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ON SUMMANDS OF GENERAL PARTITIONS*

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Dedicated to our friend Jean-Marc Deshouillers
at the occasion of his sixtieth birthday

Abstract: It is proved that if $\mathcal{A}$ is a set of positive integers with $1 \in \mathcal{A}$ then almost all partitions of n into the elements of $\mathcal{A}$ contain the summand 1.
Keywords: partitions, distribution of summands.

1. Introduction

The set of the positive integers will be denoted by $\mathbb{N}$. If $\mathcal{A} = \{a_1, a_2, \dots\}$ (with $a_1 < a_2 < \dots$) is a non-empty set of positive integers then let $p(\mathcal{A}, n)$ denote the number of solutions of

$$x_1 a_1 + x_2 a_2 + \dots + x_k a_k + \dots = n \tag{1}$$

in non-negative integers $x_1, x_2, \dots$ As usual, we set $p(\mathcal{A}, 0) = 1$. A solution of (1) is said to be an $\mathcal{A}$-*partition* of n, and the a_k's with $x_k > 0$ (counted with multiplicity x_k) are called the *parts* or *summands* of the partition. If $a_1 = 1$, then let $p_1(\mathcal{A}, n) = p(\mathcal{A}, n-1)$ denote the number of $\mathcal{A}$-partitions (1) of n with $x_1 > 0$, i.e., containing 1 as a part, and let $\overline{p}_1(\mathcal{A}, n)$ denote the number of $\mathcal{A}$-partitions (1) with $x_1 = 0$, i.e.,

$$\overline{p}_1(\mathcal{A}, n) = p(\mathcal{A} \setminus \{1\}, n) = p(\mathcal{A}, n) - p_1(\mathcal{A}, n) = p(\mathcal{A}, n) - p(\mathcal{A}, n-1). \tag{2}$$

In particular, we write $p(\mathbb{N}, n) = p(n)$, $p_1(\mathbb{N}, n) = p_1(n)$ and $\overline{p}_1(\mathbb{N}, n) = \overline{p}_1(n)$. C will denote the constant

$$C = \pi\sqrt{\frac{2}{3}} = 2.565\dots \tag{3}$$

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Then, by a classical theorem of Hardy and Ramanujan [6] we have

$$p(n) = \frac{1}{4\sqrt{3}n} e^{C\sqrt{n}} \left(1 + O\left(n^{-1/2}\right)\right). \tag{4}$$

In 1941, Erdős and Lehner [3] studied the distribution of the greatest part of partitions of n: they showed that for $k = C^{-1}n^{1/2}\log n + xn^{1/2}$, the number of partitions of n with greatest part not greater than k is $(1+o(1))\exp\left(-2C^{-1}e^{-(C/2)x}\right)$ $p(n)$. Since that, many results have been proved on statistical properties of partitions by Bateman, Erdős, Szalay, Szekeres, Turán, Dixmier, Nicolas, Sárközy, Mosaki and others (cf. [1,5,12,2,4,7,8,9,10,11] and the references quoted in them). In particular, Szalay and Turán [12] studied the distribution of other large parts of partitions of n. In [5] (p. 193), Erdős and Szalay showed that it follows from (4) that the part 1 occurs in almost every partition of n, more precisely, we have

$$\overline{p}_1(n) = p(n) - p(n-1) = \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)\frac{\pi}{\sqrt{6n}}\, p(n) \tag{5}$$

(cf. (2)). (5) also follows from a result of Dixmier and Nicolas [2]: for $m \leqslant n^{1/4}$, they gave an asymptotic formula for the function $r(n,m)$ which counts the number of partitions of n into parts not smaller than m, and clearly we have $\overline{p}_1(n) = r(n,2)$. The behaviour of $r(n,m)$ for larger m has been studied in [9, 7, 8].

