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Some Open Questions

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Abstract. In this paper, several longstanding problems that the author has tried to solve, are described. An exposition of these questions was given in Luminy in January 2002, and now three years later the author is pleased to report some progress on a couple of them.

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1. Introduction

In January 2002, Christian Mauduit, Joel Rivat and András Sárközy kindly organized in Luminy a meeting for my sixtieth birthday and asked me to give a talk on the 17th, the day of my birthday. I presented six problems on which I had worked unsuccessfully. In the next six Sections, these problems are exposed.

It is a good opportunity to thank sincerely all the mathematicians with whom I have worked, with a special mention to Paul Erdős from whom I have learned so much.

1.1. Notation

p, q, q_i denote primes while p_i denotes the i-th prime. The p-adic valuation of the integer N is denoted by $v_p(N)$: it is the largest exponent α such that p^α divides N. The following classical notation for arithmetical function is used: φ for Euler's function and, for the number and the sum of the divisors of N,

$$\tau(N) = \sum_{d \mid N} 1, \quad \sigma_r(N) = \sum_{d \mid N} d^r, \quad \sigma(N) = \sigma_1(N). \tag{1.1}$$

When $x \to x_0$, $f(x) \ll g(x)$ means $f(x) = \mathcal{O}(g(x))$.

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2. Highly composite numbers

Let f be an arithmetic function. An integer N is called f-champion if

$$M < N \Rightarrow f(M) < f(N). \tag{2.1}$$

The notion of champion number was first introduced by S. Ramanujan in his thesis [35] where he defined and studied *highly composite* numbers as τ-champion numbers. In particular, he has shown that, if N_i denotes the i-th highly composite number, then $N_{i+1}/N_i = 1 + \mathcal{O}(\frac{(\log\log\log N_i)^{3/2}}{\sqrt{\log\log N_i}})$ from which it follows that the number $Q_\tau(X)$ of highly composite numbers up to X is not of the form

$$Q_\tau(X) = o\left(\frac{\log X\sqrt{\log\log X}}{(\log\log\log X)^{3/2}}\right).$$

P. Erdős has shown in [9] that

$$Q_\tau(X) \gg (\log X)^{1+\delta}$$

for some $\delta > 0$ by using the recent Theorem of Hoheisel [19]: there exists λ, $0 < \lambda < 1$ such that

$$\pi(x + x^\lambda) - \pi(x) \gg \frac{x^\lambda}{\log x} \tag{2.2}$$

where $\pi(x)$ is the number of primes up to x. The value of λ given by Hoheisel was very close to 1, but the best known value of λ for which (2.2) holds is now $\lambda = 0.525$ (cf. [2]).

I proved in [30] (see also [45]) that

$$Q_\tau(X) = \mathcal{O}(\log X)^c \tag{2.3}$$

for some c. One of the tools was the recent progress on linear forms of logarithms and more precisely the result of Feldman [13] that

$$\theta = \frac{\log 3}{\log 2} \tag{2.4}$$

cannot be approximated too closely by rational numbers. The value $c = 1.71$ is given in [31], and it is conjectured in [30] that (2.3) holds for any $c > \frac{\log 30}{\log 16} = 1.227\ldots$

A multiplicative function f satisfies property $\mathcal{P}$ if $f(p^\alpha)$ does not depend on the prime p but only on the exponent α. The number $\tau(N)$ of divisors of N satisfies property $\mathcal{P}$ since $\tau(p^\alpha) = \alpha + 1$.

In (2.4), 3 and 2 occur as the values of $\tau(p^2)$ and $\tau(p)$. Let us consider a function f satisfying $\mathcal{P}$ and such that $f(p^\alpha)$ grows like $\tau(p^\alpha) = \alpha + 1$, but such that $\frac{\log f(p^2)}{\log f(p)}$ is a Liouville number. Then the method of proof of (2.3) given in [30] (see also [45]) no longer works, and it is not known whether the number $Q_f(X)$ of f-champion numbers up to X

satisfies

$$Q_f(X) = \mathcal{O}(\log X)^{c_f} \tag{2.5}$$

for some constant c_f.

