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ON TWO PARTITION PROBLEMS

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1. Introduction

The set of the positive integers will be denoted by **N** and we shall write $\lfloor x \rfloor$ for the integral part of x. For $n \in \mathbf{N}$, $x > 0$, let $r(n,x)$ denote the number of partitions of n into parts $\geqq x$, and let $\rho(n,x)$ denote the number of partitions of n into distinct parts $\geqq x$ so that we have

$$\rho(n,x) \leqq r(n,x) \tag{1}$$

for all n and x.

The quantity $\rho(n,x)$ has already been studied. In [7], and [6], the asymptotic behaviour of $\rho(n,x)$ is investigated for $x = O(n^{1/5})$, and for $x = O(n^{3/8-\varepsilon})$, respectively. In [9], G. Freiman and J. Pitman, by the saddle point method, gave the following estimation for $x = o\big(n(\log n)^{-9}\big)$:

$$\rho(n,x) \sim \frac{1}{\sqrt{2\pi B^2}} e^{\sigma n} \prod_{x \leqq j \leqq n} (1 + e^{-j\sigma}) \tag{2}$$

where $\sigma = \sigma(n,x)$ is the root of the equation

$$n = \sum_{x \leqq j \leqq n} \frac{j}{1 + e^{\sigma j}} \tag{3}$$

and

$$B^2 = \sum_{x \leqq j \leqq n} \frac{j^2 e^{\sigma j}}{(1 + e^{\sigma j})^2}. \tag{4}$$

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Dixmier and Nicolas [1,2] studied the function $r(n,x)$ and, in particular, in [2] they showed that for fixed $\lambda > 0$ we have

$$\log r\left(n, \lambda\sqrt{n}\right) \sim g(\lambda)\sqrt{n} \quad \text{as} \quad n \to +\infty \tag{5}$$

where the function $g(\lambda)$ is analytic for $\lambda > 0$, $g'(\lambda) < 0$ and $g''(\lambda) > 0$ for $\lambda > 0$; $g(\lambda)$ satisfies the differential equation

$$\lambda^2 g''(\lambda) + \lambda g'(\lambda) - g(\lambda) - \frac{2g''(\lambda)}{1 - e^{-g'(\lambda)}}$$

and we have

$$g(\lambda) = \frac{2\log\lambda - \log\log\lambda + 1 - \log 2}{\lambda} + \frac{\log\log\lambda + \log 2}{2\lambda\log\lambda} + O\left(\frac{(\log\log\lambda)^2}{\lambda\log^2\lambda}\right) \quad \text{as} \quad \lambda \to +\infty. \tag{6}$$

In this paper, first we will study the analogous problem with $\rho(n,x)$ in place of $r(n,x)$ (note that in this case the methods used in [2] cannot be applied) and, indeed, we will prove the following theorem:

THEOREM 1. *Let us define*

$$F(x) = \int_x^\infty \frac{u}{1+e^u}\,du. \tag{7}$$

For all $x \geqq 0$, there exists a unique function H satisfying $H(x) \geqq 0$ and

$$H(x)^2 = x^2 F\left(H(x)\right). \tag{8}$$

For $\lambda > 0$, and n tending to infinity, one has

$$\log\rho\left(n, \lambda\sqrt{n}\right) \sim h(\lambda)\sqrt{n} \tag{9}$$

where

$$h(\lambda) = \frac{2H(\lambda)}{\lambda} - \lambda\log\left(1 + e^{-H(\lambda)}\right). \tag{10}$$

It would not be difficult to give a proof of Theorem 1 by using (2). In particular, for $x = \lambda\sqrt{n}$, one has first to deduce from (3) that $\sigma \sim H(\lambda)/\left(\lambda\sqrt{n}\right)$. In Section 3, we shall prove the lower bound (that is $\log\rho\left(n, \lambda\sqrt{n}\right)$

$\geqq h(\lambda)\sqrt{n}(1+o(1))$) based on a method of combinatorial nature which is certainly not so accurate as the saddle point method but which can be used more widely when the generating function is not easy to deal with.

In the second half of this paper we shall consider the following problem of Erdős: For each (unrestricted) partition Π of n, $n = n_1 + n_2 + \ldots + n_k$ with $n_1 \leqq n_2 \leqq \ldots \leqq n_k$, we say that an integer a is represented by Π if a can be written as $a = \varepsilon_1 n_1 + \ldots + \varepsilon_k n_k$ with $\varepsilon_i = 0$ or 1. For each Π denote the set of these integers by $T(\Pi)$. We shall call it the set represented by the partition Π. For fixed n, let $\hat{p}(n)$ denote the number of different sets amongst the sets $T(\Pi)$ (where Π varies over the $p(n)$ partitions of n). From [8] (see also [1]) we know that most of the partitions do represent all integers between 1 and n so that $\hat{p}(n)$ is smaller than $p(n)$. Erdős' problem is to estimate the function $\hat{p}(n)$ and indeed we will prove:

THEOREM 2. *There exist two constants* $0 < c_1 \leqq c_2 < 1$ *such that for n large enough one has*

$$p(n)^{c_1} \leqq \hat{p}(n) \leqq p(n)^{c_2} \tag{11}$$

where $p(n)$ *is the number of unrestricted partitions of n. Moreover, one can choose in* (11)

$$c_1 = 0.361 \tag{12}$$

and

$$c_2 = 0.948. \tag{13}$$

The proof of the lower bound in (11) with a small constant c_1 is not difficult: Let $E = \{e_1, e_2, \ldots, e_k\}$ be any set of integers between $\sqrt{n}/2$ and $\sqrt{n}$, and let us set

$$e_{k+1} = n - (e_1 + e_2 + \ldots + e_k) > \sqrt{n}.$$

The integers a represented by the partition $n = e_1 + e_2 + \ldots + e_{k+1}$ and included between $\sqrt{n}/2$ and $\sqrt{n}$ are exactly the elements of E and so $\hat{p}(n) \gg 2^{\sqrt{n/2}}$. From the famous result of Hardy and Ramanujan [11], it is known that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(C\sqrt{n}\right) \tag{14}$$

where $C = \pi\sqrt{2/3} = 2.565\ldots$, and thus any $c_1 < \frac{\log 2}{2C} = 0.135\ldots$ can be chosen in (11).

Let us denote by $\rho_\lambda(n)$ the number of partitions of n into distinct parts belonging to the interval $\left]\lambda\sqrt{n}, 2\lambda\sqrt{n}\right]$. If $\lambda < \sqrt{2/3}$, the sum of all the integers of this interval is smaller than n, and therefore $\rho_\lambda(n) = 0$. If Π denotes the partition

$$n = n_1 + n_2 + \ldots + n_k$$

with $\lambda\sqrt{n} < n_1 < n_2 < \ldots < n_k \leqq 2\lambda\sqrt{n}$, then the set $T(\Pi)$ represented by the partition Π will satisfy

$$T(\Pi) \cap \left]0, 2\lambda\sqrt{n}\right] = \{n_1, n_2, \ldots, n_k\}$$

and thus

$$\hat{p}(n) \geqq \rho_\lambda(n) \tag{15}$$

for any choice of λ. In Section 5, we shall give a lower bound for $\hat{p}(n)$ not so accurate as (15) (though it would be possible to show with some more effort that $\lim \log \rho_\lambda(n)/\sqrt{n} = Q\big(\lambda, \mu(\lambda)\big)$, where Q is given by (50) below). For a good choice of λ this lower bound will yield the value of c_1 as announced by (12).