In this paper, our goal is to extend the study of the distribution of parts of partitions from the special case of the classical partitions of n to the general case of $\mathcal{A}$-partitions of n. The simplest and most natural question of this type is the following: as we have seen (cf. (5)) almost all partitions of n contain the part 1; if $1 \in \mathcal{A}$, then do the $\mathcal{A}$-partitions also have this property? First we will show that the answer to this question is affirmative:

Theorem 1. *If $\mathcal{A} \subset \mathbb{N}$ is a set containing 1 then we have*

$$\lim_{n\to\infty} \frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} = 0. \tag{6}$$

Moreover, for any integers k and j satisfying

$$ka_k \leqslant \frac{n}{e} \tag{7}$$

and

$$ja_j \leqslant \sqrt{n}, \tag{8}$$

respectively, we have

$$\frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} \leqslant \min\left(\frac{10\log n}{k}, \frac{9}{j}\right). \tag{9}$$

Note that for "dense" $\mathcal{A}$ the first upper bound is sharp while, for "thin" $\mathcal{A}$, the second one is better but the inequality is not sharp. We will be able to improve it only for infinitely many values of n:

Theorem 2. *If $\mathcal{A} \subset \mathbb{N}$ is a set containing 1 then we have*

$$\liminf_{n\to\infty} \frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} n^{1/2} \leqslant \frac{C}{2} \tag{10}$$

(where C is the constant defined by (3) so that $\frac{C}{2} = \frac{\pi}{\sqrt{6}}$). More precisely, there exists an increasing sequence $(n_i)_{i\geqslant 1}$ such that

$$\overline{p}_1(\mathcal{A},n_i) \leqslant \frac{C}{2\sqrt{n_i}}\, p(\mathcal{A},n_i), \qquad i = 1,2,\ldots \tag{11}$$

Note that the upper bound (10) is the best possible in the sense that, as by (5) the special case $\mathcal{A} = \mathbb{N}$ shows, the constant on the right hand side cannot be replaced by a smaller one. On the other hand, we do not know whether one can make the upper bound (11) uniform in n, i.e., we have not been able to settle Problem 1 (see § 4).

Note moreover that Theorem 2 provides a partial answer to a conjecture of Bateman and Erdős [1], p. 12.

On the other hand, no non-trivial uniform *lower* bound can be given for $\overline{p}_1(\mathcal{A},n)$:

Example 1. Let $d \in \mathbb{N}, d > 1$ and $\mathcal{A} = \{1, d, 2d, \ldots, kd, \ldots\}$. For this set $\mathcal{A}$ we have $\overline{p}_1(\mathcal{A},n) = 0$ for all $d \nmid n$.

We can avoid this type of counterexamples by assuming that $\mathcal{A}$ satisfies the regularity condition of Bateman and Erdős (cf. [1])

$$\forall (a_{i_1}, a_{i_2}, \ldots, a_{i_k}) \in \mathcal{A}^k, \quad \gcd\left(\mathcal{A} \setminus \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\}\right) = 1 \tag{12}$$

which implies that the k-th difference $\Delta^k p(\mathcal{A},n) = \sum_{j=0}^{k}(-1)^j\binom{k}{j}p(\mathcal{A},n-j)$ is positive for n large enough. Then, for $k \geqslant 2$, it follows from (12) that $p(\mathcal{A},n) \gg n^k$.

2. Proof of Theorem 1

We will use a sharper version of the argument given by Bateman and Erdős in the proof of Theorem 4 in [1]. We start with a classical lemma:

Lemma 1. *Let $r, a_1, a_2, \ldots, a_r$ be positive integers, $a_1 < a_2 < \ldots < a_r$, and $S = a_1 + a_2 + \ldots + a_r$. The number $N(n)$ of integer solutions of the inequality*

$$a_1x_1 + a_2x_2 + \ldots + a_rx_r \leqslant n$$

satisfies

$$\left(\frac{n}{ra_r}\right)^r \leqslant \frac{n^r}{r!a_1a_2\ldots a_r} \leqslant N(n) \leqslant \frac{(n+S)^r}{r!a_1a_2\ldots a_r}.$$