Let us call $\mathcal{W}$ the set of positive integers with non increasing exponents in their standard factorization into primes, more precisely

$$\mathcal{W} = \{N > 0; \quad p \mid N,\ q \mid N \quad \text{and} \quad p < q \Rightarrow v_p(N) \geq v_q(N)\}. \tag{2.6}$$

The number $W(X)$ of elements of $\mathcal{W}$ up to X has been investigated by G. H. Hardy and S. Ramanujan who show in [17] that

$$\log W(X) = (1 + o(1))\frac{2\pi}{\sqrt{3}}\sqrt{\frac{\log X}{\log\log X}}, \quad X \to +\infty.$$

If f is a multiplicative function satisfying property $\mathcal{P}$, then, clearly, f-champion numbers belong to the set $\mathcal{W}$, and therefore, $W(X)$ is an upper bound for $Q_f(X)$. Is it possible to find a smaller upper bound for $Q_f(X)$ which is valid for any f?

Let $\mathcal{N}_k$ the set of integers of the form $2^{\alpha_1}3^{\alpha_2}\dots p_k^{\alpha_k}$ with $\alpha_i \geq 0$. Let us call an integer $N \in \mathcal{N}_k$ *k-highly composite* if

$$M < N \quad \text{and} \quad M \in \mathcal{N}_k \Rightarrow \tau(M) < \tau(N)$$

and let $\mathcal{Q}_k(X)$ be the number of k-highly composite numbers up to X. In [3], by using the continued fraction expansion of $\theta = \frac{\log 3}{\log 2}$, it was shown that

$$\frac{4}{3} \leq \underline{\lim}\frac{\log \mathcal{Q}_2(X)}{\log X} \leq \overline{\lim}\frac{\log \mathcal{Q}_2(X)}{\log X} \leq \frac{3}{2} \tag{2.7}$$

but no estimation for $\mathcal{Q}_k(X)$ is given for $k \geq 3$.

For $k \geq 2$, let us define the multiplicative function f_k by

$$f_k(p^\alpha) = \begin{cases} \tau(p^\alpha) = \alpha + 1 & \text{for } \alpha \leq k \\ 0 & \text{for } \alpha > k. \end{cases}$$

Perhaps, the method of [3] can be used to investigate $Q_{f_2}(X)$, but it seems difficult to find a good estimate for $\overline{\lim}_{X\to\infty}\frac{\log Q_{f_k}(X)}{\log X}$ when $k > 2$.

A superabundant number is a champion for the function $\sigma_{-1}(n) = \frac{\sigma(n)}{n}$. In my first joint paper with P. Erdős [10], we proved that there exists a positive constant δ such that the number of superabundant numbers up to X satisfies $Q_{\sigma_{-1}}(X) \geq (\log X)^{1+\delta}$ for X large enough, but we were unable to show that $Q_{\sigma_{-1}}(X) \ll (\log X)^{\Delta}$ for some Δ.

3. Hyper champions

Let f be a non negative multiplicative function, and assume that there are only finitely many pairs (q, k) such that $f(q^k) > 1$:

$$\begin{aligned} &q_1;\ k_{1,1}, k_{1,2}, \dots, k_{1,\ell_1} \\ &q_2;\ k_{2,1}, k_{2,2}, \dots, k_{2,\ell_2} \\ &\ \ \vdots \\ &q_r;\ k_{r,1}, k_{r,2}, \dots, k_{r,\ell_r}. \end{aligned}$$

For each i, $1 \le i \le r$, let us denote by α_i one of the indices j, $1 \le j \le \ell_i$ such that $f(q_i^{k_{i,j}})$ is maximal on j. Then, clearly, f will be maximal on N,

$$N = \prod_{i=1}^{r} q_i^{\alpha_i}$$

(see [18], chap. 18).

For $\varepsilon > 0$, let us set $f(n) = \frac{\sigma(n)}{n^{1+\varepsilon}}$ where $\sigma(n) = \sum_{d \mid n} d$; f is multiplicative,

$$f(q^k) = \frac{1 + q + \dots + q^k}{q^{k(1+\varepsilon)}} = \frac{1 + \frac{1}{q} + \dots + \frac{1}{q^k}}{q^{k\varepsilon}} < \frac{2}{(q^k)^\varepsilon}$$

and, in order that $f(q^k) > 1$, one should have $q^k < 2^{1/\varepsilon}$, which is satisfied by only finitely many pairs (q, k) and, therefore, $f(n)$ is bounded.