The proof of the upper bound of (11) with a constant c_2 close to 1 is easy: Let us denote by $R(n,a)$ the number of unrestricted partitions of n which do not represent a. Clearly $R(n,a) = R(n, n-a)$. Let us choose b, $1 \leqq b < n/2$ and let $\mathcal{R}_b(n)$ denote the set of partitions of n which represent all integers a, with $b < a < n-b$. The number of different sets represented by the partitions of $\mathcal{R}_b(n)$ is certainly at most 2^b, while the number of partitions of n which are not in $\mathcal{R}_b(n)$ is

$$\leqq \sum_{b<a<n-b} R(n,a) \leqq n \max_{b<a<n-b} R(n,a).$$

So we have proved that

$$\hat{p}(n) \leqq 2^b + n \max_{b<a<n-b} R(n,a). \tag{16}$$

In [3], [4] and [5] different estimations of $R(n,a)$ are given. Specially, Lemma 2.1 of [3] claims that for all $\varepsilon > 0$, there exists $\delta \in]0,1[$ such that

$$\varepsilon\sqrt{n} < a < n - \varepsilon\sqrt{n} \Rightarrow R(n,a) \leqq p(n)^\delta.$$

This lemma, and (16) allow us to choose $c_2 < 1$ in (11). In order to get a constant c_2 as small as possible, we shall prove:

THEOREM 3. (i) *Let $\alpha \leqq 0.54$. When n tends to infinity one has*

$$\log\Big(R\big(n, \lfloor \alpha\sqrt{n}\rfloor\big)\Big) \leqq \big(g(\alpha/2) + (\alpha/2)(1+\log 2) + o(1)\big)\sqrt{n} \tag{17}$$

where g is defined by (5).

(ii) *When n tends to infinity, one has for $0.18\sqrt{n} \leqq a \leqq n - 0.18\sqrt{n}$*

$$\log R(n,a) \leqq \big(2.43 + o(1)\big)\sqrt{n}. \tag{18}$$

The value of $c_2 = 2.43/C \leqq 0.948$ given by (13) follows immediately from (16) (by choosing $b = 0.18\sqrt{n}$), (18) and (14).

We know from [2, Théorème 2.15] that

$$g(\lambda) = \lambda\log\lambda + \sum_{n=0}^{\infty} g_n\lambda^n \tag{19}$$

where this power series expansion has a positive radius of convergence and $g_0 = C = \pi\sqrt{2/3}$, $g_1 = \log(C/2) - 1$, $g_2 = -C/8 - 1/(2C)$.

The result of [4, Théorème 1, i] gives for $\alpha \leqq \lambda_0$ (this λ_0 could be calculated)

$$\log R\big(n, \lfloor \alpha\sqrt{n}\rfloor\big) \leqq \left(C + \frac{\alpha}{2}\log\left(\frac{C\alpha}{2}\right) + o(1)\right)\sqrt{n}. \tag{20}$$

Noting that

$$\frac{\lambda}{2}\log\frac{\lambda}{2} + g_0 + g_1\frac{\lambda}{2} + \frac{\lambda}{2}(1+\log 2) = C + \frac{\lambda}{2}\log\left(\frac{C\lambda}{2}\right)$$

and since g_2 is negative, (17) is a little better than (20).

The proof of Theorem 3 will be given in Section 7. It is similar to the proof of Theorem 1(i) of [4]. Let $\mathcal{A} = \{a_1, a_2, \dots, a_j\}$ be a finite set of distinct integers, and let us denote by $r(n,\mathcal{A})$ the number of partitions of n into parts not belonging to $\mathcal{A}$. In particular, one has $r(n, m+1) = r\big(n, \{1, 2, \dots, m\}\big)$. In the first step, an upper bound of $r(n,\mathcal{A})$ is needed and is obtained here, in a different way as in [4], by an argument of convexity. The second step of the proof of Theorem 3 is exactly the same as in [4].

The argument of convexity is interesting in itself, and is made precise in the following statement:

THEOREM 4. *Let δ be a positive real number, $\delta < 0.133$. There exists a number n_0 depending only on δ such that if $\mathcal{A} = \{a_1, a_2, \dots, a_j\}$ is a set of* *integers,* $1 \leqq a_1 < a_2 < \ldots < a_j$, *then*

(i) *the sequence* $z(n) = r(n, \mathcal{A})$ *is convex for* $n \geqq \max(n_0, s/\delta)$ *where*

$$s = s(\mathcal{A}) = a_1 + a_2 + \ldots + a_j + 4j + 8. \tag{21}$$

Whenever $\delta = 0.11$, n_0 *can be taken equal to* 13 000.

(ii) *For* $n \geqq \max(n_0 + a_j, s/\delta)$, *the inequality*

$$z(n) = r\big(n, \{a_1, a_2, \ldots, a_j\}\big) \leqq \frac{\prod_{i=1}^{j} a_i}{j!} r(n, j+1) \tag{22}$$

holds.

We may observe that (22) does not hold unconditionally: for instance

$$r\big(10, \{1, 3\}\big) = 8 > \frac{3}{2} r(10, 3) = \frac{15}{2}.$$

The proof of Theorem 4 will be given in Section 6. It is similar to the proofs used by A. Odlyzko in [13] to study the k-th difference of the partition function, but it is not so deep: in terms of m we do not make precise the smallest N for which the function $r(n, m)$ is convex for $n \geqq N$, and this is made possible by the methods of [13].

At the end of the paper, a table of $\hat{p}(n)$ for $n \leqq 121$ can be found. It has been calculated by Marc Deléglise by a method shortly explained in Section 8.

What is the good exponent in Theorem 2? From the table, one might guess 0.7. Clearly, the upper bound given for $R\big(n, \lfloor \alpha\sqrt{n} \rfloor\big)$ in Theorem 3 is not very sharp, unless α is very small. So the constant c_2 in (13) is too large. On the other hand, since $R(n, a) \geqq r\big(n, \{1, 2, \ldots, \lfloor a/2 \rfloor, a\}\big)$ and we may think that $\log r\big(n, \{1, 2, \ldots, \lfloor a/2 \rfloor, a\}\big) \sim g(\alpha/2)\sqrt{n}$ for $a \sim \alpha\sqrt{n}$, we cannot hope to deduce from (16) a better upper bound than

$$\frac{\log \hat{p}(n)}{\sqrt{n}} \leqq \big(\max\big(\alpha \log 2, g(\alpha/2)\big)\big).$$

This right hand side is minimal for $\alpha = 2.02$, and is approximately 1.40, so that it is not possible to prove $c_2 < \dfrac{1.40}{2.56} = 0.55$ by this method.

We are pleased to warmly thank Marc Deléglise for kindly computing the table of $\hat{p}(n)$ and Paul Erdős for valuable discussions about this paper.

