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MÉTHODES DE VARIÉTÉS INVARIANTES POUR LES ÉQUATIONS DE SAINT VENANT ET LES SYSTÈMES HAMILTONIENS DISCRETS

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Introduction

Depuis ces quinze dernières années, les techniques issues des systèmes dynamiques en dimension infinie ont eu de nombreuses applications en mécanique des fluides ou encore dans l'étude de la dynamique des réseaux discrets. Pour la mécanique des fluides, on peut citer l'existence d'ondes progressives internes dans des fluides stratifiés ([24], [23]) comme les solitons ou encore les nanoptérons, l'existence et la stabilité d'ondes progressives pour des fluides ayant une faible tension de surface ([18],[20],[6]) par des techniques de réduction à une variété invariante (variété centrale, variétés stables et instables) et des techniques de bifurcations en dimension infinie. Pour la dynamique des systèmes discrets, ces techniques ont permis d'étudier l'existence et la stabilité d'ondes progressives dans les réseaux de Klein Gordon ou de Fermi-Pasta-Ulam ([21], [22], [15]).
Dans ce travail, on va d'abord appliquer ces techniques pour démontrer l'existence et la stabilité des "roll-waves". C'est un phénomène apparaissant généralement dans les écoulements en eaux peu profondes sur une pente faible (le fond pouvant être plat ou périodiquement modulé). On utilise ensuite la réduction à une variété centrale pour des mappings quasilinéaires en dimension infinie pour démontrer l'existence de "breathers" dans les réseaux Fermi-Pasta-Ulam diatomiques et dans les réseaux de spins. Les breathers sont des oscillations périodiques localisées spatialement apparaissant dans les réseaux discrets du fait de la non linéarité des équations décrivant la dynamique de la chaîne et le caractère discret du réseau [11].

0.1 Existence et stabilité des roll-waves dans un canal à fond plat ou périodique

L'apparition des roll-waves est un phénomène fréquemment rencontré en hydraulique [50],[44]. On les observe généralement dans des constructions comme des barrages, des passes à poissons ou encore des rivières de faible profondeur [34], [35]. Dressler [7] les décrit de la manière suivante : "toute onde progressive, périodique en espace, apparaissant quand un liquide s'écoule de manière turbulente sur un plan incliné, le profil de l'onde descendant à vitesse constante et sans déformation, la vitesse du liquide étant toujours inférieure à celle de l'onde". C'est un phénomène indésirable puisque un ouvrage destiné à un flot normal ne remplira plus son rôle si le flot se transforme en roll-waves. Dans ce cas, l'eau peut déborder du canal. L'apparition des roll-

waves dépend essentiellement de la friction provoquée par la rugosité des parois composant le canal et de la turbulence qui en résulte. Cornish a pris différents clichés de roll-waves apparaissant dans un long canal rectangulaire qui alimente le lac Thune dans les Alpes [4]. D'un point de vue expérimental, on peut simuler ce type d'écoulement dans des bancs inclinables, un fond mobile entrainé par un moteur simulant le lit du canal. Ce dispositif a été très tôt mis en place pour étudier les conditions d'apparitions des roll-waves [40],[41],[46]. Plus récemment, on peut citer les expériences menées à l' Institut de Mécanique des Fluides de Toulouse pour simuler des écoulements dans un canal à fond périodique [35].
Pour décrire ces écoulements, on choisit le modèle de Saint Venant auquel on rajoute un terme empirique de friction dû à Chézy [55]

$$\begin{array}{l} h_t + (hu)_x = 0, \\ (hu)_t + (hu^2 + G\frac{h^2}{2})_x = GhS - C_f u^2, \end{array} \tag{1}$$

où h désigne la hauteur de fluide en x, u la vitesse du fluide. On a $S = tan(\theta)$ où θ représente l'inclinaison du canal et C_f la résistance des parois. Enfin on a $G = g\cos(\theta)$ où g représente la constante de gravité. On peut écrire les équations de Saint Venant à partir des équations d'Euler à frontière libre au moyen de développements multiéchelles dans la limite des grandes longueurs d'onde [57], [45]. Des modèles plus simples possèdent aussi des solutions roll-waves : par exemple l'équation de Burgers avec ou sans viscosité

$$u_t + (u + c)u_x = u + \epsilon u_{xx}. \tag{2}$$

Pour ce modèle, Novik [43] a démontré l'existence de roll-waves continues convergeant vers des roll-waves discontinues lorsque la viscosité ϵ devient petite. Pour le modèle de Saint Venant (1), Dressler [7] a établi l'existence de roll-waves discontinues en reliant sur une période deux solutions continues particulières par une discontinuité satisfaisant les conditions de Rankine-Hugoniot. Needham et Merkin [42] ont analysé un modèle de Saint Venant "visqueux" en rajoutant dans la deuxième équation de (1) un terme du type $\epsilon h u_{xx}$ et prouvé par une bifurcation de Hopf l'existence à ϵ fixé de solutions roll-waves continues de petite amplitude.
Dans le premier chapitre, on va faire tendre la viscosité ϵ vers 0 dans le modèle de Saint Venant "visqueux" et faire le lien avec le système non visqueux. Pour le système (1), on peut généraliser l'approche de Dressler et construire toute une famille de solutions entropiques de type onde progressive périodique discontinue en mêlant deux ou plusieurs périodes ou encore des solutions présentant des points d'accumulation de discontinuités. Ces solutions entropiques n'ont aucun sens physique et il apparaît que le critère

d'entropie pour sélectionner des solutions de systèmes hyperboliques physiquement acceptables est insuffisant. On introduit donc un critère plus fin de limite visqueuse.

Définition 1 *On dit que u solution d'onde progressive de vistesse c du sytème*

$$(f'(u) - c)u' = g(u),$$

est limite visqueuse si il existe une suite ϵ_n qui converge vers 0, une suite c_n qui tend vers c et u_{ϵ_n} de solutions du système

$$(f'(u) - c_n)u' = g(u) + \epsilon_n u_{xx},$$

qui converge vers u lorsque $n \to \infty$.

On prouve que les roll-waves construites par Dressler sont les seules solutions d'ondes progressives périodiques de type limite visqueuse des équations de Saint Venant. On démontrera l'existence de roll-waves continues de taille $O(1)$ proches des roll-waves construites par Dressler pour le système de Saint Venant visqueux lorsque ϵ est proche de 0. Le problème se réduit à l'étude d'une équation différentielle de la forme

$$\epsilon u'' = p(u)u' - q(u). \tag{3}$$

Lorsque $\epsilon = 0$, le système possède une variété $M_0 = \{(x, y)/y = \frac{q(x)}{p(x)}\}$ invariante. Sous une hypothèse d'hyperbolicité normale, il existe une variété "lente" invariante M_ϵ pour ϵ proche de 0 et on peut décrire précisément le comportement des solutions au voisinage de cette variété [27]. En particulier, on montre que la variété lente s'écrit $M_\epsilon = \{(x, y)/y = \frac{q(x)}{p(x)} + O(\epsilon)\}$. L'équation issue du système de Saint Venant présente cependant une perte d'hyperbolicité normale (appelé "canard" en analyse non standard). Dans ce cas, on peut montrer l'existence de variétés localement invariantes M_ϵ^+ et M_ϵ à droite et à gauche de ce point singulier dans un voisinage d'ordre $O(\sqrt{\epsilon})$ de la variété lente de référence M_0. En général ces variétés ne coincident pas en ce point. De plus, lorsqu'on prolonge ces variétés au dela de la singularité, elles sont très rapidement éjectées de M_0. Cependant, on va montrer qu'en perturbant le système initial de la manière suivante

$$\epsilon u'' = (p(u) + \epsilon s)u' - q(u), \tag{4}$$

on peut faire coincider ces deux variétés pour une valeur donnée de $s = s(\epsilon)$ et construire une variété lente invariante M_{ep} pour le système (4) et dans un voisinage d'ordre $O(\sqrt{\epsilon})$ de la variété M_0. De plus, dans le cas où $s \approx s(\epsilon)$, les variétés $M_\epsilon^\pm$ ne coincident pas mais on peut contrôler leur éjection de M_0.

On applique d'abord ces résultats à l'étude des limites visqueuses pour les équations hyperboliques scalaires avec un terme source. Pour le système de Saint Venant, on construira une application de Poincaré d'abord en étudiant le système lent par les techniques de variété invariante puis en étudiant le système rapide gouverné par les conditions de Rankine Hugoniot. On montrera alors l'existence de points fixes qui correspondent à des solutions roll-waves. Des théorèmes plus généraux ([32], [33]) permettent d'étudier ce type de problème ([16], [17]) mais l'approche présentée ici est plus élémentaire et permet des démonstrations plus courtes.
Dans le deuxième chapitre, on étudie le problème de la stabilité linéaire des roll-waves de Dressler. C'est un problème classique qui a été peu traité. Tamada et Tougou [52] ont réalisé une première analyse de stabilité linéaire de roll-waves pour des flots laminaires de faible profondeur. Ils ont calculé une relation de dispersion impliquant la stabilité linéaire des roll-waves de grande longueur d'onde. Cependant leur approche n'est pas rigoureuse d'un point de vue mathématique car ils négligent les effets du mouvement des discontinuités et travaillent dans des espaces de perturbations non physiques. Kevorkian et Yu [57] ont réalisé des calculs asymptotiques lorsque le flot uniforme est faiblement instable et ont écrit des équations approchées de profils des solutions pour des conditions initiales périodiques ou localisées. Ils établissent en particulier que pour une condition initiale somme de fonctions périodiques, la solution converge vers la roll-wave de Dressler de plus grande période spatiale ce qui tend à confirmer que les roll-waves de grandes longueurs d'ondes sont stables. Plus récemment, Sinestrari a démontré l'instabilité de solutions de type roll-waves pour les équations hyperboliques scalaires avec terme source [48].
Le problème essentiel pour l'étude de la stabilité linéaire des roll-waves est due à la présence d'une distribution infinie de discontinuités. Pour contourner cette difficulté, on utilise une nouvelle approche inspirée de la démarche de Majda pour l'étude de la stabilité linéaire des chocs dans les systèmes hyperboliques [39] (voir également [47] et [5] pour des références plus générales sur les systèmes hyperboliques non linéaires). Dans ce cas, on travaille dans l'espace des fonctions avec une discontinuité proche de la discontinuité du choc, ces fonctions étant régulières en dehors de la discontinuité. Au moyen d'un changement de repère, on fixe la discontinuité à l'origine : le choc devient alors une solution stationnaire du système étudié. On linéarise les équations au voisinage de ce choc sur les domaines où les fonctions sont régulières d'une part et les conditions de Rankine-Hugoniot d'autre part. On peut alors écrire le problème spectral associé et étudier la stabilité linéaire du choc. Cette approche a également été utilisée avec succès pour l'étude de la stabilité des transitions de phases [2], [3]. Pour le problème des roll-waves, on choisit l'es-

pace des fonctions régulières par morceaux avec une distribution des chocs proche de ceux de la roll-wave considérée. On fixe les chocs à l'aide d'un changement de variable affine par morceaux : les roll-waves deviennent alors des solutions stationnaires et on linéarise le système de Saint Venant sur les zones régulières et les conditions de Rankine-Hugoniot pour les discontinuités. On obtient ainsi le problème spectral associé à la stabilité linéaire d'une roll-wave de longueur d'onde L.

Trouver $\lambda \in \mathbb{C}$, (h,v) deux fonctions C^1 par morceaux avec des discontinuités aux points $\{iL, i \in \mathbb{Z}\}$, une suite bornée $(\epsilon_i)_{i\in\mathbb{Z}} \in \mathbb{R}$ tel que $\lim_{\xi\to\pm\infty} \|(h,v)(\xi)\| = 0$ et (h,v,ϵ_i) vérifiant le système différentiel

$$\begin{aligned}
&(v-ch)' + \lambda h = \lambda\big(\epsilon_i + \frac{\xi - iL}{L}(\epsilon_{i+1}-\epsilon_i)\big)H' \\
&\big((GH - \frac{V^2}{H^2})h \quad + \quad (\tfrac{2V}{H} - c)v\big)' + \\
&\qquad\qquad + \quad \lambda v = (GS + \tfrac{2C_f V^2}{H^3})h - \tfrac{2C_f V}{H^2}v + \\
&\qquad\qquad\qquad + \tfrac{\epsilon_{i+1}-\epsilon_i}{L}\big(\tfrac{GH^2}{2} + \tfrac{K^2}{H}\big)' + \\
&\qquad\qquad\qquad + \big(\epsilon_i + \tfrac{\xi - iL}{L}(\epsilon_{i+1}-\epsilon_i)\big)\lambda V'
\end{aligned} \tag{5}$$

$\forall \xi \in]iL,(i+1)L[, i \in \mathbb{Z}$ *et les valeurs aux points de discontinuité* $\{iL, i \in \mathbb{Z}\}$ *vérifient les conditions de Rankine-Hugoniot linéarisées*

$$\begin{aligned}
&[v-ch]_{iL} = \epsilon_i \lambda [H]_{iL}, \\
&\left[(GH - \frac{V^2}{H^2})h + (\frac{2V}{H} - c)v\right]_{iL} = \epsilon_i \lambda c [H]_{iL}.
\end{aligned} \tag{6}$$

La fonction H, V désigne l'équation du profil continu de la roll-wave construite par Dressler sur les intervalles $I_i =]iL,(i+1)L[$.
Le problème spectral (5,6) obtenu est un ensemble infini de systèmes différentiels ayant tous une singularité. On ne peut intégrer explicitement ce problème. Pour poursuivre le calcul, une direction est la recherche d'asymptotique. Un premier résultat concerne la stabilité linéaire des roll-waves dans la limite des grandes longueurs d'onde.

Théorème 1 *Il existe deux fonctions r_1 et r_2, croissantes, et L_0 tel que pour tout (λ, L) vérifiant $Re(\lambda) > r_1(L)$ ou $L > L_0$ et $Im(\lambda) > r_2(L)$, λ n'est pas valeur propre du problème spectral (5,6).*

Autrement dit, le problème spectral associé à la stabilité linéaire des roll-waves ne possède pas de valeurs propres instables de grand module. On peut donc envisager une étude numérique sur les valeurs propres instables dans un domaine borné. Pour démontrer le théorème 1, on raisonne par l'absurde et on suppose l'existence de valeurs propres instables arbitrairement grandes.

On calcule alors un développement asymptotique des vecteurs propres associés. La contradiction vient de l'incompatibilité des valeurs des vecteurs propres aux points de discontinuités avec les conditions de Rankine-Hugoniot linéarisées. Pour conclure cette étude, on s'intéresse à deux situations où on peut mener des calculs explicites. On s'intéresse d'abord aux équations de Burgers et on prouve que les roll-waves sont linéairement instables. On analyse ensuite la stabilité des roll-waves de Dressler quand la pente θ tend vers 0 dans les équations de Saint Venant (1). A l'aide du théorème 1, on démontre le résultat suivant.

Théorème 2 *Pour $\theta \approx 0$, il n'y a pas de valeurs propres instables $O(1)$ et pas de valeurs propres instables $O(\theta)$. Il existe $\theta_0 > 0$ tel que pour $0 < \theta < \theta_0$, les roll-waves sont linéairement stables.*

Le troisième chapitre est consacré à l'étude d'écoulements de faible profondeur dans un canal avec une pente faible et un fond périodique faiblement modulé. Cette étude est motivée par les différentes expériences et simulations numériques menées à l'Institut de Mécanique des Fluides de Toulouse [35] qui ont mis en évidence l'existence de formations de type roll-waves et d'une sélection de la longueur d'onde correspondant au double de la période du fond. On étudiera dans ce chapitre l'existence du phénomène de roll-waves pour le système de Saint Venant, le fil conducteur étant l'étude des instabilités lorsque le nombre de Froude de l'écoulement dépasse un seuil critique. On étudiera également un système proche : le p-système avec terme source lorsqu'on ajoute une faible modulation spatiale

$$\begin{aligned} &u_t + v_x = 0, \\ &v_t + p(u)_x = f(u) + \epsilon b'(x) - v. \end{aligned} \tag{7}$$

Pour le p-système, on peut citer entre autres les travaux de Vila [55] sur l'existence de certaines solutions globales C^1 (voir également [19]).
On diagonalise le système en introduisant le changement de variable $r = v + \phi(u)$ et $s = v - \phi(u)$ dans (7)

$$\begin{aligned} r_t + \sqrt{p' \circ \phi^{-1}(\frac{r-s}{2})}r_x = \epsilon b'(x) + f \circ \phi^{-1}(\frac{r-s}{2}) - \frac{r+s}{2}, \\ s_t - \sqrt{p' \circ \phi^{-1}(\frac{r-s}{2})}s_x = \epsilon b'(x) + f \circ \phi^{-1}(\frac{r-s}{2}) - \frac{r+s}{2}. \end{aligned} \tag{8}$$

Pour démontrer l'existence de roll-waves, on s'inspire des méthodes utilisées pour étudier la stabilité linéaire des ondes de chocs [39], [47], des transitions de phases [2], [3] et développées dans le cadre de la stabilité linéaire des roll-waves dans le chapitre 2. On se place donc dans l'espace des fonctions régulières par morceaux avec des chocs se propageant à une vitesse presque constante. On fait alors un changement de variable pour fixer les chocs qui doivent vérifier les conditions de Rankine Hugoniot.

$$c'(t) = \frac{[p \circ \phi^{-1}(\frac{r-s}{2})]}{[\frac{r+s}{2}]} = \frac{[\frac{r+s}{2}]}{[\phi^{-1}(\frac{r-s}{2})]}.$$

On se fixe une longueur d'onde L pour la roll-wave (r, s) et on travaille sur l'ensemble $C^1(]0, L[\times\mathbb{T})$ des fonctions périodiques en temps. On remplace c' dans les équations en fonction des valeurs de (r, s) aux bords. Le problème est le suivant.

Trouver $(r, s) \in C^1(]0, L[\times\mathbb{T})^2$ *tel que*

$$\begin{aligned} r_t + \Big(\sqrt{p' \circ \phi^{-1}(\frac{r-s}{2})} - \frac{[\frac{r+s}{2}]_0^L}{[\phi^{-1}(\frac{r-s}{2})]_0^L}\Big) r_x &= \epsilon b'(x) + f \circ \phi^{-1}(\frac{r-s}{2}) - \frac{r+s}{2}, \\ s_t - \Big(\sqrt{p' \circ \phi^{-1}(\frac{r-s}{2})} + \frac{[\frac{r+s}{2}]_0^L}{[\phi^{-1}(\frac{r-s}{2})]_0^L}\Big) s_x &= \epsilon b'(x) + f \circ \phi^{-1}(\frac{r-s}{2}) - \frac{r+s}{2}. \end{aligned} \tag{9}$$

Les valeurs aux bords doivent vérifier

$$\frac{[p \circ \phi^{-1}(\frac{r-s}{2})]_0^L}{[\frac{r+s}{2}]_0^L} = \frac{[\frac{r+s}{2}]_0^L}{[\phi^{-1}(\frac{r-s}{2})]_0^L}.$$

On obtient alors la solution roll-wave en translatant le profil obtenu. La démarche est strictement identique pour le système de Saint Venant avec fond périodique.

Dans un premier temps, on essaie d'aborder la question des roll-waves de taille $O(1)$. On suit une démarche similaire à celle introduite par Dressler et on analyse la stabilité de solutions stationnaires pour les équations de Saint Venant à fond périodique (voir également [53]). On montre d'abord l'existence d'une solution stationnaire 1-périodique en espace. Ensuite on étudie la stabilité linéaire de cette solution stationnaire pour des perturbations n-périodiques et on dérive une relation de dispersion. Lorsque $\epsilon \approx 0$, on montre par un théorème des fonctions implicites l'existence d'un nombre de Froude critique $F_c = F_c(\epsilon, n)$ au dela duquel la solution stationnaire devient instable dans l'espace des fonctions n-périodiques. On montre d'abord que le nombre de Froude critique est indépendant de n pour $\epsilon = 0$ et $F_c(0, n) = 2$. On donnera également un développement limité de F_c au voisinage de $\epsilon = 0$ et on montrera que les plus grandes longueurs d'ondes sont les plus instables.

Pour aller plus loin dans l'analyse, on s'intéresse aux instabilités de petite amplitude lorsque le nombre de Froude $F > 2$. Pour trouver une asymptotique pertinente, on s'inspire des méthodes de relaxation utilisées par Jin [28]. On recherche des solutions $r = r_0 + \epsilon R(\xi, t, \epsilon)$ et $s = s_0 + \epsilon S(\xi, t, \epsilon)$ où $\xi = \frac{x-c(t)}{\epsilon}$ et $R(t, .), S(t, .) \in C^1(-1, 1)$. Pour le p-système, le couple (R, S) vérifie le système

$$R_t + \frac{p''(u_0)}{4p'(u_0)}(R - \frac{R(1,t) + R(-1,t)}{2} + O(\epsilon))R_\xi = AR - \\ - BS + b'(c_0 t) + O(\epsilon) \quad (10)$$

$$S_t - \frac{2c_0}{\epsilon}(1 + O(\epsilon))S_\xi = AR - BS + b'(c_0 t + \epsilon\xi) + O(\epsilon) \quad (11)$$

et la condition aux bords

$$S(1,t) = S(-1,t) + \epsilon\mathcal{RH}(S(-1,t), R(-1,t), R(1,t), \epsilon) \quad (12)$$

où $A = \frac{1}{2}(\frac{F}{2} - 1)$, $B = \frac{1}{2}(\frac{F}{2} + 1)$ et $c_0 = \sqrt{p'(u_0)}$.
Le petit paramètre ϵ induit une dynamique lente/rapide sur le système (10),(11). Ce dernier est écrit en variable lente. Pour $\epsilon = 0$, la "variété" lente est donnée par $S(t, \xi) = S(t)$ et le flot sur cette variété est donnée par une équation de Burgers

$$R_t + \frac{p''(u_0)}{4p'(u_0)}(R - \frac{R(1,t) + R(-1,t)}{2})R_\xi = AR - BS + b'(c_0 t) \quad (13)$$

qui possède des solutions de type roll-waves si et seulement si $A > 0$ soit $F > 2$.
Pour démontrer l'existence de roll-waves pour $\epsilon \neq 0$, on suit la démarche de Fenichel (on est cette fois en dimension infinie). En fixant R, on démontre qu'il existe S solution de la deuxième équation de (11) avec la condition aux bords (12) et $S(\xi, t) = S(R)(t) + O(\epsilon)$. On démontre ainsi l'existence d'un "graphe" invariant. On calcule le flot sur ce graphe en injectant la relation obtenue dans la première équation. On obtient alors l'existence d'une solution de type roll-waves par une méthode de point fixe.

0.2 Existence de breathers dans les réseaux FPU diatomiques et les chaînes de spins

Dans cette deuxième partie, on va montrer l'existence d'oscillations non linéaires localisées appelées "breathers" pour différents systèmes discrets de

particules interagissant non linéairement avec leurs plus proches voisins. On s'intéressera en particulier aux réseaux FPU diatomiques et aux chaînes de spins décrites classiquement par les équations de Landau-Lifshitz.
Les breathers sont des oscillations périodiques, localisées spatialement dans des réseaux de particules couplées non linéairement. Elles apparaissent dans beaucoup de systèmes du fait de l'interaction entre la non linearité du système et son caractère discret. On trouve donc de nombreuses applications en physique notamment la physique de la matière condensée ou encore la biophysique. D'un point de vue expérimental, ces oscillations localisées ont été observées dans des systèmes couplés de jonctions de Josephson [54], dans des réseaux de fibres optiques couplées faiblement [8], [13], des cristaux de faible dimension [51] ou dans des systèmes biologiques [56]. De nombreuses simulations numériques sur les modèles de Fermi-Pasta-Ulam ou encore de Klein Gordon ont également mis en évidence ce phénomène. La première preuve rigoureuse d'existence de breathers pour les systèmes de Klein Gordon repose sur la méthode de la limite anticontinue (voir Aubry et Mac Kay [37]). On part d'un système de particules découplées : on a alors trivialement l'existence de breathers avec un ou plusieurs sites excités, les autres étant au repos. Le couplage est considéré comme un paramètre. Pour des couplages petits, on peut au moyen d'un théorème des fonctions implicites prolonger de manière continue ces solutions breathers. Cette méthode fonctionne également dans les réseaux Fermi Pasta Ulam diatomiques pour des rapports de masses très grands : dans la limite du rapport de masse infini, les grosses masses sont immobiles alors que les petites bougent indépendamment [36]. Cette méthode ne fonctionne pas pour les systèmes de Fermi Pasta Ulam monoatomiques ou pour des rapports de masses intermédiaires. James a prouvé l'existence de breathers de petite amplitude pour des potentiels durs en utilisant des techniques de réduction à une variété centrale pour des mappings ([25],[26]). Dans le chapitre 4, présenté sous forme d'article, on adopte cette technique pour démontrer l'existence de breathers de petite amplitude dans des réseaux FPU alternant grandes masses m_2 et petites masses m_1. Ce système est décrit par l'ensemble infini d'équations

$$m_n \frac{d^2}{dt^2} x_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \tag{14}$$

où $m_{2n+1} = m_1$, $m_{2n} = m_2$, x_n représente le déplacement des masses par rapport à leur position d'équilibre et V est un potentiel d'interaction régulier vérifiant en plus $V'(0) = 0$ et $V''(0) > 0$. On formule le problème comme un mapping dans des espaces de fonctions périodiques et on utilise une technique de réduction à une variété centrale pour des mappings quasilinéaires en dimension infinie. Les solutions d'ondes progressives du système linéarisé possèdent des fréquences dans deux zones : une bande optique et une bande acoustique correspondant à des fréquences inférieures. Lorsque les fréquences sont proches des bords de ces bandes toutes les solutions bornées de petite amplitude appartiennent à une variété de dimension finie ce qui réduit l'étude à celle d'un mapping en dimension finie. Lorsque le potentiel vérifie la condition de potentiel dur $\frac{1}{2}V''(0)V^{(4)}(0) - V^{(3)}(0)^2 > 0$, on trouve des breathers discrets de fréquences au dessus de la bande optique et au dessus de la bande acoustique. Lorsque le potentiel vérifie la condition opposée de potentiel mou $\frac{1}{2}V''(0)V^{(4)}(0) - V^{(3)}(0)^2 < 0$, on trouve des breathers de fréquence juste en dessous de la bande optique. Les breathers possèdent des symétries héritées du système initial qui permet en plus de décrire leur géométrie. C'est un élément important pour étudier leur stabilité linéaire. On montre ainsi le théorème suivant.

Théorème 3 *On fixe $B = \frac{1}{2}V''(0)V^{(4)}(0) - V^{(3)}(0)^2$ et m_1, m_2 avec $m_2 > m_1$. Le problème (14) possède les familles de breathers paramétrées par leur fréquence ω.*

i) Pour $B < 0$, $\omega \approx \omega_c = \left(\frac{2\kappa}{m_1}\right)^{1/2}$ et $\omega < \omega_c$, il existe des solutions breathers discrets x_n^1, x_n^2 ayant les symétries

$$x_{-n}^1(t) = -x_{n-2}^1(t + \frac{\pi}{\omega}), \quad x_{-n}^2(t) = -x_n^2(t).$$

ii) Pour $B > 0$, $\omega \approx \omega_c = \left[2\kappa(\frac{1}{m_1} + \frac{1}{m_2})\right]^{1/2}$ et $\omega > \omega_c$, on a des solutions breathers discrets x_n^1, x_n^2 avec les symétries

$$x_{-n}^1(t) = -x_{n-2}^1(t + \frac{\pi}{\omega}), \quad x_{-n}^2(t) = -x_n^2(t + \frac{\pi}{\omega}).$$

iii) Si $B > 0$, $\frac{m_2}{m_1} \in (k^2, k(k+2))$ (pour un $k \geq 1$), $\omega \approx \omega_c = \left(\frac{2\kappa}{m_2}\right)^{1/2}$ et $\omega > \omega_c$, les solutions breathers discrets x_n^1, x_n^2 existent et ont les symétries

$$x_{-n}^1(t) = -x_{n-2}^1(t), \quad x_{-n}^2(t) = -x_n^2(t + \frac{\pi}{\omega}).$$

Dans tous les cas, ces solutions ont la forme

$$x_n^i(t) = d_n + X_n^i(t)$$

où X_n^i a une moyenne temporelle nulle et $\|X_n^i\|_{L^\infty}$ décroît exponentiellement quand $n \to \pm\infty$. Le terme stationnaire d_n vérifie $d_n = O(|\omega - \omega_c|)$ pour tout n fixé, $\lim_{n\to\pm\infty} d_n = O(|\omega - \omega_c|^{1/2})$ et a la forme d'un kink si $V^{(3)}(0) \neq 0$.

Les parties oscillantes X_n^i ont la forme suivante.
Dans le cas i)

$$X_{2n}^i(t) = O(|\omega-\omega_c|), \quad X_{2n-1}^i(t) = A_n \cos(\omega t) + O(|\omega-\omega_c|), \tag{15}$$

dans le cas ii)

$$X_{2n}^i(t) = -m\,A_n \cos(\omega t) + O(|\omega-\omega_c|), \quad X_{2n-1}^i(t) = A_n \cos(\omega t) + O(|\omega-\omega_c|), \tag{16}$$

dans le cas iii)

$$X_{2n}^i(t) = A_n \cos(\omega t) + O(|\omega-\omega_c|), \quad X_{2n-1}^i(t) = O(|\omega-\omega_c|), \tag{17}$$

où $0 < A_n \leq C\,|\omega-\omega_c|^{1/2}\,\nu^{-|n|}$, $\nu = 1 + O(|\omega-\omega_c|^{1/2}) > 1$.

Les résultats obtenus sont cohérents avec les simulations numériques menées par Kiselev et al.[30], Franchini et al. [14], sur des modèles de cristaux K-Br et Li-I (avec des rapports de masses $\frac{1}{2}$ et $\frac{1}{18}$). Ils complètent l'approche du problème par Livi, Spicci et MacKay [36] où on examinait la limite anticontinue des grands rapports de masse.

Dans le chapitre 5, également présenté sous forme d'article, on étudie des réseaux de spins classiques ferromagnétiques avec une anisotropie locale. Le système est décrit classiquement par les équations de Landau-Lifshtiz :

$$\begin{aligned}
\dot{x}_n &= \frac{1}{2}[J_y z_n(y_{n-1}+y_{n+1}) - J_z y_n(z_{n-1}+z_{n+1})] - 2D y_n z_n, \\
\dot{y}_n &= \frac{1}{2}[J_z x_n(z_{n-1}+z_{n+1}) - J_x z_n(x_{n-1}+x_{n+1})] + 2D x_n z_n, \\
\dot{z}_n &= \frac{1}{2}[J_x y_n(x_{n-1}+x_{n+1}) - J_y x_n(y_{n-1}+y_{n+1})]\ ,\ n \in \mathbb{Z},
\end{aligned} \tag{18}$$

où x_n, y_n, z_n désignent les composantes du n-ième spin vérifiant la condition de normalisation

$$x_n^2 + y_n^2 + z_n^2 = 1\ ,\ n \in \mathbb{Z}. \tag{19}$$

Les constantes J_x, J_y, J_z sont les intégrales d'échange et D représente la constante d'anisotropie locale. On va considérer $J_x \geq 0$, $J_y \geq 0$ et $J_z > 0$. On va étudier la dynamique au voisinage de deux équilibres du réseau (i.e. une solution statique de (18) minimisant une fonctionnelle d'énergie). Dans un premier temps, on s'intéresse à une chaîne ferromagnétique avec une anisotropie "easy plane" où l'état d'équilibre est donné par $x_n = \pm 1, y_n = z_n = 0$. On utilisera la méthode de la limite anticontinue pour prouver rigoureusement l'existence des breathers "out of plane" dans lesquels un ou plusieurs spins tournent autour d'un axe principal ("hard axis" qu'on supposera être l'axe Z) hors du plan déquilibre XY, les autres restant proches du plan d'équilibre.

Ces breathers possèdent un seuil en énergie et une magnétisation localisée. On étudiera ensuite la dynamique dans une chaîne ferromagnétique avec une anisotropie "easy axis" où l'état d'équilibre est donné par $z_n = \pm 1$ et $x_n = y_n = 0$. On va utiliser les techniques de variété centrale pour étendre les résultats d'existence obtenus dans la limite anticontinue par Flach et Speight dans le cas d'une forte anisotropie d'échange $J_x, J_y << J_z$ [12] ou dans la limite d'échanges isotropes très petits $J_x, J_y, J_z << |D|$ [49]. Le résultat obtenu est le suivant.

