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An Extremal Problem

Problem 92-9*, *by* PAUL ERDÖS (Hungarian Academy of Sciences, Budapest, Hungary).

Let

$$f(n) = \max\left\{\sum_{i>j}\frac{1}{x_i - x_j}\Big/\sum_i \frac{1}{x_{i+1}-x_i}\right\},$$

where the maximum is taken over all real sequences $x_1 < x_2 < \cdots < x_n$. It is easy to see that

$$f(n) = \ln n + O(1) \qquad (n \to \infty).$$

(a) Find

$$\limsup_{n\to\infty}\{f(n) - \ln n\}.$$

(b) Describe the extremal sequences.

This problem admits an electrostatic interpretation. Suppose n unit charges are placed on a line. Find the positions of these charges which maximize the ratio of the total potential energy to that part which comes from nearest neighbor interaction alone.

Partial solution by JEAN-LOUIS NICOLAS (Université Claude Bernard, Lyon, France).

Let H_n denote the nth harmonic number. It is well known that $H_{n-1} < \ln n + \gamma < H_n$ for each $n \geq 2$, where $\gamma = 0.5772156649\ldots$ is Euler's constant. First we prove that for all $n \geq 2$,

$$f(n) > H_n - 1 > \ln n + \gamma - 1 \tag{1}$$

and

$$f(n) \leq H_{n-1} < \ln n + \gamma. \tag{2}$$

Let $F(x_1, x_2, \ldots, x_n)$ denote the quotient to be maximized. Then

$$\begin{aligned} F(1,2,\ldots,n) &= \frac{1}{n-1}\left(\frac{n-1}{1} + \frac{n-2}{2} + \cdots + \frac{1}{n-1}\right)\\ &= \frac{nH_{n-1} - (n-1)}{n-1}\\ &= \frac{n(H_n - 1)}{n-1}\\ &> H_n - 1\\ &> \ln n + \gamma - 1.\end{aligned}$$

This proves (1). The proof of (2) uses the well-known inequality between the arithmetic and harmonic means

$$\frac{1}{a_1+a_2+\cdots+a_k} \le \frac{1}{k^2}\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}\right). \tag{3}$$

Let $\Delta x_i = x_{i+1} - x_i$ for $i = 1, 2, \ldots, n-1$. Using (3) we obtain

$$\begin{aligned}\sum_{i<j}\frac{1}{x_j - x_i} &= \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\frac{1}{x_{i+k}-x_i}\\ &= \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\frac{1}{\Delta x_i + \Delta x_{i+1} + \cdots + \Delta x_{i+k-1}}\\ &\le \sum_{k=1}^{n-1}\frac{1}{k^2}\sum_{i=1}^{n-k}\left(\frac{1}{\Delta x_i}+\frac{1}{\Delta x_{i+1}}+\cdots+\frac{1}{\Delta x_{i+k-1}}\right).\end{aligned}$$

Note that

$$\sum_{i=1}^{n-k}\left(\frac{1}{\Delta x_i}+\frac{1}{\Delta x_{i+1}}+\cdots+\frac{1}{\Delta x_{i+k-1}}\right) = \sum_{j=1}^{n-1}\frac{M_j}{\Delta x_j},$$

where M_j is the number of indices i between 1 and $n-k$ for which $i \le j \le i+k-1$. Clearly, $M_j \le k$ for each j, so

$$\sum_{i<j}\frac{1}{x_j - x_i} \le \sum_{k=1}^{n-1}\frac{1}{k}\sum_{j=1}^{n-1}\frac{1}{\Delta x_j} = H_{n-1}\sum_{j=1}^{n-1}\frac{1}{x_{j+1}-x_j}.$$

Hence for all choices of $(x_1, x_2, \ldots, x_n)$,

$$F(x_1, x_2, \ldots, x_n) \le H_{n-1} < \ln n + \gamma,$$

and we have (2).