2. The functions F, G, H and h

In this section we shall study the real functions introduced in the statement of Theorem 1. Since these functions are closely related to the functions studied and used in [14], [15], and [2], and they can be analyzed by using the same tools, we will leave some details to the reader. For $x \geqq 0$, let

$$F(x) = \int_x^{+\infty} \frac{u}{1+e^u}\,du \tag{23}$$

$$= \int_x^{+\infty} u \sum_{k=1}^{+\infty} (-1)^{k-1} e^{-ku}\,du = \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{kx+1}{k^2} e^{-kx}$$

so that

$$F(x) = \int_0^{+\infty} \frac{u}{1+e^u}\,du - \int_0^x \frac{u}{1+e^u}\,du \tag{24}$$

$$= \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{1}{k^2} - \int_0^x \frac{u}{1+e^u}\,du = \frac{\pi^2}{12} - \int_0^x \frac{u}{1+e^u}\,du$$

and

$$F(x) = (x+1)e^{-x} + O(xe^{-2x}) \quad \text{as} \quad x \to +\infty. \tag{25}$$

From the identity

$$\frac{u}{e^u+1} = \frac{u}{e^u-1} - \frac{2u}{e^{2u}-1}$$

and from the definition of Bernoulli numbers

$$\frac{u}{e^u-1} = 1 + \sum_{n\geqq 1} \frac{b_n}{n!} u^n, \quad b_1 = -\frac{1}{2}, \quad b_2 = \frac{1}{6}, \quad b_4 = -\frac{1}{30}, \dots$$

one deduces from (24)

$$F(x) = \frac{\pi^2}{12} + \sum_{m\geqq 2} (2^{m-1}-1) b_{m-1} x^m / m! \tag{26}$$

which for $|x| < \pi$ can be used to get numerical values of $F(x)$, and shows that $F(x) - \pi^2/12 + x^2/4$ is odd. The function $F(x)$ is analytic, and it decreases from $\frac{\pi^2}{12}$ to 0 on $[0+\infty)$. For $x \geqq 0$, set $G(x) = xF(x)^{-1/2}$. Then the function $G(x)$ is analytic, and differentiating the equation $G(x)^{-2} = x^{-2}F(x)$, we obtain

$$-2G(x)^{-3}\frac{dG}{dx} = -2x^{-3}F(x) - x^{-2}\frac{x}{1+e^x} = -x^3\left(2F(x) + \frac{x^2}{1+e^x}\right) < 0$$

for $x > 0$. It follows that $dG/dx > 0$ whence the function $G(x)$ is increasing on $[0,+\infty)$ from $G(0) = 0$ to

$$\lim_{x\to+\infty} G(x) = \lim_{x\to+\infty} xF(x)^{-1/2} = \lim_{x\to+\infty} \big(1+o(1)\big)\sqrt{x}e^{x/2} = +\infty$$

(in view of (25)). Thus the function $G(x)$ has a unique analytic inverse $H(x)$ on $(0,+\infty)$ whose derivative $H'(x)$ is positive. Moreover, by (25) we have

$$\begin{aligned}(27)\qquad G(x) &= xF(x)^{-1/2} = x^{1/2}e^{x/2}\big(1+x^{-1}+O(e^{-x})\big)^{-1/2}\\ &= x^{1/2}e^{x/2}\left(1-\frac{1}{2}x^{-1}+\frac{3}{8}x^{-2}+O(x^{-3})\right)\\ &= \left(x^{1/2}-\frac{1}{2}x^{-1/2}+\frac{3}{8}x^{-3/2}+O(x^{-5/2})\right)e^{x/2} \quad \text{as} \quad x\to+\infty.\end{aligned}$$

When x tends to zero, one deduces from (26), with the help of MAPLE

$$(28)\quad G(x) = \frac{2\sqrt{3}}{\pi}x + \frac{3\sqrt{3}}{\pi^3}x^3 - \frac{\sqrt{3}}{\pi^3}x^4 + \frac{27\sqrt{3}}{4\pi^5}x^5 + \frac{\sqrt{3}(\pi^2-90)}{20\pi^5}x^6 + O(x^7).$$

Clearly the inverse $H(x)$ of the function $G(x)$ satisfies the equation

$$(29)\qquad x^{-2}H(x)^2 = \int_{H(x)}^{+\infty}\frac{u}{1+e^u}\,du = F\big(H(x)\big).$$

It can be derived easily from (27) that

$$\begin{aligned}(30)\qquad H(x) &= 2\log x - \log\log x - \log 2 + \frac{\log\log x + \log 2 + 1}{2\log x}\\ &+ \frac{(\log\log x)^2 + 2\log 2\log\log x + \log^2 2 - 3}{8\log^2 x}\\ &+ O\left(\frac{(\log\log x)^3}{\log^3 x}\right) \quad \text{as} \quad x\to+\infty,\end{aligned}$$

and from (28)

$$(31)\quad H(x)=\frac{\pi}{2\sqrt{3}}x-\frac{\pi}{16\sqrt{3}}x^3+\frac{\pi^2}{288}x^4+\frac{\pi\sqrt{3}}{256}x^5-\frac{\pi^2(\pi^2+120)}{69120}x^6+O(x^7)$$

as $x\to 0$.

The numerical calculation of $H(x)$ can be carried out by solving (29) as explained at the end of [2].

Further, it follows from (29) that

$$2x^{-1}H(x)\big(x^{-1}H(x)\big)'=-\frac{H(x)H'(x)}{1+e^{H(x)}}$$

whence

$$(32)\qquad \big(x^{-1}H(x)\big)'=-\frac{xH'(x)}{2\big(1+e^{H(x)}\big)}$$

so that $H(x)$ satisfies the differential equation

$$(33)\qquad 2xH'-2H=-\frac{x^3H'}{e^H+1}.$$

From (30) it is easy to find the asymptotic expansion of $h(\lambda)$ defined by (10) as $\lambda\to+\infty$. It turns out that this asymptotic expansion does coincide with the one of $g(\lambda)$ as given in (6). In fact it is possible to prove that

$$g(\lambda)-h(\lambda)\sim(\log\lambda)/\lambda^3\quad\text{as}\quad\lambda\to+\infty$$

and we shall return to this question in an other paper. From (10) and (32), one has

$$h'(\lambda)=-\log\left(1+e^{-H(\lambda)}\right)$$

which together with (10) yields

$$(34)\qquad \lambda h(\lambda)-\lambda^2h'(\lambda)=2H(\lambda)$$

and

$$h(\lambda)-\lambda h'(\lambda)-\lambda^2h''(\lambda)=2H'(\lambda).$$

Substituting these values of $H(\lambda)$ and $H'(\lambda)$ in (33) gives

$$(35)\qquad \lambda^2h''+\lambda h'-h=-2h''\left(1+\exp\left(\frac{1}{2}\lambda h-\frac{1}{2}\lambda^2h'\right)\right).$$

From (10) and (31) one gets the asymptotic expansion

$$h(x)=\frac{\pi}{\sqrt{3}}-(\log 2)x+\frac{\pi\sqrt{3}}{24}x^2-\frac{\pi^2}{288}x^3-\frac{\sqrt{3}\pi}{384}x^4+O(x^5)$$

as $x\to 0$. Due to (34) this can be written as

$$-\frac{2H(\lambda)}{\lambda^3}=\frac{d}{d\lambda}\left(\frac{h(\lambda)}{\lambda}\right),$$

for $n\geqq 2$ the coefficient of x^n in $h(x)$ is equal to $\frac{-2}{n-1}$ times the coefficient of x^{n+1} in $H(x)$. Note that h is analytic for $x=0$, while, from (19), $g(x)$ is not.