In [1] (see also [36, 10]), an integer N for which there exists $\varepsilon > 0$ such that for all $M \ge 1$

$$\frac{\sigma(M)}{M^{1+\varepsilon}} \le \frac{\sigma(N)}{N^{1+\varepsilon}}$$

holds, is called *colossally abundant*. For instance, 60 is colossally abundant for any ε satisfying

$$0.09954\ldots = \frac{\log 15/14}{\log 2} \le \varepsilon \le \frac{\log 6/5}{\log 5} = 0.11328\ldots$$

It is a classical result that (see [18], chap. 18)

$$\overline{\lim} \frac{\sigma(n)}{n \log\log n} = e^\gamma \tag{3.1}$$

where $\gamma = 0.577\ldots$ is Euler's constant. From (3.1), it follows that for all $u > 0$,

$$\lim \frac{\sigma(n)}{n(\log n)^u} = 0$$

and thus, $\frac{\sigma(n)}{n(\log n)^u}$ is bounded. Let us call *hyper abundant* an integer N for which there exists u such that for all $M \geq 1$

$$\frac{\sigma(M)}{M(\log M)^u} \leq \frac{\sigma(N)}{N(\log N)^u}. \tag{3.2}$$

It is easy to see that a hyper abundant number is colossally abundant: we have

$$\log \frac{\sigma(n)}{n} - \varepsilon \log\log n = \left[\log \frac{\sigma(n)}{n} - u \log\log n \right] + \{u \log\log n - \varepsilon \log n\}. \tag{3.3}$$

If N is a hyper abundant number with parameter u, the bracket is maximal on N, and, if we choose $\varepsilon = \frac{u}{\log N}$, the function $t \mapsto u \log\log t - \varepsilon \log t$, and thus also the above curly bracket, is maximal on N; therefore, (3.3) implies that N is colossally abundant with parameter ε.

S. Ramanujan first introduced the above notion (see [35, 36]): he defined N to be *superior highly composite* if there exists $\varepsilon > 0$ such that for all $M \geq 1$

$$\frac{\tau(M)}{M^\varepsilon} \leq \frac{\tau(N)}{N^\varepsilon}$$

holds. Hyper composite numbers were defined by G. Robin in [39] as the numbers maximizing

$$\log \tau(n) - \alpha \frac{\log n}{\log\log n} \tag{3.4}$$

for some $\alpha > 0$. G. Robin shows that there are infinitely many superior highly composite numbers which are not hyper composite, and that the number of hyper composite numbers up to X is larger than $(\log X)^{5/12}$ for X large enough (as shown by S. Ramanujan in [35], the number of superior highly composite numbers up to X is asymptotically equal to $\log X$). P. Erdős was very interested by these hyper composite numbers which have not yet been investigated too much. Probably, the method used by G. Robin in [39] can be adapted to study hyper abundant numbers defined by (3.2), but, as far as I know, nobody did it.

The formula

$$\overline{\lim} \frac{\log \tau(n)}{\log n / \log\log n} = \log 2$$

is often enounced by saying that the *maximal order* of $\log \tau(n)$ is $\frac{\log 2 \log n}{\log\log n}$ (see [18], Th. 317 and [31], §VIII). But this notion is not very precise: it is not easy to select a set of numbers for which $\tau(n)$ is very large: the set of highly composite numbers contains the set of superior highly composite numbers which contains the set of hyper composite numbers maximizing (3.4). And, in (3.4), we may change the function $\frac{\log n}{\log\log n}$ to obtain smaller and smaller sets on which $\tau(n)$ is larger and larger.

4. Does $\mathcal{H}(n) < \tau(n)$ hold for $n > 5040$?

Let us denote by $1 = d_1 < d_2 < \cdots < d_{\tau(n)} = n$ the increasing sequence of divisors of n, and by

$$\mathcal{H}(n) \stackrel{def}{=\!=} \sum_{1 \le i < j \le \tau(n)} \frac{1}{d_j - d_i}, \quad n \ge 2. \tag{4.1}$$