Théorème 4 *On note $\alpha = \frac{J_x}{J_z}$, $\beta = \frac{J_y}{J_z}$ et $d = \frac{D}{J_z}$. Dans le domaine*

$$0 < \beta < \frac{4(1+2d) - 5\alpha}{5(1+2d) - 4\alpha}(1+2d)$$

et pour $\omega \approx \omega_c$ ($\omega_c^2 = (\alpha - (1+2d))(\beta - (1+2d))$) et $\omega < \omega_c$, on a l'existence de breathers discrets de période $\frac{2\pi}{J_z\omega}$.

Dans le dernier chapitre, on étudie la stabilité linéaire des breathers type "out of plane" dans la limite anticontinue. Dans ce cas, les techniques utilisées par Aubry et MacKay pour démontrer la stabilité de breathers dans la limite anticontinue ne fonctionnent pas. En effet dans la limite $J_x = J_y = J_z = 0$, on retrouve une infinité de fois la valeur propre 1. Les études menées sur la stabilité des breathers "out of plane" avec un site principal dépendaient de la taille du réseau (voir [29] alors que les résultats classiques sont valables pour des réseaux de tailles fixées mais arbitrairement grandes. On s'intéresse au cas où les solutions breathers sont proches de solutions stationnaires type "out of plane" : on prouve (numériquement) que ces solutions stationnaires sont instables et on conclut par un argument de continuité du spectre. Les résultats obtenus ici sont indépendants de la taille du réseau.

FIG. 1 – Instabilité des ondes de surfaces sur un fond périodique
D. Astruc, N. Dolez, O. Thual
Photo : A.L. Le Fessant, thèse 2001 de l'IMFT.

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Chapitre 1

Application des théorèmes de variété lente pour deux problèmes d'ondes progressives limites visqueuses dans des systèmes hyperboliques avec terme source

Les systèmes hyperboliques avec terme source possèdent en général une grande famille de solutions d'ondes progressives entropiques. On introduit donc le critère de limite visqueuse, plus sélectif que les conditions d'entropie. Ceci nous amène à traiter des problèmes de perturbations singulières. Dans ce chapitre, on va utiliser des techniques de réduction à une variété invariante développées par Fenichel [2] pour l'étude des équations scalaires hyperboliques avec terme source et pour l'analyse du système de Saint Venant. Il existe des théorèmes de bifurcations généraux pour les problèmes de perturbations singulières dans le plan [6], [7]. Haerterich ([3], [4]) a appliqué ces résultats pour l'étude du système de Saint Venant visqueux ou dans les équations hyperboliques/paraboliques scalaires. Dans notre cas, on démontrera l'existence de variétés invariantes par une méthode analytique (et non géométrique) inspirée des travaux de Sakamoto [10]. L'approche que nous proposons est plus élémentaire et permet des démonstrations plus courtes.

1.1 Solutions entropiques et critère de limite visqueuse

1.1.1 Définitions

Dans ce chapitre, on va étudier des équations scalaires ou des systèmes hyperboliques avec un terme source de la forme

$$v_t + F(v)_x = G(v) \tag{1.1}$$

où $v \in \mathbb{R}^n$, F et G sont des fonctions d'un ouvert $\mathcal{O}$ de $\mathbb{R}^n$ dans $\mathbb{R}^n$ "suffisamment" régulières (dans les cas traités, elles sont de classe C^∞) et $n = 1$ ou $n = 2$. On étudie les solutions d'ondes progressives entropiques de (1.1) et leurs limites visqueuses. Dans tout le chapitre, on adopte les définitions suivantes.

Définition 2 *Une* **onde progressive entropique** *est une solution de (1.1) de la forme $v(x,t) = v(x - ct)$ pour une certaine vitesse $c \in \mathbb{R}$ qui a les propriétés suivantes :*

- *(i) v est C^1 par morceaux sur $\mathbb{R}$. Dans les régions où v est C^1, la fonction v vérifie le système différentiel ordinaire*

$$(DF(v(\xi)) - cId)v'(\xi) = G(v(\xi)).$$

- *(ii) Les discontinuités vérifient les conditions de* **Rankine Hugoniot** *:*

$$F(v(\xi^+)) - F(v(\xi^-)) = c(v(\xi^+) - v(\xi^-)).$$

Il faut rajouter en plus dans la définition une condition d'entropie. Dans le cas scalaire $n = 1$, cette condition est donnée par la condition d'Oleinik $v(\xi^+) \leq v(\xi^-)$. On ne donnera pas ici de conditions générales d'entropie pour $n = 2$. Cependant pour le cas particulier de l'écoulement de Saint Venant, celle ci est donnée par $h(\xi^+) \leq h(\xi^-)$ où h désigne la hauteur de fluide au dessus du canal (voir le paragraphe sur l'étude des roll-waves).

Remarque : il existe des définitions plus générales pour les solutions entropiques de systèmes hyperboliques, les conditions de Rankine-Hugoniot et d'entropie découlant alors directement de ces définitions (voir [11]).

On rappelle la définition de limite visqueuse

Définition 3 *Une onde progressive v solution de (1.1) avec une vitesse c_0 est dite* **admissible** *ou* **admet un profil visqueux** *s'il existe une suite de solutions régulières (u_{ϵ_n}) solution de*

$$\epsilon_n v'' = (DF(v) - c_n Id)v' - G(v)$$

où $\epsilon_n \to 0$, $c_n \to c_0$ et tel que $\|v - v_{\epsilon_n}\|_{L^1(\mathbb{R})} \to 0$. La notation v' désigne la dérivation par rapport à la variable $\xi = x - ct$.

1.1.2 Exemple des équations hyperboliques scalaires avec terme source

On s'intéresse à l'équation scalaire hyperbolique avec un terme source

$$u_t + f(u)_x = g(u),$$

où g possède trois zéros (u_r, u_m, u_l) et $f'' > 0$.
Les solutions d'ondes progressives avec une vitesse c satisfont l'équation

$$(f'(u) - c)u' = g(u). \tag{1.2}$$

En choisissant $c = f'(u_m)$, on peut écrire l'équation (1.2) sous la forme

$$u' = (u - u_r)(u - u_l)h(u)$$

où la fonction h est de signe constant. A une translation près, l'équation (1.2) possède une unique solution hétérocline U reliant les points u_r, u_l. On choisit U tel que $U(0) = u_m$. La fonction constante $u = u_m$ est également solution de l'équation (1.2). On peut construire une famille de solutions entropiques U_λ paramétrée par $\lambda \geq 0$ en superposant ces deux solutions et définie par

$$U_\lambda(\xi) = \begin{cases} U(\xi) & \text{si } \xi \leq 0 \\ 0 & \text{si } 0 \leq \xi \leq \lambda \\ U(\xi - \lambda) & \text{si } \xi \geq \lambda \end{cases} \tag{1.3}$$

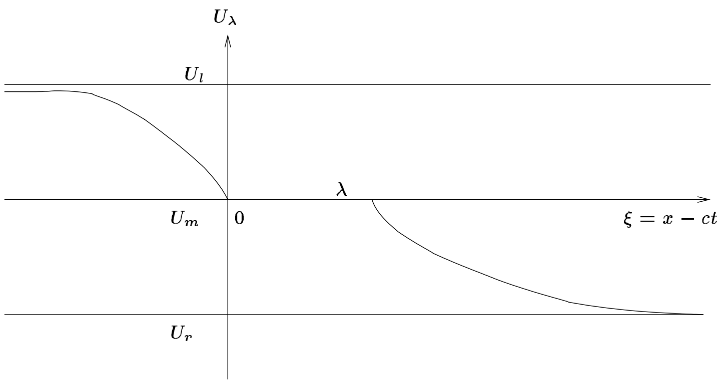


FIG. 1.1 – Allure de U_λ si $h > 0$

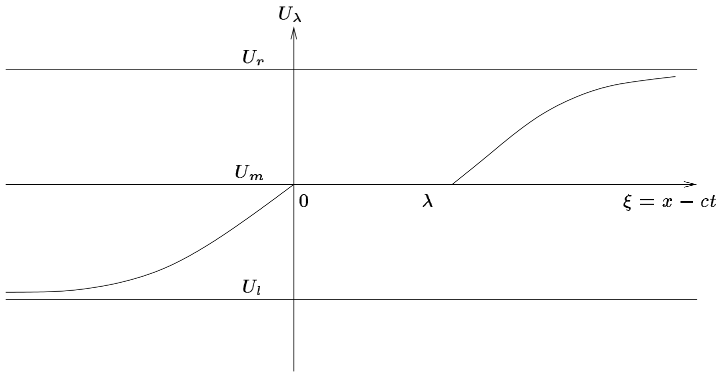


FIG. 1.2 – Allure de U_λ si $h < 0$

On désire sélectionner parmi cette famille de solutions entropiques une solution limite visqueuse qui sera physiquement acceptable. On étudie donc les solutions d'ondes progressives de l'équation

$$u_t + f(u)_x = g(u) + \epsilon u_{xx}.$$

Les ondes progressives avec une vitesse c vérifient l'équation différentielle

$$\epsilon u'' = (f'(u) - c)u' - g(u),$$

que l'on peut écrire

$$\begin{aligned} x' &= y, \\ \epsilon y' &= (f'(x) - c)y - g(x). \end{aligned}$$

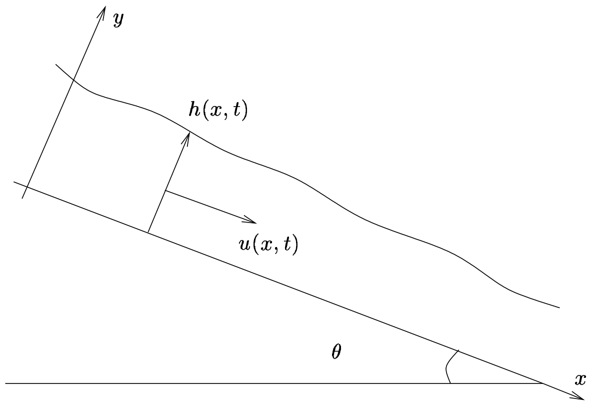
Haerterich [3] a démontré que la solution entropique pour $\lambda = 0$ est la seule qui soit limite visqueuse. On va dans la suite de ce chapitre simplifier cette preuve.

1.1.3 Roll waves et autres solutions entropiques du système de Saint Venant

On rappelle le système de Saint Venant

$$\begin{aligned} & h_t + (hu)_x = 0, \\ & (hu)_t + (hu^2 + g\cos\theta \tfrac{h^2}{2})_x = gh\sin\theta - C_f u^2, \end{aligned} \tag{1.4}$$

où θ représente l'inclinaison du canal et C_f le frottement dû aux parois.



Pour simplifier les notations, on pose $G = g\cos(\theta)$ et $S = \tan(\theta)$. Le système (1.5) s'écrit

$$\begin{aligned} & h_t + (hu)_x = 0, \\ & (hu)_t + (hu^2 + G\tfrac{h^2}{2})_x = GhS - C_f u^2, \end{aligned} \tag{1.5}$$

Les solutions d'ondes progressives (h, u) ayant une vitesse c vérifient le système

$$\begin{aligned} & (h(c-u))' = 0, \\ & (u-c)u' + Gh' = GS - C_f \tfrac{u^2}{h}. \end{aligned} \tag{1.6}$$

Soit $K = h(c-u))$, en remplaçant $u = c - \frac{K}{h}$ dans la deuxième équation de (1.6), on obtient

$$h' = -\frac{h(GS - C_f \frac{(ch-K)^2}{h^3})}{(\frac{K^2}{h^2}) - Gh} = P(h). \tag{1.7}$$

En intégrant cette équation sous la forme $\xi - \xi_0 = \int \frac{dh}{P(h)}$, on peut montrer que cette équation différentielle ne possède pas de solutions périodiques continues (voir [1]). On recherche donc des roll-waves discontinues, les discontinuités vérifiant des conditions d'entropie. En particulier, si on note H_+ la hauteur après un choc et H_- la hauteur avant un choc, ces discontinuités vérifient les conditions de Rankine Hugoniot qui se réduisent à

$$\frac{G}{2K}H_+^2 + \frac{K}{H_+} = \frac{G}{2K}H_-^2 + \frac{K}{H_-}$$

puisque la condition sur le débit est automatiquement vérifiée. Le système de Saint Venant écrit en variables conservatives $(h, q = hu)$ sont analogues aux équations d'Euler pour un gaz parfait isentropique avec $\gamma = 2$ en remplaçant la densité ρ par la hauteur h. Les conditions d'entropie sont équivalentes aux conditions de choc de Lax et

$$U_+ + \sqrt{GH_+} < c < U_- + \sqrt{GH_-}.$$

En utilisant la conservation du débit relatif, on obtient la condition d'entropie

$$\sqrt{GH_+} - \frac{K}{H_+} < 0 < \sqrt{GH_-} - \frac{K}{H_-} \tag{1.8}$$

soit $H_+ < H_-$. On retrouve ici le fait que la vitesse relative est surcritique après le choc $|U_+ - c| > \sqrt{GH_+}$ et sous critique $|U_- - c| < \sqrt{GH_-}$ (voir [12]). Dressler [1] a démontré le résultat.

Théorème 5 *Pour tout $F = \frac{\sin(\theta)}{C_f} > 4$ et (L, c) il existe une solution d'onde progressive entropique de vitesse c, C^1 par morceaux, périodique de période spatiale L satisfaisant le système de Saint Venant (1.4).*

Preuve. Soit H_- la hauteur avant le choc et H_+ la hauteur après le choc. Les hauteurs (H_+, H_-) satisfont les conditions de Rankine-Hugoniot

$$\frac{G}{2K}H_+^2 + \frac{K}{H_+} = \frac{G}{2K}H_-^2 + \frac{K}{H_-}$$

et la condition d'entropie (1.8)

$$\sqrt{GH_+} - \frac{K}{H_+} < 0 < \sqrt{GH_-} - \frac{K}{H_-}.$$

Il existe donc un point H_0 tel que $H_+ < H_0 < H_-$ et

$$\frac{G}{K}H_0 - \frac{K}{H_0^2} = 0. \tag{1.9}$$

De plus, on a nécessairement $\frac{dh}{d\xi}|_{H_0} > 0$ soit

$$F = \frac{U_0^2}{GH_0} = \frac{\sin(\theta)}{C_f} > 4. \tag{1.10}$$

Le dénominateur de la fraction (1.7) s'annule au point H_0. Pour traverser continuement cette valeur, le numérateur de (1.7) doit aussi s'annuler en H_0

$$GS - C_f\frac{(cH_0 - K)^2}{H_0^3} = 0. \tag{1.11}$$

L'équation (1.7) devient

$$h' = S\frac{h^2 + (H_0 - \frac{c^2}{GF})h + \frac{H_0^2}{F}}{h^2 + H_0h + H_0^2} = P(h). \tag{1.12}$$

On veut maintenant construire une roll-wave de longueur d'onde L. Soit H la solution continue de (1.12) telle que $H(0) = H_0$ qu'on obtient de manière implicite $\xi(H) = \int_{H_0}^{H} \frac{dh}{P(h)}$. Plus précisément, on a

$$\begin{aligned} S\xi(h) = h - H_0 + \frac{H_a^2 + H_0 H_a + H_0^2}{H_a - H_b} \ln(\frac{h - H_a}{H_0 - H_a}) - \\ - \frac{H_b^2 + H_0 H_b + H_0^2}{H_a - H_b} \ln(\frac{h - H_b}{H_0 - H_b}) \end{aligned} \quad (1.13)$$

où $H_a > H_b$ désignent les zéros P. On définit la suite de fonctions $(H_n)_{n\in\mathbb{N}}$ par $H_n(\xi) = H(\xi - nL)$. On construit alors la solution périodique en reliant H_n et H_{n+1} au moyen d'un choc entropique. En éliminant le cas $H_+ = H_-$, on obtient la première relation

$$H_+ + H_- = \frac{2H_0^3}{H_+ H_-}. \quad (1.14)$$

On détermine la position du choc ξ_c en écrivant $H_n(\xi_c) = H_-$ et $H_{n+1}(\xi_c) = H_+$. C'est équivalent à la relation

$$\int_{H_0}^{H^-} \frac{dh}{P(h)} = L + \int_{H_0}^{H_+} \frac{dh}{P(h)}. \quad (1.15)$$

En éliminant H_+ entre les équations (1.14) et (1.15), on obtient la longueur d'onde L en fonction de H_- la hauteur maximale de la roll-wave. On a ainsi construit une solution d'onde progressive périodique et entropique du système de Saint Venant (1.4). □

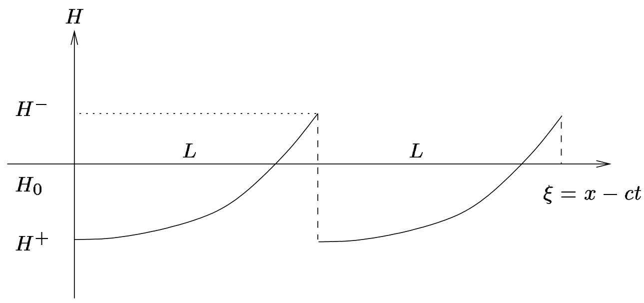


FIG. 1.3 – Profil d'une roll-wave de Dressler de longueur d'onde L

On peut utiliser le procédé de Dressler pour construire d'autres types de solutions périodiques par exemple en superposant deux périodes. La vitesse

c étant fixée, on choisit $L_2 < L_1$. On note H_1 la roll-wave de Dressler de longueur d'onde L_1 tel que $H_1(0) = H_0$ et H_2 la roll-wave de Dressler de longueur d'onde L_2 tel que $H_2(0) = H_0$. On a alors $H_1(kL_1) = H_0$ et $H_2(kL_2) = H_0$ pour tout $k \in \mathbb{Z}$. On définit alors la "roll-wave" H_{12} $L_1 + L_2$-périodique par

$$H_{12}(\xi) = \begin{cases} H_1(\xi) & \text{si } 0 \leq \xi \leq L_1 \\ H_2(\xi - L_1) & \text{si } L_1 < \xi \leq L_1 + L_2 \end{cases} \tag{1.16}$$

Cette solution est clairement C^1 aux points de raccord $\{L_1 + k(L_1 + L_2),\ k \in \mathbb{Z}\}$ et $\{k(L_1 + L_2),\ k \in \mathbb{Z}\}$.

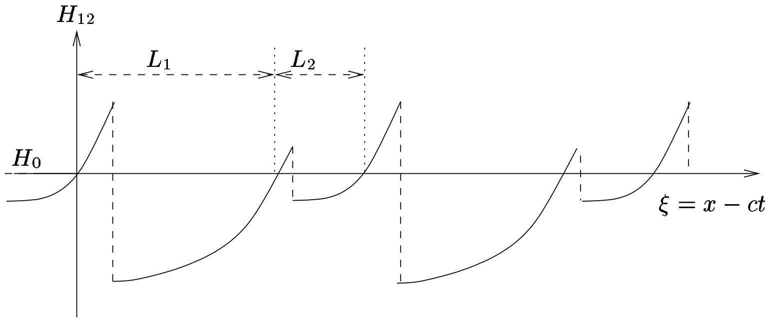


FIG. 1.4 – Profil de H_{12}

En s'inspirant de ce procédé de "mélange", on peut naturellement construire des solutions vérifiant les conditions d'entropie et présentant des points d'accumulations de discontinuités. Si $(L_n)_{n\in\mathbb{N}}$ est une suite strictement positive telle que $L = \sum_n L_n < +\infty$, on construit une solution $2L$-périodique en reliant continuement les profils des solutions de Dressler L_n-périodiques comme suit.

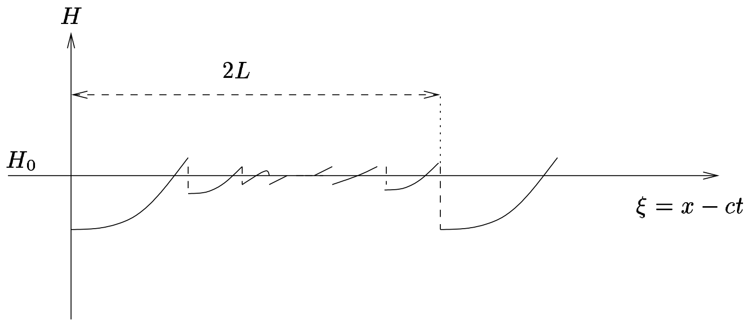


FIG. 1.5 – Exemple de "roll-wave" avec points d'accumulation de discontinuités

Pour sélectionner les solutions physiquement acceptables, on utilise le critère plus fin de limite visqueuse. On étudie le modèle de Saint Venant avec un terme de viscosité

$$\begin{aligned} &h_t + (hu)_x = 0, \\ &(hu)_t + (hu^2 + G\tfrac{h^2}{2})_x = GhS - C_f u^2 + \epsilon(hu_x)_x. \end{aligned} \tag{1.17}$$

Remarque : on a choisi ici un critère de limite visqueuse légèrement différent de celui énoncé dans l'introduction afin d'avoir une perturbation plus physique du système. En effet, l'utilisation du premier critère conduirait à rajouter un terme visqueux dans l'équation de conservation de la masse

$$h_t + (hu)_x = \epsilon h_{xx},$$

ce qui n'a aucun sens physiquement.

Needham et Merkin [8] ou encore Hwang et Chang [13] ont étudié le système (1.17) pour ϵ fixé et prouvé au moyen d'une bifurcation de Hopf l'existence de roll-waves continues de petite amplitude mais ils n'ont pas étudié le cas où ϵ tend vers 0. Les solutions d'ondes progressives (h, u) ayant une vitesse c vérifient le système

$$\begin{aligned} &h(c-u) = K, \\ &(u-c)u' + Gh' = GS - C_f\tfrac{u^2}{h} + \tfrac{\epsilon}{h}(hu')'. \end{aligned} \tag{1.18}$$

En injectant la relation $u = c - \frac{K}{h}$ dans la deuxième équation de (1.18), on obtient

$$\epsilon(\frac{h'}{h})' = (\frac{G}{2K}h^2 + \frac{K}{h})' - \frac{1}{K}(GhS - C_f(c - \frac{K}{h})^2). \tag{1.19}$$

En posant $h = x$ et $h' = xy$, on peut mettre l'équation (1.19) sous la forme

$$\begin{aligned} &x' = xy, \\ &\epsilon y' = p(x)y - q(x). \end{aligned} \tag{1.20}$$

On va montrer que seules les roll-waves construites par Dressler sont des solutions limite visqueuse. Au passage, on démontre pour ϵ proche de 0, l'existence de roll-waves continues qui convergent vers les roll-waves de Dressler quand ϵ tend vers 0.

1.2 Théorèmes de Fenichel et perte d'hyperbolicité normale

Dans cette partie, on présente les outils qui permettent d'analyser le système (1.20) lorsque $\epsilon \to 0$.

1.2.1 Rappel des théorèmes de Fenichel

Pour ces rappels, on fait référence au cours de C. Jones [5] sur les outils géométriques pour les problèmes de perturbations singulières. Considérons le système différentiel

$$\begin{aligned} \dot{y} &= f(x, y, \epsilon), \\ \dot{x} &= \epsilon g(x, y, \epsilon), \end{aligned} \tag{1.21}$$

où f et g sont régulières, $x \in \mathbb{R}^m$ et $y \in \mathbb{R}^n$. En posant $\tau = \epsilon t$, on obtient un nouveau système équivalent à (1.21) si $\epsilon \neq 0$

$$\begin{aligned} \epsilon y' &= f(x, y, \epsilon), \\ x' &= g(x, y, \epsilon), \end{aligned} \tag{1.22}$$

où $x' = \frac{dx}{d\tau}$ désigne la nouvelle dérivation. Le système (1.21) correspond à une dynamique rapide et le système (1.22) à une dynamique lente. Pour $\epsilon = 0$, on obtient deux systèmes non équivalents

$$\begin{aligned} \dot{y} &= f(x, y, 0), \\ \dot{x} &= 0 \end{aligned} \tag{1.23}$$

et

$$\begin{aligned} 0 &= f(x, y, 0), \\ x' &= g(x, y, 0). \end{aligned} \tag{1.24}$$

Pour le système (1.23), x ne varie pas et c'est y qui varie : y est la variable rapide et x est la variable lente. On note M_0 l'ensemble des points (x, y) tel que $f(x, y, 0) = 0$ que l'on peut écrire $y = h_0(x)$ (au moins localement). On voit que les solutions du système (1.24) sont sur M_0 et en remplaçant $y = h_0(x)$ dans la deuxième équation de (1.24), on connait la dynamique sur cette variété. On a donc trivialement un résultat de réduction à une variété invariante et une forme réduite du système différentiel. Les théorèmes de Fenichel relient cette situation simplifiée au système complet pour ϵ proche de 0. La variété M_0 est appelée la variété lente. On se restreint au cas où M_0 est un graphe. Dans ce système de coordonnées, on s'intéresse à la coordonnée normale à M_0 i.e. y. En faisant le changement de variables $y = h_0(x) + z$, on obtient pour la deuxième équation de (1.22)

$$\epsilon z' = D_y f(x, h_0(x), \epsilon) z + N(y, z, \epsilon) \ ,$$

où $N(0, 0, 0) = 0$ et $D_{(x,z)} N(0, 0, 0) = 0$. On a alors la définition suivante

Définition 4 *La variété M_0 est dite normalement hyperbolique si $D_y f(\overline{x}, \overline{y}, 0)$ n'a que des valeurs propres non imaginaires pures pour tout $(\overline{x}, \overline{y}) \in M_0$.*

Si la variété M_0 est compacte (on travaille avec des variétés à bord en particulier des restrictions de graphes à un compact) et vérifie l'hypothèse d'hyperbolicité normale, on a les théorèmes suivants.

Théorème (Fenichel) 1 *Pour $\epsilon > 0$ et assez proche de 0, il existe une variété M_ϵ localement invariante sous le flot associé au système (1.22) et qui se trouve dans un voisinage $O(\epsilon)$ de M_0. De plus, il existe un difféomorphisme entre M_0 et M_ϵ.*

Remarque : dans le cas où M_0 est un graphe, on peut montrer que M_ϵ est aussi un graphe. Dans ce cas on a $M_\epsilon = \{(x, y)/y = h_\epsilon(x) = h_0(x) + O(\epsilon)\}$. En remplaçant $y = h_\epsilon(x)$ dans la deuxième équation de (1.22), on obtient le système réduit sur cette variété.

Ce premier théorème nous donne l'existence d'une variété invariante et le comportement du système sur celle-ci mais on désire connaître le comportement au voisinage de celle ci. On a le résultat suivant.

Théorème (Fenichel) 2 *Pour $\epsilon > 0$ et assez proche de 0, il existe une variété $W^s(M_\epsilon)$ et une variété $W^u(M_\epsilon)$ respectivement positivement et négativement localement invariantes sous le flot associé au système (1.21) et qui sont dans un voisinage d'ordre $O(\epsilon)$ de $W^s(M_0)$ et $W^u(M_0)$ et il existe un difféomorphisme entre $W^s(M_\epsilon)$ et $W^s(M_0)$ d'une part et entre $W^u(M_\epsilon)$ et $W^u(M_0)$ d'autre part.*

Le dernier théorème donne des renseignements très précis sur le comportement du système en dehors de M_ϵ. En particulier, on peut fibrer les variétés stables et instables de M_ϵ par des trajectoires.

Théorème (Fenichel) 3 *Pour tout $v_\epsilon \in M_\epsilon$, il existe une variété localement positivement invariante*

$$W^s(v_\epsilon) \subset W^s(M_\epsilon)$$

et une variété localement négativement invariante

$$W^u(v_\epsilon) \subset W^u(M_\epsilon)$$

qui se trouve à $O(\epsilon)$ de $W^s(v_0)$ et $W^u(v_0)$ et il existe un difféomorphisme entre $W^s(v_\epsilon)$ et $W^s(v_0)$ d'une part et entre $W^u(v_\epsilon)$ et $W^u(v_0)$ d'autre part.

On étudie maintenant un cas où il y a une perte d'hyperbolicité normale qu'on appelle un "canard" en analyse non standard.

1.2.2 Un cas de perte d'hyperbolicité normale

On considère le système différentiel bidimensionel suivant

$$\begin{aligned} x' &= y, \\ \epsilon y' &= f(x)y - g(x), \end{aligned} \tag{1.25}$$

où f et g sont régulières, f et g ont un zéro commun en x_0. De plus la fonction f vérifie $f'(x) > 0$ et $f(x) \geq f'(x_0)(x - x_0)$ pour tout x. Cette dernière hypothèse est clairement vérifiée pour les deux situations qu'on étudie puisque dans ce cas la fonction f est alors strictement convexe. On suppose également que $\frac{g}{f}$ est minorée par une constante strictement positive et bornée.
Dans ce cas, le graphe invariant M_0 pour $\epsilon = 0$ est donné par $M_0 = \{(x,y)/y = \frac{g}{f}(x)\}$. La variété M_0 est normalement hyperbolique si et seulement si $f(x) \neq 0$. Tant que la variété M_0 vérifie l'hypothèse d'hyperbolicité normale, les théorèmes de Fenichel assurent l'existence d'un graphe M_ϵ localement invariant et proche de M_0 pour ϵ suffisamment petit et précisent le comportement des solutions de (1.25) au voisinage de M_ϵ. Mais ces théorèmes ne s'appliquent pas au voisinage de $(x_0, \frac{g'(x_0)}{f'(x_0)})$ car M_0 perd son hyperbolicité normale en ce point. Cependant, on peut encore démontrer la persistance de variétés invariantes $M_\epsilon^\pm$ à droite et à gauche de ce point respectivement positivement et négativement invariantes et dans un voisinage d'ordre $O(\sqrt{\epsilon})$ de M_0. En général, ces variétés ne coincident pas au point de perte d'hyperbolicité sont très rapidement éjectées de la variété lente de référence lorsqu'on les prolonge au dela de la singularité.
En perturbant le système (1.25), on va montrer d'une part qu'on peut faire coincider ces deux variétés et en faire une variété invariante dans un voisinage $O(\sqrt{\epsilon})$ de la variété lente M_0. D'autre part, on va montrer lorsque les variétés lentes à droite et à gauche ne coincident pas qu'on peut contrôler leur éjection de M_0.

Persistance d'une variété invariante

On montre le théorème suivant.

Théorème 4 *Si $\epsilon > 0$, mais assez petit, il existe deux graphes $M_\epsilon^\pm$ à droite et à gauche de x_0 respectivement positivement et négativement invariant qui se trouvent dans un voisinage d'ordre $O(\sqrt{\epsilon})$ de la variété lente M_0.*

Preuve. On va faire la preuve de l'existence de M_ϵ^+, la preuve d'existence de M_ϵ^- étant analogue. Dans cette démonstration, on désignera par K toute constante liée aux normes $\|.\|_\infty$ et normes lipschitz des fonctions mises en jeu et ne poseront aucun problème pour les estimations que l'on fera.