3. Proof of Theorem 1: the lower bound

Throughout the proof, L will denote a large but fixed positive integer; k is a positive integer large in terms of L; fixing L and k, n will denote a positive integer with $n\to+\infty$. Let $y_1,y_2,\dots,y_{Lk}$ be non-negative real numbers satisfying the inequality

$$\sum_{j=1}^{Lk} y_j\left(\lambda+\frac{j}{k}\right)\leqq k-1; \tag{36}$$

these numbers will be defined later optimally (by using the Lagrange multiplier method). For $j=1,2,\dots,Lk$, write

$$I_j=\left\{\lfloor\lambda\sqrt{n}\rfloor+(j-1)\left\lfloor\frac{\sqrt{n}}{k}\right\rfloor+1,\right.$$

$$\left.\lfloor\lambda\sqrt{n}\rfloor+(j-1)\left\lfloor\frac{\sqrt{n}}{k}\right\rfloor+2,\dots,\lfloor\lambda\sqrt{n}\rfloor+j\left\lfloor\frac{\sqrt{n}}{k}\right\rfloor\right\}.$$

Let Γ denote the family of the sets S of the form $S=\bigcup_{j=1}^{Lk}S_j$, where $S_j\subset I_j$ and

$$|S_j|=\left\lfloor y_j\frac{\sqrt{n}}{k}\right\rfloor\quad (j=1,2,\dots,Lk).$$

Then by (36), for all $S \in \Gamma$ we have

$$\sum_{s\in S} s = \sum_{j=1}^{Lk} \sum_{s\in S_j} s \leqq \sum_{j=1}^{Lk} \left\lfloor y_j \frac{\sqrt{n}}{k} \right\rfloor \left(\lfloor \lambda\sqrt{n} \rfloor + j \left\lfloor \frac{\sqrt{n}}{k} \right\rfloor \right)$$

$$\leqq \frac{n}{k} \sum_{j=1}^{Lk} y_j \left(\lambda + \frac{j}{k} \right) \leqq n - \frac{n}{k}.$$

It follows that if n is large enough, then the number

$$s_0 \overset{\text{def}}{=} n - \sum_{s\in S} s \left(\geqq \frac{n}{k} \right)$$

is greater than the greatest element of S. Thus the elements of $S \cup \{s_0\}$ determine a partition of n into distinct parts $> \lambda\sqrt{n}$; moreover, distinct sets $S \in \Gamma$ determine distinct partitions. It follows that we have

$$|\Gamma| \leqq \rho\left(n, \lambda\sqrt{n}\right). \tag{37}$$

Clearly, by Stirling's formula we have

$$|\Gamma| = \prod_{j=1}^{Lk} \left(\binom{\lfloor \sqrt{n}/k \rfloor}{\lfloor y_j\sqrt{n}/k \rfloor} \right) \tag{38}$$

$$= \exp\left((1+o(1)) \left(-\sum_{j=1}^{Lk} \left(y_j \log y_j + (1-y_j)\log(1-y_j) \right) \right) \frac{\sqrt{n}}{k} \right).$$

It remains to maximize the lower bound given by (37) and (38) for $\rho\left(n, \lambda\sqrt{n}\right)$; this can be done by computing

$$\min \sum_{j=1}^{Lk} \left(y_j \log y_j + (1-y_j)\log(1-y_j) \right)$$

under the constraint (36). The Lagrange multiplier method gives that the optimal choice of the parameters y_j is

$$y_j = \left(1 + e^{(\lambda + j/k)\mu} \right)^{-1}$$

where μ is a parameter independent of j, whose value can be determined by substituting into the constraint (36):

$$\sum_{j=1}^{Lk}\left(\lambda+\frac{j}{k}\right)\left(1+e^{(\lambda+j/k)\mu}\right)^{-1} \leqq k-1.$$

We are looking for a μ independent of both k and L: Dividing by k and taking the limit when $k\to\infty$ on both sides we obtain

$$\int_{\lambda}^{\lambda+L}\frac{x}{1+e^{\mu x}}\,dx \leqq 1.$$

Taking the limit as $L\to+\infty$ we obtain

$$\int_{\lambda}^{+\infty}\frac{x}{1+e^{\mu x}}\,dx \leqq 1$$

or, in equivalent form,

$$\mu^{-2}\int_{\lambda\mu}^{+\infty}\frac{t}{1+e^{t}}\,dt \leqq 1.$$

Using the notation introduced in Section 2, this can be rewritten as

$$\mu^{-2}F(\lambda\mu)\leqq 1,\quad G(\lambda\mu)=\lambda\mu\big(F(\lambda\mu)\big)^{-1/2}\geqq\lambda$$

whence

$$\lambda\mu\geqq H(\lambda),\quad \mu\geqq\frac{H(\lambda)}{\lambda}.$$

For $\mu=\frac{H(\lambda)}{\lambda}$ we have $\int_{\lambda}^{\lambda+L}\frac{x}{1+e^{\mu x}}\,dx<1$ which implies (36) for large $k\geqq k_0(L,\lambda)$. Substituting this value of μ in the definition of y_j, and taking the limit first as $k\to+\infty$ and then $L\to+\infty$, we obtain that

$$\frac{1}{k}\sum_{j=1}^{Lk}\big(y_j\log y_j+(1-y_j)\log(1-y_j)\big)$$

$$=\frac{1}{k}\sum_{j=1}^{Lk}\left(\frac{\log\left(1+e^{(\lambda+j/k)\mu}\right)}{1+e^{(\lambda+j/k)\mu}}\right.$$

$$\left.-\frac{e^{(\lambda+j/k)\mu}}{1+e^{(\lambda+j/k)\mu}}\left((\lambda+j/k)\mu-\log\left(1+e^{(\lambda+j/k)\mu}\right)\right)\right)$$

$$= \frac{1}{k}\sum_{j=1}^{Lk}\left(\log\left(1+e^{(\lambda+j/k)\mu}\right) - \frac{(\lambda+j/k)\mu e^{(\lambda+j/k)\mu}}{1+e^{(\lambda+j/k)\mu}}\right)$$

$$\to \int_{\lambda}^{\lambda+L}\left(\log(1+e^{\mu x}) - \frac{x\mu e^{\mu x}}{1+e^{\mu x}}\right)dx$$

$$\to \int_{\lambda}^{+\infty}\left(\log(1+e^{\mu x}) - \frac{x\mu e^{\mu x}}{1+e^{\mu x}}\right)dx$$

$$= \frac{1}{\mu}\int_{\lambda\mu}^{+\infty}\left(\log(1+e^{\mu}) - \frac{ue^{u}}{1+e^{u}}\right)du$$

$$= \frac{\lambda}{H(\lambda)}\int_{H(\lambda)}^{+\infty}\left(\log(1+e^{u}) - \frac{ue^{u}}{1+e^{u}}\right)du$$

$$= \frac{\lambda}{H(\lambda)}\int_{H(\lambda)}^{\infty}\left(\log(1+e^{-u}) + \frac{u}{1+e^{u}}\right)du = h(\lambda),$$

by integrating by parts the logarithm and by (29). Since all the functions above are continuous, and the improper integrals are convergent, it follows from (37) and (38) that

$$\liminf_{n\to\infty} \log\left(\rho\left(n,\lambda\sqrt{n}\right)\right)/\sqrt{n} \geqq h(\lambda). \tag{39}$$

4. Proof of Theorem 1: the upper bound

Clearly, the proof of Theorem 1 will follow from (39) and from the following proposition:

PROPOSITION 1. *Let λ be a real positive number and define $h(\lambda)$ by (10). For $n \geqq 1/\lambda^2$ the following inequality holds:*

$$\log\rho\left(n,\lambda\sqrt{n}\right) \leqq h(\lambda)\sqrt{n}+1.$$

PROOF. From the generating function

$$\sum_{n=0}^{\infty}\rho(n,x)z^n = \prod_{m\geqq x}(1+z^m) \tag{40}$$

one deduces for $0 < R < 1$ that

$$\rho(n, \lambda\sqrt{n}) \leqq R^{-n} \prod_{m \geqq \lambda\sqrt{n}} (1 + R^m). \tag{41}$$

Minimizing the right hand side of (41) would drive us to choose $R = e^{-\sigma}$, where σ is defined by (3). But (41) holds for any $R < 1$, and, with the notation of Section 2, we shall choose

$$R = e^{-s}, \quad s = H(\lambda)/(\lambda\sqrt{n}) \tag{42}$$

so that (41) yields

$$\log \rho(n, \lambda\sqrt{n}) \leqq \frac{H(\lambda)}{\lambda}\sqrt{n} + \sum_{m \geqq \lambda\sqrt{n}} \log(1 + \exp(-ms)).$$

Since the function $x \mapsto \log(1 + \exp(-ax))$ is decreasing for any positive a, it follows

$$\log \rho(n, \lambda\sqrt{n}) \leqq \frac{H(\lambda)}{\lambda}\sqrt{n} + \int_{\lambda\sqrt{n}-1}^{\infty} \log(1 + e^{-sx})\, dx. \tag{43}$$

Now, one has for $n \geqq 1/\lambda^2$ (that is $\lambda\sqrt{n} \geqq 1$)

$$\int_{\lambda\sqrt{n}-1}^{\lambda\sqrt{n}} \log(1 + e^{-sx})\, dx \leqq \log\Big(1 + \exp\big(-s(\lambda\sqrt{n} - 1)\big)\Big)$$

$$\leqq \exp\big(-s(\lambda\sqrt{n} - 1)\big) = \exp(s - H(\lambda)) \leqq 1.$$

Further, observing that from (42)

$$\int_{\lambda\sqrt{n}}^{\infty} \log(1 + e^{-sx})\, dx = \frac{\lambda\sqrt{n}}{H(\lambda)} \int_{H(\lambda)}^{\infty} \log(1 + e^{-u})\, du$$

$$= \frac{\lambda\sqrt{n}}{H(\lambda)}\Big(-H(\lambda)\log\big(1 + e^{-H(\lambda)}\big) + F\big(H(\lambda)\big)\Big),$$

one easily completes from (43) the proof of Proposition 1 with the help of (29).

5. The lower bound in Theorem 2

First we will prove the lower bound in (11) with the value of c_1 given by (12) by modifying the method used in Section 3 slightly. Throughout the proof, z will denote a fixed positive number to be chosen optimally; k is a large positive integer; fixing k, n will denote a positive integer with $n \to +\infty$. Since the method will be similar to the one used in Section 3, some details will be left to the reader. Let $y_1, y_2, \dots, y_{\lfloor kz \rfloor}$ be non-negative real numbers satisfying the inequality

$$\sum_{j=1}^{\lfloor kz \rfloor} y_j\left(z + \frac{j}{k}\right) \leqq k - 1. \tag{44}$$

For $j = 1, 2, \dots, \lfloor kz \rfloor$, write

$$I_j = \left\{ \lfloor z\sqrt{n} \rfloor + (j-1)\left\lfloor \frac{\sqrt{n}}{k} \right\rfloor + 1, \right.$$

$$\left. \lfloor z\sqrt{n} \rfloor + (j-1)\left\lfloor \frac{\sqrt{n}}{k} \right\rfloor + 2, \dots, \lfloor z\sqrt{n} \rfloor + j\left\lfloor \frac{\sqrt{n}}{k} \right\rfloor \right\}.$$

Let Γ denote the family of the sets S of the form $S = \bigcup_{j=1}^{\lfloor kz \rfloor} S_j$, where $S_j \subset I_j$ and $|S_j| = \left\lfloor y_j \frac{\sqrt{n}}{k} \right\rfloor$ for $j = 1, 2, \dots, \lfloor kz \rfloor$. It follows from (44) that

$$s_0 \overset{\text{def}}{=} n - \sum_{s \in S} s \geqq \frac{n}{k}$$

so that for sufficiently large n we have $s_0 > 2z\sqrt{n}$($\geqq$ the elements of S). Thus the elements of $S \cup \{s_0\}$ determine a partition $\Pi(S)$ of n into distinct parts $> z\sqrt{n}$, and clearly, for this partition $\Pi(S)$ we have

$$T\big(\Pi(S)\big) \cap \big]0, 2z\sqrt{n}\big] = S.$$

Thus for $S \in \Gamma, S' \in \Gamma, S \neq S'$ we have $T\big(\Pi(S)\big) \neq T\big(\Pi(S')\big)$. It follows that

$$\hat{p}(n) \geqq |\Gamma|. \tag{45}$$

Clearly, by Stirling's formula we have

$$|\Gamma| = \Pi_{j=1}^{\lfloor kz \rfloor} \binom{\lfloor \sqrt{n}/k \rfloor}{\lfloor y_j \sqrt{n}/k \rfloor} \tag{46}$$

$$= \exp\left(\left(1+o(1)\right)\left(-\sum_{j=1}^{\lfloor kz\rfloor}\left(y_j\log y_j+(1-y_j)\log(1-y_j)\right)\right)\frac{\sqrt{n}}{k}\right).$$

Next, we maximize the lower bound for $\hat{p}(n)$, given by (45) and (46), under the constraint (44). By using the Lagrange multiplier method as in Section 3, finally we obtain the estimate

$$\text{(47)}\qquad \liminf_{n\to+\infty}\frac{\log\hat{p}(n)}{\sqrt{n}} \geqq \int_z^{2z}\left(\log(1+e^{\mu x})-\frac{\mu x e^{\mu x}}{1+e^{\mu x}}\right)dx$$

$$= \frac{1}{\mu}\int_{z\mu}^{2z\mu}\left(\log(1+e^{u})-\frac{ue^{u}}{1+e^{u}}\right)du \stackrel{\text{def}}{=} Q(z,\mu)$$

where $\mu=\mu(z)$ is defined by

$$\text{(48)}\qquad \int_z^{2z}\frac{x}{1+e^{\mu x}}\,dx = 1$$

or, in equivalent form,

$$\text{(49)}\qquad \mu^{-2}\int_{z\mu}^{2z\mu}\frac{t}{1+e^{t}}\,dt = \mu^{-2}\left(F(z\mu)-F(2z\mu)\right) = 1.$$

Clearly the integral in (48) is a decreasing function of μ which is equal to $3z^2/2$ for $\mu=-\infty$, and tends to 0 when μ tends to $+\infty$. Thus, (48) defines $\mu(z)$ for $z>\sqrt{2/3}$ ($\mu(z)>0$ for $z>2/\sqrt{3}$). Moreover, by integrating by parts the logarithm, and by using (49), (47) yields

$$\text{(50)}\qquad Q\left(z,\mu(z)\right) = 2\mu(z) - z\log\left(1+e^{-z\mu(z)}\right) + 2z\log\left(1+e^{-2z\mu(z)}\right).$$

Now, in view of (47), to choose c_1 in (11) optimally, we have to find an approximate value of the maximum of $Q\left(z,\mu(z)\right)$ as given in (50). This can be done: for different values of z, one calculates $\mu(z)$ from (49) and then $Q\left(z,\mu(z)\right)$ by (50). A slightly better method is to set $x=\mu z$: we select several (positive) values of x. For each of them we compute the function $K(x)=F(x)-F(2x)$ (by using the series expansion of $F(x)$ given in Section 2). Then we compute $\mu=K(x)^{1/2}$ from (49), $z=x/\mu$, and, from (50)

$$Q = 2\mu - \frac{x}{\mu}\left(\log(1+e^{-x})-2\log(1+e^{-2x})\right).$$

Choosing $x=0.791$ yields $\mu=0.463\ldots$, $z=1.708\ldots$, $Q=0.92614\ldots$ and $c_1=Q/C=0.361$ as announced in (12).