Proof. For a proof, see for instance [13], III.5.2. ■

The proof of Theorem 1 will be based on the following proposition:

Proposition 1. *Let $\mathcal{A} = \{a_1 = 1 < a_2 < \ldots < a_i < \ldots\}$. For any positive integer ℓ and for $n \geqslant 2$ we have*

$$\overline{p}_1(\mathcal{A}, n) = p(\mathcal{A}, n) - p(\mathcal{A}, n-1) \leqslant \frac{1}{\ell+1}\, p(\mathcal{A}, n) + \frac{4}{3}\, n^{2\ell}. \tag{13}$$

Proof. We split the partitions counted by $\overline{p}_1(\mathcal{A}, n)$ into two classes: let $q_\ell^-(\mathcal{A}, n)$ (resp. $q_\ell^+(\mathcal{A}, n)$) denote the number of $\mathcal{A}$-partitions of n into at most ℓ (resp. more than ℓ) distinct a_k's greater than 1 so that

$$\overline{p}_1(\mathcal{A}, n) = q_\ell^-(\mathcal{A}, n) + q_\ell^+(\mathcal{A}, n). \tag{14}$$

Consider a partition counted in $q_\ell^-(\mathcal{A}, n)$ into parts $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$ occurring with (positive) multiplicity $y_1, y_2, \ldots, y_t$, respectively, so that $1 < a_2 \leqslant a_{i_1} < a_{i_2} < \ldots < a_{i_t} \leqslant n$ and

$$a_{i_1} y_1 + a_{i_2} y_2 + \ldots + a_{i_t} y_t = n, \qquad t \leqslant \ell. \tag{15}$$

In (15), each of $a_{i_1}, a_{i_2}, \ldots, a_{i_t}, y_1, y_2, \ldots, y_t$ can be chosen in at most n ways and thus for fixed t the number of these partitions is not greater than n^{2t}. It follows that, for $n \geqslant 2$,

$$q_\ell^-(\mathcal{A}, n) \leqslant \sum_{t=1}^{\ell} n^{2t} \leqslant n^{2\ell}\left(1 + \frac{1}{4} + \frac{1}{16} + \ldots\right) = \frac{4}{3}\, n^{2\ell}. \tag{16}$$

Next we will show that

$$q_\ell^+(\mathcal{A}, n) \leqslant \frac{p(\mathcal{A}, n)}{\ell+1}. \tag{17}$$

Consider an $\mathcal{A}$-partition of n counted on the left hand side of (17) into parts $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$ occurring with (positive) multiplicities $y_1, y_2, \ldots, y_t$:

$$a_{i_1} y_1 + a_{i_2} y_2 + \ldots + a_{i_t} y_t = n \tag{18}$$

where now

$$t \geqslant \ell + 1.$$

For each of $r = 1, 2, \ldots, t$, replace one part a_{i_r} by a_{i_r} parts equal to $a_1 = 1$ in equation (18); we get the partition of n:

$$a_1 a_{i_r} + a_{i_1} y_1 + a_{i_2} y_2 + \ldots + a_{i_r}(y_r - 1) + \ldots + a_{i_t} y_t = n. \tag{19}$$

The partition in (19) determines the partition in (18) uniquely, since we obtain the latter from the first one by replacing the parts equal to 1 by their sum a_{i_r}.

Thus, the partitions (19) are all distinct; their number is at least $(\ell+1) q_\ell^+(\mathcal{A}, n)$ and at most $p(\mathcal{A}, n)$, which proves (17).