Supposons $x_0 = 0$. Partant du système (1.25), on fait le changement de variable $y = \frac{g}{f}(x) + z$. Le couple (x, z) vérifie le système

$$\begin{aligned} x' &= \frac{g}{f}(x) + z, \\ z' &= \frac{f(x)}{\epsilon} z - (\frac{g}{f})'(x)(z + \frac{g}{f}(x)). \end{aligned} \tag{1.26}$$

On recherche une solution de (1.26) avec z bornée. On remplace le problème (1.26) par le problème tronqué

$$\begin{aligned} x' &= \gamma(\frac{g}{f}(x) + z), \\ z' &= \gamma\big(\frac{f(x)}{\epsilon} z - (\frac{g}{f})'(x)(z + \frac{g}{f}(x))\big). \end{aligned} \tag{1.27}$$

où γ désigne la fonction caractéristique de l'intervalle $(0, t_0)$ et t_0 est à préciser. On intègre (1.27) en utilisant le principe de Duhamel et on obtient

$$\begin{aligned} x(t) &= \int_0^t \gamma(s)(z(s) + \frac{g}{f}(x(s)))ds = T_1(z)(t), \\ z(t) &= \int_t^\infty \exp(-\frac{\int_t^s \gamma(u) f(x(u)) du}{\epsilon}) \gamma(s) (\frac{g}{f})'(x(s))(z(s) + \frac{g}{f}(x(s)))ds. \end{aligned} \tag{1.28}$$

La première équation de (1.28) définit un opérateur $x = T_1(z)$. En injectant cette relation dans la deuxième équation de (1.28), la fonction z apparaît comme le point fixe d'un opérateur T dont on va montrer qu'il est contractant. On précise les espaces fonctionnels. On recherche $x \in \mathbb{X}_{\alpha,M}$ où

$$\mathbb{X}_{\alpha,M} = \{x \in C^0(0, t_0)/x(0) = 0 \ \ et \ \ \alpha \leq \frac{|x(t)|}{|t|} \leq M, \forall t \in (0, t_0)\}$$

pour un certain $\alpha > 0$ et $M > 0$ à déterminer. On munit cet espace de la norme $\|x\| = \sup_{(0,t_0)}(\frac{|x(t)|}{|t|})$. On recherche $z \in \mathbb{Y}$ où

$$\mathbb{Y}_C = \{z \in C^0(0, t_0)/\|z\|_\infty = \sup_{(0,t_0)} |z(t)| \leq C\sqrt{\epsilon}\}.$$

Comme la fonction $\frac{g}{f}$ est minorée et bornée, pour ϵ suffisamment petit, on a $T_1 : \mathbb{Y}_C \to \mathbb{X}_{\alpha,M}$ où $\alpha \geq \min(\frac{g}{2f}) > 0$ et $M \leq 2\max(\frac{g}{f})$. Ensuite, en utilisant l'hypothèse de croissance sur f, on a

$$\begin{aligned} T(z)(t) &= \int_t^{t_0} \exp(-\frac{\int_t^s f(x)}{\epsilon})(\frac{g}{f})'(x)(z + \frac{g}{f}(x)), \\ |T(z)(t)| &\leq (\int_t^{t_0} \exp(-\frac{f'(0)\alpha}{\epsilon} \int_t^s u du))(KC\sqrt{\epsilon} + K), \\ |T(z)(t)| &\leq K\sqrt{\epsilon}(C\sqrt{\epsilon} + 1). \end{aligned} \tag{1.29}$$

Soit $C = 2K$,si ϵ est petit, on aura bien $|T(z)(t)| \leq 2K\sqrt{\epsilon}$ et on a alors $T : \mathbb{Y}_C \to \mathbb{Y}_C$. Montrons que l'opérateur T est contractant sur $\mathbb{Y}_C$.
On note $x_1 = T_1(z_1)$ et $x_2 = T_1(z_2)$. On a la majoration

$$\begin{aligned} |x_1(t) - x_2(t)| &\leq \int_0^t |z_1 - z_2| + |\frac{g}{f}(x_1) - \frac{g}{f}(x_2)|, \\ |x_1(t) - x_2(t)| &\leq t\|z_1 - z_2\|_\infty + K\|x_1 - x_2\| \int_0^t sds, \\ \|x_1 - x_2\| &\leq 2\|z_1 - z_2\|_\infty \quad \text{si } t_0 \leq \frac{1}{K}. \end{aligned} \tag{1.30}$$

On va se servir de la majoration (1.30) pour montrer que T est contractant. On rappelle que

$$T(z)(t) = \int_t^{t_0} \exp(-\frac{\int_t^s f(x(u))du}{\epsilon})(\frac{g}{f})'(x(s))(z(s) + \frac{g}{f}(x(s)))ds \tag{1.31}$$

où $x = T_1(z)$. On a

$$\begin{aligned} &|Tz_1(t) - Tz_2(t)| \leq \\ &\int_t^{t_0} \exp(-\frac{\int_t^s f(x_1)}{\epsilon})\Big|\frac{g}{f}'(x_1)(z_1 + \frac{g}{f}(x_1)) - \frac{g}{f}'(x_2)(z_2 + \frac{g}{f}(x_2))\Big|ds+ \\ &+ \int_t^{t_0} \Big|\exp(-\frac{\int_t^s f(x_1)}{\epsilon}) - \exp(-\frac{\int_t^s f(x_2)}{\epsilon})\Big|\Big|(\frac{g}{f})'(x_2)(z_2 + \frac{g}{f}(x_2))\Big|ds. \end{aligned} \tag{1.32}$$

Soit $A(t)$ le premier terme dans le second membre de (1.32) et $B(t)$ le second terme, on a les majorations

$$\begin{aligned} B(t) &\leq \int_t^{t_0} \frac{K}{\epsilon}|\int_t^s f(x_1) - f(x_2)| \int_0^1 \exp(-\frac{\int_t^s uf(x_1) + (1-u)f(x_2)}{\epsilon})duds, \\ B(t) &\leq \int_t^{t_0} \frac{K}{2\epsilon}(s^2 - t^2)\|x_1 - x_2\| \exp(-\frac{f'(0)\alpha}{2\epsilon}(s^2 - t^2))ds, \\ B(t) &\leq Kt_0\|x_1 - x_2\| \leq 2Kt_0\|z_1 - z_2\|_\infty. \end{aligned} \tag{1.33}$$

Pour t_0 assez petit, on aura $B(t) \leq \frac{1}{2}\|z_1 - z_2\|_\infty$. Pour $A(t)$, on a l'inégalité

$$A(t) \leq \int_t^{t_0} \exp(-\frac{\int_t^s f(x_1)}{\epsilon})K\|z_1 - z_2\|_\infty$$

Si $t \geq t_\epsilon = \frac{\sqrt{\epsilon}}{\alpha f'(0)}$ alors en utilisant l'inégalité $f(x) > f'(0)x$ pour tout $x \in \mathbb{R}$, on obtient $f(x(t)) \geq \sqrt{\epsilon}$ et

$$\begin{aligned} A(t) &\leq (\int_t^{t_0} \exp(-\frac{s-t}{\sqrt{\epsilon}})ds)K\|z_1 - z_2\|_\infty, \\ A(t) &\leq K\sqrt{\epsilon}\|z_1 - z_2\|_\infty. \end{aligned} \tag{1.34}$$

Si $t \leq t_\epsilon$ alors

$$A(t) \leq (\int_t^{t_\epsilon} exp(-\frac{\int_t^s f(x_1)}{\epsilon})ds + \int_{t_\epsilon}^{t_0} exp(-\frac{\int_t^s f(x_1)}{\epsilon})ds)K\|z_1 - z_2\|_\infty,$$

$$A(t) \leq K\|z_1 - z_2\|_\infty(K_1\sqrt{\epsilon} + \int_{t_\epsilon}^{t_0} exp(-\frac{\int_{t_\epsilon}^s f(x_1)}{\epsilon})exp(-\frac{\int_t^{t_\epsilon} f(x_1)}{\epsilon})ds),$$

soit finalement avec la majoration faite pour $t \geq t_\epsilon$

$$A(t) \leq K\sqrt{\epsilon}\|z_1 - z_2\|_\infty.$$

Donc pour ϵ assez petit l'application T est contractante. On conclut en appliquant le théorème du point fixe. □

Remarque : On a démontré l'existence de variétés lentes M_{ϵ^+} et M_ϵ^- à droite et à gauche de x_0 (respectivement positivement et négativement invariantes) dans un voisinage indépendant de ϵ. En dehors de ce point, on peut prolonger ces variétés en utilisant le premier théorème de Fenichel. Généralement, ces variétés ne coincident pas au point x_0 et sont très rapidement éjectées de la variété lente de référence lorsqu'on les prolonge au dela de la singularité (voir la figure 1.6).

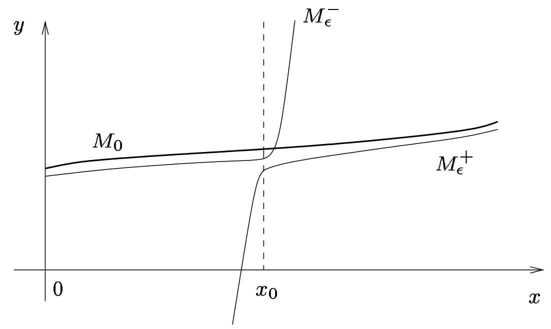


FIG. 1.6 – Ejection de $M_\epsilon^\pm$ loin de M_0

On va montrer qu'en modifiant légèrement la forme du système différentiel (1.25), on peut faire coincider ces variétes et montrer l'existence d'une variété *invariante* pour le système perturbé, qui se trouve dans un voisinage d'ordre $O(\sqrt{\epsilon})$ de M_0. Ensuite on montrera qu'on a également un contrôle sur le phénomène d'éjection loin de la variété de référence M_0 lorsque les variétés $M_\epsilon^\pm$ ne coincident pas en x_0.

Phénomène d'éjection des variétés $M_\epsilon^\pm$ loin de la variété lente M_0

Considérons le système perturbé

$$\begin{aligned} x' &= y, \\ \epsilon y' &= (f(x)+\epsilon c)y - g(x), \end{aligned} \tag{1.35}$$

où f et g vérifient les conditions $f(0) = g(0) = 0$, $f'(0) > 0$, $g'(0) > 0$ et $c \in \mathbb{R}$. On rappelle qu'on a également $f'' > 0$, $h(x) = \frac{g}{f}(x) \geq \alpha > 0$. On va montrer au moyen d'un calcul à la Melnikov, qu'on peut trouver des variétés invariantes pour le système (1.35) dont on peut contrôler l'éjection de la variété lente de référence. En particulier, on prouvera l'existence d'une variété invariante qui est dans un voisinage d'ordre $O(\sqrt{\epsilon})$ de M_0.
La fonction f étant un difféomorphisme local au voisinage de 0, il existe un point x_ϵ unique tel que

$$f(x_\epsilon) + \epsilon c = 0.$$

On étudie le comportement des solutions au voisinage de la variété lente

$$W^s = \{(x,y) \in \mathbb{R}^2 / \, y = \frac{g(x) - g(x_\epsilon)}{f(x)+\epsilon c} = h^\epsilon(x)\}. \tag{1.36}$$

La variété W^s est dans un voisinage d'ordre $O(\epsilon)$ de la variété lente de référence $M_0 = \{(x,y)/y = \frac{g(x)}{f(x)} = h(x)\}$. En appliquant le théorème 4, on peut montrer l'existence d'une variété lente positivement invariante M_ϵ^+ pour $x > x_\epsilon$. On introduit alors le changement de variables $y = h^\epsilon(x) + z$. On obtient un nouveau système

$$\begin{aligned} x' &= h^\epsilon(x) + z, \\ \epsilon z' &= (f(x)+\epsilon c)z + \epsilon k^\epsilon(x,z), \end{aligned} \tag{1.37}$$

où $\epsilon k^\epsilon(x,z) = -g(x_\epsilon) - \epsilon \frac{dh^\epsilon}{dx}(x)(z + h^\epsilon(x))$. Comme dans la section précédente, la variété lente invariante M_ϵ^+ est obtenue par une méthode de point fixe sur l'équation

$$z^+(t) = -\int_t^{+\infty} \exp\Big(-\int_t^s \frac{f(x(\sigma))+\epsilon c}{\epsilon} d\sigma\Big) k^\epsilon(x(s), z(s)) ds. \tag{1.38}$$

En particulier, on a

$$\overline{z}(c,\epsilon)^+ = z^+(0) = -\int_0^{+\infty} \exp\Big(-\int_0^s \frac{f(x(\sigma))+\epsilon c}{\epsilon} d\sigma\Big) k^\epsilon(x(s), z(s)) ds. \tag{1.39}$$

De la même manière, on peut montrer l'existence d'une variété lente négativement invariante M_ϵ^- pour $x < x_\epsilon$. On obtient

$$\overline{z}(c,\epsilon)^- = -\int_{-\infty}^{0} \exp\Big(\int_s^0 \frac{f(x(\sigma))+\epsilon c}{\epsilon} d\sigma\Big) k^\epsilon(x(s),z(s))ds. \tag{1.40}$$

Pour obtenir une variété invariante régulière au voisinage de x_ϵ, on doit avoir $\overline{z}(c,\epsilon)^+ = \overline{z}(c,\epsilon)^-$. On calcule le développement asymptotique de $\overline{z}(c,\epsilon)^\pm$

$$\overline{z}(c,\epsilon)^\pm = \sqrt{\epsilon}A(c) + \epsilon D_p(c,\epsilon) \pm \epsilon D_i(c,\epsilon)$$

avec $A(c) = \int_0^{+\infty} \exp(-g'(0)\frac{t^2}{2})(h(0)h'(0) - c\frac{g'(0)}{f'(0)})dt$. Les fonctions D_p et D_i sont respectivement les intégrales d'une fonction paire et d'une fonction impaire. En particulier, on peut montrer que

$$\begin{aligned} D_i(c,0) &= \int_0^\infty \exp\Big(-g'(0)\frac{t^2}{2}\Big)\Big(h(0)h'(0) - c\frac{g'(0)}{f'(0)}\Big)\frac{1}{6}f'(0)h'(0)^2t^3dt \\ &+ \int_0^\infty \exp\Big(-g'(0)\frac{t^2}{2}\Big)\Big(h'(0)^2 + h(0)h''(0)\frac{g'(0)}{f'(0)}\Big)tdt. \end{aligned} \tag{1.41}$$

On obtient donc $\overline{z}(c,\epsilon)^+ = \overline{z}(c,\epsilon)^-$ si et seulement si $D_i(c,\epsilon) = 0$. On voit aisément qu'il existe un unique c_0 tel que $D_i(c_0,0) = 0$. De plus on a

$$\frac{\partial D_i}{\partial c}(c_0,0) = -\frac{1}{6}g'(0)h'(0)^2\int_0^{+\infty} \exp\Big(-g'(0)\frac{t^2}{2}\Big)t^3dt \neq 0.$$

On peut donc appliquer le théorème des fonctions implicites : il existe $\delta > 0$ et une fonction C tel que $C(0) = c_0$ et pour tout $\epsilon < \delta$, $D_i(C(\epsilon),\epsilon) = 0$. Donc pour ce choix de $c = C(\epsilon)$, on a montré l'existence d'une variété lente invariante M_ϵ dans un voisinage de W^s d'ordre $O(\sqrt{\epsilon})$ qui coincident avec M_ϵ^+ si $x > x_\epsilon$ et avec M_ϵ^- si $x < x_\epsilon$.
On examine maintenant le phénomène d'éjection de M_ϵ^+ lorsqu'on passe la singularité pour $c \neq C(\epsilon)$. On cherche donc à prolonger M_ϵ^+ pour des temps $t < 0$. On calcule donc

$$z^+(-t) = -\int_{-t}^{+\infty} \exp(-\int_{-t}^{s} \frac{f(x(\sigma))+\epsilon c}{\epsilon} d\sigma) k^\epsilon(x(s),z(s))ds, \tag{1.42}$$

pour $t > 0$. On peut montrer aisément que

$$\begin{aligned} z^+(-t) &= \big(\overline{z}(c,\epsilon)^+ - \overline{z}(c,\epsilon)^-\big)\exp\Big(-\int_{-t}^{0}\frac{f(x(s))+\epsilon c}{\epsilon}ds\Big) + \\ &+ \int_{-\infty}^{-t} \exp\Big(-\int_s^{-t} \frac{f(x(\sigma))+\epsilon c}{\epsilon} d\sigma\Big) k^\epsilon(x(s),z(s))ds \end{aligned} \tag{1.43}$$

Le deuxième terme de l'égalité correspondant à l'équation de M_ϵ^-, on a

$$z^+(-t) = \big(\overline{z}(c,\epsilon)^+ - \overline{z}(c,\epsilon)^-\big) \exp\Big(- \int_{-t}^{0} \frac{f(x(s)) + \epsilon c}{\epsilon} ds\Big) + O(\sqrt{\epsilon}). \quad (1.44)$$

On voit aisément que si on est loin de $c = C(\epsilon)$, on a un éjection de M_ϵ^+ loin de M_0 pour $t = O(\epsilon)$. Par contre pour $c = C(\epsilon)$, la variété M_ϵ^+ reste dans un voisinage d'ordre $O(\sqrt{\epsilon})$. On a donc une situation intermédiaire où on a éjection pour $t = O(1)$. Pour fixer les idées, on choisit $t = 1$.

$$z^+(-1) = 2\epsilon D_i(c,\epsilon) \exp\Big(- \int_{-1}^{0} \frac{f(x(s)) + \epsilon c}{\epsilon} ds\Big) + O(\sqrt{\epsilon}). \quad (1.45)$$

On a clairement

$$\exp\Big(- \int_{-1}^{0} \frac{f(x(s)) + \epsilon c}{\epsilon} ds\Big) \sim C \exp(\frac{K}{\epsilon})$$

La fonction $D_i(.,\epsilon)$ étant un difféomorphisme local au voisinage du point $c = C(\epsilon)$, on peut trouver

$$c = C(\epsilon) + O\Big(\frac{\exp(-\frac{K}{\epsilon})}{\epsilon}\Big),$$

tel que $z^+(-1) \sim M$ où M est un constante arbitraire.
On a ainsi démontré la proposition suivante.

Proposition 1 *Soit $x_1 < 0 < x_2$, il existe $c_1(\epsilon)$ et $c_2(\epsilon)$ tel que les systèmes perturbés*

$$\begin{aligned} &x' = y, \\ &\epsilon y' = (f(x) + \epsilon c_i(\epsilon))y - g(x),\ i = 1, 2. \end{aligned} \quad (1.46)$$

possèdent chacun un graphe lent invariant $M_i^\epsilon = \{(x,y)/\ y = h_i^\epsilon(x)\}$ et tel que $(x_i, 0) \in M_i^\epsilon$. De plus, pour tout $\delta > 0$, on a $h_1^\epsilon(x) - h(x) = O(\sqrt{\epsilon})$ si $x > x_1 - \delta$ et $h_2^\epsilon(x) - h(x) = O(\sqrt{\epsilon})$ si $x < x_2 - \delta$. Enfin on a $c_i(\epsilon) - C(\epsilon) = O\Big(\frac{\exp(-\frac{K}{\epsilon})}{\epsilon}\Big)$.

Les différents cas d'éjection des variétés $M_\epsilon^\pm$ loin de M_0 sont présentés sur la figure 1.7.
Remarque : Tant que les variétés $M_\epsilon^\pm$ sont proches de la variété lente de référence M_0, le théorème de fibration de Fenichel reste valable et on peut ainsi décrire le comportement du système perturbé en dehors de ces variétés.

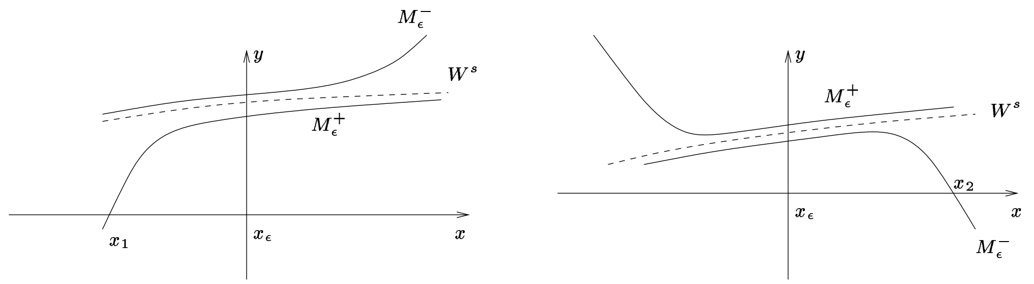


FIG. 1.7 – Diagrammes d'éjection quand $\overline{z}(c,\epsilon)^+ < \overline{z}(c,\epsilon)^-$ et $\overline{z}(c,\epsilon)^- < \overline{z}(c,\epsilon)^+$.

1.3 Application des techniques de variété lente

1.3.1 Profils visqueux pour des ondes progressives solutions d'équations hyperboliques/paraboliques avec terme source.

On rappelle qu'on étudie les solutions d'ondes progressives de

$$u_t + (f(u))_x = g(u) + \epsilon u_{xx}.$$

Elles doivent vérifier $\big(f'(u) - c\big)u' = g(u) + \epsilon u''$. Dans ce cas, on perturbe le système avec $c = f'(u_m) + \epsilon c_1$. On écrit alors la variété lente invariante M_0 sous la forme

$$u' = (u - u_r)(u - u_l)h(u).$$

On applique le théorème 4 : il existe deux variétés $M_\epsilon^\pm$ invariantes pour le système perturbé à droite et à gauche du point u_m. On choisit alors $c_1 = c_1(\epsilon)$ de telle sorte que les variétés $M_\epsilon^\pm$ coincident en $u = u_m$. On a donc l'existence d'une variété invariante M_ϵ donnée par

$$u' = (u - u_r)(u - u_l)(h(u) + O(\sqrt{\epsilon})).$$

Cette variété lente invariante relie les points critiques $(u_l, 0)$ et $(u_r, 0)$ et lorsque ϵ tend vers 0, celle ci reste loin du point critique $(u_m, 0)$ (voir la figure 1.8). En conséquence, on a démontré le résultat suivant.

Proposition 2 *Les profils entropiques U_λ sont solutions de type limite visqueuse de l'équation*

$$u_t + (f(u))_x = g(u)$$

si et seulement si $\lambda = 0$.

Contrairement à l'approche de Haerterich, le point critique $(u_m, 0)$ où il y a perte d'hyperbolicité n'est pas sur le graphe invariant et la preuve est plus simple.

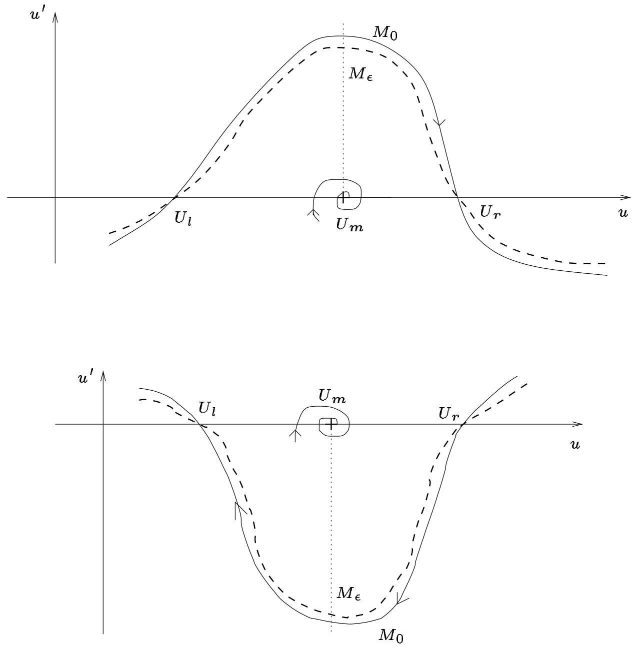


FIG. 1.8 – Plans de phases pour le système issu des équations hyperboliques/paraboliques scalaires

1.3.2 Le problème des roll-waves

Dans cette partie, on étudie les solutions d'ondes progressives périodiques du système de Saint Venant visqueux qu'on peut réduire à l'étude des solutions périodiques de l'équation

$$\epsilon(\frac{h'}{h})' = (\frac{Gh}{2K} - \frac{K}{h^2})h' - \frac{1}{K}(GhS - C_f(c - \frac{K}{h})^2). \tag{1.47}$$

On va faire apparaître les roll-waves continues comme les points fixes d'une application de Poincaré. Ce système possède une dynamique lente/rapide. On considère d'abord le système lent : on a une variété lente correspondant au profil calculé par Dressler $h' = P(h)$. Cette variété est attractive si $h < H_0$ et répulsive si $h > H_0$. En utilisant le théorème de fibration de Fenichel, on définit une application d'entrée-sortie ES sur la section $\Sigma = \{(h, k\log(\epsilon)), h > 0\}$ du plan des phases tel que $h_- = ES(h_+)$ et

$h_+ < H_0 < h_-$. On analyse ensuite la dynamique du système rapide essentiellement définie par les conditions de Rankine-Hugoniot. On obtient alors une application $h^+ = RH(h^-)$ où $h^+ < h^-$. On obtient alors une application de Poincaré $\mathcal{P}$ donnée par $\mathcal{P} = RH \circ ES$. Pour obtenir des roll-waves continues de taille $O(1)$, il faut aussi perturber la variété lente en considérant des vitesses $c(\epsilon) = c + O(\epsilon)$ et utiliser le critère d'éjection étudié dans la section 1.2.2. On montrera enfin que les roll-waves sont les seules limites visqueuses possibles.

Analyse du système lent

Le système lent est donné par

$$\epsilon(\frac{h'}{h})' = (\frac{G}{2K}h^2 + \frac{K}{h})' - \frac{1}{K}(GhS - C_f(c - \frac{K}{h})^2) \tag{1.48}$$

et le système rapide par

$$(\frac{h'}{h})' = (\frac{G}{2K}h^2 + \frac{K}{h})' - \frac{\epsilon}{K}(GhS - C_f(c - \frac{K}{h})^2). \tag{1.49}$$

Le graphe invariant M_0 a pour équation

$$h' = -\frac{h(GS - C_f\frac{(ch-K)^2}{h^3})}{(\frac{K^2}{h^2}) - Gh} \tag{1.50}$$

et l'hypothèse d'hyperbolicité normale n'est pas vérifiée en H_0. Le système issu des équations de Saint Venant est de la forme

$$\begin{aligned} x' &= y, \\ \epsilon(\tfrac{y}{x})' &= f(x)y - g(x). \end{aligned} \tag{1.51}$$

A quelques modifications près, on peut appliquer les théorèmes énoncés dans les sections précédentes au système (1.51). En effet, une fois le système intégré on a

$$\begin{aligned} x(t) &= H_0 + \int_0^t z(s) + \frac{g}{f}(x(s))ds, \\ z(t) &= \int_t^\infty \exp(-\frac{\int_t^s x(u)f(x(u))du}{\epsilon})\frac{g}{f}'(x(s))\big(z(s) + \frac{g}{f}(x(s))\big)ds + \\ &\quad + \int_t^\infty \exp(-\frac{\int_t^s x(u)f(x(u))du}{\epsilon})\frac{(z(s) + \frac{g}{f}(x(s)))^2}{x(s)}ds. \end{aligned} \tag{1.52}$$

Par des calculs analogues à ceux des sections précédentes, on prouve l'existence de variétés lentes invariantes $M_\epsilon^\pm$ à droite et à gauche de H_0. En perturbant le système initial en remplaçant K par $K+\epsilon K_1$, on peut contrôler l'éjection de $M_\epsilon^\pm$ loin de M_0. On garde la même notation pour les variétés prolongées au dela de H_0. Tant que ces variétés restent proches de M_0, on peut appliquer les théorèmes classiques de Fenichel. On utilise le troisième théorème de Fenichel sur les fibrations pour connaître le comportement en dehors des variétés lentes $M_\epsilon^\pm$. Si $h < H_0$, $f(h) < 0$ et la variété lente est attractive. Si $h > H_0$, $f(h) > 0$ et la variété lente est répulsive. Pour quantifier ce phénomène, on pose le changement de variable

$$y = h_s(x) + z$$

et on a de nouvelles équations du type

$$\begin{aligned} x' &= z + h_s(x), \\ z' &= \tfrac{1}{\epsilon}(f(x) - \epsilon h_s'(x))z + (\tfrac{1}{\epsilon}(f(x)h_s(x) - g(x)) - h_s(x)h_s'(x)). \end{aligned} \tag{1.53}$$

Comme $z = 0$ est la nouvelle variété lente par ce changement de variable, alors $z = 0$ implique $z' = 0$ et la deuxième parenthèse est toujours nulle donc il reste

$$z' = \frac{1}{\epsilon}(f(x) - \epsilon h_s'(x))z,$$

qui s'intègre en

$$z(t) = z_0 \exp(\frac{1}{\epsilon}\int_0^t (f(x) - \epsilon h_s'(x)))dt.$$

On va construire l'application ES définissant le critère d'entrée sortie pour le système lent sur la section $\Sigma = \{(h, k\log\epsilon)/h > 0\}$. Partant de $(h_1, z = k\log(\epsilon))$ avec $h_1 < H_0$, on est attiré par la variété lente puis passé H_0 on est éjecté de la variété lente au temps $t(h_2)$ tel que asymptotiquement lorsque ϵ tend vers 0

$$|\int_{t(h_1)}^{t(H_0)} f(x(t))dt| = |\int_{t(H_0)}^{t(h_2)} f(x(t))dt|$$

Ce qui donne après le changement de variable $x' = \frac{g(x)}{f(x)}$ dans l'intégrale

$$|\int_{h_1}^{H_0} \frac{f^2(x)}{g(x)}dx| = |\int_{H_0}^{h_2} \frac{f^2(x)}{g(x)}dx|$$

On a la propriété

Proposition 3 *Soit $(x(t), z(t))$ solution de (1.52) avec*

$$\begin{aligned} x(0) &= h_1, \\ z(0) &= k\log(\epsilon). \end{aligned} \tag{1.54}$$

Il existe t^ϵ, premier instant tel que $z(t^\epsilon) = 10k\log(\epsilon)$. Et on a

$$lim_{\epsilon\to 0} x(t^\epsilon) = h_2$$

avec h_2 tel que

$$|\int_{h_1}^{H_0} \frac{f^2(x)}{g(x)}dx| = |\int_{H_0}^{h_2} \frac{f^2(x)}{g(x)}dx|.$$

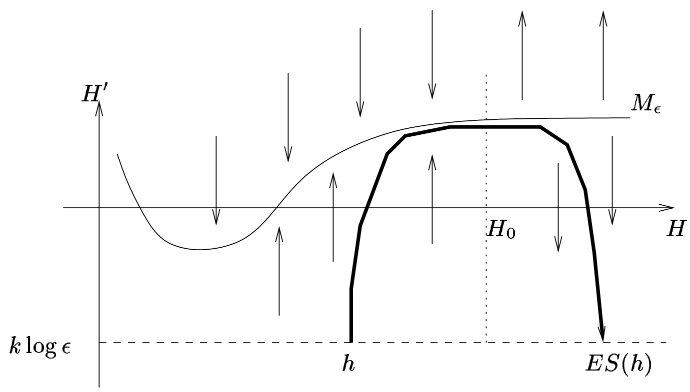
Preuve. Soit $t(h_1, h_2)$ le premier instant tel que la solution de $x' = h_s(x)$ vérifie : $x(t(h_1, h_2)) = h_2$. On montre , en imitant la démonstration du théorème de Fenichel , que pour $\epsilon > 0$ et $t \geq t(h_1, h_2) + 10^{10}\epsilon|\log(\epsilon)|$, alors on a

$$10^5\epsilon|\log(\epsilon)| \leq |z(t)| \leq 10^{10}\epsilon|\log(\epsilon)|$$

Ceci implique l'existence de t^ϵ tel que $z(t^\epsilon) = 10k\log(\epsilon)$. De plus on a :

$$t^\epsilon \in [t(h_1, h_2) - 10^{10}\epsilon|\log(\epsilon)|, t(h_1, h_2) + 10^{10}\epsilon|\log(\epsilon)|]$$

ce qui donne du même coup le critère. □
On peut alors construire une application $ES(., \epsilon)$ associée au système lent et au critère d'entrée sortie tel que $ES(h_1, 0) = h_2$ (voir le graphe).



Etude du système rapide

Le système rapide

$$(\frac{h'}{h})' = (\frac{G}{2K}h^2 + \frac{K}{h})' - \frac{\epsilon}{K}(GhS - C_f(c - \frac{K}{h})^2) \tag{1.55}$$

vérifie la propriété suivante

Proposition 4 *Soit h_r la solution du système rapide pour $\epsilon = 0$, et ξ_1 et ξ_2 deux points tels que $\xi_2 - \xi_1 = k\log(\epsilon)$. Pour $\epsilon > 0$ suffisamment petit, il existe une solution h du système rapide (1.55) qui vérifie les conditions aux limites $h(\xi_2) = h_r(\xi_2)$ et $h(\xi_1) = h_r(\xi_1)$.*

Remarque : on a choisi $\xi_2 - \xi_1 = k\log(\epsilon)$ qui correspond à un ordre de grandeur intermediaire entre $O(1)$ et $O(\frac{1}{\epsilon})$.