Let $x = (x_1, x_2, \ldots, x_n)$. Noting that $F(x)$ is invariant under the transformation $x \mapsto ax + b$, we may fix $x_1 = 1$ and $x_n = n$. Now if x optimizes $F(x)$, it must satisfy the system of equations

$$\frac{\partial F}{\partial x_i} = 0, \qquad 2 \le i \le n-1.$$

We apply Newton's method to this system of equations, starting with $x_k^{(0)} = k$ for $k = 2, 3, \ldots, n-1$. Let $A = [a_{ij}]$ where

$$a_{ij} = \left.\frac{\partial^2 F}{\partial x_i \partial x_j}\right|_{x^{(0)}},$$

with $x_1 = x_1^{(0)} = 1$ and $x_n = x_n^{(0)} = n$ and define the vector $b = (b_2, b_3, \ldots, b_{n-1})^T$ by

$$b_i = \left.\frac{\partial F}{\partial x_i}\right|_{x^{(0)}}, \qquad (i = 2, 3, \ldots, n-1).$$

The first iteration of Newton's method yields $x_1^{(1)} = 1$, $x_n^{(1)} = n$, and

$$\begin{pmatrix} x_2^{(1)} \\ x_3^{(1)} \\ \vdots \\ x_{n-1}^{(1)} \end{pmatrix} = \begin{pmatrix} x_2^{(0)} \\ x_3^{(0)} \\ \vdots \\ x_{n-1}^{(0)} \end{pmatrix} - A^{-1}b.$$

This algorithm has been implemented using MAPLE. It turns out that Newton's method converges very quickly to a vector $\hat{x}$, for which the matrix A is negative definite. The program was run on an IBM 3090 for $n \le 19$. For all $n \le 19$, the solution was found to be symmetric, that is to say $\hat{x}_k + \hat{x}_{n-k} = n+1$ for all k. Thus when n is odd, $\hat{x}_{(n+1)/2} = (n+1)/2$. For $n = 19$, the algorithm finds

$$\begin{aligned} x_1 &= 1.00 = 20 - x_{19}, & x_6 &= 7.70 = 20 - x_{14},\\ x_2 &= 3.57 = 20 - x_{18}, & x_7 &= 8.33 = 20 - x_{13},\\ x_3 &= 5.07 = 20 - x_{17}, & x_8 &= 8.92 = 20 - x_{12},\\ x_4 &= 6.14 = 20 - x_{16}, & x_9 &= 9.47 = 20 - x_{11},\\ x_5 &= 6.99 = 20 - x_{15}, & x_{10} &= 10.00. \end{aligned}$$

The numerical results prompt the following questions.

Question 1: Is the result given by this algorithm a solution which maximizes $F(x)$?

Question 2: Is it possible to prove that the algorithm always converges to a symmetric vector?

Question 3: Is the maximum $f(n)$ attained by a symmetric vector?

Note that if the function $F(x)$ is concave for $x_1 < x_2 < \cdots < x_n$, then the maximum is unique, and thus symmetric.

Assuming that the answer to Question 3 is yes, we have implemented the above algorithm working only on symmetric vectors. It is then possible to calculate a solution for all $n \le 95$. The value obtained for $f(95)$ is 4.3777, and the first few coordinates of x are

$$x_1 = 1, \quad x_2 = 8.07, \quad x_3 = 12.40, \quad x_4 = 15.60,$$
$$x_5 = 18.13, \quad x_6 = 20.24, \quad x_7 = 22.06.$$

If our table of values for $f(n)$ is correct, the difference $f(n) - \ln n$ decreases quite regularly for $2 \le n \le 95$ and is equal to -0.176 for $n = 95$.

Editorial note. The author has kindly provided a listing of his MAPLE program. This is available on request from the editor. [C.C.R.]

A Unique Vector Sum

Problem 93-11, *by* RUDOLF X. MEYER (University of California, LA).

Let $P_1, P_2, \ldots, P_m$ and $Q_1, Q_2, \ldots, Q_n$ be two sets of distinct real-valued vectors in $\mathbb{E}^n$ such that $|P_i| = P$, $|Q_j| = Q$ for all i, j with $P \ne Q$. Among the mn sums $P_i + Q_j (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$, show that for any P_i there is a Q_k such that $P_i + Q_k$ is unique.