6. Proof of Theorem 4

The proof is similar to the proof of Theorem 1 of [1] or of the Proposition of [4]. We shall use the same notation. If f is any real function, and $u_1, u_2, \dots, u_m$ are positive real numbers, the operator $D^{(m)}$ is defined by induction:

$$D^{(1)}(u_1; f, x) = f(x) - f(x - u_1),$$

$$D^{(m)}(u_1, \dots, u_m; f, x) = D^{(m-1)}(u_1, \dots, u_{m-1}; f, x)$$
$$-D^{(m-1)}(u_1, \dots, u_{m-1}; f, x - u_m)$$

and it follows from the generating functions that

$$z(n) = r(n, \mathcal{A}) = D^{(j)}(a_1, a_2, \dots, a_j; p, n)$$

and

(51) $w(n) = z(n) - 2z(n-1) + z(n-2) = D^{(j+2)}(a_1, a_2, \dots, a_j, 1, 1; p, n).$

Now, writing $P(x) = \exp\left(\sqrt{x}\right)/\sqrt{x}$, the classical result of Hardy and Ramanujan can be written as (cf. [1, (1)])

$$p(n) = \frac{C^3}{2\pi\sqrt{2}} P'\left(C^2(n - 1/24)\right) + f_1(n) \tag{52}$$

with $C = \pi\sqrt{2/3} = 2.56\dots$ and

$$\left| f_1(n) \right| \leqq \frac{0.11}{n} \exp\left(\frac{C\sqrt{n}}{2}\right), \quad n \geqq 1. \tag{53}$$

Furthermore, using Lemma 3 of [1], (51), (52) and (53) give

$$w(n) = M_n + R_n \tag{54}$$

where

$$M_n = \left(\prod_{i=1}^{j} a_i\right) \frac{C^{2j+7}}{2\pi\sqrt{2}} P^{(j+3)}\left(C^2(\xi - 1/24)\right) \tag{55}$$

with

$$n - (a_1 + \dots + a_j + 2) \leqq \xi \leqq n \tag{56}$$

and

$$|R_n| \leqq 0.11 \frac{2^{j+2}}{n} \exp\left(\frac{C\sqrt{n}}{2}\right). \tag{57}$$

Let us observe here that, from (21), (56) implies

$$n - s \leqq \xi - 1/24 \leqq n \tag{58}$$

and also from (21), one has

$$s \geqq 1 + 2 + \ldots + j + 4j + 8 = (j^2 + 9j + 16)/2 > (j+4)^2/2. \tag{59}$$

Now, it is known that

$$p^{(m)}(x) = \frac{\exp\left(\sqrt{x}\right)}{2^m x^{(m+1)/2}} y_m\left(-\frac{1}{\sqrt{x}}\right)$$

where y_m is the m^{th} Bessel polynomial. So, (55) becomes

$$M_n = \left(\prod_{i=1}^{j} a_i\right) \frac{(C/2)^{j+3}}{2\pi\sqrt{2}} \frac{\exp\left(C\sqrt{\xi - 1/24}\right)}{(\xi - 1/24)^{(j+4)/2}} y_{j+3}\left(\frac{-1}{C\sqrt{\xi - 1/24}}\right). \tag{60}$$

To get a lower bound for y_{j+3}, we shall apply Lemma 3 of [4] which asserts that, for $0 \leqq mx \leqq 1/\sqrt{2}$, the inequality

$$y_m(-x) \geqq \left(1 - \frac{m(m+1)}{2} x^2\right) \exp\left(-\frac{m(m+1)}{2} x\right) \tag{61}$$

holds. We set $x = 1/C\sqrt{\xi - 1/24}$, and $m = j + 3$. From (58) and (59) we have

$$mx = \frac{j+3}{C\sqrt{\xi - 1/24}} \leqq \frac{\sqrt{2s}}{C\sqrt{n-s}} \leqq \frac{\sqrt{2\delta}}{C\sqrt{1-\delta}} \leqq 0.216 < 1/\sqrt{2} \tag{62}$$

since, from the hypothesis, we know that $s \leqq \delta n$ which implies

$$\frac{s}{n-s} \leqq \frac{\delta}{1-\delta} \leqq \frac{0.133}{1 - 0.133} \leqq 0.154. \tag{63}$$

In view of (62), (61) may be applied and with (58), (59) and (63) yields:

$$(64)\qquad y_{j+3}(-x) \geqq \left(1-\frac{(j+4)^2}{2}x^2\right)\exp\left(-\frac{(j+4)^2}{2}x\right)$$

$$\geqq \left(1-\frac{s}{C^2(n-s)}\right)\exp\left(\frac{-s}{C\sqrt{n-s}}\right)$$

$$\geqq (1-\lambda)\exp\left(-\lambda C\sqrt{n-\delta n}\right) = (1-\lambda)\exp\left(-\lambda C\left(\sqrt{1-\delta}\right)\sqrt{n}\right)$$

with

$$(65)\qquad \lambda = \frac{\delta}{C^2(1-\delta)} \leqq \frac{0.154}{C^2} \leqq 0.024.$$

To get a lower bound for M_n, we first observe that the function $t \to \dfrac{\exp\left(\sqrt{t}\right)}{t^{(j+4)/2}}$ is increasing for $t \geqq (j+4)^2$, and since one has from (58), (59) and (63)

$$(j+4)^2 \leqq 2s \leqq 2\delta n \leqq C^2(1-\delta)n \leqq C^2(n-s) \leqq C^2(\xi - 1/24),$$

it follows that

$$(66)\qquad \frac{\exp\left(C\sqrt{\xi-1/24}\right)}{(\xi-1/24)^{(j+4)/2}} \geqq \frac{\exp\left(C\sqrt{1-\delta}\sqrt{n}\right)}{\left((1-\delta)n\right)^{(j+4)/2}}.$$

Now, from (64), (66) and the inequality $\Pi_{i=1}^{j} a_i \geqq j! \geqq (j/e)^j$, (60) yields

$$(67)\quad M_n \geqq \frac{C^3(1-\lambda)}{16\pi\sqrt{2}(1-\delta)^2 n^2}\left(\frac{Cj}{2e\sqrt{1-\delta}\sqrt{n}}\right)^j \exp\left((1-\lambda)C\sqrt{1-\delta}\sqrt{n}\right).$$

We want to prove the convexity of $z(n)$, that is to prove that $w(n)$ is positive. From (54), it suffices to show that $M_n > |R_n|$. From (57) and (67), one has

$$(68)\qquad \frac{M_n}{|R_n|} \geqq \frac{C^3(1-\lambda)}{(0.44)16\pi\sqrt{2}(1-\delta)^2 n}\left(\frac{Cj}{4e\sqrt{1-\delta}\sqrt{n}}\right)^j$$

$$\cdot \exp\left(C\left((1-\lambda)\sqrt{1-\delta} - 1/2\right)\sqrt{n}\right).$$