Preuve. Le système rapide est donné par

$$(\frac{h'}{h})' = (\frac{G}{2K}h^2 + \frac{K}{h})' - \frac{\epsilon}{K}(GhS - C_f(c - \frac{K}{h})^2).$$

On recherche h sous la forme $h = h_r + \epsilon h_1$ avec h_1 nulle aux bords. La fonction h_1 vérifie l'équation différentielle

$$\begin{aligned}(\frac{h_1}{h_r})'' &= (\frac{G}{K}h_r h_1 - \frac{Kh_1}{h_r^2})' - \\ &\quad - P_1(h_r) + f(\epsilon, h_1, h_1') \end{aligned} \tag{1.56}$$

où

$$\begin{aligned}\epsilon f(\epsilon, h_1, h_1') &= [\frac{K}{h_r}(\frac{1}{1+\epsilon\frac{h_1}{h_r}} - 1 + \epsilon\frac{h_1}{h_r})]' - \\ &\quad - \frac{\epsilon}{h_r}c_f\big(2c - \frac{K}{h_r}(1 + \frac{1}{1+\epsilon\frac{h_1}{h_r}})\big)(\frac{1}{1+\epsilon\frac{h_1}{h_r}} - 1) + \\ &\quad + \epsilon^2((\frac{g}{2K}h_1^2)' - \frac{GS}{K}h_1) - \\ &\quad - [\frac{h_r' + \epsilon h_1'}{h_r + \epsilon h_1} - \frac{h_r'}{h_r} - \epsilon(\frac{h_1}{h_r})']' \end{aligned} \tag{1.57}$$

et

$$P_1(h_r) = \frac{1}{K}(Gh_rS - C_f(c - \frac{K}{h_r})^2).$$

Donc f est d'ordre $O(\epsilon)$. Une base de solution de l'équation linéaire homogène pour $\epsilon = 0$ est donnée par $\{h_r'(\xi); h_r'(\xi)\phi(\xi)\}$ où $\phi(\xi) = \int_{\xi_1}^{\xi}\frac{h_r(u)}{h_r'(u)}du$. On résout le problème non homogène pour $\epsilon = 0$ par une méthode de variation de la constante. On trouve

$$\begin{aligned}h_1 &= h_{1p} \\ &= h_r'(\xi)\phi(\xi)\int_{\xi_1}^{\xi} P(h_r)(u)du - \\ &\quad -h_r'(\xi)\int_{\xi_1}^{\xi} P(h_r)(u)\phi(u)du + C_1h_r'(\xi)\phi(\xi) \end{aligned} \tag{1.58}$$

où C_1 est une constante ajustée pour que $h_{1p}(\xi_2) = 0$. De la même manière, pour $\epsilon \neq 0$ on montre que la fonction h_1 solution de (1.56) vérifie

$$h_1 = h_{1p} + h_r'(\xi)\phi(\xi) \int_{\xi_1}^{\xi} f_1(u)du - h_r'(\xi) \int_{\xi_1}^{\xi} f_1(u)\phi(u)du + C_2(h_1)h_r'(\xi)\phi(\xi) \tag{1.59}$$

où $C_2(h_1)$ est une constante ajustée pour que $h_1(\xi_2) = 0$ et pour simplifier les notations, on pose $f_1(x) = f(\epsilon, h_1, h_1')(x)$. La constante C_2 est donnée par

$$C_2(h_1) = \frac{1}{\phi(\xi_2)} \int_{\xi_1}^{\xi_2} f_1(u)(\phi(u) - \phi(\xi_2))du. \tag{1.60}$$

L'équation (1.59) peut s'écrire

$$h_1 = h_{1p} + \mathfrak{F}^\epsilon(h_1) = \mathfrak{T}^\epsilon(h_1). \tag{1.61}$$

Montrons que $\mathfrak{T}^\epsilon$ est un opérateur contractant pour la norme $\|.\|_{C^1}$ sur $\mathrm{B}_{\|.\|_{C^1}}(h_{1p}, 1)$. On procède en deux étapes

Première étape : $\mathfrak{T}^\epsilon(\mathrm{B}_{\|.\|_{C^1}}(h_{1p}, 1)) \subset \mathrm{B}_{\|.\|_{C^1}}(h_{1p}, 1)$

On a

$$\mathfrak{T}^\epsilon(h_1) - h_{1p} = h_r'(\xi)\Big(\phi(\xi) \int_{\xi_1}^{\xi} f_1(u)du - \int_{\xi_1}^{\xi} f_1(u)\phi(u)du + C_2\phi(\xi)\Big). \tag{1.62}$$

D'où la majoration

$$\begin{aligned} |\mathfrak{T}^\epsilon(h_1)(\xi) - h_{1p}(\xi)| \quad &\leq \quad \|h_r'\|_\infty \Bigg[\Bigg(|C_2| + \Big|\int_{\xi_1}^{\xi} f_1(u)du\Big|\Bigg)\|\phi\|_\infty + \\ &\quad + \Big|\int_{\xi_1}^{\xi} f_1(u)\phi(u)du\Big|\Bigg]. \end{aligned} \tag{1.63}$$

D'autre part, $\|\phi\|_\infty \leq \|\frac{h_r}{h_r'}\||\xi_2 - \xi_1| \leq A|\log(\epsilon)|$ donc

$$\begin{aligned} \Big|\int_{\xi_1}^{\xi} f_1(u)du\Big| \quad &\leq \quad \frac{2}{\epsilon}\left\|\frac{K}{h_r}(\frac{1}{1+\epsilon\frac{h_1}{h_r}} - 1 + \epsilon\frac{h_1}{h_r})\right\|_\infty + \\ &\quad + \frac{C}{\min(h_r)}\left\|\frac{1}{1+\epsilon\frac{h_1}{h_r}} - 1\right\|_\infty |\xi_2 - \xi_1| + \\ &\quad + 2\epsilon\left\|\frac{G}{2K}h_1^2\right\|_\infty + \epsilon\frac{GS}{K}\|h_1\|_\infty|\xi_2 - \xi_1| + \\ &\quad + \frac{2}{\epsilon}\left\|\frac{h_r' + \epsilon h_1'}{h_r + \epsilon h_1} - \frac{h_r'}{h_r} - \epsilon(\frac{h_1}{h_r})'\right\|_\infty. \end{aligned} \tag{1.64}$$

On regarde ensuite les termes en ϵ et $\frac{1}{\epsilon}$ à l'aide des formules de Taylor :

$$\Big\|\frac{1}{1+\epsilon\frac{h_1}{h_r}}-1+\epsilon\frac{h_1}{h_r}\Big\|_\infty \le \epsilon^2(\frac{h_1}{h_r}^2)\sup_{x\in[0,\epsilon\frac{h_1}{h_r}]}(\frac{2}{(1+x)^3})\le C\epsilon^2, \tag{1.65}$$

$$\Big\|\frac{1}{1+\epsilon\frac{h_1}{h_r}}-1\Big\|_\infty \le \epsilon\|\frac{h_1}{h_r}\|_\infty \sup_{x\in[0,\epsilon\frac{h_1}{h_r}]}(\frac{1}{(1+x^2)})\le C\epsilon, \tag{1.66}$$

De même, on peut montrer que

$$\Big\|\frac{h_r'+\epsilon h_1'}{h_r+\epsilon h_1}-\frac{h_r'}{h_r}-\epsilon(\frac{h_1}{h_r})'\Big\|_\infty \le C\epsilon^2. \tag{1.67}$$

Des trois inégalités, on déduit que :

$$\Big|\int_{\xi_1}^{\xi} f_1(u)du\Big| \le C\epsilon(1+|\log(\epsilon)|). \tag{1.68}$$

De même

$$\begin{aligned}\Big|\int_{\xi_1}^{\xi} f_1(u)\phi(u)du\Big| \le{}& \frac{2}{\epsilon}\Big\|\frac{K}{h_r}(\frac{1}{1+\epsilon\frac{h_1}{h_r}}-1+\epsilon\frac{h_1}{h_r})\phi\Big\|_\infty+\\ &+\frac{1}{\epsilon}\int_{\xi_1}^{\xi_2}\Big|\frac{K}{h_r}(\frac{1}{1+\epsilon\frac{h_1}{h_r}}-1+\epsilon\frac{h_1}{h_r})\Big|\Big|\frac{1}{h_r'}\Big|du+\\ &+C\|\phi\|_\infty\Big\|\frac{1}{1+\epsilon\frac{h_1}{h_r}}-1\Big\|_\infty|\xi_2-\xi_1|+\\ &+2\epsilon\bigg(\frac{G}{K}\|h_1^2\phi\|_\infty+|\xi_2-\xi_1|\Big(\frac{GS}{K}\|h_1^2\|_\infty+\|\frac{Gh_1^2h_r}{2Kh_r'}\|_\infty\Big)\bigg)+\\ &+\frac{2}{\epsilon}\Big\|\Big(\frac{h_r'+\epsilon h_1'}{h_r+\epsilon h_1}-\frac{h_r'}{h_r}-\epsilon(\frac{h_1}{h_r})'\Big)\phi\Big\|_\infty+\\ &+|\xi_2-\xi_1|\Big\|\Big(\frac{h_r'+\epsilon h_1'}{h_r+\epsilon h_1}-\frac{h_r'}{h_r}-\epsilon(\frac{h_1}{h_r})'\Big)\frac{h_r}{h_r'}\Big\|_\infty.\end{aligned} \tag{1.69}$$

On montre aisément que

$$\Big|\int_{\xi_1}^{\xi} f_1(u)\phi(u)du\Big| \le C\epsilon(1+|\log\epsilon|)^2. \tag{1.70}$$

Enfin pour la constante C_2, on a la majoration

$$|C_2|\le C\epsilon(1+|\log\epsilon|)^2. \tag{1.71}$$

Les calculs pour l'estimation de la norme C^1 sont de la même nature et pour ϵ suffisamment petit, on obtient le résultat

$$\mathfrak{T}^\epsilon(\mathrm{B}_{\|\cdot\|_{C^1}}(h_{1p},1))\subset \mathrm{B}_{\|\cdot\|_{C^1}}(h_{1p},1)$$

Deuxième étape : $\mathfrak{T}^\epsilon$ est un opérateur contractant.
Soit h_1 et h_2 dans $\mathrm{B}_{\|.\|_{C^1}}(h_{1p}, 1)$, on a

$$\begin{aligned}(\mathfrak{F}^\epsilon(h_2) - \mathfrak{F}^\epsilon(h_1))(\xi) &= h_r'(\xi)\phi(\xi)\int_{\xi_1}^{\xi}(f_1 - f_2)(u)du - \\ &\quad -h_r'(\xi)\int_{\xi_1}^{\xi}(f_1 - f_2)(u)\phi(u)du + \\ &\quad + h_r'(\xi)\phi(\xi)(C_2(h_2) - C_2(h_1)). \end{aligned}\tag{1.72}$$

Donc

$$\begin{aligned}\left|\mathfrak{F}^\epsilon(h_2)(\xi) - \mathfrak{F}^\epsilon(h_1)(\xi)\right| &\leq \|h_r'\phi\|_\infty\Big|\int_{\xi_1}^{\xi}(f_1(u) - f_2(u))du\Big| + \\ &\quad +\|h_r'\|_\infty\Big|\int_{\xi_1}^{\xi}(f_1(u) - f_2(u))\phi(u)du\Big| + \\ &\quad + |C_2(h_2) - C_2(h_1))(\xi)|\|h_r'\|_\infty\|\phi\|_\infty \end{aligned}\tag{1.73}$$

On a

$$\begin{aligned}\Big|\int_{\xi_1}^{\xi}(f_2(u) - f_1(u))du\Big| &\leq \frac{2}{\epsilon}\Big\|\frac{K}{h_r}\big(\frac{1}{1+\epsilon\frac{h_1}{h_r}} - \frac{1}{1+\epsilon\frac{h_2}{h_r}} + \epsilon\frac{h_2 - h_1}{h_r}\big)\Big\|_\infty + \\ &\quad + 2\epsilon\Big\|\frac{G}{2K}(h_1^2 - h_2^2)\Big\|_\infty + \\ &\quad +\epsilon\frac{GS}{K}\|h_1 - h_2\|_\infty|\xi_2 - \xi_1| + \\ &\quad + \frac{2}{\epsilon}\Big\|\frac{h_r' + \epsilon h_1'}{h_r + \epsilon h_1} - \frac{h_r' + \epsilon h_2'}{h_r + \epsilon h_2} + \epsilon\big(\frac{h_2 - h_1}{h_r}\big)'\Big\|_\infty + \\ &\quad + C\Big\|\frac{K}{h_r + \epsilon h_1} - \frac{K}{h_r + \epsilon h_2}\Big\|_\infty|\xi_2 - \xi_1| \\ &= A_1 + A_2 + A_3 + A_4 + A_5 \end{aligned}\tag{1.74}$$

On majore d'abord A_1

$$\begin{aligned}A_1 &\leq \frac{2}{\epsilon}\frac{K}{\min|h_r|}\big(\epsilon^2\frac{\|h_1 - h_2\|_\infty}{\min|h_r|} + O(\epsilon^2\|h_2 - h_1\|_\infty)\big) \\ &\leq C\epsilon\|h_2 - h_1\|_\infty \end{aligned}\tag{1.75}$$

Il est clair que $A_2 \leq C\epsilon\|h_2 - h_1\|_\infty$ et $A_3 \leq C\epsilon|\log(\epsilon)|\|h_2 - h_1\|_\infty$.
On majore ensuite A_4

$$\begin{aligned}A_4 &\leq \frac{2}{\epsilon}\Big\|\epsilon^2\frac{h_r'}{h_r}(h_1^2 - h_2^2) + \epsilon^2\frac{h_1h_1' - h_2h_2'}{h_r^2} + O(\epsilon^2\|h_1 - h_2\|_{C^1})\Big\|_\infty \\ &\leq C\epsilon\|h_1 - h_2\|_{C^1} \end{aligned}\tag{1.76}$$

De même $A_5 \leq C\epsilon|\log(\epsilon)|\|h_1 - h_2\|_{C^1}$.
On fait le même type d'estimation pour

$$\Big|\int_{\xi_1}^{\xi}(f_2(u) - f_1(u))\phi(u)du\Big| \leq C(1 + |\log(\epsilon|)\|h_1 - h_2\|_\infty.$$

Enfin l'estimation sur la constante C_2 donne

$$|C_2(h_1) - C_2(h_2)| < C\epsilon(1 + |\log\epsilon|)^2\|h_1 - h_2\|_\infty. \tag{1.77}$$

En réunissant ces inégalités, on obtient

$$\big\|\mathfrak{F}^\epsilon(h_2) - \mathfrak{F}^\epsilon(h_1)\big\|_\infty \leq C\epsilon(1 + |\log(\epsilon)|)^2\|h_1 - h_2\|_{C^1}. \tag{1.78}$$

Considérant les dérivées, on a des termes du même type à majorer et on prouve

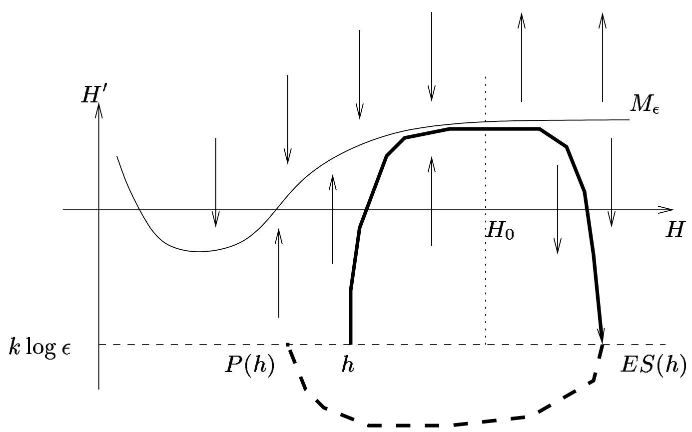
$$\big\|\mathfrak{F}^\epsilon(h_2)' - \mathfrak{F}^\epsilon(h_1)'\big\|_\infty \leq C\epsilon(1 + |\log(\epsilon)|)^2\|h_1 - h_2\|_{C^1}. \tag{1.79}$$

Pour ϵ suffisamment petit, $\mathfrak{F}^\epsilon$ est contractant. Donc $\mathfrak{T}^\epsilon$ est contractant. Il admet un unique point fixe dans $\mathrm{B}_{\|.\|_{C^1}}(h_{1p}, 1)$ et la proposition est démontrée. □

Soit $h^+ < h^-$ vérifiant les conditions de Rankine-Hugoniot

$$\frac{G}{2K}(h^+)^2 + \frac{K}{h^+} = \frac{G}{2K}(h^-)^2 + \frac{K}{h^-}.$$

Partant du point $(h^-, k\log\epsilon)$, la proposition donne l'existence d'une fonction h telle que $h(k|\log\epsilon|) = h^+$. On peut définir alors la fonction RH associée au système rapide sur la section $\Sigma = \{(h, k\log(\epsilon),\ h > 0\}$ par $RH(h^-) = h^+$. On définit l'application de Poincaré $\mathcal{P}$ sur la section Σ par $\mathcal{P} = RH \circ ES$ (voir le graphe).



1.3.3 Existence d'orbites périodiques

On veut prouver l'existence de roll-waves visqueuses proches de celles construites par Dressler. On se fixe une roll-wave de Dressler de hauteur maximum h_- et de hauteur minimum h_+. On applique la proposition 1 et on perturbe le système de Saint Venant de la manière évoquée dans le phénomène d'éjection de la variété invariante (par exemple en changeant K par $K + \epsilon K^{\pm}$) de telle sorte qu'on ait une variété M_ϵ^+ qui passe par le point $(h^+, 0)$ ou une variété M_ϵ^- qui passe par le point $(h_-, 0)$. Les théorèmes de fibration de Fenichel restent valables tant que les variétés $M_\epsilon^{\pm}$ restent proches de M_0. La fonction de Poincaré est donc bien définie (voir section précédente). Supposons que $\mathcal{P}(h) < h$. Dans ce cas, le point critique $(H_0, 0)$ est instable. On fait une perturbation de la variété lente M_ϵ^+ pour obtenir la situation suivante.

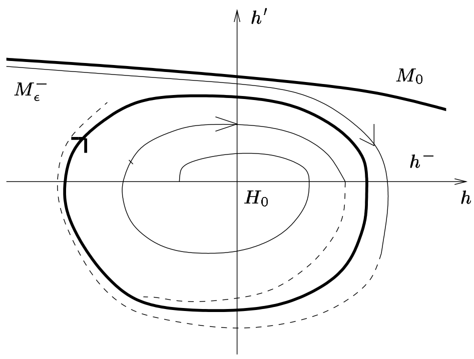


FIG. 1.9 – Existence de roll-waves visqueuses si $P(h) < h$.

Notons $h_+ = RH(h_-)$, soit $h_0 > h_+$. On considère la suite $h_n = \mathcal{P}^n(h_0)$. Tant que le théorème de fibration de Fenichel est valable et pour $\mathcal{P}(h) < h$, la suite $(h_n)_{n\in\mathbb{N}}$ est décroissante et minorée par h_+. Cette suite converge donc vers un point h_+^ϵ point fixe de $\mathcal{P}$ tel que $h_+ < h_+^\epsilon < h_0$. Comme l'éjection de la variété lente M_ϵ^- vers $(h_-, 0)$ est très rapide pour $\epsilon \to 0$, on voit que l'application de Poincaré est bien définie sur des intervalles $]h_+, h_- - \delta[$ où $\delta > 0$ est arbitrairement petit. On peut donc choisir un point $h_0 = ES^{-1}(h_- - 2\delta)$ arbitrairement proche de h_+. Donc on a une famille de roll-waves continues paramétrées par ϵ qui converge vers la roll-wave de Dressler. De plus en appliquant le théorème de Poincaré Bendixon, en suivant la trajectoire correspondant à la variété M_ϵ^-, pour $t \to +\infty$, on voit qu'on va s'enrouler autour de la roll-wave continue.

Dans le cas où $\mathcal{P}(h) > h$, on perturbe la variété lente de telle sorte que la variété lente M_ϵ^+ passe par le point $(h_+, 0)$.

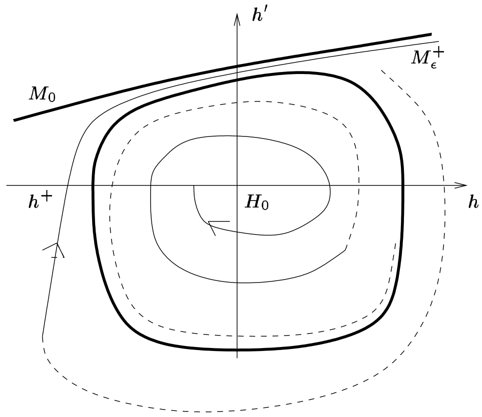


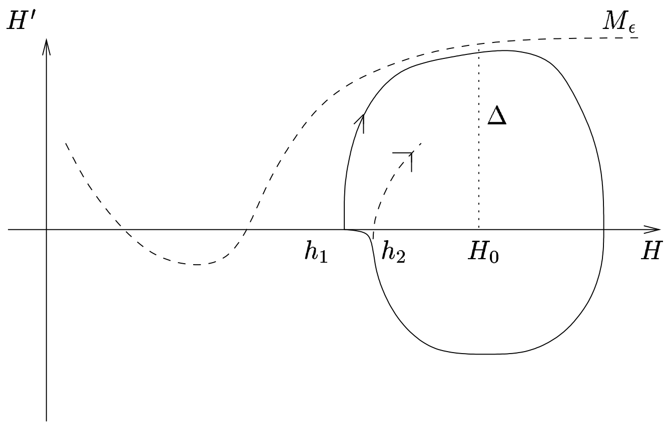
FIG. 1.10 – Existence de roll-wave visqueuse si $\mathcal{P}(h) > h$.

Dans ce cas ci, on considère la suite $(h_n = \mathcal{P}^{-n}(h_0),\ \ n \in \mathbb{N})$ et $h_0 > h^+$ arbitrairement proche de h_+ (pour ϵ suffisamment petit). En suivant la variété M_ϵ^- pour $t \to -\infty$, on va converger vers une orbite continue correspondant à une roll-wave.

Reamrque : Dans un article antérieur, l'auteur concluait plutôt , au moyen de simulations numériques, à la non-existence de roll-waves continues proches des roll-waves de Dressler. Ce n'est pas contradictoire car, on montrait essentiellement que l'application de Poincaré était strictement monotone et ne possédait pas de point fixe lorsqu'on considérait un système uniquement perturbé par une petite viscosité (on gardait une vitesse fixe du type $c = c_0$ à la place de $c = c_0 + \epsilon c_1 + O(\epsilon^2)$) et dans le cas où la variété lente restait proche de M_0. En perturbant cette variété lente (c'est à dire en considérant des vitesses $c = c_0 + O(\epsilon)$, on arrive à construire des bornes aux suites d'itérées et ces suites convergent vers des points fixes correspondant à des roll-waves continues.

1.3.4 Elimination des "autres" solutions entropiques

On considère d'abord les solutions périodiques superposition de deux solutions de périodes $L_1 > L_2$. Supposons l'existence d'une solution continue proche de la solution entropique. Dans ce cas, le plan de phase a l'allure suivante



Les hauteurs h_1 et h_2 correspondent respectivement aux hauteurs minimales de la roll-wave de période L_1 et de la roll-wave de période L_2. En appliquant le lemme de Jordan et l'unicité pour le problème de Cauchy, la région Δ est invariante et on ne peut pas revenir au point initial $(h_1, 0)$. On n'a donc pas de solutions périodiques continues proche de la solution entropique. On raisonne de manière identique pour les solutions entropiques périodiques ayant des points d'accumulations de discontinuités. Les roll-waves de Dressler sont les seules limites visqueuses possibles dans la classe des ondes progressives périodiques.

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Chapitre 2

Etude de la stabilité linéaire des Roll-Waves

Dans ce chapitre, on étudie la stabilité linéaire des roll-waves apparaissant dans les écoulements de faible profondeur. Les roll-waves sont des ondes progressives discontinues solutions entropiques des équations de Saint Venant. Ce système décrit un écoulement de faible profondeur sur une pente faible. L'existence de telles solutions a été démontré par Dressler.
Le but de ce chapitre est d'abord de formuler de manière rigoureuse le problème spectral associé à la stabilité linéaire des roll-waves de Dressler. On prouve alors analytiquement que le problème spectral obtenu ne possède pas de valeurs propres instables de grand module lorsque la longueur d'onde des roll-waves considérées deviennent grandes. On s'intéresse également à deux cas particuliers où les calculs sont explicites. D'abord on montre que les roll-waves solutions de l'équation de Burgers (décrivant de manière satisfaisante les roll-waves de Dressler lorsque l'amplitude tend vers 0) sont linéairement instables. On démontre ensuite que dans la limite d'une pente nulle, les roll-waves construites par Dressler sont linéairement stables.

On the linear stability of Roll-Waves

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Abstract

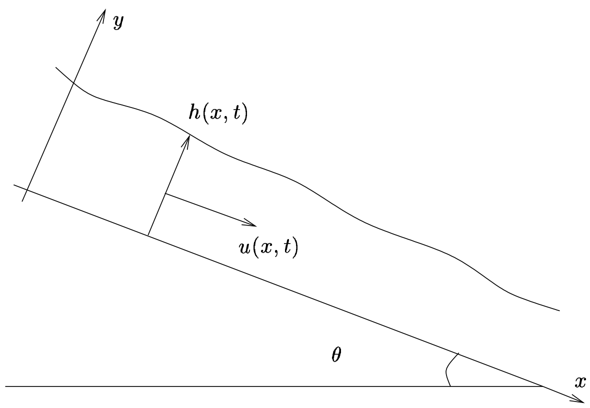
In this paper, we study the linear stability of "roll-waves" occurring in shallow water flows. Roll-waves are periodic, discontinuous travelling waves solutions of the Saint Venant equations. This system describes a shallow water flow downstream to a dam with a small slope. The existence of such profiles is well established due to a classical work of Dressler.
The purpose of this article is to give a good formulation of the spectral problem related to the linear stability of these roll-waves in a suitable space of functions. Then it is proved analytically that there are no unstable eigenvalues of large modulus when the wavelength is large. We carry out a complete study for the stability analysis of roll-waves in Burgers equations and for the stability of Dressler roll-waves when the slope of the channel goes to zero in Saint Venant equations. When the slope is sufficiently small, Dressler roll-waves are proved to be linearly stable.

2.1 Introduction

Roll-waves are well known hydrodynamic instabilities appearing in shallow water flows. It is in general of practical importance to know their characteristics (wavelength, depth,...) in order to conceive suitable protection devices (for example in the case of a river downstream to a dam). See Stoker [21].
One mathematical model for this flow down an open inclined channel is given by the Saint Venant equations augmented with a Chezy friction term

$$\begin{aligned}
&h_t + (hu)_x = 0, \\
&(hu)_t + (hu^2 + g\cos\theta \tfrac{h^2}{2})_x = gh\sin\theta - C_f u^2,
\end{aligned} \tag{2.1}$$

where θ represents the inclination of the channel and C_f its bottom resistance. In order to simplify the notations, let us define $S = \tan(\theta)$ and $G = g\cos(\theta)$.



Travelling waves $(h(x,t), u(x,t)) = \big(H(x-ct), U(x-ct)\big)$ solutions of (2.1) with wave speed c satisfy the system

$$\begin{aligned} &\Big(HU(U-c) + G\tfrac{H^2}{2}\Big)' = GHS - C_f U^2, \\ &\Big(H(c-U)\Big)' = 0. \end{aligned} \tag{2.2}$$

System (2.2) has no continuous periodic solutions. The idea is to search for periodic discontinuous waves with entropic shocks. The solution must satisfy the Rankine-Hugoniot jump conditions

$$\begin{aligned} &H_+(c-U)_+ = H_-(c-U)_- \equiv K, \\ &K[U]_-^+ = G[\tfrac{H^2}{2}]_-^+, \end{aligned} \tag{2.3}$$

where $+$ sign denotes the value of the functions after the shocks and $-$ sign for the functions values before the shocks.
Dressler [5] has proved the existence of such periodic travelling waves (with arbitrary wavelength and wave speed) by fitting together special continuous solutions of the system (2.2) when $F = \frac{\sin(\theta)}{C_f} > 4$. The condition $F > 4$ is also a condition for the uniform flow (H_0, U_0) (satisfying $GH_0 = C_f H_0^2$) to be unstable [5]. See Mathieu [13] for results on the control of the Saint Venant equations and the existence aspects.
Other models are available to describe the roll-wave phenomenon. Novik [17] proposed a model equation (the viscous Burgers equation) to obtain continuous solutions of roll-wave type. Following Whitham [26], Needham and Merkin [16] considered a Saint Venant model augmented with a viscous term and proved by a bifurcation analysis that there exist small amplitude and continuous roll-waves. See also Wang and Chang [25] for a complete bifurcation analysis of the system. Maciel and Vila [11] studied the situation for

a non Newtonian fluid and proved the existence of roll-waves for Bingham fluids with the Dressler method. Kevorkian [27] and Kranenburg [7] carried out asymptotic and numerical studies for the evolution of small periodic perturbations of the constant state for $0 < F - 4 << 1$. They used multiple time scale expansions to derive the dominant evolution equations that govern the solution behavior for long times. They give asymptotic and numerical results for periodic and isolated perturbations. They found in particular that an initial *sinus* perturbation converges numerically to the Dressler roll-wave with the same spatial period. Moreover for a perturbation which is a superposition of two periodic perturbations, the solution converges to the Dressler roll-wave with the largest wavelength. These observations are coherent with experimentations led in artificial channels [14]. See Vila [24], Mathieu [13] and Jin [8] for precise numerical schemes for the computation of roll-waves. There are not so much mathematical studies devoted to the roll-waves stability. Tamada and Tougou [22] carried out a first study on roll-waves linear stability for thin laminar flow. They found a dispersion relation law which implied that roll-waves were linearly stable for large wavelength. But their approach was not rigorous from a mathematical point of view: in fact they didn't take into account the shock movement effects in their linearization which leads to another spectral problem. On the other hand, discontinuous periodic traveling waves are proved to be unstable for hyperbolic scalar balance laws [20],[9], [10]. The purpose of the paper is to examine the linear stability of Dressler roll-waves when their wavelength is large. The starting point of our study is inspired by Majda's analysis of shock fronts linear stability in hyperbolic systems [12]. The context is the following. We study the hyperbolic system

$$u_t + (f(u))_x = 0, \tag{2.4}$$

and the stability of a shock wave $\overline{u}$ defined by

$$\begin{aligned} \overline{u}(x,t) &= u_r \text{ if } x > ct, \\ \overline{u}(x,t) &= u_l \text{ if } x < ct, \end{aligned}$$

where (u_r, u_l) satisfy the Rankine-Hugoniot conditions $f(u_r) - f(u_l) = c(u_r - u_l)$. We are looking for a solution $u(x,t)$ close to $\overline{u}$. The perturbation u must be regular on $\mathbb{R} - \{X(t)\}$ with $X(t)$ close to ct (see figure 2.1).
For $x \geq 0$, we introduce

$$\begin{aligned} u^+(x,t) &= u(x + X(t), t), \\ u^-(x,t) &= u(X(t) - x, t). \end{aligned}$$

The perturbation u is solution of (2.4) iff

$$\begin{aligned} &\partial_t u^\pm + \partial_t X \partial_x u^\pm \pm df(u^\pm)\partial_x u^\pm = 0 \text{ for } x \neq 0, \\ &f(u^+) - f(u^-) = \partial_t X(u^+ - u^-) \text{ for } x = 0. \end{aligned}$$

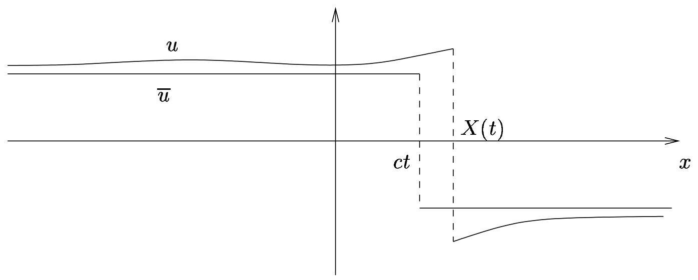


Figure 2.1: Shape of the perturbation u

We focus on the linearization

$$L\begin{pmatrix} u^+ \\ u^- \\ X \end{pmatrix} = \begin{pmatrix} \partial_t u^+ + (df(u_r)+c)\partial_x u^+ \\ \partial_t u^- - (df(u_l)-c)\partial_x u^- \end{pmatrix},$$

and

$$B\begin{pmatrix} u^+ \\ u^- \\ X \end{pmatrix} = \partial_t X(u_r - u_l) + (df(u_l)-c)u^- - (df(u_r)-c)u^+.$$

Then we write a Lopatinskii condition for the linear operator (L, B) and define the stability of a shock wave with this condition. See [12], [19] and the bibliography therein. This framework is also useful in hyperbolic balance laws for the linear stability analysis of detonation waves [4] and the stability of phase transitions in van der Waals fluids [2], [3]. In the case of roll-wave linear stability, we will work in a space of piece wise regular functions with a discrete (but infinite) distribution of moving shocks close to the shocks of a particular roll-wave with wavelength L and wave speed c. In our context, due to the presence of infinitely many shocks, a new ingredient is required. Loosely speaking, the idea is to fix all the shocks by means of a Lipschitz change of coordinates, and to linearize the equations in this new reference frame. More precisely, after the change of coordinates, the roll-wave becomes a steady state. Then, we proceed as explained in Serre [19] and linearize both the differential system and the Rankine-Hugoniot jump conditions. The whole procedure is explained in section 2.2. This leads to the spectral problem associated to Dressler roll-wave with a wavelength L.