It is conjectured in [11] (cf. also [25], problem 23, p. 200) that

$$\mathcal{H}(n) < \tau(n) \quad \text{for } n \notin \mathcal{S} \tag{4.2}$$

with $\mathcal{S} = \{12, 24, 60, 120, 180, 240, 360, 420, 720, 840, 1260, 1680, 2520, 5040\}$. Conjecture (4.2) has been checked up to $n = 10^6$ (for $200000 \le n \le 10^6$, $\mathcal{H}(n) \le \frac{4}{5}\tau(n)$ holds). The following very nice upper bound for $\mathcal{H}(n)$ is given by G. Tenenbaum and R. de la Bretèche (cf. [44], Lemme 1, and [4], Théorème 2):

$$\mathcal{H}(n) \le (2\log\tau(n) + 1 - \log 2)\frac{\sigma(n)}{n}\tau(n)^c \tag{4.3}$$

where

$$c = \frac{\log 3}{\log 2} - \frac{2}{3} = 0.9182958\ldots \tag{4.4}$$

In fact, Tenenbaum and de la Bretèche prove

$$\mathcal{H}(n) \le \sum_{m \mid n} \frac{1}{m}\left(\tau\left(\frac{n}{m}\right)\right)^c \left(\sum_{1 \le s \le (\tau(n/m))^2/2} \frac{1}{s}\right) \tag{4.5}$$

which, through the inequality $\sum_{1 \le s \le t} \frac{1}{s} \le 1 + \log t$, yields (4.3).

G. Tenenbaum observes in [44] that

$$\frac{\sigma(n)}{n} \ll \log\log(3\tau(n)) \tag{4.6}$$

which, with (4.3), shows that, for $\tau(n)$ large enough, $\mathcal{H}(n) \le \tau(n)$ holds. Effective forms of (4.6) can be found in [33].

Let ε be a fixed positive real number. For which n is the multiplicative function $f(n) = \frac{n}{\varphi(n)\tau(n)^\varepsilon}$ maximal? Since $f(p^\alpha) = \frac{p}{(p-1)(\alpha+1)^\varepsilon}$ is decreasing in α, such an n should be squarefree; and, as $f(p) = \frac{p}{(p-1)2^\varepsilon}$ is smaller than 1 if $p > \frac{1}{1-2^{-\varepsilon}}$, the maximum is attained in

$$N_\varepsilon = \prod_{p \le \frac{1}{1-2^{-\varepsilon}}} p \tag{4.7}$$

and one has, for all $n \geq 1$,

$$\frac{n}{\varphi(n)\tau(n)^{\varepsilon}} \leq \frac{N_{\varepsilon}}{\varphi(N_{\varepsilon})\tau(N_{\varepsilon})^{\varepsilon}} = \prod_{p \leq \frac{1}{1-2^{-\varepsilon}}} \frac{p}{(p-1)2^{\varepsilon}}. \tag{4.8}$$

Let us choose $\varepsilon = 0.00186$. From the classical inequality $\frac{\sigma(n)}{n} \leq \frac{n}{\varphi(n)}$ and from (4.8), we have, for $n \geq 1$,

$$\frac{\sigma(n)}{n} \leq \frac{n}{\varphi(n)} \leq \left(\prod_{p \leq 773} \frac{p}{(p-1)2^{\varepsilon}} \right) \tau(n)^{\varepsilon} \leq 9.996\, \tau(n)^{\varepsilon} \leq 10\, \tau(n)^{\varepsilon}. \tag{4.9}$$

Since the function $t \mapsto \frac{2\log t + 1 - \log 2}{t^{1-c-\varepsilon}}$ is decreasing for $t > 2.4 \cdot 10^5$ and is smaller than $1/10$ for $t > 1.11 \cdot 10^{41}$, it follows from (4.3) and (4.9) that

$$\tau(n) > 1.11 \cdot 10^{41} \Rightarrow \mathcal{H}(n) < \tau(n). \tag{4.10}$$

The value $1.11 \cdot 10^{41}$ can be slightly shortened by applying classical techniques on champion numbers, and by using (4.5) instead of (4.3), but not below $1.77 \cdot 10^{25}$. Indeed, the upper bound in 4.5 is certainly not less than $\tau(n)^c \sum_{s \leq \tau(n)^2/2} \frac{1}{s} \geq \tau(n)^c(2\log\tau(n) - \log 2)$ which is bigger than $\tau(n)$ for $\tau(n) < 1.77 \cdot 10^{25}$. As there are infinitely many n's such that $\tau(n) \leq 1.77 \cdot 10^{25}$, the question "*are there finitely many exceptions to the inequality* $\mathcal{H}(n) < \tau(n)$" remains open.