(13) follows from (14), (16) and (17) and the proof of Proposition 1 is completed. ■

Proof of Theorem 1. If $\mathcal{A} = \{a_1 = 1 < a_2 < \ldots < a_m\}$ is finite, by studying the partial fraction decomposition of the generating function $\prod_{i=1}^m (1 - X^{a_i})^{-1}$, it is easy to show that (cf. [1], Lemma 1)

$$p(\mathcal{A}, n) = \frac{n^{m-1}}{(m-1)!\, a_1 a_2 \ldots a_m} + \mathcal{O}(n^{m-2})$$

and

$$p(\mathcal{A} \setminus \{1\}, n) = \frac{n^{m-2}}{(m-2)!\, a_2 a_3 \ldots a_m} + \mathcal{O}(n^{m-2}) = \mathcal{O}(n^{m-2}).$$

Therefore,

$$\frac{\overline{p}_1(\mathcal{A}, n)}{p(\mathcal{A}, n)} = \frac{p(\mathcal{A} \setminus \{1\}, n)}{p(\mathcal{A}, n)} = \mathcal{O}\left(\frac{1}{n}\right) \tag{20}$$

which proves (6).

When $\mathcal{A}$ is infinite, (6) will follow from (9), since, from (8), j tends to infinity with n; so it remains to prove (9). Clearly we have for any $r \geqslant 1$ and $n \geqslant 1$:

$$p(\mathcal{A}, n) \geqslant p(\{a_1, a_2, \ldots, a_r\}, n) \geqslant \frac{1}{n+1} N(n) \geqslant \frac{1}{2n} N(n). \tag{21}$$

Thus by Lemma 1, (7) and (8) we have

$$p(\mathcal{A}, n) \geqslant \begin{cases} \frac{1}{2n}\left(\frac{n}{k a_k}\right)^k \geqslant \frac{e^k}{2n} \\ \frac{1}{2n}\left(\frac{n}{j a_j}\right)^j \geqslant \frac{1}{2} n^{\frac{j}{2}-1}. \end{cases} \tag{22}$$

■

Case 1. Assume that (7) holds. From Proposition 1 and (22), we get

$$\frac{\overline{p}_1(\mathcal{A}, n)}{p(\mathcal{A}, n)} \leqslant \frac{1}{\ell+1} + \frac{8 n^{2\ell+1}}{3 e^k} \qquad \text{for any } \ell \geqslant 1. \tag{23}$$

If $k \geqslant 6 \log n$, we choose $\ell = \left\lfloor \frac{k}{2 \log n} \right\rfloor - 1 \geqslant 2$. Since for $a \in \mathbb{N}$ and $x \geqslant a$, $\lfloor x \rfloor \geqslant \frac{a}{a+1}\, x$, we have

$$\ell + 1 = \left\lfloor \frac{k}{2 \log n} \right\rfloor \geqslant \frac{3k}{8 \log n}. \tag{24}$$

Further, (7) implies $n \geqslant e > 2$ and

$$\ell + 1 \leqslant \frac{k}{2 \log n} \leqslant \frac{k}{2 \log 2} \leqslant k \leqslant e\, k\, a_k \leqslant n$$

so that

$$\frac{8\,n^{2\ell+1}}{3\,e^k} \leqslant \frac{8\,n^{\frac{2k}{2\log n}}}{3\,e^k n} = \frac{8}{3n} \leqslant \frac{8}{3(\ell+1)}$$

and (23) and (24) yield

$$\frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} \leqslant \frac{11}{3(\ell+1)} \leqslant \frac{88}{9}\,\frac{\log n}{k} \leqslant 10\,\frac{\log n}{k}.$$

If $k < 6\log n$, the trivial upper bound

$$\frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} \leqslant 1 \leqslant 6\,\frac{\log n}{k}$$

completes the proof of the first case.

Case 2. Assume now that inequality (8) holds. Similarly, Proposition 1 and (22) imply

$$\frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} \leqslant \frac{1}{\ell+1} + \frac{8n^{2\ell+1}}{3n^{j/2}}. \tag{25}$$

Here we choose $\ell = \lfloor \frac{j}{4} \rfloor - 1 \geqslant 1$ if $j \geqslant 8$, so that $\ell + 1 = \lfloor \frac{j}{4} \rfloor \geqslant \frac{j}{6}$ and $\ell+1 \leqslant \frac{j}{4} \leqslant \frac{ja_j}{4} \leqslant \frac{\sqrt{n}}{4}$.