Find $\lambda \in \mathbb{C}$, (h, v) *a piece wise* C^1 *function with discontinuities at points* $\{iL, i \in \mathbb{Z}\}$, *a bounded sequence* $\epsilon_i \in \mathbb{R}$ *such that* $\lim_{\xi\to\pm\infty} \|(h, v)(\xi)\| = 0$

and (h, v, ϵ_i) satisfy the differential system

$$
\begin{array}{l}
(v-ch)' + \lambda h = \lambda\big(\epsilon_i + \frac{\xi - iL}{L}(\epsilon_{i+1}-\epsilon_i)\big)H' \\
\big((GH - \frac{V^2}{H^2})h \;\; + \;\; (\frac{2V}{H} - c)v\big)' + \\
\qquad + \;\; \lambda v = (GS + \frac{2C_f V^2}{H^3})h - \frac{2C_f V}{H^2}v + \\
\qquad\qquad + \frac{\epsilon_{i+1}-\epsilon_i}{L}\big(\frac{GH^2}{2} + \frac{K^2}{H}\big)' + \\
\qquad\qquad + \big(\epsilon_i + \frac{\xi - iL}{L}(\epsilon_{i+1}-\epsilon_i)\big)\lambda V'
\end{array}
\tag{2.5}
$$

when $\xi \in]iL, (i+1)L[, i \in \mathbb{Z}$ and the shock values at points $\{iL, i \in \mathbb{Z}\}$ satisfy the linearized Rankine-Hugoniot jump conditions

$$
\begin{array}{l}
[v - ch]_{iL} = \epsilon_i \lambda \, [H]_{iL} \\
\Big[(GH - \frac{V^2}{H^2})h + (\frac{2V}{H} - c)v\Big]_{iL} = \epsilon_i \lambda c \, [H]_{iL}
\end{array}
\tag{2.6}
$$

The functions $(H, V = HU)$ are the continuous profiles of roll-waves computed by Dressler on the intervals $]iL, (i+1)L[$.
Moreover the differential system (2.5) possesses a singularity. This type of problem does not belong to the standard framework of Evans function for the stability analysis of traveling waves (see [1], [18] and bibliography therein for more details on Evans function). We cannot construct a function of λ vanishing when (h, v), $(\epsilon_n)_{n\in\mathbb{Z}}$ is a solution of (2.5), (2.6). But we cannot also integrate this system explicitly. In order to simplify the analysis, we make computations for large wavelength L. We prove analytically that there are no unstable eigenvalues of large modulus in the space of functions introduced before which moreover decrease to 0 at infinity. To obtain a contradiction, suppose there exist unstable eigenvalues of arbitrary large modulus. This assumption, together with the large wavelength assumption, provides an approximation of the resolvent matrix when the system is non singular (WKB approximation). We can compute an expansion of the possible eigenfunctions at the discontinuity points. The incompatibility of these values with the linearized Rankine-Hugoniot conditions then leads to the contradiction. The proof is divided into two parts. We first make the computation for possible unstable eigenvalues with large real parts and then for the ones with large imaginary parts. We state the main result of this paper

Theorem 1 *There exist two growing functions $r_1 \geq 0$ and $r_2 \geq 0$ and L_0 such that for any (λ, L) satisfying $Re(\lambda) > r_1(L)$ or $L > L_0$ and $|Im(\lambda)| > r_2(L)$, λ is not an eigenvalue of the spectral problem (2.5), (2.6).*

Loosely speaking, it means that for large wavelength, there are no large unstable eigenvalues for the spectral problem (2.5),(2.6).

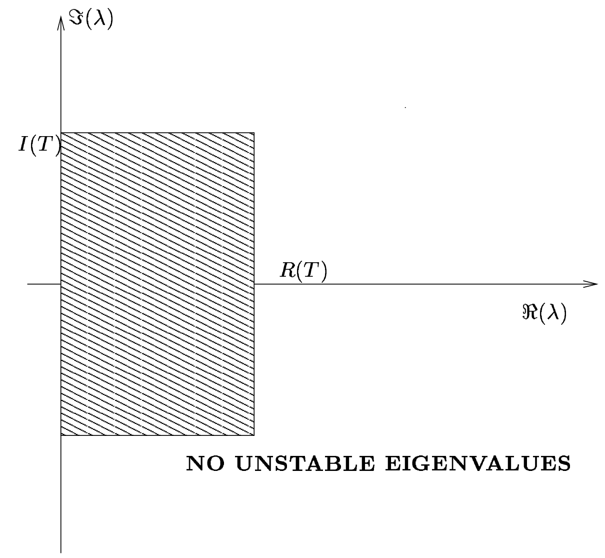


Figure 2.2: Problem (2.5),(2.6) has no large unstable eigenvalues

We focus on two cases where the computations are explicit. Firstly, we study the linear stability of roll-waves solutions of inviscid Burgers equation. Secondly, we study the linear stability of Dressler roll-waves when the slope θ goes to 0 in the Saint Venant equations. We can prove the result.

Theorem 2 *As $\theta \to 0$, there are no $O(1)$ unstable eigenvalues and no $O(\theta)$ unstable eigenvalues. There exists $\theta_0 > 0$ such that for $0 < \theta < \theta_0$, the roll-waves are linearly stable.*

The paper is organized as follows. In the next section, we recall Dressler construction of roll-waves and give their asymptotic shape as L, their wavelength, goes to infinity. Then we consider small perturbations of a roll-wave: we assume that they are piecewise C^1 functions with an infinite distribution of shocks close to those of the roll-wave considered. After a change of coordinates which fix these shocks, we obtain a new system and the roll-wave becomes a steady state. We linearize the equations and write the associated spectral problem. The section 2.3 is devoted to the analysis of a simpler problem: the linear stability of roll-waves solutions of Burgers equation. In this case, the computations are explicit and we prove that roll-waves are linearly unstable. The sections 2.4 and 2.5 are devoted to the proof of theorem 1. First, we consider possible unstable eigenvalues with a large real part and then we study possible unstable eigenvalues with large imaginary part. In both cases, the scheme of the proof is the same. We compute an expansion of a simple basis of solutions to the homogeneous problem in the areas where it is non singular. This yields an approximation of the resolvent matrix. Then, with the Duhamel formula, we calculate the shock values of the possible eigenfunctions. Finally we obtain an asymptotic development of the shock values which are proved to be incompatible with the linearized Rankine-Hugoniot conditions. The last section is dedicated to the asymptotic analysis $\theta \to 0$ and the proof of theorem 2. We restrict our attention to the space of functions with an exponential decrease (i.e. there exists K and $\alpha > 0$ such that $\max(|h(\xi)|, |v(\xi)|) \leq K \exp(-\alpha|\xi|)$. We first consider $O(S)$ unstable eigenvalues. After a rescaling $\lambda = \overline{\lambda} S$ and $\overline{\lambda}$ lying in a compact set, we have a new spectral problem involving $\overline{\lambda}$ which is simpler: we can prove that there are no unstable eigenvalues for S sufficiently small. Then, we consider $O(1)$ unstable eigenvalues. Applying theorem 1, we can prove that for S sufficiently small, there are no $O(1)$ unstable eigenvalues.

2.2 Formulation of the spectral problem

In this section, we recall Dressler computations of roll-waves (periodic and entropic travelling waves solutions of the Saint Venant equations) and present the asymptotic shape of the solution as the wavelength L goes to infinity. Then, towards this example, we present a general method to study the linear stability of discontinuous solutions of hyperbolic systems with an arbitrary (even infinite) numbers of shocks. It generalizes the approach proposed by Majda et al. [12],[4].

2.2.1 Dressler analysis

In [5], Dressler looks for periodic travelling waves solutions of the inviscid Saint Venant equations. The problem is to find periodic solutions of the equation

$$H' = -\frac{H(GS - C_f \frac{(cH-K)^2}{H^3})}{(\frac{K^2}{H^2}) - GH} = P(H). \tag{2.7}$$

Integrating (2.7) under the form $\xi = \xi_0 + \int \frac{dh}{P(h)}$, it is proved that (2.7) has no continuous periodic solution [5]. Thus, we are looking for periodic solutions with discontinuities which satisfy entropic conditions. We have the following theorem.

Theorem 3 *Given any* $F = \frac{\sin(\theta)}{C_f} > 4$ *and* (L, c) *there exists a piecewise* C^1 *periodic travelling wave with wave speed* c *and wavelength* L*, entropic solution of the inviscid Saint Venant equations (2.1).*

We recall the proof for completeness.

Proof. Let H_- be the height to the left of a shock and H_+ the height to the right of this shock. The entropic shock must satisfy the Rankine-Hugoniot jump conditions

$$\frac{G}{2K}H_+^2 + \frac{K}{H_+} = \frac{G}{2K}H_-^2 + \frac{K}{H_-} \tag{2.8}$$

The Saint Venant equations are similar to the isentropic Euler equations for a ideal gas with $\gamma = 2$, where h plays the role of the density ρ. The entropy condition is equivalent to the Lax shock condition (see [24] for more details). We find that

$$U_+ + \sqrt{GH_+} < c < U_- + \sqrt{GH_-}.$$

Since $H(c - U) = K$, we have

$$\sqrt{GH_+} - \frac{K}{H_+} < 0 < \sqrt{GH_-} - \frac{K}{H_-} \tag{2.9}$$

Hence $H_+ < H_-$ and we must have a point H_0 such that $H_+ < H_0 < H_-$ and

$$\frac{G}{K}H_0 - \frac{K}{H_0^2} = 0. \tag{2.10}$$

Moreover we must have $\frac{dH}{d\xi}|_{H_0} > 0$ or equivalently $F = \frac{\sin(\theta)}{C_f} > 4$. The denominator of the fraction in (2.7) vanishes at point H_0. To pass continuously through this value, the numerator must also vanishes at this point H_0

$$GS - C_f\frac{(cH_0 - K)^2}{H_0^3} = 0. \tag{2.11}$$

Hence equation (2.7) now reads

$$\frac{dH}{d\xi} = S\frac{H^2 + (H_0 - \frac{c^2}{GF})H + \frac{H_0^2}{F}}{H^2 + H_0H + H_0^2} = P_1(H). \tag{2.12}$$

The construction of a periodic solution with an arbitrary wavelength L is at this point straightforward. Let $H(\xi)$ be the special solution with $H(0) = H_0$. It is defined implicitly by

$$H(\xi) = h \Leftrightarrow \xi = \frac{f_1(h)}{S}, \tag{2.13}$$

where f_1 is given by

$$\begin{aligned} f_1(h) &= h - H_0 + \frac{H_a^2 + H_0 H_a + H_0^2}{H_a - H_b} \ln(\frac{h - H_a}{H_0 - H_a}) - \\ &- \frac{H_b^2 + H_0 H_b + H_0^2}{H_a - H_b} \ln(\frac{h - H_b}{H_0 - H_b}) \end{aligned} \tag{2.14}$$

and $H_a > H_b$ are the zeros of P_1. The implicit function ξ is obtained by integration of (2.12). Define $H_n(\xi) = H(\xi - nT)$. We fit H_n and H_{n+1} together by means of an entropic shock. Eliminating the solution $H_+ = H_-$, the Rankine-Hugoniot condition (2.8) now reads

$$H_+ + H_- = \frac{2H_0^3}{H_+ H_-}. \tag{2.15}$$

We determine the position of the n-th shock ξ_c^n with $H_n(\xi_c^n) = H_-$ and $H_{n+1}(\xi_c^n) = H_+$. This is equivalent to the relation

$$\int_{H_0}^{H^-} \frac{dh}{P_1(h)} = L + \int_{H_0}^{H_+} \frac{dh}{P_1(h)}. \tag{2.16}$$

Now, eliminating H_+ between (2.15) and (2.16) yields L as a function of H_-. Then we have obtained periodic and entropic solutions of the Saint Venant equations. □

It is easy to derive the asymptotic behavior of this family of roll-waves as L goes to infinity (see figure 2.3).

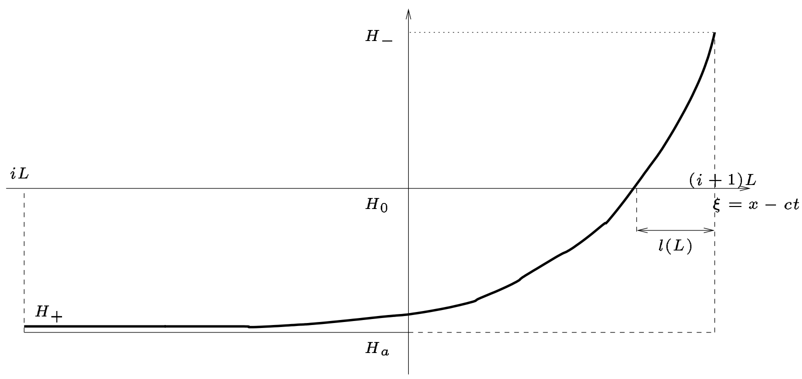


Figure 2.3: Asymptotic shape of the roll-wave

Both conditions (2.10) and (2.11) implies that the intermediate height H_0 is given by $H_0 = \frac{1}{g}(\frac{c}{1+\sqrt{F}})^2$. As $L \to \infty$, the maximum and the minimum heights of the roll-wave, H_- and H_+, have a limit: $H_+ \to H_a$ and $H_- \to H_c$. The asymptotic heights H_a and H_c have the form

$$H_a = \frac{H_0}{2F}(1 + 2\sqrt{F} + (1 + 4\sqrt{F})^{\frac{1}{2}}),$$

and H_c satisfy the Rankine Hugoniot jump condition

$$H_c + H_a = \frac{2H_0^3}{H_a H_c}.$$

Hence, as $L \to \infty$, the amplitude of Dressler roll-waves remains finite and the function l defined by $l(L) = \frac{f_1(H_-)}{S}$ has a finite limit (see the figure 2.3). We now investigate the linear stability of these roll-waves. First, we introduce a rigorous framework to study this problem.

2.2.2 Spectral problem computation

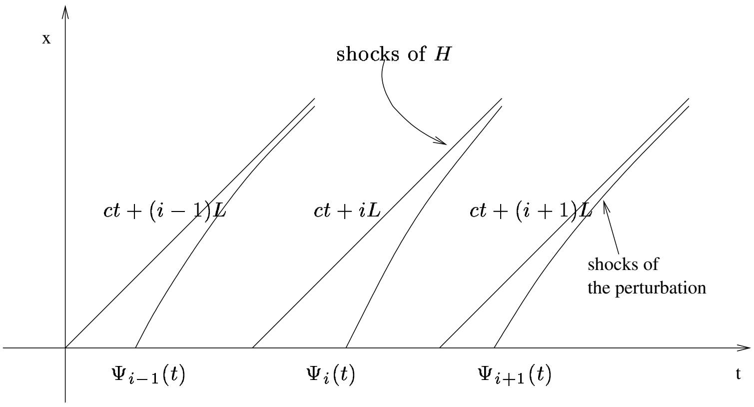
In this section, we are going to introduce a rigorous method to study the linear stability of Dressler roll-waves. These roll-waves are parametrized by L the wavelength and c the wave speed. When these parameters are fixed, we will denote by H_1 the minimum height of the roll-wave (which is also the height after a shock) and by H_2 the maximum height of the roll-wave (which is also the height before the shock). We are going to work with the variables $(h, v = hu)$. The Saint Venant equations then take the form

$$\begin{aligned} &h_t + v_x = 0, \\ &v_t + (\tfrac{Gh^2}{2} + \tfrac{v^2}{h})_x = GhS - C_f \tfrac{v^2}{h^2}. \end{aligned} \tag{2.17}$$

The derivation of the spectral problem is inspired by a paper of Majda et al. [4]. Let us fix a roll-wave $(H, V = HU)$ with wave speed c and wavelength L: the position of the shocks are given by $\phi_i(t) = ct + iL$. Now we consider a small perturbation of this solution: we can suppose it is a piecewise C^1 function with discontinuities at points $\Psi_i(t) = ct + iL + \epsilon_i(t)$. The ϵ_i are supposed to be small and we have the following distribution of shocks
Due to the Rankine Hugoniot conditions,

$$\begin{aligned} &[v]_{\Psi_i} = \dot{\Psi}_i \, [h]_{\Psi_i}, \\ &\left[\tfrac{Gh^2}{2} + \tfrac{v^2}{h}\right]_{\Psi_i} = \dot{\Psi}_i \, [v]_{\Psi_i}. \end{aligned} \tag{2.18}$$

In [4], changing the moving frame of the shock made the detonation wave steady. In our case, because of the existence of an infinite distribution of shocks, we have to fix all the discontinuities in order to make the roll-wave



steady. We fix the shocks by working in the new system of coordinates $(\xi = \xi(x,t), t)$ such that $\xi(\Psi_i(t), t) = iL$. The function ξ has the form

$$\forall x \in]\Psi_i(t), \Psi_{i+1}(t)[,\ \xi(x,t) = \frac{x - \Psi_i(t)}{\Psi_{i+1}(t) - \Psi_i(t)} L + iL \tag{2.19}$$

We can see on figure (2.4) the shape of the function

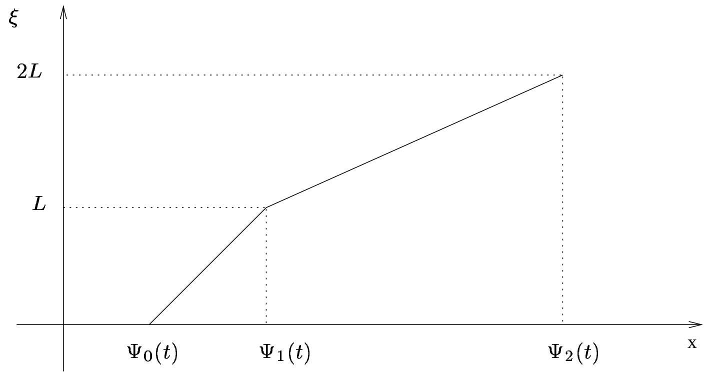


Figure 2.4: The function ξ

Remark: We can see that considering $|\epsilon_i| << 1$, we have $\xi(x,t) \approx x - ct$ at the first order: it looks like a change of reference frame.

The function ξ is Lipschitz and the system is of order one so this change of coordinates is licit and does not change the Rankine-Hugoniot jump conditions. We make the change of coordinates $h(x,t) = \overline{h}(\xi(x,t), t)$, where $\overline{h}$ is a piecewise C^1 function with discontinuities at points $\{iL, i \in \mathbb{Z}\}$. The derivation rules are given by

$$\begin{aligned} \frac{\partial h}{\partial x} &= \frac{\partial \overline{h}}{\partial \xi}\frac{L}{\Psi_{i+1}-\Psi_i}, \\ \frac{\partial h}{\partial t} &= \frac{\partial \overline{h}}{\partial t} - \frac{L}{\Psi_{i+1}-\Psi_i}\left(\dot{\Psi}_i + \frac{\xi - iL}{L}(\dot{\Psi}_{i+1}-\dot{\Psi}_i)\right)\frac{\partial \overline{h}}{\partial \xi}. \end{aligned} \tag{2.20}$$

Then we replace (2.20) into (2.17) and dropping the overlines into the equations, the Saint Venant system now reads

$$\begin{aligned} & h_t + \frac{L}{\Psi_{i+1}-\Psi_i}\Big(v_\xi - \big(\dot{\Psi}_i + \frac{\xi - iL}{L}(\dot{\Psi}_{i+1}-\dot{\Psi}_i)\big)h_\xi\Big) = 0, \\ & v_t + \frac{L}{\Psi_{i+1}-\Psi_i}\Big(\frac{Gh^2}{2}+\frac{v^2}{h}\Big)_\xi - \\ & -\frac{L}{\Psi_{i+1}-\Psi_i}\big(\dot{\Psi}_i + \frac{\xi - iL}{L}(\dot{\Psi}_{i+1}-\dot{\Psi}_i)\big)v_\xi = GhS - C_f\frac{v^2}{h^2}. \end{aligned} \tag{2.21}$$

At the discontinuity points iL, the jump conditions remain the same:

$$\begin{aligned} & [v]_{iL} = \dot{\Psi}_i\,[h]_{iL}\,, \\ & \left[\tfrac{Gh^2}{2}+\tfrac{v^2}{h}\right]_{iL} = \dot{\Psi}_i\,[v]_{iL}\,. \end{aligned} \tag{2.22}$$

The particular solution $(H, V, \phi_i(t) = ct + iL)$ computed by Dressler is now a steady solution of (2.21) and (2.22). We linearize (2.21) and (2.22) around the steady state $(h, v, \Psi_i(t)) = (H, V, \phi_i(t) = ct + iL)$, where $(H, V, ct + iL)$ is a periodic and entropic travelling wave parametrized by the wavelength L. The linearization of the equations (2.21), (2.22) yields

$$\begin{aligned} h_t + (v - ch)_\xi = {} & \Big(\dot{\epsilon}_i + \frac{\xi - iL}{L}(\dot{\epsilon}_{i+1}-\dot{\epsilon}_i)\Big)H'(\xi) + \\ & + \frac{\epsilon_{i+1}-\epsilon_i}{L}\big(V'(\xi) - cH'(\xi)\big), \end{aligned}$$

$$\begin{aligned} v_t + \Big(\big(GH - \frac{V^2}{H^2}\big)h + \big(\frac{2V}{H} - c\big)v\Big)_\xi & = \\ = \Big(GS + 2C_f\frac{V^2}{H^3}\Big)h - \frac{2C_fV}{H^2}v & + \\ + \quad \frac{\epsilon_{i+1}-\epsilon_i}{L}\Big(\frac{GH^2}{2}+\frac{K^2}{H}\Big)'(\xi) & + \\ + \quad \Big(\dot{\epsilon}_i + \frac{\xi - iL}{L}(\dot{\epsilon}_{i+1}-\dot{\epsilon}_i)\Big)V'(\xi) & \end{aligned} \tag{2.23}$$

and the linearized Rankine-Hugoniot conditions are given by

$$\begin{aligned} &\Big[v - ch\Big]_{iL} = \dot{\epsilon}_i \big[H\big]_{iL}, \\ &\Big[(GH - \tfrac{V^2}{H^2})h + (\tfrac{2V}{H} - c)v\Big]_{iL} = \dot{\epsilon}_i \big[V\big]_{iL}. \end{aligned} \tag{2.24}$$

We are looking for solutions (h, v, ϵ_i) with an exponential growth in time: $h = \underline{h}\exp(\lambda t)$, $v = \underline{v}\exp(\lambda t)$ and $\epsilon_i(t) = \underline{\epsilon_i}\exp(\lambda t)$. Dropping the underlines, the equations (2.23) and (2.24) yields the spectral problem

Spectral problem

Find $\lambda \in \mathbb{C}$, (h, v) a piecewise C^1 function with discontinuities at points $\{iL, i \in \mathbb{Z}\}$, a bounded sequence $(\epsilon_i)_{i\in\mathbb{Z}} \in \mathbb{R}$ such that $\lim_{\xi\to\pm\infty} \|(h, v)(\xi)\| = 0$ *and (h, v, ϵ_i) satisfy the differential system*

$$\begin{aligned} &(v - ch)' + \lambda h = \lambda\big(\epsilon_i + \tfrac{\xi - iL}{L}(\epsilon_{i+1} - \epsilon_i)\big)H' \\ &\big((GH - \tfrac{V^2}{H^2})h \;\; + \;\; (\tfrac{2V}{H} - c)v\big)' + \\ &\qquad + \;\; \lambda v = (GS + \tfrac{2C_f V^2}{H^3})h - \tfrac{2C_f V}{H^2}v + \\ &\qquad\qquad + \tfrac{\epsilon_{i+1} - \epsilon_i}{L}\big(\tfrac{GH^2}{2} + \tfrac{K^2}{H}\big)' + \\ &\qquad\qquad + \big(\epsilon_i + \tfrac{\xi - iL}{L}(\epsilon_{i+1} - \epsilon_i)\big)\lambda V' \end{aligned} \tag{2.25}$$

when $\xi \in]iL, (i+1)L[, i \in \mathbb{Z}$ and the shock values at points $\{iL, i \in \mathbb{Z}\}$ satisfy the linearized Rankine-Hugoniot jump conditions

$$\begin{aligned} &\Big[v - ch\Big]_{iL} = \epsilon_i \lambda \big[H\big]_{iL}, \\ &\Big[(GH - \tfrac{V^2}{H^2})h + (\tfrac{2V}{H} - c)v\Big]_{iL} = \epsilon_i \lambda c\big[H\big]_{iL}. \end{aligned} \tag{2.26}$$

In order to simplify the notations, we write the differential system (2.25) under a matricial form

$$A(H(\xi))\begin{pmatrix} h' \\ v' \end{pmatrix} + (B(H(\xi)) + \lambda Id)\begin{pmatrix} h \\ v \end{pmatrix} = \lambda H'(\xi)\begin{pmatrix} \alpha_i(H(\xi), \xi) \\ \beta_i(H(\xi), \xi) \end{pmatrix} \tag{2.27}$$

The matrix A, B and functions α_i, β_i are easily obtained by identification in (2.25). In particular, we have

$$A(H) = \begin{pmatrix} -c & 1 \\ GH - \frac{(cH-K)^2}{H^2} & 2\frac{cH-K}{H} - c \end{pmatrix}. \tag{2.28}$$

The matrix $A(H)$ is diagonalizable and the reduction gives

$$A(H(\xi)) = P^{-1}DP(H_D(\xi))$$

with

$$D(H) = \mathrm{diag}(-\frac{K}{H} + \sqrt{GH}; -\frac{K}{H} - \sqrt{GH}),$$
$$D(H) = \mathrm{diag}(d_1(H), d_2(H)).$$
$$P(H) = \begin{pmatrix} \sqrt{GH} - c + \frac{K}{H} & 1 \\ -\sqrt{GH} - c + \frac{K}{H} & 1 \end{pmatrix} \tag{2.29}$$

In order to get a simpler differential system, we introduce the new variables

$$\begin{pmatrix} g \\ r \end{pmatrix} = P(H(\xi)) \begin{pmatrix} h \\ v \end{pmatrix}, \tag{2.30}$$

The differential system satisfied by (g, r) is given by

$$\begin{aligned} d_1(H(\xi))g' + (c_1(H(\xi)) + \lambda)g + c_2(H(\xi))r = \gamma_i(H(\xi)), \\ d_2(H(\xi))r' + (c_4(H(\xi)) + \lambda)r + c_3(H(\xi))g = \delta_i(H(\xi)). \end{aligned} \tag{2.31}$$

Remark: Notice that $d_1(H_0) = 0$ and the differential system (2.31) is singular at points ξ_i^0 such that $\xi_i^0 \in]iL, (i+1)L[$ and $H(\xi_i^0) = H_0$.

Let us note $u(H) = c - \frac{K}{H}$. The coefficients c_1, c_2, c_3, c_4 and γ_i, δ_i have the form

$$\begin{aligned} \gamma_i(H) = & -\lambda P_1(H)\Big(\epsilon_i + \frac{L - l(L) + \frac{f_1(H)}{S}}{L}(\epsilon_{i+1} - \epsilon_i)\Big) d_2(H) - \\ & - \frac{\epsilon_{i+1} - \epsilon_i}{L}(GH - \frac{K^2}{H^2})P_1(H), \end{aligned}$$

$$\begin{aligned} \delta_i(H) = & -\lambda P_1(H)\Big(\epsilon_i + \frac{L - l(L) + \frac{f_1(H)}{S}}{L}(\epsilon_{i+1} - \epsilon_i)\Big) d_1(H) - \\ & - \frac{\epsilon_{i+1} - \epsilon_i}{L}(GH - \frac{K^2}{H^2})P_1(H), \end{aligned}$$

$$\begin{aligned} c_1(H) = & \frac{1}{4\sqrt{GH}}\Big((4C_f \frac{u(H)\sqrt{GH}}{H} - 2GS) + \\ & + P_1(H)(\frac{K}{H^2}(5\sqrt{GH} - 2\frac{K}{H}) + G)\Big), \end{aligned}$$

$$\begin{aligned} c_2(H) = & \frac{1}{4\sqrt{GH}}\Big((4C_f \frac{u(H)\sqrt{GH}}{H} + 2GS) + \\ & + P_1(H)(\frac{K}{H^2}(\sqrt{GH} + 2\frac{K}{H}) - G)\Big), \end{aligned}$$

$$c_3(H) = \frac{1}{4\sqrt{GH}}\Big((4C_f\frac{u(H)\sqrt{GH}}{H} - 2GS) +$$
$$+P_1(H)(\frac{K}{H^2}(\sqrt{GH} - 2\frac{K}{H}) + G)\Big),$$

$$c_4(H) = \frac{1}{4\sqrt{GH}}\Big((4C_f\frac{u(H)\sqrt{GH}}{H} + 2GS) +$$
$$+P_1(H)(\frac{K}{H^2}(7\sqrt{GH} + 2\frac{K}{H}) + G)\Big). \tag{2.32}$$

Let us recall that $l(L) = \frac{f_1(H_2)}{S}$ where H_2 is the maximum height of the roll-wave, the variable ξ can be written $\xi = \frac{f_1(H(\xi))}{S}$ and

$$P_1(H) = S\frac{H^2 + (H_0 - \frac{c^2}{GF})H + \frac{H_0^2}{F}}{H^2 + H_0H + H_0^2}.$$

With these new variables, the linearized Rankine-Hugoniot conditions read

$$\Big[E(H)\begin{pmatrix} g \\ r \end{pmatrix}\Big]_{iL} = \epsilon_i\lambda\big[H\big]_{iL}\begin{pmatrix} 1 \\ c \end{pmatrix} \tag{2.33}$$

with $E(H) = A(H)P^{-1}(H)$.
The dependence of c_1, c_2, c_3, c_4 and γ_i, δ_i with respect to $h \in [H_1, H_2]$ (which are the minimum and the maximum heights of the roll-wave) on the one hand and the characteristic numbers of the flow S, F, c, L on the other hand are complex. Hence to get a simpler form of the coefficients, we consider the functions $\underline{c}_j(h) = c_j(H_0h)$, $\underline{\gamma}_i(h) = \gamma_i(H_0h)$, $\underline{\delta}_i(h) = \delta_i(H_0h)$. A lengthy but easy computation yields

$$\underline{c}_j(h) = \frac{GS(1+\sqrt{F})}{c}C_j(h),$$

$$\underline{\gamma}_i(h) = \frac{GS(1+\sqrt{F})}{c}\Gamma_i(h),\ \underline{\delta}_i(h) = \frac{gS(1+\sqrt{F})}{c}\Delta_i(h)$$

with

$$\Gamma_i(h) = -\lambda\frac{c^2}{(1+\sqrt{F})^2}p(h)D_2(h)\Big(\epsilon_i + (1 - \frac{f(h_2) - f(h)}{f(h_2) - f(h_1)})(\epsilon_{i+1} - \epsilon_i)\Big) -$$
$$-\frac{\epsilon_{i+1} - \epsilon_i}{f(h_2) - f(h_1)}\frac{Sc}{1+\sqrt{F}}p(h)(h - \frac{1}{h^2}),$$

$$\Delta_i(h) = -\lambda\frac{c^2}{(1+\sqrt{F})^2}p(h)D_1(h)\Big(\epsilon_i + (1-\frac{f(h_2)-f(h)}{f(h_2)-f(h_1)})(\epsilon_{i+1}-\epsilon_i)\Big) -$$
$$-\frac{\epsilon_{i+1}-\epsilon_i}{f(h_2)-f(h_1)}\frac{Sc}{1+\sqrt{F}}P(h)(h-\frac{1}{h^2}),$$

$$C_1(h) = \frac{1}{4\sqrt{h}}\Big(4\frac{1+\sqrt{F}}{F\sqrt{h}}(1-\frac{1}{(1+\sqrt{F})h}) - 2 +$$
$$+p(h)(1+\frac{1}{h^2}(5\sqrt{h}-\frac{2}{h}))\Big),$$

$$C_2(h) = \frac{1}{4\sqrt{h}}\Big(4\frac{1+\sqrt{F}}{F\sqrt{h}}(1-\frac{1}{(1+\sqrt{F})h}) + 2 +$$
$$+p(h)(-1+\frac{1}{h^2}(\sqrt{h}+\frac{2}{h}))\Big),$$

$$C_3(h) = \frac{1}{4\sqrt{h}}\Big(4\frac{1+\sqrt{F}}{F\sqrt{h}}(1-\frac{1}{(1+\sqrt{F})h}) - 2 +$$
$$+p(h)(1+\frac{1}{h^2}(\sqrt{h}-\frac{2}{h}))\Big),$$

$$C_4(h) = \frac{1}{4\sqrt{h}}\Big(4\frac{1+\sqrt{F}}{F\sqrt{h}}(1-\frac{1}{(1+\sqrt{F})h}) + 2 +$$
$$+p(h)(1+\frac{1}{h^2}(7\sqrt{h}+\frac{2}{h}))\Big), \tag{2.34}$$

and

$$D_1(h) = \frac{1}{G}(\sqrt{h}-\frac{1}{h}),\ D_2(h) = -\frac{1}{G}(\sqrt{h}+\frac{1}{h}), f(h) = f_1(H_0h),$$

$$p(h) = \frac{P_1(H_0h)}{S} = \frac{h^2 + (1-(1+\frac{1}{\sqrt{F}})^2)h + \frac{1}{F}}{h^2+h+1}$$

and $h_i = \frac{H_i}{H_0}$ for $i = 1, 2$.