Choosing $\varepsilon = 0.01749$, (4.9) becomes

$$\frac{\sigma(n)}{n} \leq \frac{n}{\varphi(n)} \leq \left(\prod_{p \leq 79} \frac{p}{(p-1)2^{\varepsilon}} \right) \tau(n)^{\varepsilon} \leq 6.1541711\, \tau(n)^{\varepsilon} \tag{4.11}$$

and so,

$$\frac{\mathcal{H}(n)}{\tau(n)} \leq 6.1541711\, \frac{2\log(\tau(n) + 1 - \log 2}{\tau(n)^{1-c-\varepsilon}} \leq 71.212,$$

by considering the maximum of the function $t \mapsto \frac{2\log t + 1 - \log 2}{t^{1-c-\varepsilon}}$. Another open question is to show that $\max \frac{\mathcal{H}(n)}{\tau(n)} = \frac{\mathcal{H}(60)}{\tau(60)} = 1.126\ldots$

5. Landau's function

Let $\mathfrak{S}_n$ denote the symmetric group with n letters. The order of a permutation of $\mathfrak{S}_n$ is the least common multiple of the lengths of its cycles. Let us call $g(n)$ the maximal order of an element of $\mathfrak{S}_n$. If the standard factorization of M into primes writes $M = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, we define $\ell(M)$ to be

$$\ell(M) = q_1^{\alpha_1} + q_2^{\alpha_2} + \cdots + q_k^{\alpha_k}.$$

E. Landau proved in [20] that

$$g(n) = \max_{\ell(M) \le n} M \tag{5.1}$$

and P. Erdős and P. Turán proved in [12] that

$$M \text{ is the order of some element of } \mathfrak{S}_n \iff \ell(M) \le n. \tag{5.2}$$

E. Landau also proved in [20] that

$$\log g(n) \sim \sqrt{n \log n}, \quad n \to \infty. \tag{5.3}$$

This asymptotical result was made precise in [41, 42, 22] while effective estimations of $g(n)$ can be found in [21, 23, 16]. The survey paper [24] of W. Miller is a nice introduction to $g(n)$; it contains elegant and simple proofs of (5.1), (5.2) and (5.3).

A table of Landau's function up to 300 is given at the end of [28]. It has been computed with the algorithm described and used in [29] to compute $g(n)$ up to 8000. By using similar algorithms, a table up to 32000 is given in [26], and a table up to 500000 is mentionned in [16].

In Luminy, on January 17 2002, I asked as a challenge to compute $g(10^6)$ and $g(10^9)$. Three days after, Paul Zimmermann, who was in the audience, sent me an e-mail with the value of $g(10^6)$. In fact, I did not realize the huge progresses on the size of the storage of computers, and Paul, who was well aware of theses progresses, got the result by rediscovering an algorithm similar to that described in [29]. When giving this challenge, I had in mind the method used by G. Robin in [38] to compute the highly composite numbers of Ramanujan. I immediately offered to Paul to write the joint paper [7], where, with M. Deléglise, we propose an algorithm able to calculate $g(n)$ up to $n = 10^{15}$.

The first tool of the algorithm is to introduce the so called ℓ-superchampion numbers, which are the equivalent of the *superior highly composite numbers* introduced by S. Ramanujan.

An integer N is said ℓ-superchampion if there exists $\rho > 0$ such that, for all $M \ge 1$

$$\ell(M) - \rho \log(M) \ge \ell(N) - \rho \log(N).$$

These ℓ-superchampion numbers were already used in [27, 28, 21, 22, 23, 34] to prove theoretical properties of $g(n)$. They are easy to calculate and have the property (for $N \ge 6$) that the p-adic valuation $v_p(N)$ is a non increasing function of p. Further, if N is a ℓ-superchampion, we have:

$$N = g(\ell(N)) \tag{5.4}$$

so that, if $n = \ell(N)$, the value of $g(n)$ is given by (5.4). Unfortunately, the ℓ-superchampion numbers are rather sparse, and the number of n's up to X such that this lucky way works is $\mathcal{O}(\sqrt{\frac{X}{\log X}})$. To compute $g(n)$, the first step of the algorithm is to find two consecutive ℓ-superchampion numbers N and N' such that

