hypothesis is new (and has not yet been rediscovered!) and it will definitely be worthwhile to look into this further. I hope to do this on a later occasion.

REFERENCES

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3) S. Ramanujan, Highly composite numbers, Proc. London Math. Soc., 2, 14 (1915), p. 347-409.

4) G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. pures et appl., 63 (1984), p. 187-213.

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