These reduced expressions will take a great importance in the last section where we study the asymptotic $S \to 0$. Moreover, they clarify the dependence of the coefficients with respect to $H = H_0h$ and S, c, F, L the

characteristic numbers of the roll-wave considered.
There exists one solution of the differential system which is analytic on $]iL,(i+1)L[$. Hence in order to prove theorem 1, the task is two fold. On the one hand, we have to compute a good approximation of the resolvent matrix as $|\lambda| \to \infty$ and $L \to \infty$ when the system is non singular. On the other hand, we have to carry out a precise analysis of the analytical solution of (2.31) in the neighborhood of ξ_i^0. This is done in the forthcoming section starting with the case of possible unstable eigenvalues with large real part and then for large imaginary part.
Using the singularity of the system (2.31) at points ξ_i^0, we can prove the following lemma

Lemma 1 *The spectral problem (2.25), (2.26) has no solutions which decrease to 0 at infinity when $\epsilon_i = 0$.*

Proof. In this case, the differential system (2.25) reads

$$A(H(\xi))\begin{pmatrix} h' \\ v' \end{pmatrix} + (B(H(\xi)) + \lambda Id)\begin{pmatrix} h \\ v \end{pmatrix} = 0. \tag{2.35}$$

The differential system (2.35) has singularities at points $\xi_i^0 \in]iL,(i+1)L[$ such that $H(\xi_i^0) = H_0$ and it possesses a one dimensional vector space of analytic solutions. Let us note $(\mathcal{H},\mathcal{V})$ the analytic solution of (2.35) such that $\mathcal{H}(\xi_0^0) = 1$. We look for a solution (h,v) of (2.35) such that $h(\xi_0^0) = 1$. Then $(h,v)(\xi) = \alpha_i(\mathcal{H},\mathcal{V})(\xi)$ for all $\xi \in]iL,(i+1)L[$. The Rankine Hugoniot jump conditions (2.26) imply that

$$\alpha_{i+1}(\mathcal{V} - c\mathcal{H})^- = \alpha_i(\mathcal{V} - c\mathcal{H})^+$$

Then we obtain $\alpha_i = q^i$ with $q = \frac{(\mathcal{V}-c\mathcal{H})^+}{(\mathcal{V}-c\mathcal{H})^-}$. Then either $|q| \neq 1$ and the solution (h,v) blows up in $\pm\infty$, either $|q| = 1$ but then $(|h|,|v|)$ is a periodic function and does not decrease to 0 as $\xi \to \pm\infty$. Thus the lemma is proved.$\Box$
As a conclusion, we mention that we have introduced in this section a general framework in order to study the linear stability of entropic solutions with an infinite distribution of shocks solutions of hyperbolic systems with a source term. This method generalizes the approach proposed by Majda [12],[4] to study the linear stability of shock waves with a single shock. Before proving theorem 1, we apply this method to the analysis of roll-waves linear stability for Burgers equation. In this case, the computations are explicit and it is a good introduction before the proof of theorem 1.

2.3 Linear stability of roll-waves for Burgers equations

In this section, we study the linear stability of roll-waves solutions of the Burgers equation. It is known that scalar hyperbolic conservation laws with a source term

$$u_t + (f(u))_x = g(u) \tag{2.36}$$

and periodic initial data u_0 have a property of Poincaré Bendixon type namely any bounded solution converges either to a constant, either to a periodic travelling wave (see [6], [10]). This property characterizes the possible behavior of the solutions of (2.36). There remains the question of whether, given an initial value, the corresponding solution converges either to a constant or to a travelling wave. Lyberopoulos [9] and Sinestrari [20] have proved that periodic traveling waves are in general unstable. So the result presented here is not new but the Burgers equation provides a good example to apply the method introduced in the section 2.2 before dealing with the case of Dressler roll-waves. In the case of Burgers equation, the computations are explicit and the roll-waves are proved to be linearly unstable. The Burgers equation is given by

$$u_t + uu_x = u \tag{2.37}$$

It provides a good model of roll-waves when the wavelength is small. There exists a family of periodic and entropic steady solutions $U_L(x)$ parametrized by the wavelength L. We have $U_L(x) = x - \frac{L}{2}$ for $x \in]0, L[$ and U_L is L-periodic. The solution U_L satisfy the Rankine-Hugoniot conditions $U_L(iL)^+ = -U_L(iL)^-$. (see figure 2.5).

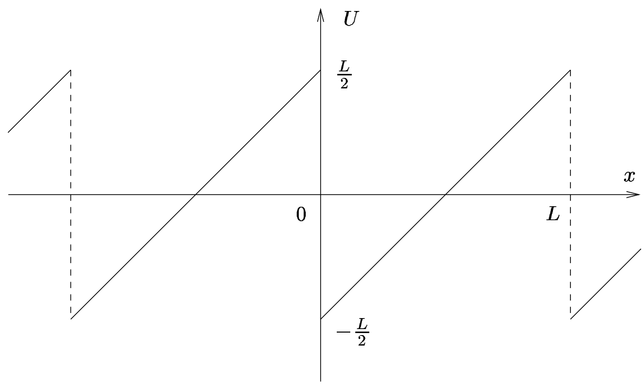


Figure 2.5: Roll-waves in the Burgers equation

Now we study their linear stability. Assume u to be a small perturbation of

U_L with shocks at points $\psi_i(t) = iL + \epsilon_i(t)$ and $|\epsilon_i(t)| << 1$. The Rankine-Hugoniot jump conditions read

$$[\frac{u^2}{2}]_{\psi_i} = \dot{\epsilon}_i[u]_{\psi_i} \tag{2.38}$$

We make the change of coordinates $(x,t) \to (\xi(x,t),t)$ introduced in section 2.2. This leads to the new equation

$$u_t + \frac{L}{\psi_{i+1} - \psi_i}(uu_\xi - (\dot{\epsilon}_i + \frac{\xi - iL}{L}(\dot{\epsilon}_{i+1} - \dot{\epsilon}_i))u_\xi) = u \tag{2.39}$$

The linearized equation around the steady state $(u, \psi_i) = (U_L, iL)$ can be written

$$u_t + U_L u_\xi = \frac{\epsilon_{i+1} - \epsilon_i}{L}U_L + (\dot{\epsilon}_i + \frac{\xi - iL}{L}(\dot{\epsilon}_{i+1} - \dot{\epsilon}_i)) \tag{2.40}$$

The linearized Rankine-Hugoniot conditions are given by

$$[U_L u]_{iL} = \dot{\epsilon}_i[U_L]_{iL} \tag{2.41}$$

We are looking for solutions with an exponential growth in time: $u = \exp(\lambda t)\underline{u}$ and $\epsilon_i(t) = \exp(\lambda t)\underline{\epsilon_i}$. Dropping the underlines, it yields the spectral problem

Find $\lambda \in \mathbb{C}$, a function u C^1 in $\mathbb{R}\backslash\{iL, i \in \mathbb{Z}\}$, a bounded sequence $(\epsilon_i)_{i\in\mathbb{Z}}$ satisfying $\lim_{\xi\to\pm\infty} u(\xi) = 0$ *and (u, ϵ_i) solution of*

$$U_L u_\xi + \lambda u = \frac{\epsilon_{i+1} - \epsilon_i}{L}U_L(1+\lambda) + \lambda\frac{\epsilon_{i+1} + \epsilon_i}{2} \tag{2.42}$$

$$\frac{u(iL)^+ + u(iL)^-}{2} = \epsilon_i\lambda \tag{2.43}$$

When $\lambda \neq 0$, the equation (2.42) has a unique analytical solution on $]iL,(i+1)L[$,

$$u(\xi) = \frac{\epsilon_{i+1} + \epsilon_i}{2} + \frac{\epsilon_{i+1} - \epsilon_i}{L}U_L(\xi). \tag{2.44}$$

Then, we find $u(iL)^+ = \epsilon_i$ and $u(iL)^- = \epsilon_i$. The linearized Rankine-Hugoniot conditions read: $\epsilon_i(\lambda - 1) = 0$. If $\lambda \neq 1$ and $\lambda \neq 0$, then $\epsilon_i = 0$ and the analytical solution u is equal to 0. Thus λ is not an eigenvalue. When $\lambda = 0$, $\epsilon_i = 0$ and the particular solution $u(\xi) = (-1)^i C$ on $]iL,(i+1)L[$ satisfy (2.42) and (2.43) but does not satisfy the growth assumptions. We now consider the case when $\lambda = 1$. It is possible to construct a family of eigenfunctions solutions of the spectral problem and associated to the unstable eigenvalue

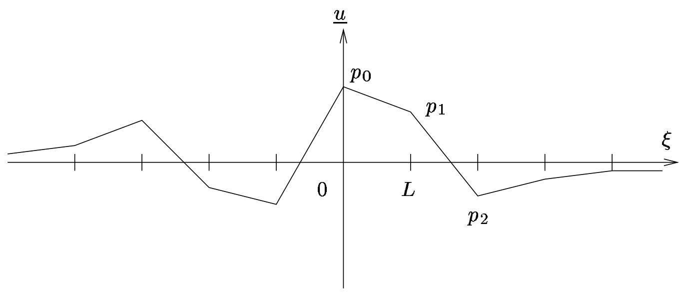


Figure 2.6: Shape of the eigenfunction $\underline{u}$

$\lambda = 1$. Let us take an arbitrary sequence $\epsilon_i \in \mathbb{R}$ satisfying $\lim_{|i| \to \pm\infty} \epsilon_i = 0$. Then consider a function $\underline{u}$ continuous and affine on the intervals $]iL, (i+1)L[$ with $\underline{u}(iL) = \epsilon_i$ (see figure 2.6).
Then the function $u(\xi, t) = \exp(t)\underline{u}(\xi)$ is an eigenfunction associated to the eigenvalue $\lambda = 1$ and solution of the spectral problem. Then the roll-waves of the Burgers equation are proved to be unstable. This result is in agreement with the results obtained in [9],[20]. The Burgers equation is a good model to describe small amplitude roll-waves solution of the Saint Venant equations and Dressler roll-waves are linearly unstable when their wavelength L goes to 0.
This instability has an easy interpretation. Integrating equation (2.37) over the whole space, we find

$$\frac{d}{dt} \int_{\mathbb{R}} u(x)dx = \int_{\mathbb{R}} u(x)dx.$$

Thus if $\int_{\mathbb{R}} u_0(x)dx \neq 0$, the spatial mean $\int_{\mathbb{R}} u(x,.)dx$ grows exponentially with time. Exploiting this property, Lyberopoulos has proved that the initial data for which the solution tends to a travelling wave are exactly those with mean zero; in all the other cases, the solution diverges [9]. Then, the roll-waves are linearly stable if we restrict the functional space and consider only functions u such that $\int_{\mathbb{R}} u(x)dx = 0$.
Now we come back to the Saint Venant system and start the proof of theorem 1 for possible eigenvalues with large real part.

2.4 Proof of theorem 1: the case of large real part

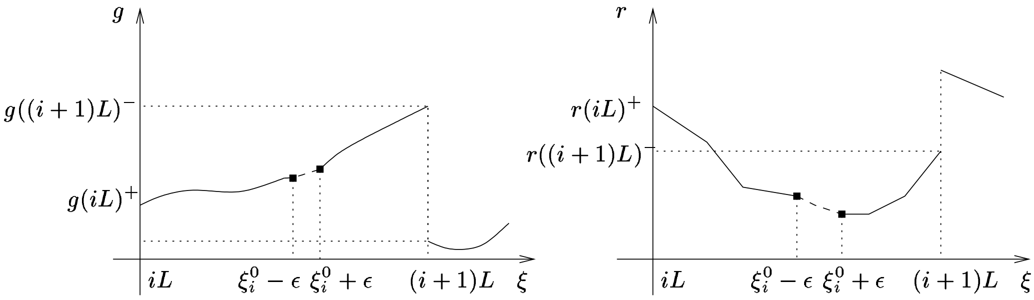
In this section, we prove the first part of theorem 1: there exists a function r_1 (r_1 is a growing function of L) such that if $Re(\lambda) \geq r_1(L)$, λ is not an eigenvalue of the spectral problem (2.25), (2.26). In order to prove this property, let us work with the diagonalized spectral problem (2.31), (2.33). To obtain a contradiction, suppose that (2.31), (2.33) has unstable eigenvalues λ with an arbitrary large real part. Under this hypothesis, we can compute an asymptotic expansion of the shock values $(g(iL)^{\pm}, r(iL)^{\pm})$, $i \in \mathbb{Z}$ where (g, r) is a possible eigenfunction associated to the eigenvalue λ. The $+$ sign denotes the value after the discontinuity and the $-$ sign the value before the discontinuity. Then we prove that these shock values are incompatible with the linearized Rankine Hugoniot conditions (2.33).
Let us fix the interval $I_i =]iL, (i+1)L[$. On I_i, the possible eigenfunction (g, r) satisfy the differential system

$$\begin{aligned} d_1(H(\xi))g' + (c_1(H(\xi)) + \lambda)g + c_2(H(\xi))r = \gamma_i(H(\xi)), \\ d_2(H(\xi))r' + (c_4(H(\xi)) + \lambda)r + c_3(H(\xi))g = \delta_i(H(\xi)). \end{aligned} \tag{2.45}$$

There exists a point $\xi_i^0 \in I_i$ such that $d_1(H(\xi_i^0)) = 0$ and the system (2.45) is singular at this point. System (2.45) possess an analytic solution g, r on I_i. We want to compute the shock values $(g(iL)^+, r(iL)^+)$ and $(g((i+1)L)^-, r((i+1)L)^-)$. The scheme of the calculation is the following. Let us fix $\epsilon > 0$ arbitrary small. We first work on the interval $I_i^+ =]\xi_i^0 + \epsilon, (i+1)L[$. We compute a simple basis of solutions of the homogeneous system

$$\begin{aligned} d_1(H(\xi))g' + (c_1(H(\xi)) + \lambda)g + c_2(H(\xi))r = 0, \\ d_2(H(\xi))r' + (c_4(H(\xi)) + \lambda)r + c_3(H(\xi))g = 0. \end{aligned} \tag{2.46}$$

In fact, we just have an asymptotic expansion of this basis as $Re(\lambda)$ goes to infinity. This yields an asymptotic expansion of the resolvent matrix $R^+(\lambda, t, s)$. With a variation of the constant we can compute an asymptotic expansion of $(g((i+1)L)^-, r((i+1)L)^-)$. These values are dependent of $g(\xi_i^0 + \epsilon)$ and $r(\xi_i^0 + \epsilon)$ but we shall see later that it has no influence on the result provided we have chosen $|i|$ sufficiently large. To compute the values $(g(iL)^+, r(iL)^+)$, we work on the interval $I_i^- =]iL, \xi_i^0 - \epsilon[$ and the scheme of the calculation remains the same.



2.4.1 Asymptotic expansion of $\Big(g\big((i+1)L\big)^-, r\big((i+1)L\big)^-\Big)$

We need a precise expansion of the resolvent matrix on I_i^+ and I_i^- to ensure that the errors we make when we take the approximated resolvent matrix in the variation of the constant are not relevant. These computations are inspired by Nayfeh [15] and Titschmarsch [23].
We first compute an approximation of the resolvent matrix $R^+(\lambda, t, s)$ associated to the homogeneous system (2.46) on I_i^+. For this purpose, we calculate a basis of solution. On the interval I_i^+, we have $d_1(H) > 0$ and $d_2(H) < 0$. Thus, we are looking for a first solution which decreases to 0 as $Re(\lambda) \to \infty$ and a second solution (independent of the first one) which blows up exponentially as $Re(\lambda) \to \infty$. This is done in the following lemmas

Proposition 5 *Consider the linear differential system*

$$\begin{array}{l} u_1' + (a(t) + \lambda e_1(t))u_1 + b(t)u_2 = 0, \\ u_2' + (d(t) - \lambda e_2(t))u_2 + c(t)u_1 = 0, \end{array} \tag{2.47}$$

for $t \in [0, T]$. *We suppose that* $a, b, c, d, e_1, e_2 \in C^0(0, T)$ *and* $\min_{(0,T)}(e_1, e_2) \geq \delta > 0$ *Then there exists a solution which has the form*

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (Id + P(\lambda, t)) \begin{pmatrix} \exp(-\int_0^t a + \lambda e_1 ds) \\ \exp(\int_t^T d - \lambda e_2 ds) \end{pmatrix} \tag{2.48}$$

with

$$\| P(\lambda, .) \|_\infty \leq \Big(\Big\| \frac{b}{e_1 + e_2} \Big\|_\infty + \Big\| \frac{c}{e_1 + e_2} \Big\|_\infty \Big) \frac{C(T)}{Re(\lambda)}$$

and

$$C(T) = O\Big(\exp\big((\|a\|_\infty + \|d\|_\infty) T \big) \Big).$$

This solution decreases to 0 *as* $Re(\lambda) \to \infty$.

Proposition 6 *There exists a solution which has the form*

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (Id + Q(\lambda, t)) \begin{pmatrix} \exp(\int_t^T \lambda e_1 + ads) \\ \exp(\int_0^t \lambda e_2 - dds) \end{pmatrix} \tag{2.49}$$

with

$$\| Q(\lambda, .) \|_\infty \leq \Big(\| \frac{b}{e_1 + e_2} \|_\infty + \| \frac{c}{e_1 + e_2} \|_\infty \Big) \frac{C(T)}{Re(\lambda)}$$

and

$$C(T) = O\Big(\exp\big((\|a\|_\infty + \|d\|_\infty) T \big) \Big).$$

This solution blows up as $Re(\lambda) \to \infty$.

We apply these propositions to the system (2.46) with

$$a = \frac{c_1}{d_1},\ b = \frac{c_2}{d_1},\ c = \frac{c_3}{d_2},$$
$$d = \frac{c_4}{d_2},\ e_1 = \frac{1}{d_1},\ e_2 = -\frac{1}{d_2}$$

and the interval $(0, T)$ is replaced by $]\xi_i^0 + \epsilon, (i+1)L[$. We find a basis of solutions $(B_1(\lambda, t), B_2(\lambda, t))$ where

$$B_1(\lambda, \xi) = (Id + P_1(\lambda, \xi)) \begin{pmatrix} \exp(-\int_{\xi_i^0+\epsilon}^{\xi} \frac{\lambda + c_1(H(s))}{d_1(H(s))} ds) \\ \exp(\int_{\xi}^{(i+1)L} \frac{\lambda + c_4(H(s))}{d_2(H(s))} ds) \end{pmatrix} \tag{2.50}$$

and

$$B_2(\lambda, \xi) = (Id + P_2(\lambda, \xi)) \begin{pmatrix} \exp(\int_{\xi}^{(i+1)L} \frac{\lambda + c_1(H(s))}{d_1(H(s))} ds) \\ \exp(-\int_{\xi_i^0+\epsilon}^{\xi} \frac{\lambda + c_4(H(s))}{d_2(H(s))} ds) \end{pmatrix} \tag{2.51}$$

where P_1 represents the matrix $P(., \lambda)$ of the proposition 5 and P_2 represents the matrix $Q(., \lambda)$ of the proposition 6. Since the estimates

$$\frac{\frac{|c_2(H(\xi))|}{|d_1(H(\xi))|}}{|\frac{1}{d_1(H(\xi))} - \frac{1}{d_2(H(\xi))}|} \leq K, \forall \xi \in I_i^+ \tag{2.52}$$

and

$$\frac{\frac{|c_3(H(\xi))|}{|d_2(H(\xi))|}}{|\frac{1}{d_1(H(\xi))} - \frac{1}{d_2(H(\xi))}|} \leq K, \forall \xi \in I_i^+ \tag{2.53}$$

are valid for all $\epsilon > 0$, the presence of the singularity does not change the final result and we have $\|P_i(\lambda, .)\| \leq \frac{C(T)}{Re(\lambda)}$ with $T = diam(I_i^+)$. Since $T = l(L)$

remains finite as L goes to infinity, these estimates are uniform with respect to L and $\|P_i(\lambda,.)\| \leq \frac{C}{Re(\lambda)}$. Let us note $B(\lambda,t) = (B_1(\lambda,t), B_2(\lambda,t))$. The resolvent matrix $R^+(\lambda,t,s)$ is then given by $R^+(\lambda,t,s) = B(\lambda,t) \times B^{-1}(\lambda,s)$. With a short computation, we can prove that

$$R^+(\lambda,t,s) = \begin{pmatrix} \exp(-\int_s^t \frac{\lambda+c_1(H(u))}{d_1(H(u))}du) & 0 \\ 0 & \exp(-\int_s^t \frac{\lambda+c_4(H(u))}{d_2(H(u))}du) \end{pmatrix} \times \\ \times \Big(Id + O(\frac{1}{Re(\lambda)})\Big) \tag{2.54}$$

We can compute an expansion of $\Big(g\big((i+1)L\big)^-, r\big((i+1)L\big)^-\Big)$. The result is given in the following lemma

Lemma 2 *For large $Re(\lambda)$, the shocks values $\Big(g\big((i+1)L\big)^-, r\big((i+1)L\big)^-\Big)$ of the possible eigenfunctions (g,r) have the expansion*

$$\begin{aligned} g\big((i+1)L\big)^- &= g_i^-\Big(1 + O(\frac{1}{Re(\lambda)})\Big), \\ r\big((i+1)L\big)^- &= r_i^-\Big(1 + O(\frac{1}{Re(\lambda)})\Big), \end{aligned} \tag{2.55}$$

with

$$\begin{aligned} g_i^- = \lim_{\epsilon\to 0}\Bigg(& \exp\Big(-\int_{H_\epsilon^+}^{H_2} \frac{c_1+\lambda}{P_1 d_1}\Big) g(\xi_i^0+\epsilon) \\ & - \int_{H_\epsilon^+}^{H_2} \exp\Big(-\int_h^{H_2} \frac{c_1+\lambda}{P_1 d_1}\Big)\frac{\lambda d_2}{d_1}\Big(\epsilon_i + \frac{L - l(L) + \frac{f_1(h)}{S}}{L}(\epsilon_{i+1}-\epsilon_i)\Big)dh \\ & - \int_{H_\epsilon^+}^{H_2} \exp\Big(-\int_h^{H_2} \frac{c_1+\lambda}{P_1 d_1}\Big)\big(\frac{\epsilon_{i+1}-\epsilon_i}{L}\big) d_2 P_1(h) dh\Bigg), \end{aligned}$$

$$\begin{aligned} r_i^- = \lim_{\epsilon\to 0}\Bigg(& \exp\Big(-\int_{H_\epsilon^+}^{H_2} \frac{c_4+\lambda}{P_1 d_2}\Big) r(\xi_i^0+\epsilon) \\ & - \int_{H_\epsilon^+}^{H_2} \exp\Big(-\int_h^{H_2} \frac{c_4+\lambda}{P_1 d_2}\Big)\frac{\lambda d_1}{d_2}\Big(\epsilon_i + \frac{L - l(L) + \frac{f_1(h)}{S}}{L}(\epsilon_{i+1}-\epsilon_i)\Big)dh \\ & - \int_{H_\epsilon^+}^{H_2} \exp\Big(-\int_h^{H_2} \frac{c_4+\lambda}{P_1 d_2}\Big)\big(\frac{\epsilon_{i+1}-\epsilon_i}{L}\big) d_1 P_1(h) dh\Bigg), \end{aligned}$$

where H_ϵ^+ is defined by $H_\epsilon^+ = H(\xi_i^0+\epsilon) = H_0 + \alpha(\epsilon)$ and α is a $O(\epsilon)$ function. The height H_2 is the maximum height of the roll-wave considered.

Proof. We make a variation of the constant with the approximated resolvent matrix $R^+(\lambda,t,s)$ on the interval $]\xi_i^0+\beta(\epsilon),(i+1)L[$. The shock values $g\big((i+1)L\big)^-, r\big((i+1)L\big)^-$ are given by the Duhamel formula

$$\begin{pmatrix} g\big((i+1)L\big)^- \\ r\big((i+1)L\big)^- \end{pmatrix} = R^+(\lambda,(i+1)L,\xi_i^0+\epsilon)\begin{pmatrix} g(\xi_i^0+\epsilon) \\ r(\xi_i^0+\epsilon) \end{pmatrix} + \\ + \int_{\xi_i^0+\epsilon}^{(i+1)L} R^+(\lambda,t,s)\begin{pmatrix} \gamma_i(H(s)) \\ \delta_i(H(s)) \end{pmatrix} ds.$$

After a change of variable $\xi \to H(\xi)$ in the integrals, we get (g_i^-, r_i^-). As $\epsilon \to 0$, the integrals considered remain convergent. The singularity does not change the final result and we can make $\epsilon \to 0$ in the formula obtained. □

Remark: In the sequel, in order to simplify the notation, we only keep the principal parts (g_i^-, r_i^-) of the shock values expansions. The other terms will not have any influence on the demonstration of the result.

We now compute an asymptotic expansion of (g_i^-, r_i^-). We make successive integration by parts.

Estimate of g_i^-. We recall that g_i^- is given by

$$g_i^- = \lim_{\epsilon\to 0}\Bigg(\exp\Big(-\int_{H_\epsilon^+}^{H_2}\frac{c_1+\lambda}{P_1d_1}\Big)g(\xi_i^0+\epsilon) \\ - \int_{H_\epsilon^+}^{H_2}\exp\Big(-\int_h^{H_2}\frac{c_1+\lambda}{P_1d_1}\Big)\frac{\lambda d_2}{d_1}\Big(\epsilon_i + \frac{L-l(L)+\frac{f_1(h)}{S}}{L}(\epsilon_{i+1}-\epsilon_i)\Big)dh \\ - \int_{H_\epsilon^+}^{H_2}\exp\Big(-\int_h^{H_2}\frac{c_1+\lambda}{P_1d_1}\Big)\big(\frac{\epsilon_{i+1}-\epsilon_i}{L}\big)d_2P_1(h)dh\Bigg).$$

We start with the first term. We are looking for possible eigenfunctions (g,r) which decrease to 0 as $|\xi| \to \infty$. Moreover we take the normalization condition

$$\max_{]-\infty,+\infty[}(|(g,r)|) < 1.$$

Let us recall that $d_1(h) = d_1'(H_0)(h-H_0) + o(h-H_0)$ and $d_1'(H_0) > 0$. We have also $P_1(H_0) > 0$. Then we can prove that

$$\int_{H_\epsilon^+}^{H_2}\frac{c_1+\lambda}{P_1d_1}dh = -\frac{c_1(H_0)+\lambda}{P_1d_1'(H_0)}\ln(\epsilon) + M + o(1).$$

Hence we have for a sufficiently large $Re(\lambda)$

$$\exp\Big(-\int_{H_\epsilon^+}^{H_2}\frac{c_1+\lambda}{P_1 d_1}\Big)g(\xi_i^0+\epsilon)=O(\epsilon).$$

Then we have $-g_i^- = \lim_{\epsilon\to 0} I_1(\epsilon)+I_2(\epsilon)+I_3(\epsilon)$ where I_i are the integrals

$$\begin{aligned}
I_1(\epsilon) &= \epsilon_i\int_{H_\epsilon^+}^{H_2}\exp\Big(-\int_h^{H_2}\frac{c_1+\lambda}{P_1 d_1}\Big)\frac{\lambda d_2}{d_1}dh,\\
I_2(\epsilon) &= (\epsilon_{i+1}-\epsilon_i)\int_{H_\epsilon^+}^{H_2}\exp\Big(-\int_h^{H_2}\frac{c_1+\lambda}{P_1 d_1}\Big)\frac{\lambda d_2}{d_1}\frac{L-l(L)+f_1(h)}{L}dh,\\
I_3(\epsilon) &= -\frac{\epsilon_{i+1}-\epsilon_i}{L}\int_{H_\epsilon^+}^{H_2}\exp\Big(-\int_h^{H_2}\frac{c_1+\lambda}{P_1 d_1}\Big)d_2 P_1(h)dh.
\end{aligned}$$

Starting with I_1, an integration by parts yields

$$I_1=\lim_{\epsilon\to 0} I_1(\epsilon)=\epsilon_i d_2 P_1(H_2)+O\Big(\frac{\max(|\epsilon_i|,|\epsilon_{i+1}|)}{|\lambda|}\Big). \tag{2.56}$$

Similarly, the same computations on $I_2=\lim_{\epsilon\to 0} I_2(\epsilon)$ and $I_3=\lim_{\epsilon\to 0} I_3(\epsilon)$ gives

$$I_2=(\epsilon_{i+1}-\epsilon_i)d_2 P_1(H_2)+O\Big(\frac{\max(|\epsilon_i|,|\epsilon_{i+1}|)}{|\lambda|}\Big) \tag{2.57}$$

and

$$I_3=O\Big(\frac{\max(|\epsilon_i|,|\epsilon_{i+1}|)}{|\lambda|}\Big). \tag{2.58}$$

We put the estimates (2.56),(2.57) and (2.58) together to get the following expansion.

$$g_i^- = -\epsilon_{i+1}d_2 P_1(H_2)+O\Big(\frac{\max(|\epsilon_i|,|\epsilon_{i+1}|)}{|\lambda|}\Big)$$

We are going to make similar computations for the next expansions. We now come to the estimate of r_i^-.

Estimate of r_i^-. We recall that

$$\begin{aligned}
r_i^- = \lim_{\epsilon\to 0}\Bigg(&\exp\Big(-\int_{H_\epsilon^+}^{H_2}\frac{c_4+\lambda}{P_1 d_2}\Big)r(\xi_i^0+\epsilon)\\
&-\int_{H_\epsilon^+}^{H_2}\exp\Big(-\int_h^{H_2}\frac{c_4+\lambda}{P_1 d_2}\Big)\frac{\lambda d_1}{d_2}\Big(\epsilon_i+\frac{L-l(L)+\frac{f_1(h)}{S}}{L}(\epsilon_{i+1}-\epsilon_i)\Big)dh\\
&-\int_{H_\epsilon^+}^{H_2}\exp\Big(-\int_h^{H_2}\frac{c_4+\lambda}{P_1 d_2}\Big)\big(\frac{\epsilon_{i+1}-\epsilon_i}{L}\big)d_1 P_1(h)dh\Bigg),
\end{aligned}$$

The computations are similar to the previous case. Let us note

$$r_i^- = \lim_{\epsilon \to 0} \exp\Big(- \int_{H_\epsilon^+}^{H_2} \frac{c_4+\lambda}{d_2 P_1} \Big) \big(r(\xi_i^0 + \epsilon) - (I_1 + I_2 + I_3)(\epsilon) \big).$$

We obtain an expansion of $(I_i, i = 1 \cdots 3)$ again with the aid of integration by parts. Starting with the first integral $I_1 = \lim_{\epsilon \to 0} I_1(\epsilon)$ given by

$$I_1 = \epsilon_i \int_{H_0}^{H_2} \exp\Big(\int_{H_0}^{h} \frac{c_4+\lambda}{d_2 P_1} \Big) \frac{\lambda d_1}{d_2},$$

we obtain the expansion

$$I_1 = -\frac{\epsilon_i}{\lambda} \Big(d_1' d_2 H'^2(H_0) + O\big(\frac{1}{|\lambda|}\big) \Big).$$

The same argument holds for the second integral I_2 and we get

$$I_2 = -\frac{L - l(L)}{L} (\epsilon_{i+1} - \epsilon_i) \frac{d_1' d_2 H'^2(H_0)}{\lambda} + O\Big(\frac{\max(|\epsilon_{i+1}|, |\epsilon_i|)}{|\lambda|^2} \Big).$$

For the last integral I_3, we prove that

$$I_3 = \frac{\epsilon_{i+1} - \epsilon_i}{L} \frac{d_1 d_2(H_0) P_1(H_0)}{\lambda} + O\Big(\frac{\max(|\epsilon_{i+1}|, |\epsilon_i|)}{|\lambda|^2} \Big).$$

We put these estimates together to obtain the expansion of r_i^-:

$$r_i^- = \exp\Big(- \int_{H_0}^{H_2} \frac{c_4+\lambda}{d_2 P_1} \Big) \Big(r(\xi_i^0) + O\Big(\frac{\max(|\epsilon_{i+1}|, |\epsilon_i|)}{|\lambda|} \Big) \Big).$$

Remark: We can see that $\exp(-\int_{H_0}^{H_b} \frac{c_4+\lambda}{d_2 P_1})$ grows up exponentially with $Re(\lambda)$.
Now the point is to compute the expansions of the values $\big(g(iL)+, r(iL)^+\big)$.