$$\ell(N) \le n < \ell(N').$$

Further, by using the *benefit method*, it is possible to show that if we write $\frac{A}{B} = \frac{g(n)}{N}$ (where the fraction $\frac{A}{B}$ is irreducible) A and B are rather small. The algorithm gives, as a result, the factorizations into primes of N and $\frac{A}{B}$ (see [7]). For instance, for $n = 10^6$,

$$N = 2^9 3^6 5^4 7^3 \prod_{11 \le p \le 41} p^2 \prod_{43 \le p \le 3923} p, \quad \ell(N) = 998093$$

and $g(10^6) = g(10^6 - 1) = \frac{43 \cdot 3947}{3847} N$, while for $n = 10^9$,

$$N = 2^{14} 3^9 5^6 7^5 11^4 13^4 \prod_{17 \le p \le 31} p^3 \prod_{37 \le p \le 263} p^2 \prod_{269 \le p \le 150989} p, \quad \ell(N) = 999969437$$

and $g(10^9) = g(10^9 - 1) = \dfrac{37 \cdot 150991}{2 \cdot 3 \cdot 148399} N$.

6. A family of fractions

Let N be a positive integer and $\mathcal{F} = \mathcal{F}(N)$ the set of mappings from $\{1, 2, \ldots, N\}$ to $\{-1, 0, 1\}$. For $\varepsilon \in \mathcal{F}$, let us write

$$\prod_{n=1}^{N} \left(\frac{n+1}{n} \right)^{\varepsilon(n)} = \frac{a}{b} = \frac{a(\varepsilon, N)}{b(\varepsilon, N)} \tag{6.1}$$

where $\frac{a}{b}$ is irreducible, and set

$$\mathcal{S} = \mathcal{S}(N) = \left\{ \frac{a(\varepsilon, N)}{b(\varepsilon, N)}; \quad \varepsilon \in \mathcal{F} \right\} \tag{6.2}$$

and

$$A = A(N) = \max_{\varepsilon \in \mathcal{F}} a(\varepsilon, N). \tag{6.3}$$

Note that, by changing ε to $-\varepsilon$, $\frac{a}{b} \in \mathcal{S} \Rightarrow \frac{b}{a} \in \mathcal{S}$ and thus, $\max_{\varepsilon \in \mathcal{F}} b(\varepsilon, N) = A(N)$.

In [32], the problem of estimating $A(N)$ was asked. Clearly, $A(N) \le (N+1)!$ so that, if we set

$$K(N) = \frac{\log A(N)}{N \log N}, \tag{6.4}$$

$\overline{\lim} K(N) \le 1$ holds. Let us write

$$\left(\frac{n+1}{n} \right)^{\varepsilon(n)} \left(\frac{2n+1}{2n} \right)^{\varepsilon(2n)} \left(\frac{2n+2}{2n+1} \right)^{\varepsilon(2n+1)} = \frac{u}{v}$$

with u/v irreducible. By studying all possible cases, it was shown in [32], Lemma 4, that $\max(u, v) \leq (2n+2)^2$ from which it was deduced that

$$\overline{\lim} K(N) \leq \frac{2}{3}. \tag{6.5}$$

On the other hand, Michel Langevin has observed that, for N even,

$$\prod_{n=1}^{N} \left(\frac{n+1}{n}\right)^{(-1)^n} = \frac{1}{2}\frac{3}{2}\frac{3}{4}\frac{5}{4}\cdots\frac{N-1}{N}\frac{N+1}{N} = \frac{N+1}{4^N}\binom{N}{N/2}^2$$

which yields $\overline{\lim} K(N) \log N \geq \log 4$.

The estimate (6.5) was used in [32] to show that the quotient of two consecutive highly composite numbers (see Section 2) can be large: Proposition 4 claims that if M is a large enough superior highly composite number and M' the highly composite number following M, then

$$\frac{M'}{M} \geq 1 + \frac{1}{(\log M)^{0.9618}}. \tag{6.6}$$

In 6.6 the exponent is greater than $\frac{2}{3\log 2} = 0.961796\ldots$ Showing that $\overline{\lim} K(N) \leq \kappa$ will allow to replace 0.9618 in 6.6 by any real number greater than $\frac{\kappa}{\log 2}$.