If $n = 1$, (8) implies $j = 1$ and $\frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} = 0 \leqslant \frac{9}{j}$ trivially holds.

For $n \geqslant 2$, we have

$$\frac{8n^{2\ell+1}}{3n^{j/2}} \leqslant \frac{8}{3n} \leqslant \frac{2}{3\sqrt{n}(\ell+1)} \leqslant \frac{2}{3\sqrt{2}(\ell+1)} \leqslant \frac{1}{2(\ell+1)}$$

and (25) yields

$$\frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} \leqslant \frac{3}{2(\ell+1)} \leqslant \frac{18}{2j} = \frac{9}{j}.$$

If $j \leqslant 7$, we trivially have $\frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} \leqslant 1 \leqslant \frac{7}{j}$, and the proof of Theorem 1 is completed.

3. Proof of Theorem 2

If $\mathcal{A}$ is finite, Theorem 2 follows from (20). If $\mathcal{A}$ is infinite, we will prove Theorem 2 by contradiction: assume that there is $n_0 \in \mathbb{N}$ so that

$$\overline{p}_1(\mathcal{A},n) > \frac{C}{2\sqrt{n}}\,p(\mathcal{A},n) \qquad \text{for } n \geqslant n_0. \tag{26}$$

By $1 \in \mathcal{A}$ we have

$$p(\mathcal{A},n) \geqslant 1 \qquad \text{for all } n \in \mathbb{N} \tag{27}$$

(every n can be represented as $1+1+\ldots+1 = n$). Thus it follows from (26) that

$$\frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} > \frac{C}{2\sqrt{n}} \qquad \text{for } n \geqslant n_0. \tag{28}$$

By (2) and (27) we have

$$0 < \frac{p(\mathcal{A},k-1)}{p(\mathcal{A},k)} = 1 - \frac{\overline{p}_1(\mathcal{A},k)}{p(\mathcal{A},k)} \qquad \text{for all } k \in \mathbb{N}. \tag{29}$$

It follows from (28) and (29) that for $n \geqslant n_0$

$$\frac{1}{p(\mathcal{A},n)} = \frac{1}{p(\mathcal{A},n_0-1)} \prod_{k=n_0}^{n} \frac{p(\mathcal{A},k-1)}{p(\mathcal{A},k)} = \frac{1}{p(\mathcal{A},n_0-1)} \prod_{k=n_0}^{n} \left(1 - \frac{\overline{p}_1(\mathcal{A},k)}{p(\mathcal{A},k)}\right)$$
$$< \frac{1}{p(\mathcal{A},n_0-1)} \prod_{k=n_0}^{n} \left(1 - \frac{C}{2\sqrt{k}}\right).$$

But,

$$\prod_{k=n_0}^{n} \left(1 - \frac{C}{2\sqrt{k}}\right) = \exp\left(\sum_{k=n_0}^{n} \log\left(1 - \frac{C}{2\sqrt{k}}\right)\right) \leqslant \exp\left(-\frac{C}{2}\sum_{k=n_0}^{n} \frac{1}{\sqrt{k}}\right)$$
$$\leqslant \exp\left(-\frac{C}{2}\int_{n_0}^{n} \frac{dx}{\sqrt{x}}\right) = \frac{\exp\left(C\sqrt{n_0}\right)}{\exp\left(C\sqrt{n}\right)}$$

whence

$$p(\mathcal{A},n) > \frac{p(\mathcal{A},n_0-1)}{\exp\left(C\sqrt{n_0}\right)}\,\exp\left(C\sqrt{n}\right). \tag{30}$$

On the other hand, by (4) we have

$$p(\mathcal{A},n) \leqslant p(\mathbb{N},n) < \frac{\exp\left(Cn^{1/2}\right)}{n} \qquad \text{for } n \geqslant n_1. \tag{31}$$

However, for n large enough, (31) contradicts (30) and this completes the proof of Theorem 2.