2.4.2 Asymptotic expansion of $\big(g(iL)^+, r(iL)^+\big)$

We first compute an approximation of the resolvent matrix $R^-(\lambda, t, s)$ associated to the homogeneous system (2.46) on I_i^-. For this purpose, we calculate a basis of solution. On the interval I_i^-, we have $d_1(H(\xi)) < 0$ and $d_2(H(\xi)) < 0$. Thus, we are looking for a basis of solution which decreases to 0 as $Re(\lambda) \to \infty$. Moreover, since we want to investigate a large range of L, we try to find expansions with better error estimates than those proposed in proposition 5 and 6. This is done in the following proposition

Proposition 7 *Consider the linear differential system*

$$\begin{cases} u_1' + (a(t) - \lambda e_1(t))u_1 + b(t)u_2 = 0, \\ u_2' + (d(t) - \lambda e_2(t))u_2 + c(t)u_1 = 0, \end{cases} \tag{2.59}$$

for $t \in [-T, 0]$. *We suppose that* $a, b, c, d, e_1, e_2 \in C^0(-T, 0)$ *and* $e_1 > e_2$ *Then there exists a basis of solutions which has the form*

$$\begin{array}{l} (u_1, v_1) = \big(Id + P(\lambda, t)\big)\big(\exp \int_t^0 a - \lambda e_1, \exp \int_{-T}^t d - \lambda e_2\big) \\ (u_2, v_2) = \big(Id + Q(\lambda, t)\big)\big(0, \exp \int_t^0 d - \lambda e_2\big) \end{array} \tag{2.60}$$

with

$$\|P(\lambda, .)\| \leq \Big(\|\frac{b}{e_1 + e_2}\|_\infty + \|\frac{c}{e_1 + e_2}\|_\infty\Big)\frac{m_1(T)}{Re(\lambda)}$$

and

$$\|Q(\lambda, .)\| \leq \Big(\|\frac{b}{e_1 + e_2}\|_\infty + \|\frac{c}{e_1 + e_2}\|_\infty\Big)\frac{m_2}{Re(\lambda)}.$$

The constant m_1 *grows exponentially with* T*:*

$$m_1(T) = O\Big(\exp\big((\|a\|_\infty + \|d\|_\infty)T\big)\Big).$$

These solutions decrease to 0 *as* $Re(\lambda) \to \infty$.

Proof. The proof is similar to the proof of proposition 5: see the appendix.□

We apply this result on the interval $]iL, \xi_i^0 - \epsilon[$ with

$$\begin{array}{l} a = -\dfrac{c_1}{d_1},\ b = -\dfrac{c_2}{d_1},\ c = -\dfrac{c_3}{d_2}, \\ d = -\dfrac{c_4}{d_2},\ e_1 = -\dfrac{1}{d_1},\ e_2 = -\dfrac{1}{d_2}. \end{array}$$

We find an approximation of the resolvent matrix $R^-(\lambda, t, s)$ on $]iL, \xi_i^0 - \epsilon[$. Since the estimates

$$\frac{\frac{|c_2(H(\xi))|}{|d_1(H(\xi))|}}{|\frac{1}{d_1(H(\xi))} + \frac{1}{d_2(H(\xi))}|} \leq K, \forall \xi \in I_i^- \tag{2.61}$$

and

$$\frac{\frac{|c_3(H(\xi))|}{|d_2(H(\xi))|}}{|\frac{1}{d_1(H(\xi))} + \frac{1}{d_2(H(\xi))}|} \leq K, \forall \xi \in I_i^- \tag{2.62}$$

are valid for all $\epsilon > 0$, the presence of the singularity does not change the final result. The approximation is valid on I_i^- and we have $\|P(\lambda, .)\| \leq \frac{m_1(T)}{Re(\lambda)}$

and $\|Q(\lambda,.)\| \leq \frac{m_2}{Re(\lambda)}$ with $T = diam(I_i^+)$. We will consider large wavelength L and $diam(]iL, \xi_i^0[) \to \infty$. Hence we must have $Re(\lambda)$ growing faster than $\exp(kL)$ in order to keep the estimates valid in proposition 7 and the function r_1 in theorem 1 would have an exponential growth. In order to get a better result, we take into account the fact that the coefficients in (2.46) tend to constants as $L \to \infty$. We state the lemma.

Lemma 3 *Consider the differential system with constant coefficients*

$$\begin{aligned} u' + (a + \lambda e_1)u + bv = 0, \\ v' + (d + \lambda e_2)v + cu = 0 \end{aligned} \tag{2.63}$$

with $e_1 \neq e_2$ on the interval $[0,T]$.As $Re(\lambda) \to \infty$, the resolvent matrix has the following form

$$R(\lambda, t) = R_0(\lambda, t)\Big(Id + O\Big(\frac{T}{Re(\lambda)}\Big)\Big) \tag{2.64}$$

with

$$R_0(\lambda, t) = diag\big(\exp -(a + \lambda e_1)t, \exp -(d + \lambda e_2)t\big).$$

Back to the initial problem, the coefficients tend to a constant exponentially so we must add a $O(\exp(-kL))$ term in the error estimate but it does not change the final result: the constant m_1 in the proposition 7 is $O(T)$. Applying these propositions to the initial problem, we have found an approximation $R^-(\lambda, t, s)$ of the resolvent matrix on the interval $]iL, \xi_i^0 - \epsilon[$ with a precision up to order $O\Big(\frac{L}{Re(\lambda)}\Big)$.

We now can compute an expansion of $\big(g(iL)^+, r(iL)^+\big)$. The proof of the following lemma is analogous to the proof of lemma 2.

Lemma 4 *For large $Re(\lambda)$, the shocks values $\big(g(iL)^+, r(iL)^+\big)$ of the possible eigenfunctions (g,r) have the expansion*

$$\begin{aligned} g(iL)^+ = g_i^+\Big(1 + O\big(\frac{L}{Re(\lambda)}\big)\Big), \\ r(iL)^+ = r_i^+\Big(1 + O\big(\frac{L}{Re(\lambda)}\big)\Big) \end{aligned} \tag{2.65}$$

with

$$g_i^+ = \lim_{\epsilon\to 0}\Bigg(\exp\Big(\int_{H_1}^{H_\epsilon^-} \frac{c_1+\lambda}{P_1 d_1}\Big) g(\xi_i^0-\epsilon)$$

$$+ \int_{H_1}^{H_\epsilon^-} \exp\Big(\int_{H_1}^{h} \frac{c_1+\lambda}{P_1 d_1}\Big) \frac{\lambda d_2}{d_1}\Big(\epsilon_i + \frac{L-l(L)+\frac{f_1(h)}{S}}{L}(\epsilon_{i+1}-\epsilon_i)\Big) dh$$

$$- \int_{H_1}^{H_\epsilon^-} \exp\Big(\int_{H_1}^{h} \frac{c_1+\lambda}{P_1 d_1}\Big) (\frac{\epsilon_{i+1}-\epsilon_i}{L}) d_2 P_1(h) dh\Bigg),$$

$$r_i^+ = \lim_{\epsilon\to 0}\Bigg(\exp\Big(\int_{H_1}^{H_\epsilon^-} \frac{c_4+\lambda}{P_1 d_2}\Big) r(\xi_i^0-\epsilon)$$

$$+ \int_{H_1}^{H_\epsilon^-} \exp\Big(\int_{H_1}^{h} \frac{c_4+\lambda}{P_1 d_2}\Big) \frac{\lambda d_1}{d_2}\Big(\epsilon_i + \frac{L-l(L)+\frac{f_1(h)}{S}}{L}(\epsilon_{i+1}-\epsilon_i)\Big) dh$$

$$- \int_{H_1}^{H_\epsilon^-} \exp\Big(\int_{H_1}^{h} \frac{c_4+\lambda}{P_1 d_2}\Big) (\frac{\epsilon_{i+1}-\epsilon_i}{L}) d_1 P_1(h) dh\Bigg).$$

where H_ϵ^- *is defined by* $H_\epsilon^- = H_D(\xi_i^0-\epsilon) = H_0 - \beta(\epsilon)$ *and* β *is a* $O(\epsilon)$ *function. The height* H_1 *is the minimum height of the roll-wave.*

In the sequel, we only keep the principal parts g_i^+, r_i^+ of the shock values expansion. The other terms will not have any influence on the final result. We now compute an expansion of g_i^+, r_i^+ as $Re(\lambda) \to \infty$ with several integration by parts

Estimate of g_i^+. We first remind that g_i^+ has the form

$$g_i^+ = \lim_{\epsilon\to 0}\Bigg(\exp\Big(\int_{H_1}^{H_\epsilon^-} \frac{c_1+\lambda}{P_1 d_1}\Big) g(\xi_i^0+\epsilon) +$$

$$+ \int_{H_1}^{H_\epsilon^-} \exp\Big(\int_{H_1}^{h} \frac{c_1+\lambda}{P_1 d_1}\Big) \frac{\lambda d_2}{d_1}\Big(\epsilon_i + \frac{L-l(L)+\frac{f_1(h)}{S}}{L}(\epsilon_{i+1}-\epsilon_i)\Big) dh$$

$$- \int_{H_1}^{H_\epsilon^-} \exp\Big(\int_{H_1}^{h} \frac{c_1+\lambda}{P_1 d_1}\Big) (\frac{\epsilon_{i+1}-\epsilon_i}{L}) d_2 P_1(h) dh\Bigg),$$

We make an expansion of the first term. Similarly to the case of g_i^-, we can prove that

$$\int_{H_1}^{H_\epsilon^-} \frac{c_1+\lambda}{P_1 d_1} = \frac{c_1(H_0)+\lambda}{P_1 d_1'(H_0)} \ln(\epsilon) + M + o(1).$$

Then we get for a sufficiently large $Re(\lambda)$

$$\exp\Big(\int_{H_1}^{H_\epsilon^-}\frac{c_1+\lambda}{P_1d_1}\Big)g(\xi_i^0+\epsilon)=O(\epsilon).$$

Let us set $g_i^+=\lim_{\epsilon\to 0}I_1(\epsilon)+I_2(\epsilon)+I_3(\epsilon)$, where I_1,I_2,I_3 are defined by

$$\begin{aligned}
I_1&=\epsilon_i\lambda\int_{H_1}^{H_\epsilon^-}\exp\Big(\int_{H_1}^{h}\frac{c_1+\lambda}{d_1P_1}\Big)\frac{d_2}{d_1}dh,\\
I_2&=(\epsilon_{i+1}-\epsilon_i)\int_{H_1}^{H_\epsilon^-}\exp\Big(\int_{H_1}^{h}\frac{c_1+\lambda}{P_1d_1}\Big)\frac{\lambda d_2}{d_1}\frac{L-l(L)+f_1(h)}{L}dh,\\
I_3&=-\int_{H_1}^{H_\epsilon^-}\exp\Big(\int_{H_1}^{h}\frac{c_1+\lambda}{P_1d_1}\Big)\frac{\epsilon_{i+1}-\epsilon_i}{L}d_2P_1(h)dh.
\end{aligned}$$

An integration by parts on the first integral $I_1=\lim_{\epsilon\to 0}I_1(\epsilon)$ yields

$$I_1=-\epsilon_id_2P_1(H_1)+O\Big(P_1(H_1)\frac{\max(|\epsilon_i|,|\epsilon_{i+1}|)}{|\lambda|}\Big).$$

It is important to keep $P_1(H_1)$ in the computations since $\lim_{L\to\infty}P_1(H_1)=P_1(H_a)=0$ and it could change the result. We now compute the expansions of I_2 and I_3. An integration by parts yields

$$I_2=O\Big(P_1(H_1)\frac{\max(|\epsilon_i|,|\epsilon_{i+1}|)}{|\lambda|}\Big)$$

and

$$I_3=O\Big(P_1(H_1)\frac{\max(|\epsilon_i|,|\epsilon_{i+1}|)}{|\lambda|}\Big).$$

We put these expansions together to get the following result

$$g_i^+=-\epsilon_id_2P_1(H_1)+O\Big(P_1(H_1)\frac{\max(|\epsilon_i|,|\epsilon_{i+1}|)}{|\lambda|}\Big).$$

We finish with the computation of r_i^+ expansion.

Estimate of r_i^+. Let us recall that r_i^+ read

$$\begin{aligned}
r_i^+=\lim_{\epsilon\to 0}\Bigg(&\exp\Big(\int_{H_1}^{H_\epsilon^-}\frac{c_4+\lambda}{P_1d_2}\Big)r(\xi_i^0+\epsilon)\\
&+\int_{H_1}^{H_\epsilon^-}\exp\Big(\int_{H_1}^{h}\frac{c_4+\lambda}{P_1d_2}\Big)\frac{\lambda d_1}{d_2}\Big(\epsilon_i+\frac{L-l(L)+\frac{f_1(h)}{S}}{L}(\epsilon_{i+1}-\epsilon_i)\Big)dh\\
&-\int_{H_1}^{H_\epsilon^-}\exp\Big(\int_{H_1}^{h}\frac{c_4+\lambda}{P_1d_2}\Big)\Big(\frac{\epsilon_{i+1}-\epsilon_i}{L}\Big)d_1P_1(h)dh\Bigg).
\end{aligned}$$

The computations of the r_i^+ expansion are similar to the previous ones and we get

$$r_i^+ = -\epsilon_i d_1(H_1)P_1(H_1) + O\Big(\frac{P_1(H_1)\max(|\epsilon_i|, |\epsilon_{i+1}|)}{|\lambda|}\Big).$$

We are now in a position to prove theorem 1 when the possible eigenvalues have a large real part.

Conclusion and proof of theorem 1 for large $Re(\lambda) > 0$

In the previous section, we have obtained an expansion of the shock values $g(iL)^{\pm}, r(iL)^{\pm}$. We are going to prove that they are incompatible with the linearized Rankine Hugoniot jump conditions (2.26). According to lemma 1, we choose a shock point $\xi = (i+1)L$ such that $\epsilon_{i+1} \neq 0$. Let us consider the linearized Rankine-Hugoniot conditions (2.26) at the point $\xi = (i+1)L$. We have the following expansions

$$\begin{aligned} r\big((i+1)L\big)^+ &= -\epsilon_{i+1}d_1(H_1)P_1(H_1)\Big(1 + O\Big(\frac{L}{Re(\lambda)}\Big)\Big), \\ g\big((i+1)L\big)^+ &= -\epsilon_{i+1}d_2(H_1)P_1(H_1)\Big(1 + O\Big(\frac{L}{Re(\lambda)}\Big)\Big) \end{aligned} \tag{2.66}$$

and

$$\begin{aligned} g((i+1)L)^- &= -\epsilon_{i+1}d_2(H_2)P_1(H_2)\Big(1 + O\Big(\frac{1}{Re(\lambda)}\Big)\Big), \\ r\big((i+1)L\big)^- &= r(\xi_i^0)\exp\Big(-\int_{H_0}^{H_2}\frac{c_4+\lambda}{d_2P_1}\Big)\Big(1 + O\Big(\frac{1}{Re(\lambda)}\Big)\Big). \end{aligned} \tag{2.67}$$

At point $\xi = (i+1)L$, the linearized Rankine-Hugoniot conditions read

$$\begin{aligned} &\Big[v - ch\Big]_{(i+1)L} = \epsilon_{i+1}\lambda[H]_{(i+1)L}, \\ &\Big[\big(gH - (\frac{V}{H})^2\big)h + \big(2\frac{V}{H} - c\big)v\Big]_{(i+1)L} = \epsilon_{i+1}\lambda c[H]_{(i+1)L}. \end{aligned} \tag{2.68}$$

With the notations introduced in section 2.2, we have

$$\begin{aligned} A(H_2)\begin{pmatrix} h((i+1)L)^- \\ v((i+1)L)^- \end{pmatrix} - A(H_1)\begin{pmatrix} h((i+1)L)^+ \\ v((i+1)L)^+ \end{pmatrix} = \\ = \epsilon_{i+1}\lambda[H]\begin{pmatrix} 1 \\ c \end{pmatrix}. \end{aligned} \tag{2.69}$$

Let us introduce the matrix $E(H) = A(H)P^{-1}(H)$, the linearized Rankine Hugoniot conditions then read

$$
\begin{aligned}
E(H_2)\begin{pmatrix} g((i+1)L)^- \\ r((i+1)L)^- \end{pmatrix} - E(H_1)\begin{pmatrix} g((i+1)L)^+ \\ r((i+1)L)^+ \end{pmatrix} &= \\
= \epsilon_{i+1}\lambda(H_2 - H_1)\begin{pmatrix} 1 \\ c \end{pmatrix}. \qquad (2.70)
\end{aligned}
$$

Now, let us multiply the equation (2.70) and focus on the first equation given by (2.70). We obtain

$$
\begin{aligned}
g((i+1)L)^- = L_1(g((i+1)L)^+, r((i+1)L)^+) + \\
-\epsilon_{i+1}\lambda(H_2 - H_1)\frac{d_2(H_2)}{d_1(H_2)}, \qquad (2.71)
\end{aligned}
$$

where L_1 is a bounded linear form. We put the expansions (2.66), (2.67) into (2.71). Keeping only the leading order terms and after the simplification by p_{i+1}, we get:

$$
-d_2 P_1(H_2) = -P_1(H_1)L_1(d_2(H_1), d_1(H_1)) - \lambda(H_2 - H_1)\frac{d_2(H_2)}{d_1(H_2)}.
$$

This yields the contradiction by comparing the growth in λ of the different terms. This result is valid if the terms we have neglected remain small. Hence we must have $\frac{L}{Re(\lambda)}$ small. We have proved the following result

Proposition 8 *There exists $r_1(L) > 0$ such that for any (λ, L) satisfying $Re(\lambda) > r_1(L)$, λ is not an eigenvalue of the spectral problem (2.25),(2.26) associated to the linear stability of Dressler roll-wave with wavelength L.*

We can see that all these computations are uniform with respect to $Im(\lambda)$. Hence the first part of the theorem 1 is proved. We now come to the case of potential unstable eigenvalues with large imaginary part.

2.5 Proof of theorem 1 for large $Im(\lambda)$

We have proved that there exists $r_1(L)$ such that for any λ satisfying $Re(\lambda) \geq r_1(L)$, λ is not an eigenvalue of the spectral problem (2.31), (2.33). This result is uniform with respect to $Im(\lambda)$. It remains to deal with the case $0 \leq Re(\lambda) \leq r_1(L)$. Let us note $\lambda = \alpha + i\omega$: we are going to prove that there exists $r_2(L)$ such that for any $|\omega| \geq r_2(L)$, $\lambda = \alpha + i\omega$ is not an eigenvalue. The scheme of the proof is similar to the previous case. To obtain a contradiction, suppose that (2.31), (2.33) has unstable eigenvalue with an arbitrary large $Im(\lambda)$. We first compute an asymptotic expansion of the resolvent matrix in the area where the system is non singular. At this point, the demonstration is different. In section 2.3, due to the singularity, the integrals involved in the expansions of $(g(iL)^{\pm}, r(iL)^{\pm})$ are convergent as long as $\Re(\lambda + c_1(H_0)) > 0$. This condition was easily fulfilled since we considered possible eigenvalue λ with a large real part. But now, $c_1(H_0) < 0$ (or equivalently $C_1(1) < 0$: see the graph) and we have to distinguish two cases.

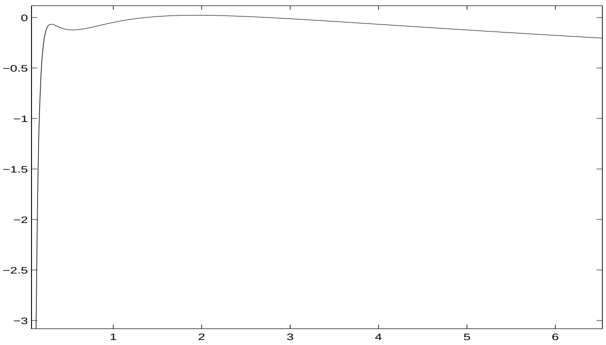


Figure 2.7: Graph of the rescaled coefficient C_1

As long as $Re(\lambda)+c_1(H_0) > 0$, the scheme of the proof is identical to the previous section and for sufficiently large $|\omega|$ and $Re(\lambda) \geq -c_1(H_0)$, $\lambda = \alpha + i\omega$ is not an eigenvalue of the spectral problem (2.31), (2.33). Now consider the case $Re(\lambda) < -c_1(H_0)$. In order to deal with convergent integrals, we will use backward integration. Taking $\big(g(iL)^+, r(iL)^+\big)$ as initial condition, the Duhamel principle gives us a first estimate of $\big(g(\xi_i^0 - \epsilon), r(\xi_i^0 - \epsilon)\big)$. As $\epsilon \to 0$, the integrals remain convergent and we get a first expansion of $\big(g(\xi_i^0), r(\xi_i^0)\big)$ namely $\big(g_i^{0-}, r_i^{0-}\big)$. Then taking $\Big(g\big((i+1)L\big)^-, r\big((i+1)L\big)^-\Big)$ as initial condition and using a backward integration, we get an expansion of $\big(g(\xi_i^0 + \epsilon), r(\xi_i^0 + \epsilon)\big)$. As $\epsilon \to 0$, the integrals involved in the expansions of $\big(g(\xi_i^0 + \epsilon), r(\xi_i^0 + \epsilon)\big)$ are still convergent and we get another expansion (g_i^{0+}, r_i^{0+}) of $\big(g(\xi_i^0), r(\xi_i^0)\big)$. To get a solution passing continuously through ξ_i^0, these values must coincide and lie on an affine space. This yields a first relation between $\big(g(iL)^+, r(iL)^+\big)$ and $\Big(g\big(i+1)L\big)^-, r\big((i+1)L\big)^-\Big)$. We shall see that this relation is incompatible with the linearized Rankine-Hugoniot condition and the growth properties of (g, r): $\lim_{\xi\to\pm\infty} \|(g, r)(\xi)\| = 0$. We first start by calculating an approximation of the resolvent matrix in the areas where the system (2.31) is non singular i.e. on the intervals $I_i^{\epsilon+} =]\xi_i^0 + \epsilon, (i+1)L[$ and $I_i^{\epsilon-} =]iL, \xi_i^0 - \epsilon[$. We must take into account that

$\mathrm{diam I}_{\mathrm{i}}^{\epsilon+} \to \infty$ as $L \to \infty$ and find uniform estimates with respect to L. The fact that the coefficients in (2.31) tend to a constant as $L \to \infty$ will have a great importance.

2.5.1 Asymptotic expansion of the resolvent matrix

We first compute an approximation of the resolvent matrix $R(\lambda, t, s)$ on the intervals $I_i^{\epsilon+} =]\xi_i^0 + \epsilon, (i+1)L[$. For this purpose, we calculate a basis of solution. The following proposition is proved in the appendix.

Proposition 9 *Consider the differential system*

$$\left\{ \begin{array}{l} u' + (a + i\omega e_1)u + bv = 0, \\ v' + (d + i\omega e_2)v + cu = 0, \end{array} \right. \tag{2.72}$$

$\forall t \in (0,T)$. *Assume* $(a, b, c, d, e_1, e_2) \in C^0(0,T)$ *and* $\min_{t\in(0,T)} |e_1(t) - e_2(t)| \geq \delta > 0$. *There exists a solution* (u, v) *of (2.72) which has the form*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \exp(-\int_0^t (a + i\omega e_1)(s)ds) \\ 0 \end{pmatrix} (Id + P(\omega, t)) \tag{2.73}$$

with

$$\|P(\omega, .)\| \leq C\Big(\|\frac{b}{e_1 - e_2}\|_\infty \|c\|_\infty\Big) \frac{\exp(kL)}{\omega}.$$

Remark: we can see that the estimate highly depends on the interval length. We shall see that we can improve this result if the coefficients are nearly a constant.

Applying this result, we have found a basis of solutions to (2.72) and the resolvent matrix has the following structure

$$R(\omega, x) = \mathrm{diag}(\exp - \int_0^x a + i\omega e_1, \exp - \int_0^x d + i\omega e_2)\Big(Id + Q(\omega, x)\Big), \tag{2.74}$$

with

$$\|Q(\omega, .)\| \leq C\Big(\|\frac{b}{e_1 - e_2}\|_\infty \|c\|_\infty\Big) \frac{\exp(kL)}{\omega}.$$

It is possible to improve this result if we further assume that the coefficients are almost constant. This is done in the following lemma.

Lemma 5 *Consider the differential system with constant coefficients on* $(0, T)$

$$\begin{array}{l} u' + (a + i\omega e_1)u + bv = 0, \\ v' + (d + i\omega e_2)v + cu = 0 \end{array} \tag{2.75}$$

with $e_1 \neq e_2$ *and* $T = o(\omega)$, *we have an approximation of the resolvent matrix*

$$R(\omega,x) = \begin{pmatrix} \exp -(a+i\omega e_1)x & 0 \\ 0 & \exp -(d+i\omega e_2)x \end{pmatrix} (Id + Q(\omega,x)) \quad (2.76)$$

with

$$\| Q(\omega,.) \|_\infty = O\Big(\frac{T}{\omega}\Big)$$

Then the error estimate is better. In the case of the Saint Venant system, the coefficients converge exponentially to constant as $L \to \infty$ (see section 2.2): we must include an error of order $O(\exp(-kL))$ for a suitable k. Applying the proposition 9 and the lemma to the homogeneous system

$$\begin{array}{l} d_1(H(\xi))g' + (c_1(H(\xi)) + \alpha + i\omega)g + c_2(H(\xi))r = 0, \\ d_2(H(\xi))r' + (c_4(H(\xi)) + \alpha + i\omega)r + c_3(H(\xi))g = 0, \end{array} \quad (2.77)$$

we find an approximation of the resolvent matrix on the interval $I_i^{\epsilon+}$:

$$R^+(\omega,t) = \begin{pmatrix} \exp \int_t^{(i+1)L} \frac{c_1(H(\xi))+\lambda}{d_1(H(\xi))} d\xi & 0 \\ 0 & \exp \int_t^{(i+1)L} \frac{c_4(H(\xi))+\lambda}{d_2(H(\xi))} d\xi \end{pmatrix} \times$$
$$\times (Id + Q_+(\lambda,t)), \quad \forall t \in I_i^{\epsilon+}$$

with $\lambda = \alpha + i\omega$ and $\alpha + c_1(H_0) < 0$. As in section 2.3, for $\epsilon \to 0$,

$$\sup_{I_i^{\epsilon+}} \frac{\left|\frac{c_2(H(\xi))+\alpha}{d_1(H(\xi))}\right|}{\left|\frac{1}{d_1(H(\xi))} - \frac{1}{d_2(H(\xi))}\right|} \leq K,$$

and

$$\sup_{I_i^{\epsilon+}} \frac{|c_3(H(\xi)) + \alpha|}{|d_2(H(\xi))|} \leq K,$$

Thus the presence of the singularity has no influence in the expansion of the resolvent matrix and we can prove that

$$\|Q_+(\lambda,.)\| \leq C\frac{\exp(kl(L))}{\omega}.$$

Since l is a bounded function, we have found a uniform approximation of the resolvent matrix on $I_i^+ =]\xi_i^0, (i+1)L[$. On the interval $I_i^{\epsilon-}$, the resolvent matrix has the form

$$R^-(\lambda,t) = diag(\exp - \int_{iL}^t \frac{c_1(H(\xi)) + \lambda}{d_1(H(\xi))} d\xi, \exp \int_{iL}^t \frac{c_4(H(\xi)) + \lambda}{d_2(H(\xi))} d\xi)$$
$$\times \Big(Id + Q_-(\lambda,t)\Big), \quad \forall t \in]iL, \xi_i^0 - \epsilon[. \quad (2.78)$$

As previously since

$$\sup_{I_i^{\epsilon -}} \frac{\frac{|c_2(H(\xi))+\alpha|}{|d_1(H(\xi))|}}{|\frac{1}{d_1(H(\xi))}-\frac{1}{d_2(H(\xi))}|} \leq K,$$

and

$$\sup_{I_i^{\epsilon -}} \frac{|c_3(H(\xi))+\alpha|}{|d_2(H(\xi))|} \leq K,$$

for all $\epsilon > 0$, the singularity has no influence and we can prove (applying the lemma) that

$$\|Q_-(\lambda, .)\| = O\Big(\frac{L}{\omega}\Big)$$

provided that $L = o(\omega)$.
Now we are in a position to compute $(g_i^{0\pm}, r_i^{0\pm})$ and to prove the second part of theorem 1.

2.5.2 Conclusion and proof of theorem 1 for large $Im(\lambda)$

Using the Duhamel principle on $]\xi_i^0, (i+1)L[$, we compute an expansion of (g_i^{0+}, r_i^{0+}) with the resolvent R^+ and $\Big(g\big((i+1)L\big)^-, r\big((i+1)L\big)^-\Big)$ as initial condition. Then we compute (g_i^{0-}, r_i^{0-}) with the resolvent R^- and $\big(g(iL)^+, r(iL)^+\big)$ as initial conditions on the interval $]iL, \xi_i^0[$. These values are proved to be incompatible with the analycity of (g,r) at points $\xi_i^0, i \in \mathbb{Z}$ and the linearized Rankine Hugoniot jump conditions. Let us note $\lambda = \alpha + i\omega$ with $0 < \alpha < -c_1(H_0)$. With the Duhamel principle, we get the principal part of $(g_i^0, r_i^0)^\pm$. We have the following lemma.

Lemma 6 *Let us note $\lambda = \alpha + i\omega$. Suppose $L = o(\omega)$. On the one hand, the values $\big(g(\xi_i^0), r(\xi_i^0)\big)$ of the possible eigenfunctions (g,r) have the expansion*

$$\begin{aligned} g(\xi_i^0) &= g_i^{0+}\Bigg(1 + O\Big(\frac{1}{\omega}\Big)\Bigg), \\ r(\xi_i^0) &= r_i^{0+}\Bigg(1 + O\Big(\frac{1}{\omega}\Big)\Bigg), \end{aligned} \tag{2.79}$$

with

$$\begin{aligned} g_i^{0+} = \lim_{\epsilon \to 0} \Bigg(& \exp\Big(\int_{H_0+\epsilon}^{H_2} \frac{c_1+\lambda}{P_1 d_1}\Big) g\big((i+1)L\big)^- + \\ & + \int_{H_0+\epsilon}^{H_2} \exp\Big(\int_{H_0+\epsilon}^{h} \frac{c_1+\lambda}{P_1 d_1}\Big) \frac{\lambda d_2}{d_1}\Big(\epsilon_i + \frac{\xi - iL}{L}(\epsilon_{i+1} - \epsilon_i)\Big) dh + \\ & + \int_{H_0+\epsilon}^{H_2} \exp\Big(\int_{H_0+\epsilon}^{h} \frac{c_1+\lambda}{P_1 d_1}\Big) \big(\frac{\epsilon_{i+1} - \epsilon_i}{L}\big) d_2(h) P_1(h) dh \Bigg), \end{aligned}$$

$$
\begin{aligned}
r_i^{0+} = \lim_{\epsilon\to 0}\Bigg(& \exp\Big(\int_{H_0+\epsilon}^{H_2} \frac{c_4+\lambda}{P_1 d_2}\Big) r\big((i+1)L\big)^- + \\
& + \int_{H_0+\epsilon}^{H_2} \exp\Big(\int_{H_0+\epsilon}^{h} \frac{c_4+\lambda}{P_1 d_2}\Big) \frac{\lambda d_1}{d_2}\Big(\epsilon_i + \frac{\xi - iL}{L}(\epsilon_{i+1}-\epsilon_i)\Big) dh + \\
& + \int_{H_0+\epsilon}^{H_2} \exp\Big(\int_{H_0+\epsilon}^{h} \frac{c_4+\lambda}{P_1 d_2}\Big) \big(\frac{\epsilon_{i+1}-\epsilon_i}{L}\big) d_1(h) P_1(h) dh \Bigg).
\end{aligned}
$$

On the other hand, the values $(g(\xi_i^0), r(\xi_i^0))$ *of the possible eigenfunctions* (g,r) *have the expansion*

$$
\begin{aligned}
g(\xi_i^0) &= g_i^{0-}\Big(1 + O\Big(\frac{L}{\omega}\Big)\Big), \\
r(\xi_i^0) &= r_i^{0-}\Big(1 + O\Big(\frac{L}{\omega}\Big)\Big),
\end{aligned}
\tag{2.80}
$$

with

$$
\begin{aligned}
g_i^{0-} = \lim_{\epsilon\to 0}\Bigg(& \exp\Big(-\int_{H_1}^{H_0-\epsilon} \frac{c_1+\lambda}{P_1 d_1}\Big) g(iL)^+ - \\
& - \int_{H_1}^{H_0-\epsilon} \exp\Big(-\int_{h}^{H_0-\epsilon} \frac{c_1+\lambda}{P_1 d_1}\Big) \frac{\lambda d_2}{d_1}\Big(\epsilon_i + \frac{\xi - iL}{L}(\epsilon_{i+1}-\epsilon_i)\Big) dh - \\
& - \int_{H_1}^{H_0-\epsilon} \exp\Big(-\int_{h}^{H_0-\epsilon} \frac{c_1+\lambda}{P_1 d_1}\Big) \big(\frac{\epsilon_{i+1}-\epsilon_i}{L}\big) d_2(h) P_1(h) dh \Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
r_i^{0-} = \lim_{\epsilon\to 0}\Bigg(& \exp\Big(-\int_{H_1}^{H_0-\epsilon} \frac{c_4+\lambda}{P_1 d_2}\Big) r(iL)^+ - \\
& - \int_{H_1}^{H_0-\epsilon} \exp\Big(-\int_{h}^{H_0-\epsilon} \frac{c_4+\lambda}{P_1 d_2}\Big) \frac{\lambda d_1}{d_2}\Big(\epsilon_i + \frac{\xi - iL}{L}(\epsilon_{i+1}-\epsilon_i)\Big) dh - \\
& - \int_{H_1}^{H_0-\epsilon} \exp\Big(-\int_{h}^{H_0-\epsilon} \frac{c_4+\lambda}{P_1 d_2}\Big) \big(\frac{\epsilon_{i+1}-\epsilon_i}{L}\big) d_1(h) P_1(h) dh \Bigg).
\end{aligned}
$$

Proof. We first use the Duhamel principle on the interval $]\xi_i^0+\epsilon, (i+1)L[$ with $\Big(g\big((i+1)L\big)^-, r\big((i+1)L\big)^-\Big)$ as initial condition to get an expansion of $\big(g(\xi_i^0+\epsilon), r(\xi_i^0+\epsilon)\big)$. As $\epsilon \to 0$ since $\frac{\alpha + c_1(H_0)}{P_1(H_0) d_1(H(\xi))} < 0$ for $\xi > \xi_i^0$ and

$\xi \approx \xi_i^0$, the integrals involved in the expressions of $\big(g(\xi_i^0+\epsilon), r(\xi_i^0+\epsilon)\big)$ have a limit and we get a first expansion of $\big(g(\xi_i^0), r(\xi_i^0)\big)$. We make the same computation on the interval $]iL, \xi_i^0-\epsilon[$.$\square$

We now compute an asymptotic expansion of the principal parts $\big(g_i^{0\pm}, r_i^{0\pm}\big)$ of $\big(g(\xi_i^0), r(\xi_i^0)\big)$. We can prove that at first order, $g_i^{0+} = g_i^{0-}$ and in order to find a contradiction, we focus on the expansions of $r_i^{0\pm}$.