I spoke about this problem in my talk in Luminy, on January 2002, and soon after, R. de la Bretèche, C. Pomerance and G. Tenenbaum wrote the paper [5], in which it is shown that $\underline{\lim} K(N) > 0$.

Below is a table of $A(N)$ and $K(N)$. For all values of N up to 40, the maximum in (6.3) is attained on only one value of ε, and $B(N)$ denotes the corresponding value of $b(\varepsilon, N)$. The exponents of A/B are the exponents of the first primes in the standard factorization of A/B: for $N = 6$, $\frac{576}{175} = 2^6 3^2 5^{-2} 7^{-1}$.

For small N's, the set $\mathcal{S}(N)$ has been computed by induction by the formula

$$\mathcal{S}(N) = \frac{N+1}{N}\mathcal{S}(N-1) \cup \mathcal{S}(N-1) \cup \frac{N}{N-1}\mathcal{S}(N-1).$$

For larger N, a sequence $N_0 = 0 < N_1 < \cdots < N_{k-1} < N_k = N$ is chosen. The sets

$$\mathcal{S}(N_i, N_{i+1}) = \left\{ \prod_{n=N_i+1}^{N_{i+1}} \left(\frac{n+1}{n}\right)^{\varepsilon(n)}; \quad \varepsilon \in \mathcal{F}(N) \right\}$$

are built and sorted in decreasing order of the numerator of their elements. $A(N)$ is equal to the maximum of the numerator of $\prod_{i=1}^{k-1} \alpha_i$ when α_i runs through $\mathcal{S}(N_i, N_{i+1})$. Actually, most of the numerators of the α_i's are small, and since the numerator of a product of fractions is not larger than the product of the numerators of the factors, most of the comparisons can be avoided. The value $k = 3$ seems a good choice.

N	$A(N)$	Exponents of A/B	$K(N)$	$\|\mathcal{S}(N)\|$
1	2	1		3
2	4	2,−1	1.0000	9
3	16	4,−2	0.8407	19
4	64	6,−2,−1	0.7500	57
5	128	7,−1,−2	0.6031	115
6	576	6,2,−2,−1	0.5909	345
7	4608	9,2,−2,−2	0.6195	691
8	16384	14,−2,−2,−2	0.5833	1221
9	64000	9,−6,3,−2	0.5596	2023
10	640000	10,−6,4,−2,−1	0.5806	6069
11	2560000	12,−5,4,−2,−2	0.5592	11639
12	10240000	14,−4,4,−2,−2,−1	0.5414	34917
13	54080000	9,−6,4,−3,−2,2	0.5341	69047
14	135200000	8,−5,5,−4,−2,2	0.5067	116429
15	1554251776	18,−8,−4,2,2,−2	0.5208	170383
16	24868028416	22,−8,−4,2,2,−2,−1	0.5397	511149
17	49736056832	23,−6,−4,2,2,−2,−2	0.5114	955223
18	557256278016	20,12,−2,−2,−2,−2,−2,−1	0.5199	2865669

N	Exponents of A/B	$K(N)$
19	20,−8,−5,2,2,−2,−2,2	0.5086
20	22,−3,6,−5,−2,2,−2,−2	0.5013
21	19,10,−4,4,−3,−2,−2,−2	0.4995
22	20,−10,−4,−4,4,2,−2,2,−1	0.5070
23	23,−9,−4,−4,4,2,−2,2,−2	0.5070
24	26,12,−6,4,−2,−2,−2,−2,−2	0.5111
25	27,−8,−8,−4,4,3,−2,2,−2	0.5206
26	28,−11,−8,−4,4,4,−2,2,−2	0.5329
27	30,−14,−8,−3,4,4,−2,2,−2	0.5227
28	28,−14,−8,−4,4,4,−2,2,−2,1	0.5199
29	27,−15,−9,−4,4,4,−2,2,−2,2	0.5241
30	26,−16,−10,−4,4,4,−2,2,−2,2,1	0.5286
31	33,−14,−8,−4,4,4,−2,2,−2,2,−2	0.5200
32	38,−15,−8,−4,3,4,−2,2,−2,2,−2	0.5088
33	37,−14,−8,−4,4,4,−3,2,−2,2,−2	0.5036
34	28,−20,−7,−5,−4,4,4,2,2,2,−2	0.4996
35	30,−18,−8,−6,−4,4,4,2,2,2,−2	0.4925
36	28,−20,−8,−6,−4,4,4,2,2,2,−2,1	0.4923
37	33,−16,−8,−6,−4,4,4,3,2,2,−2,−2	0.4963
38	36,−17,12,6,−6,−3,−2,2,2,−2,−2,−2	0.4926
39	39,−18,13,6,−6,−4,−2,2,2,−2,−2,−2	0.5023
40	42,−18,14,6,−6,−4,−2,2,2,−2,−2,−2,−1	0.5115