4. Problems

Problem 1. *Is the statement of Theorem 2 still true if we replace the* lim inf *in (10) by* lim sup*? Or, at least, can one show that* $\frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} n^{1/2} = \mathcal{O}(1)$*?* (see also Bateman and Erdős [1], p. 12.)

Problem 2. *A problem closely related to problem 1: Under what conditions can one control the rate of growth of the difference* $p(\mathcal{A},n) - p(\mathcal{A},n-1)$*?*

Problem 3. (i) *Show: if $\mathcal{A} \subset \mathbb{N}$ is infinite (one may also assume $1 \in \mathcal{A}$), then there are infinitely many n so that, for almost all $\mathcal{A}$-partitions of n, the greatest summand is* $> n^{\frac{1}{2}-\varepsilon}$ (perhaps, even $> n^{1/2}$).

(ii) *Show: Under (possibly general) regularity condition, the conclusion of* (i) *holds for* all $n \to \infty$.

(iii) *What condition is needed to ensure that, for almost all $\mathcal{A}$-partitions of n, the greatest part λ satisfies* $\frac{\lambda}{\sqrt{n}} \to \infty$ (like for the classical partitions).

Problem 4. *What about the number of parts for a random $\mathcal{A}$-partition? What about the number of* distinct *parts for a random $\mathcal{A}$-partition?*

Problem 5. *If the density of $\mathcal{A}$ is oscillating, then how and where is "an accumulation point" of $\mathcal{A}$ reflected in the behaviour of $p(\mathcal{A}, n)$?*

Added in proofs. Actually, Problem 1 under its weak form $\frac{\overline{p}_1(\mathcal{A},n)}{p(\mathcal{A},n)} n^{1/2} = O(1)$, has been proved by T.P. Bell, "A proof of a Partition Conjecture of Bateman and Erdš", *J. Number Theory*, **87** (2001), 144–153.

References

[1] P. Bateman and P. Erdős, Monotonicity of partition functions, *Mathematika*, **3** (1956), 1–14.

[2] J. Dixmier and J.-L. Nicolas, Partitions without small parts, *Colloquia Mathematica Soc. János Bolyai* **51**, Number Theory, vol. I (Budapest, 1987), 9–33, North-Holland, 1990.

[3] P. Erdős and J. Lehner, The distribution of the number of summands in the partitions of a positive integer, *Duke Math. J.*, **8** (1941), 335–345.

[4] P. Erdős and J.-L. Nicolas, On practical partitions, *Collectanea Mathematica*, **46** (1995), 57–76.

[5] P. Erdős and M. Szalay, On some problems of J. Dénes and P. Turán, *Studies in Pure Mathematics dedicated to the memory of Paul Turán*, 187–212, Birkhäuser, 1983.

[6] G.H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, *Proc. London Math. Soc.*, **17** (2) (1918), 75–115 and Collected Papers of Ramanujan, §36, 276–309.

[7] E. Mosaki, J.-L. Nicolas and A. Sárközy, Partitions sans petites parts, *J. de Théorie des Nombres de Bordeaux*, **16** (2004), 607–638.

[8] E. Mosaki, Partitions sans petites parts, II, to be published.

[9] J.-L. Nicolas and A. Sárközy, On partitions without small parts, *J. de Théorie des Nombres de Bordeaux*, **12** (2000), 227–254.

[10] J.-L. Nicolas and A. Sárközy, On the Asymptotic Behaviour of General Partition Function, *The Ramanujan J.*, **4** (2000), 29-30.

[11] J.-L. Nicolas and A. Sárközy, On the Asymptotic Behaviour of General Partition Function, II, *The Ramanujan J.*, **7** (2003), 279–298.

[12] M. Szalay and P. Turán, On some problems of the statistical theory of partitions with application to characters of the symmetric group, III, *Acta Mathematica Sci. Hungar.* **32** (1978), 129–155.

[13] G. Tenenbaum, Introduction à la théorie analytique et probabiliste des nombres, S.M.F., Paris, 1995; Introduction to analytic and probabilistic number theory, Cambridge studies in advanced mathematics, **46**, Cambridge University Press, 1995.

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