From (59), it follows that $j \leqq j+4 \leqq \sqrt{2s} \leqq \sqrt{2\delta n}$, and since the function $t \mapsto (at)^t$ is decreasing for $0 < t < 1/(ae)$ and $\sqrt{2\delta} < 4\sqrt{1-\delta}/C$, it follows from (68) that

$$(69)\qquad \frac{M_n}{|R_n|} \geqq \frac{C^3(1-\lambda)}{(0.44)16\pi\sqrt{2}(1-\delta)^2 n}$$

$$\cdot \exp\left(\sqrt{2\delta n}\log\frac{C\sqrt{2\delta}}{4e\sqrt{1-\delta}} + C\big((1-\lambda)\sqrt{1-\delta}-1/2\big)\sqrt{n}\right).$$

With λ given by (65), it is easy to see that the coefficient of $\sqrt{n}$ inside the exponential is positive for $\delta \leq 0.133$. So the right hand side of (68) tends to infinity with n. For n large enough, $M_n/|R_n|$ is greater than 1. This implies from (54) that $w(n)$ is positive, and from (51) that $z(n) = r(n,\mathcal{A})$ is convex.

Choosing $\delta = 0.11$ yields $n_0 = 13\,000$ in Theorem 4. In order to prove (ii) in Theorem 4, let us define

$$r_1(m) = r\big(m, \{a_2, a_3, \dots, a_j\}\big)$$

for $j \geqq 2$, and $r_1(m) = p(m)$ if $j = 1$. Similarly, for $2 \leqq i \leqq j$, let us set

$$r_i(m) = r\big(m, \{1, 2, \dots, i-1, a_{i+1}, \dots, a_j\}\big).$$

We shall prove that for $1 \leqq i \leqq j$, $r_i(m)$ is convex for $m \geqq n - a_i$, where $n \geqq \max(n_0 + a_j, s/\delta)$. This point will follow from (i), since

$$m \geqq n - a_i \geqq n_0 + a_j - a_i \geqq n_0$$

and

$$\begin{aligned} m &\geqq n - a_i \geqq s(\mathcal{A})/\delta - a_i \\ &\geqq \big(s\big(\{1,2,\dots,i-1,a_{i+1},\dots,a_j\}\big) + a_i\big)/\delta - a_i \\ &\geqq s\big(\{1,2,\dots,i-1,a_{i+1},\dots,a_j\}\big)/\delta. \end{aligned}$$

Note that the above proof works also whenever $j = 1$, since the proof of (i) is still valid for $\mathcal{A} = \emptyset$, $j = 0$. Anyway, it is known that $p(n)$ is convex for $n \geqq 2$ (cf.[10]).

From the convexity of $r_1(m)$ for $m \geqq n - a_1$, it follows

$$\begin{aligned} r(n,\mathcal{A}) &= r_1(n) - r_1(n-a_1) \\ &\leqq a_1\big(r_1(n) - r_1(n-1)\big) = a_1 r\big(n, \{1, a_2, \dots, a_j\}\big). \end{aligned}$$

Similarly, for $i \geqq 2$, from the convexity of $r_i(m)$, one has

$$\begin{aligned} r\big(n, \{1,2,\dots,i-1,a_i,a_{i+1},\dots,a_j\}\big) &= r_i(n) - r_i(n-a_i) \\ \leqq \frac{a_i}{i}\big(r_i(n) - r_i(n-i)\big) &= \frac{a_i}{i} r\big(n, \{1,2,\dots,i,a_{i+1},\dots,a_j\}\big) \end{aligned}$$

and applying this inequality for $2 \leqq i \leqq j$ completes the proof of Theorem 4.

7. Proof of Theorem 3

First we shall prove:

LEMMA 1. *For a large enough and $n \geqq 3a^2 + 112a$, the number $R(n,a)$ of partitions of n which do not represent a satisfies*

$$R(n,a) \leqq \begin{cases} \dfrac{a^{(a+1)/2}}{((a+1)/2)!} r\left(n, \dfrac{a+3}{2}\right) & \textit{(for a odd)} \\ \dfrac{a^{\frac{a}{2}+1}}{2(a/2+1)!}\left(r\left(n, \dfrac{a}{2}+2\right) + r\left(n-\dfrac{a}{2}, \dfrac{a}{2}+2\right)\right) & \textit{(for a even).} \end{cases}$$

PROOF. It follows the proof of Theorem 1 in [4, p.162]. Let us suppose a is odd. If a partition of n does not represent a, its parts cannot include simultaneously i and $a-i$, so that,

$$R(n,a) \leqq \sum_{\varepsilon_1,\dots,\varepsilon_{\lfloor a/2\rfloor}} r\left(n, \left(\bigcup_{i=1}^{\lfloor a/2\rfloor} \left\{ i^{\varepsilon_i}(a-i)^{1-\varepsilon_i}\right\}\right) \cup \{a\}\right)$$

where, in the summation, $\varepsilon_i \in \{0,1\}$. We apply Theorem 4(ii), with $\delta = 1/8$ and $j = \dfrac{a+1}{2}$. For any choice of ε_i, one has

$$a_1 + a_2 + \ldots + a_j \leqq \sum_{i=\frac{a+1}{2}}^{a} i = \frac{3a^2+4a+1}{8}$$

and so,

$$a_1 + \ldots + a_j + 4j + 8 \leqq \frac{3a^2+4a+1}{8} + 2(a+1) + 8 \leqq \frac{3a^2}{8} + 14a \leqq n/8.$$

Moreover, for a large enough, $n - a_j = n - a$ will certainly be greater than n_0. So Theorem 4 yields

$$R(n,a) \leqq \sum_{\varepsilon_1,\dots,\varepsilon_{\lfloor a/2\rfloor}} a \frac{\prod_{i=1}^{\lfloor a/2\rfloor} i^{\varepsilon_i}(a-i)^{1-\varepsilon_i}}{((a+1)/2)!} r\left(n, \frac{a+3}{2}\right)$$

$$= \frac{a^{(a+1)/2}}{((a+1)/2)!} r\left(n, \frac{a+3}{2}\right).$$

Whenever a is even, the part $a/2$ can occur but only once. So we have

$$R(n,a) \leqq \sum_{\varepsilon_1,\dots,\varepsilon_{\frac{a}{2}-1}} r\Big(n, \Big(\bigcup_{i=1}^{\frac{a}{2}-1} \{ i^{\varepsilon_i}(a-i)^{1-\varepsilon_i} \} \Big) \cup \Big\{\frac{a}{2}\Big\} \cup \{a\}\Big)$$

$$+ \sum_{\varepsilon_1,\dots,\varepsilon_{\frac{a}{2}-1}} r\Big(n-\frac{a}{2}, \Big(\bigcup_{i=1}^{\frac{a}{2}-1} \{ i^{\varepsilon_i}(a-i)^{1-\varepsilon_i} \} \Big) \cup \Big\{\frac{a}{2}\Big\} \cup \{a\}\Big)$$

and the rest of the proof runs in the same way as for a odd.