7. $c(n)$

In the paper [8], written jointly with J. Dixmier, we have considered the function (cf. p. 98 and 111)

$$c(n) = 2^{v_2(1)} + 2^{v_2(2)} + \cdots + 2^{v_2(n)} \tag{7.1}$$

where v_2 denotes the 2-adic valuation. This function appears as the exponent of $1 + X$ in the decomposition of the polynomial $(1 + X)(1 + X^2) \ldots (1 + X^n)$ into irreducible factors over $\mathbb{F}_2[X]$. By separating odd and even integers in (7.1), it is easy to see that $c(2n) = 2c(n) + n$ which yields

$$c(2^r) = 2^{r-1}(r + 2), \quad c(2^r - 1) = r2^{r-1}. \tag{7.2}$$

It follows from (7.1) that

$$c(2^n + s) = c(2^n) + c(s), \quad s < 2^n \tag{7.3}$$

and this implies that, if the binary expansion of n is $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}$ with $n_1 > n_2 > \cdots > n_k$ then

$$\begin{aligned} c(n) &= c(2^{n_1}) + c(2^{n_2}) + \cdots + c(2^{n_k}) \\ &= 2^{n_1-1}(n_1 + 2) + 2^{n_2-1}(n_2 + 2) + \cdots + 2^{n_k-1}(n_k + 2). \end{aligned} \tag{7.4}$$

If we denote by $d(n)$ the sum of the binary digits of n, the summatory function $S(n) = \sum_{k=1}^{n} d(k)$ satisfies

$$S(2^r - 1) = c(2^r - 1) = r2^{r-1} \tag{7.5}$$

but, as it can be seen on the following table, the two functions c and S are not equal.

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c(n)$	1	3	4	8	9	11	12	20	21	23	24	28	29	31	32	48
$S(n)$	1	2	4	5	7	9	12	13	15	17	20	22	25	28	32	33

There are several nice papers about the asymptotic behaviour of $S(n)$, for instance [6], [14] and [15], which probably can be adapted to $c(n)$. But here, the question is: *How often 2^m belongs to the image of c?*

It follows from (7.2) that, when r or $r + 2$ is a power of 2, then $c(2^r)$ or $c(2^r - 1)$ is also a power of 2 while, if $r + 1 = 2^a$, the equation $c(x) = 2^{2^a+a-2}$ has no solution.

From (7.4), we have

$$c(2^k + 2^{k-1} - 1) = (3k + 3)2^{k-2} \quad et \quad c(2^k + 2^{k-1}) = (3k + 5)2^{k-2}.$$

Clearly, $3k+3$ is never a power of 2, but $3k+4$ or $3k+5$ can be equal to 2^a, so that, for a odd, $a+\frac{2^a-11}{3}$ belongs to the image of c, while, for a even, $a+\frac{2^a-10}{3}$ does not belong to it.

Let us define $\mathcal{A}=\{a, 2^a \in \text{Image}(c)\}$ and $A(x)=\text{Card}(\{a \in \mathcal{A}, a \leq x\})$. It has just been shown that both $A(x)$ and $x-A(x)$ tend to infinity with x. Is it possible to make precise the behaviour of $A(x)$ when $x \to \infty$? Below is a table of values of $A(x)$.

$x=$	1000	2000	3000	4000	5000	6000	7000	8000	9000	10000
$A(x)$	635	1374	1787	2405	2748	3078	3646	4380	5227	5963
$A(x)/x$	0.63	0.69	0.60	0.60	0.55	0.51	0.52	0.55	0.58	0.60

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