PROOF OF THEOREM 3(i). Let us set $a = \lfloor \alpha\sqrt{n} \rfloor$. Since $\alpha \leqq 0.54$, for n large enough, the inequality $n \geqq 3a^2 + 112a$ holds, and Lemma 1 may be applied. Further, it follows from the definition of g, and of its continuity that, if $m_n \sim \lambda\sqrt{n}$, then $\log r(n, m_n) \sim g(\lambda)\sqrt{n}$ (see [2], remark 2.17). So, when a is odd, one has $\log r(n, (a+3)/2) \sim g(\alpha/2)\sqrt{n}$, and a classical estimation of $a^{(a+1)/2}/((a+1)/2)!$ by Stirling's formula completes the proof of (i). When a is even, the proof is similar.

PROOF OF THEOREM 3(ii). We shall need the following lemma (cf. Lemma 2.1 of [3]):

LEMMA 2. *Let us suppose that for some $\varepsilon > 0$, one has when n tends to infinity*

$$\varepsilon\sqrt{n} \leqq a \leqq 3\varepsilon\sqrt{n} \Rightarrow R(n,a) \leqq \exp\big((\eta + o(1))\sqrt{n}\big)$$

with $\eta > \pi/\sqrt{3}$. Then for all a such that $\varepsilon\sqrt{n} \leqq a \leqq n - \varepsilon\sqrt{n}$ one has $R(n,a) \leqq \exp\big((\eta + o(1))\sqrt{n}\big)$.

In fact, the above lemma is not exactly Lemma 2.1 of [3], but the proof of [3] is valid for our Lemma 2 here.

Now, let us observe that the function $\hat{g}$ defined by

$$\hat{g}(\alpha) = g(\alpha/2) + (\alpha/2)(1 + \log 2)$$

is convex and has a minimum for $\alpha = \alpha_0$. This follows from the facts that $g'(\alpha) < 0$, $g''(\alpha) > 0$ (cf. [2], Théorème 2.5) and $\lim_{\alpha\to\infty} g(\alpha) = 0$ (cf. (6)) which imply $\lim_{\alpha\to\infty} g'(\alpha) = 0$.

A numerical calculation yields $\alpha_0 = 0.33740\dots$.

We choose in Lemma 2, $\varepsilon = 0.17779$, $3\varepsilon = 0.53337$, so that $\hat{g}(\varepsilon) = \hat{g}(3\varepsilon) = 2.42971$, and from Theorem 3(i), one can take $\eta = 2.43$, since for $\varepsilon \leqq \alpha \leqq 3\varepsilon$, $\hat{g}(\alpha) \leqq \hat{g}(\varepsilon)$. Applying Lemma 2 completes the proof of Theorem 3(ii).

8. A numerical table

The table below has been built by Marc Deléglise and we thank him very much for allowing us to include it here.

Any integer B, written in base 2 as

$$B = b_0 + 2b_1 + 4b_2 + \dots + 2^r b_r,$$

defines a set $E = E(B)$ of non negative integers: $i \in E \Leftrightarrow b_i = 1$. The union or intersection of sets can be easily carried out with operators OR and AND.

The partitions of n are generated by backtracking. For each partition

$$\Pi : n = n_1 + n_2 + \dots + n_k,$$

the set $T(\Pi)$ is represented as explained above by an integer B:

1) initialisation: $B := 1$; $\big(T = \{0\}\big)$.

2) For $1 \leqq i \leqq k$ do $B := B$ OR $2^{n_i} B$;

To improve the running-time, the practical partitions (that is the partitions which represent all integers between 1 and n) are not generated.

n	$\hat{p}^{(n)}$	$p(n)$	$\log\big(\hat{p}^{(n)}\big) / \log\big(p(n)\big)$
3	2	3	0.631
4	4	5	0.861
5	4	7	0.712
6	6	11	0.747
7	8	15	0.768
8	11	22	0.776
9	12	30	0.731
10	17	42	0.758
11	21	56	0.756
12	27	77	0.759
13	32	101	0.751
14	41	135	0.757
15	47	176	0.745
16	60	231	0.752
17	69	297	0.744
18	87	385	0.750
19	102	490	0.747
20	126	627	0.751
21	143	792	0.744
22	174	1002	0.747

n	$\hat{p}^{(n)}$	$p(n)$	$\log\left(\hat{p}^{(n)}\right) / \log\left(p(n)\right)$
23	201	1255	0.743
24	245	1575	0.747
25	276	1958	0.742
26	330	2436	0.744
27	376	3010	0.740
28	446	3718	0.742
29	506	4565	0.739
30	604	5604	0.742
31	674	6842	0.738
32	790	8349	0.739
33	892	10143	0.736
34	1040	12310	0.738
35	1169	14883	0.735
36	1362	17977	0.737
37	1519	21637	0.734
38	1757	26015	0.735
39	1965	31185	0.733
40	2273	37338	0.734
41	2510	44583	0.731
42	2900	53174	0.733
43	3202	63261	0.730
44	3683	75175	0.731
45	4071	89134	0.729
46	4640	105558	0.730
47	5096	124754	0.727
48	5839	147273	0.729
49	6423	173525	0.727
50	7324	204226	0.728
51	7991	239943	0.725
52	9066	281589	0.726
53	9907	329931	0.724
54	11254	386155	0.725
55	12274	451276	0.723
56	13851	526823	0.724
57	15079	614154	0.722
58	17031	715220	0.723
59	18454	831820	0.721
60	20845	966467	0.722
61	22535	1121505	0.720
62	25395	1300156	0.720
63	27484	1505499	0.719

n	$\hat{p}^{(n)}$	$p(n)$	$\log\left(\hat{p}^{(n)}\right) / \log\left(p(n)\right)$
64	30781	1741630	0.719
65	33313	2012558	0.717
66	37325	2323520	0.718
67	40228	2679689	0.716
68	44908	3087735	0.717
69	48502	3554345	0.715
70	54217	4087968	0.716
71	58141	4697205	0.714
72	64950	5392783	0.715
73	69700	6185689	0.713
74	77656	7089500	0.714
75	83109	8118264	0.712
76	92436	9289091	0.713
77	99100	10619863	0.711
78	110143	12132164	0.712
79	117749	13848650	0.710
80	130631	15796476	0.711
81	139523	18004327	0.709
82	154585	20506255	0.710
83	164788	23338469	0.708
84	182200	26543660	0.709
85	194386	30167357	0.707
86	215091	34262962	0.708
87	228382	38887673	0.706
88	252396	44108109	0.707
89	268200	49995925	0.705
90	296210	56634173	0.706
91	314451	64112359	0.704
92	346085	72533807	0.705
93	366990	82010177	0.703
94	404647	92669720	0.704
95	427890	104651419	0.702
96	470379	118114304	0.703
97	498771	133230930	0.701
98	548306	150198136	0.702
99	579191	169229875	0.700
100	635657	190569292	0.701
101	671494	214481126	0.699
102	736462	241265379	0.700
103	777683	271248950	0.699
104	851930	304801365	0.699
105	897631	342325709	0.698

n	$\hat{p}^{(n)}$	$p(n)$	$\log\left(\hat{p}^{(n)}\right) / \log\left(p(n)\right)$
106	984866	384276336	0.698
107	1036224	431149389	0.697
108	1132926	483502844	0.697
109	1194496	541946240	0.696
110	1306138	607163746	0.696
111	1371450	679903203	0.695
112	1500266	761002156	0.695
113	1576460	851376628	0.694
114	1720941	952050665	0.695
115	1806268	1064144451	0.693
116	1971049	1188908248	0.694
117	2068577	1327710076	0.692
118	2257942	1482074143	0.693
119	2365295	1653668665	0.691
120	2573572	1844349560	0.692
121	2702926	2056148051	0.691

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