Estimate of r_i^{0+}. We recall that

$$\begin{aligned} r_i^{0+} = \lim_{\epsilon\to 0}\Bigg(& \exp\Big(\int_{H_0+\epsilon}^{H_2} \frac{c_4+\lambda}{P_1 d_2}\Big) r\big((i+1)L\big)^- + \\ & + \int_{H_0+\epsilon}^{H_2} \exp\Big(\int_{H_0+\epsilon}^{h} \frac{c_4+\lambda}{P_1 d_2}\Big)\frac{\lambda d_1}{d_2}\Big(\epsilon_i + \frac{\xi - iL}{L}(\epsilon_{i+1}-\epsilon_i)\Big)dh + \\ & + \int_{H_0+\epsilon}^{H_2} \exp\Big(\int_{H_0+\epsilon}^{h} \frac{c_4+\lambda}{P_1 d_2}\Big)\big(\frac{\epsilon_{i+1}-\epsilon_i}{L}\big) d_1(h) P_1(h) dh\Bigg) \end{aligned}$$

As in section 2.3, we break the integral into three parts

$$r_i^{0+} = \exp\Big(\int_{H_0}^{H_2} \frac{c_4+\lambda}{P_1 d_2}\Big)\Big(r\big((i+1)L\big)^- + I_1 + I_2 + I_3\Big),$$

and make several integration by parts. This yields the following expansions.

$$I_1 = (\alpha + i\omega)\epsilon_i \int_{H_0}^{H_2} \exp\Big(-\int_h^{H_2} \frac{c_4+\alpha+i\omega}{d_2 P_1}\Big)\frac{d_1}{d_2}$$

Two integration by parts yield

$$I_1 = \epsilon_i d_1 P_1(H_2) + O\Big(\frac{\max(|\epsilon_i|, |\epsilon_{i+1}|)}{|\omega|}\Big)$$

We now treat the integral I_2

$$I_2 = (\alpha + i\omega)(\epsilon_{i+1}-\epsilon_i)\int_{H_0}^{H_2} \exp\Big(-\int_h^{H_2} \frac{c_4+i\omega}{d_2 P_1}\Big)\frac{d_1}{d_2}\frac{L - l(L) + \frac{f_1(h)}{S}}{L} dh.$$

After an integration by parts, we obtain

$$I_2 = (\epsilon_{i+1}-\epsilon_i)\Big(\big(1 + O(\frac{1}{L})\big) d_1 P_1(H_2) + O\big(\frac{1}{|\omega|}\big)\Big).$$

Finally, we focus on I_3

$$I_3 = \frac{\epsilon_{i+1} - \epsilon_i}{L} \int_{H_0+\epsilon}^{H_2} \exp(\int_{H_0+\epsilon}^{h} \frac{c_4 + i\omega}{d_2 P_1}) d_1 P_1(h) dh.$$

This gives after integration by parts

$$I_3 = O\Big(\frac{\max(|\epsilon_i|, |\epsilon_{i+1}|)}{\omega}\Big).$$

Hence we find that

$$r_i^{0+} \sim_{(L,|\omega|\to+\infty} \exp\Big(\int_{H_0}^{H_2} \frac{c_4 + \alpha + i\omega}{d_2 P_1}\Big)\Big(\epsilon_{i+1} d_1 P_1(H_2) + r\big((i+1)L\big)^-\Big).$$

Estimate of r_i^{0-}. Recall that

$$\begin{aligned} r_i^{0-} &= \lim_{\epsilon\to 0}\Bigg(\exp\Big(-\int_{H_1}^{H_0-\epsilon} \frac{c_4+\lambda}{P_1 d_2}\Big) r(iL)^+ - \\ &- \int_{H_1}^{H_0-\epsilon} \exp\Big(-\int_{h}^{H_0-\epsilon} \frac{c_4+\lambda}{P_1 d_2}\Big)\frac{\lambda d_1}{d_2}\Big(\epsilon_i + \frac{\xi - iL}{L}(\epsilon_{i+1}-\epsilon_i)\Big) dh - \\ &- \int_{H_1}^{H_0-\epsilon} \exp\Big(-\int_{h}^{H_0-\epsilon} \frac{c_4+\lambda}{P_1 d_2}\Big)\big(\frac{\epsilon_{i+1}-\epsilon_i}{L}\big) d_1(h) P_1(h) dh\Bigg). \end{aligned}$$

We focus on the expansion of the integral

$$I_1 = \lambda \epsilon_i \int_{H_1}^{H_0} \exp\Big(-\int_{h}^{H_0} \frac{c_4+\lambda}{d_2 P_1}\Big)\frac{d_1}{d_2} dh.$$

After two integration by parts, we get

$$\begin{aligned} I_1 &= -\epsilon_i d_1 P_1(H_1) \exp\Big(-\int_{H_1}^{H_0} \frac{c_4+\lambda}{d_2 P_1}\Big) + \\ &+ \frac{\epsilon_i}{i\omega} \exp\Big(-\int_{H_1}^{H_0} \frac{c_4 + i\omega}{d_2 P_1}\Big)\Big(c_4 d_1 P_1(H_1) + d_2 P_1 (d_1 P_1)'(H_1)\Big) - \\ &- \frac{\epsilon_i}{i\omega} d_1' d_2 P_1^2(H_0) + O\Big(\frac{\max(|\epsilon_{i+1}|, |\epsilon_i|)}{\omega^2}\Big). \end{aligned}$$

We now come to the computation of I_2

$$I_2 = \lambda(\epsilon_{i+1} - \epsilon_i) \int_{H_1}^{H_0} \exp\Big(- \int_h^{H_0} \frac{c_4 + \lambda}{d_2 P_1}\Big) \frac{d_1}{d_2} \frac{L - l(L) + \frac{f_1}{S}}{L} dh$$

An integration by parts yields

$$\begin{aligned} I_2 = \frac{\epsilon_{i+1} - \epsilon_i}{i\omega} \Big(& - d_2 P_1 \Big(d_1 P_1 (\frac{L - l(L) + \frac{f_1}{S}}{L}) \Big)'(H_0) \\ & + \exp\Big(- \int_{H_1}^{H_0} \frac{c_4 + i\omega}{d_2 P_1} \Big) d_2 P_1 \Big(d_1 P_1 (\frac{L - l(L) + \frac{f_1}{S}}{L}) \Big)'(H_2) + \\ & + O\Big(\frac{1}{|\omega|}\Big)\Big). \end{aligned}$$

The last term is given by

$$I_3 = \frac{\epsilon_{i+1} - \epsilon_i}{L} \int_{H_1}^{H_0} \exp\Big(- \int_h^{H_0} \frac{c_4 + i\omega}{d_2 P_1}\Big) d_1 P_1 dh.$$

We can prove that

$$I_3 = O\Big(\frac{\max(|\epsilon_{i+1}|, |\epsilon_i|)}{|\omega|}\Big).$$

This yields

$$r_i^{0-} \sim \Big(r(iL)^+ + \epsilon_i d_1 P_1(H_1) \Big) \exp\Big(- \int_{H_1}^{H_0} \frac{c_4 + \lambda}{d_2 P_1} \Big).$$

We come to the proof of theorem 1 in the case of large $Im(\lambda)$. The continuity of r at points ξ_i^0 reads

$$r((i+1)L)^- + \epsilon_{i+1} d_1 P_1(H_2) = \exp - \int_{H_1}^{H_2} \frac{c_4 + \lambda}{P_1 d_2} (r(iL)^+ + p_i d_2 P_1(H_1)) \quad (2.81)$$

On the other hand, the linearized Rankine Hugoniot conditions yield:

$$\begin{aligned} r((i+1)L)^- &= c(H_1, H_2) g((i+1)L)^+ + d(H_1, H_2) r((i+1)L)^+ \\ &- \epsilon_{i+1} \frac{d_1}{d_2}(H_2)(H_2 - H_1)\lambda, \end{aligned}$$

where c is a continuous function of H_1, H_2 and d is given by

$$d(H_1, H_2) = \frac{\frac{d_1}{d_2}(H_2) - 1}{\frac{d_1}{d_2}(H_1) - 1}.$$

Since

$$d_1(H) - d_2(H) \geq 2\sqrt{GH_1}, \quad \forall H \in (H_1, H_2),$$

the function d is well defined, continuous and does not vanish on (H_1, H_2). Inserting (2.82) into (2.81) yields the following recurrence relation for the sequence $r(iT)^+$:

$$d(H_1, H_2)r((i+1)L)^+ - \exp - \int_{H_1}^{H_2} \frac{c_4 + \lambda}{P_1 d_2} r(iL)^+ = d_i \tag{2.82}$$

with

$$d_i = -\epsilon_{i+1}\frac{d_1}{d_2}(H_2)(H_2 - H_1)\lambda - \epsilon_{i+1}d_1 P_1(H_2) +$$
$$+ \epsilon_i \exp(-\int_{H_1}^{H_2} \frac{c_4 + \lambda}{P_1 d_2}) d_1 P_1(H_1) - c(H_1, H_2) g((i+1)L)^+.$$

This relation can be written in a simpler way

$$r((i+1)L)^+ - q(H_1, H_2) r(iL)^+ = c_i \tag{2.83}$$

where $q(H_1, H_2)$ and c_i are easily obtained by identification. Let us normalize $r(0)^+ = 1$, we are looking for an eigenvector (g, r, ϵ_i) such that $lim_{\xi \to \pm\infty} \|(g, r)\|(\xi) = 0$. Then we can see that $\lim_{i \to \pm\infty} |r(iL)^+| = 0$ and $\lim_{i \to \pm\infty} |c_i| = 0$. On the other hand, since $Re(c_4(H_a) + \lambda) > 0$ (see the graph), we can prove that $\lim_{L \to \infty} |q(H_1, H_2)| = +\infty$.

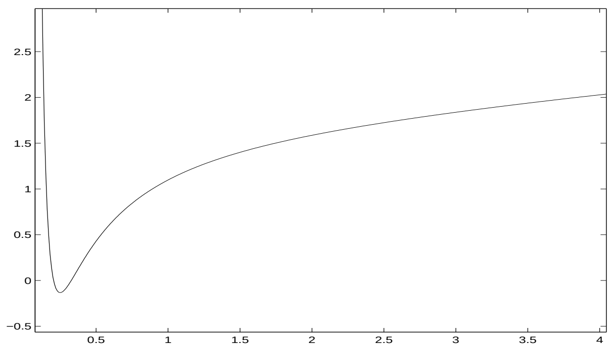


Figure 2.8: Graph of the rescaled coefficient C_4

Hence for a sufficiently large L, we have $|q(H_1, H_2)| > 1$. A short computation gives

$$r(nL)^+ = q(H_1, H_2)^n + \sum_{i=1}^{n} q(H_1, H_2)^{n-i} c_{i-1}.$$

Then for large i, we have $r(iL)^+ \sim q(H_1, H_2)^i$ which is in contradiction with the growth assumption on r. And the theorem is proved provided that we have chosen $L = o(\omega)$ to ensure that all the others terms involved in the different expansions are sufficiently small and do not have any influence. We have proved

Proposition 10 *There exists a function $r_2(L) > 0$ satisfying $\lim_{L\to\infty} \frac{r_2(L)}{L} = +\infty$ and a constant L_0 such that for any (λ, L) satisfying $L > L_0$ and $Im(\lambda) > r_2(L)$, there are no unstable eigenvalues λ for the spectral problem (2.25),(2.26).*

The proof of theorem 1 is complete. This result will be helpful to prove theorem 2: when $S = \tan(\theta) \to 0$, the spectral problem (2.25),(2.26) has no unstable eigenvalues.

2.6 Stability of Dressler roll-waves when $S \to 0$

We treat here a case where the computations are simpler. We come to the proof of theorem 2. We restrict our attention to the spaces of functions (h, v) with an exponential decrease (there exists K and $\alpha > 0$ such that $\max(|h(\xi)|, |v(\xi)|) \leq K \exp(-\alpha|\xi|)$. As $S \to 0$, Dressler roll-waves are proved to be linearly stable. We divide the proof into two parts. On the one hand, we consider $O(S)$ unstable eigenvalues. After a rescaling $\lambda = \overline{\lambda} S$, we have a new spectral problem involving $\overline{\lambda}$ which is simpler: we can prove that there are no unstable eigenvalues $\overline{\lambda}$ for S sufficiently small. On the other hand, we consider $O(1)$ unstable eigenvalues. Applying theorem 1, we can prove that for S sufficiently small, there are no $O(1)$ unstable eigenvalues.

2.6.1 The case of $O(S)$ unstable eigenvalues

Let us suppose the existence of $O(S)$ unstable eigenvalues. After the rescaling, $\lambda = \overline{\lambda} S$ ($\overline{\lambda}$ lying in a compact set), the spectral problem (2.31),(2.33) reads:

$$\begin{aligned} D_1(h)P(h)g' + (C_1(h) + \overline{\lambda})g + C_2(h)r &= S\overline{\Gamma}_i, \\ D_2(h)P(h)g' + (C_4(h) + \overline{\lambda})r + C_3(h)g &= S\overline{\Delta}_i, \end{aligned} \tag{2.84}$$

The functions $\overline{\Gamma}_i$, $\overline{\Delta}_i$ can be written

$$\begin{aligned} \overline{\Gamma}_i &= \epsilon_i a_1(\overline{\lambda}, h) + \epsilon_{i+1} a_2(\overline{\lambda}, h), \\ \overline{\Delta}_i &= \epsilon_i b_1(\overline{\lambda}, h) + \epsilon_{i+1} b_2(\overline{\lambda}, h), \end{aligned}$$

where a_i, b_i are obtained with an easy identification. Let us note $\mathcal{U}_j = (g_j, r_j)$ the unique analytic solution of

$$\begin{array}{l} D_1(h)P(h)g' + (C_1(h) + \overline{\lambda})g + C_2(h)r = a_j(\overline{\lambda}, h), \\ D_2(h)P(h)g' + (C_4(h) + \overline{\lambda})r + C_3(h)g = b_j(\overline{\lambda}, h), \end{array} \tag{2.85}$$

for $j = 1, 2$, passing through the singularity ξ_0^0 such that $g_j(\xi_0^0) = 0$.
Let us note also $\mathcal{U}_0 = (\mathcal{G}_0, \mathcal{R}_0)$ the unique analytic solution of

$$\begin{array}{l} D_1(h)P(h)g' + (C_1(h) + \overline{\lambda})g + C_2(h)r = 0, \\ D_2(h)P(h)g' + (C_4(h) + \overline{\lambda})r + C_3(h)g = 0, \end{array} \tag{2.86}$$

such that $\mathcal{G}_0(\xi_0^0) = 1$. Then, the solution $\mathcal{U} = (g, r)$ of the complete differential system (2.31) can be written on the interval $]iL, (i+1)L[$

$$\mathcal{U}(\xi) = \alpha_i \mathcal{U}_0(H(\xi)) + S\epsilon_i \mathcal{U}_1(H(\xi)) + \epsilon_{i+1}\mathcal{U}_2(H(\xi)). \tag{2.87}$$

The exponential decrease assumption on $\mathcal{U}$ imply that for some K and $\alpha > 0$

$$|\alpha_i| = |g(\xi_i^0)| \leq K\exp(-\alpha|i|), \quad |S\epsilon_i| \leq K\exp(-\alpha|i|). \tag{2.88}$$

We normalize this solution with $g(\xi_0^0) = \alpha_0 = 1$. The linearized Rankine-Hugoniot conditions read

$$[E(H)\begin{pmatrix} g \\ r \end{pmatrix}]_{H_1}^{H_2} = \epsilon_i S\overline{\lambda}\begin{pmatrix} 1 \\ c \end{pmatrix}. \tag{2.89}$$

We replace (2.87) into (2.89) and find that

$$\begin{aligned} \alpha_{i+1}E(H_1)\mathcal{U}_0(H_1) - \alpha_i E(H_2)\mathcal{U}_0(H_2) = {} & S\epsilon_{i-1}C_1(H_1, H_2) + S\epsilon_i C_2(H_1, H_2) + \\ & + S\epsilon_{i+1}C_3(H_1, H_2), \end{aligned} \tag{2.90}$$

where C_i are bounded functions of H_1, H_2. Now we focus on the first line of the system (2.90). Since the function

$$\begin{pmatrix} \mathcal{H}(H(\xi)) \\ \mathcal{V}(H(\xi)) \end{pmatrix} = P^{-1}(H(\xi))\begin{pmatrix} \mathcal{G}_0 \\ \mathcal{R}_0 \end{pmatrix})$$

is an analytic solution of the original homogeneous system (2.35), we find

$$(\mathcal{V} - c\mathcal{H})' = O(S)$$

Thus the first line of

$$E(H_1)\mathcal{S}_0(H_1) - E(H_2)\mathcal{S}_0(H_2) = A(H_1)\begin{pmatrix} \mathcal{H}(H_1) \\ \mathcal{V}(H_1) \end{pmatrix} - A(H_2)\begin{pmatrix} \mathcal{H}(H_2) \\ \mathcal{V}(H_2) \end{pmatrix}$$

can be written

$$(\mathcal{V}(H_2) - c\mathcal{H}(H_2)) - (\mathcal{V}(H_1) - c\mathcal{H}(H_1)) = O(S).$$

Then the first line of (2.90) reads

$$\begin{aligned}\alpha_{i+1} - (1 + k(S))\alpha_i &= S\epsilon_{i-1}c_1(H_1, H_2) + \\ &\quad + S\epsilon_i c_2(H_1, H_2) + S\epsilon_{i+1}c_3(H_1, H_2).\end{aligned} \tag{2.91}$$

where c_j are bounded functions and $k(S) = O(S)$. Let us write (2.91) into a simpler form

$$\alpha_{i+1} - (1 + k(S))\alpha_i = Sv_i,$$

where $|Sv_i| \leq K\exp(-\alpha|i|)$ and $\max_i(|v_i|) < C$.
Suppose $|1 + k(S)| \geq 1$. Then we can prove that

$$\alpha_n = \Big(\alpha_0 + \sum_{j=0}^{n-1} Sv_j(1 + k(S))^{-j-1}\Big)(1 + k(S))^n.$$

We have $|Sv_j(1 + k(S))^{-j-1}| \leq S|v_j| \leq K\exp(-\alpha|j|)$ and $\lim_{S\to 0} Sv_i = 0$ then

$$\lim_{S\to 0} \sum_{j=0}^{+\infty} Sv_j(1 + k(S))^{-j-1} = 0.$$

Thus we have proved that $\alpha_n = (\alpha_0 + o(1))(1 + k(S))^n$ which is in contradiction with the growth assumption (2.88). To study the case $|1+k(S)| \leq 1$, we change i into $-i$ in (2.91) to find the contradiction. Thus for S sufficiently small, we have proved that there are no $O(S)$ unstable eigenvalues. $\square$

2.6.2 The case of $O(1)$ unstable eigenvalues

Let us consider the spectral problem (2.31),(2.33) and work on the interval $]iL, (i+1)L[$. After the change of coordinates $\xi \to H(\xi)$ and $H \to H_0 h$, we find the system (see computations in section 2.2)

$$\begin{aligned}&SP(h)D_1(h)g' + (SC_1(h) + \lambda)g + SC_2(h)r = S\Gamma_i(h),\\ &SP(h)D_2(h)r' + (SC_4(H(\xi)) + \lambda)r + SC_3(H(\xi))g = S\Delta_i(h).\end{aligned} \tag{2.92}$$

For $S = 0$, we find $\lambda(g, r) = 0$. Thus, we have $\lambda = 0$ eigenvalue. Now we come to the case $0 < S << 1$. Dividing by S, the differential system (2.92) now reads

$$\begin{aligned}&P(h)D_1(h)g' + (C_1(h) + \frac{\lambda}{S})g + C_2(h)r = \Gamma_i(h),\\ &P(h)D_2(h)r' + (C_4(H(\xi)) + \frac{\lambda}{S})r + C_3(H(\xi))g = \Delta_i(h).\end{aligned} \tag{2.93}$$

The computations of sections 2.3 and 2.4 remain valid: when $\frac{\lambda}{S}$ is sufficiently large and $Re(\lambda) \geq 0$, λ is not an eigenvalue of the spectral problem (2.31), (2.33). Thus for S sufficiently small, we have proved that there are no $O(1)$ unstable eigenvalues.$\square$

The proof of theorem 2 is complete.

Appendix

Proposition 5 *Consider the linear differential system*

$$\begin{array}{c} u_1' + (a(t) + \lambda e_1(t))u_1 + b(t)u_2 = 0, \\ u_2' + (d(t) - \lambda e_2(t))u_2 + c(t)u_1 = 0, \end{array} \tag{2.94}$$

for $t \in [0, T]$. *We suppose that* $a, b, c, d, e_1, e_2 \in C^0(0, T)$ *and* $\min_{(0,T)}(e_1, e_2) \geq \delta > 0$ *Then there exists a solution which has the form*

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (Id + P(\lambda, t)) \begin{pmatrix} \exp(-\int_0^t a + \lambda e_1 ds) \\ \exp(\int_t^T d - \lambda e_2 ds) \end{pmatrix} \tag{2.95}$$

with

$$\| P(\lambda, .) \|_\infty \leq \Big(\| \frac{b}{e_1 + e_2} \|_\infty + \| \frac{c}{e_1 + e_2} \|_\infty \Big) \frac{C(T)}{Re(\lambda)}$$

and

$$C(T) = \exp\Big((\|a\|_\infty + \|d\|_\infty) T \Big).$$

This solution decrease to 0 *as* $Re(\lambda) \to \infty$.

Proof. Assume (u_1, u_2) to have the form

$$\begin{cases} u_1(t) = \exp(-\int_0^t a + \lambda e_1 ds) + v_1(t) \\ u_2(t) = \exp(\int_t^T d - \lambda e_2 ds) + v_2(t) \end{cases} \tag{2.96}$$

with $v_1(0) = 0$ and $v_2(T) = 0$. Plugging (2.96) into (2.94), we can see that (v_1, v_2) satisfy the system

$$\begin{aligned} v_1(t) = & - \int_0^t b(s) \exp(- \int_s^t a + \lambda e_1 du) \exp(\int_s^T d - \lambda e_2 du) ds - \\ & - \int_0^t b(s) \exp(- \int_s^t a + \lambda e_1 du) v_2(s) ds = \mathcal{R}(v_2)(t) \end{aligned}$$

$$v_2(t) = \int_t^T c(s)\exp(-\int_0^s a+\lambda e_1 du)\exp(\int_t^s d-\lambda e_2 du)ds+$$
$$+\int_t^T c(s)\exp(\int_s^t d-\lambda e_2 du)v_1(s)ds = \mathcal{S}(v_1)(t). \quad (2.97)$$

These formula are obtained with a classical variation of the constant for inhomogeneous linear differential equations. Hence v_1 can be seen as a fixed point of the operator $\mathcal{R}\circ\mathcal{S}$. We want to prove that $\mathcal{R}\circ\mathcal{S}$ is a contraction in a suitable Banach space in order to use the fixed point theorem. Let us introduce the following weighted norm

$$\|v\| = \sup_{t\in(0,T)}\left(\frac{|v(t)|}{(\exp(-Re(\lambda)\int_0^t e_1 du)+\exp(-Re(\lambda)\int_t^T e_2 du)}\right).$$

The space $\mathcal{B}=(C^0(0,T),\|.\|)$ is a Banach space. We can prove that

$$|R(v)(t)| \le C(T)\Bigg(\|\frac{b}{e_1+e_2}\|_\infty \frac{\exp(-Re(\lambda)\int_t^L e_2 du)}{Re(\lambda)}+$$
$$+\int_0^t |b(s)|\exp(-Re(\lambda)\int_s^t e_1 du)|v(s)|ds\Bigg), \quad (2.98)$$

$$|S(v)(t)| \le C(L)\Bigg(\|\frac{c}{e_1+e_2}\|_\infty \frac{\exp(-Re(\lambda)\int_0^t e_1 du)}{Re(\lambda)}+$$
$$+\int_t^T |c(s)|\exp(-Re(\lambda)\int_s^L e_2 du)|v(s)|ds\Bigg) \quad (2.99)$$

Hence with these estimates, we have

$$\begin{aligned}\| \mathcal{R}(v) \| &\le C(T)\|\frac{b}{e_1+e_2}\|_\infty\left(\frac{1}{Re(\lambda)}+\frac{\| v \|}{Re(\lambda)}\right),\\ \| \mathcal{S}(v) \| &\le C(T)\|\frac{c}{e_1+e_2}\|_\infty\left(\frac{1}{Re(\lambda)}+\frac{\| v \|}{Re(\lambda)}\right).\end{aligned} \quad (2.100)$$

Hence for large $Re(\lambda)$, $\mathcal{R}\circ\mathcal{S}$ preserves a ball $B_\lambda = B(0,\frac{2C(T)M}{Re(\lambda)})$ where $M=\|\frac{b}{e_1+e_2}\|_\infty+\|\frac{c}{e_1+e_2}\|_\infty$. It is easily proved that $\mathcal{R}\circ\mathcal{S}$ is a contraction in B_λ and then have a fixed point lying in B_λ. And this completes the proof.□

Proposition 6 *There exists a solution of (2.94) which has the form*

$$\begin{pmatrix}u_1\\u_2\end{pmatrix} = (Id+Q(\lambda,t))\begin{pmatrix}\exp(\int_t^T \lambda e_1+ads)\\ \exp(\int_0^t \lambda e_2-dds)\end{pmatrix} \quad (2.101)$$

with

$$\| Q(\lambda,.) \|_\infty \leq \Big(\| \frac{b}{e_1+e_2} \|_\infty + \| \frac{c}{e_1+e_2} \|_\infty \Big) \frac{C(T)}{Re(\lambda)}$$

and

$$C(T) = \exp \Big((\|a\|_\infty + \|d\|_\infty) T \Big).$$

Proof. Making the change of variable $t \to -t$ on the interval $]-T,0[$, the proof is similar to the proof of proposition 5.$\square$

Lemma

Consider the differential system with constant coefficients

$$\begin{array}{l} u' + (a+\lambda e_1)u + bv = 0, \\ v' + (d+\lambda e_2)v + cu = 0, \end{array} \tag{2.102}$$

with $e_1 \neq e_2$ *on the interval* $[0,T]$. *As* $Re(\lambda) \to \infty$*, the resolvent matrix has the form*

$$R(\lambda,t) = R_0(\lambda,t)\Big(Id + O\big(\frac{T}{Re(\lambda)}\big)\Big) \tag{2.103}$$

with

$$R_0(\lambda,t) = diag(\exp -(a+\lambda e_1)t, \exp -(d+\lambda e_2)t).$$

Proof. We write the system under a matricial form $X' = M(\lambda)X$ with $X = (u,v)$. As $Re(\lambda) \to \infty$, the matrix $M(\lambda)$ is diagonalizable and the resolvent matrix has the form

$$R(\lambda,t) = diag(\exp -\alpha_1(\lambda)t, \exp -\alpha_2(\lambda)t),$$

where α_i are the eigenvalues of $M(\lambda)$. The expansion of the α_i are given by

$$\begin{aligned} \alpha_1(\lambda) &= \lambda e_1 + a + O\Big(\frac{1}{Re(\lambda)}\Big) \\ \alpha_2(\lambda) &= \lambda e_2 + d + O\Big(\frac{1}{Re(\lambda)}\Big) \end{aligned} \tag{2.104}$$

Then we have

$$R(\lambda,t) = R_0(\lambda,t) diag\Big(\exp O\big(\frac{t}{Re(\lambda)}\big), \exp O\big(\frac{t}{Re(\lambda)}\big)\Big).$$

This completes the proof.$\square$

Proposition 9 *Consider the differential system*

$$\begin{cases} u' + (a + i\omega e_1)u + bv = 0, \\ v' + (d + i\omega e_2)v + cu = 0, \end{cases} \tag{2.105}$$

$\forall t \in (0,T)$. *Assume* $(a,b,c,d,e_1,e_2) \in C^0(0,T)$ *and* $\min_{t\in(0,T)} |e_1(t) - e_2(t)| \geq \delta > 0$. *Then there exists a solution* (u,v) *of (2.105) which has the form*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \exp(-\int_0^t (a+i\omega e_1)(s)ds) \\ 0 \end{pmatrix} (Id + Q(\omega,t)) \tag{2.106}$$

with

$$\|Q(\omega,.)\|_\infty \leq \Big(\|c\|_\infty \|\frac{b}{e_1 - e_2}\|_\infty\Big)\frac{C(T)}{\omega}$$

and

$$C(T) = \exp\Big((\|a\|_\infty + \|d\|_\infty)T\Big).$$

Proof. We assume that (u,v) has the following form

$$(u,v)(t) = \exp(-\int_0^t (a + i\omega e_1)(s)ds)(1 + U(t), V(t)), \tag{2.107}$$

with $U(0) = V(0) = 0$. We plug the ansatz (2.107) into the differential system (2.72) and we get:

$$\begin{array}{l} U(x) = -\int_0^x b(s)V(s)ds = \mathcal{R}(V)(x), \\ V(x) = \int_0^x \exp(-\int_s^x (d - a + i\omega(e_2 - e_1)))c(s)(1 + U(s))ds = \mathcal{S}(U)(x) \end{array}$$

Then V can be seen as a fixed point of $\mathcal{S}\circ\mathcal{R}$ in the Banach space $C^0(0,L)$ with the norm $\|.\|_\infty$. Let us prove that $\mathcal{S}\circ\mathcal{R}$ is a contraction. The operator $\mathcal{S}\circ\mathcal{R}$ is affine and can be written $\mathcal{S}\circ\mathcal{R}(V)(x) = \alpha(x) + M(V)(x)$ where M is a linear operator and α is given by:

$$\alpha(x) = \int_0^x \exp(-\int_s^x (d - a + i\omega(e_2 - e_1)))c(s)ds$$

We make an integration by parts to obtain an estimate of α:

$$\| \alpha \|_\infty \leq \frac{\| c \|}{\omega\delta}(1 + \exp((\| d - a \|)T)) \leq (\frac{2 \| c \|}{\omega\delta})\frac{\exp(kT)}{\omega} \tag{2.108}$$

Then we make an estimate of $M(v)$. We have:

$$M(V)(x) = \int_0^x \exp(-\int_s^x (d-a+i\omega(e_2-e_1)))c(s)(\int_0^s b(u)V(u)du)ds \tag{2.109}$$

Applying Fubini's theorem, we get:

$$\begin{aligned} M(V)(x) = \int_0^x \Big(\int_u^x \exp(-\int_s^u (d-a+i\omega(e_2-e_1))c(s)ds) \Big) \times \\ \times b(u)V(u)\exp(-\int_u^x (d-a+i\omega(e_2-e_1))du \end{aligned} \tag{2.110}$$

Then, after an integration by parts, we find that:

$$\| M(V) \|_\infty \leq \Big(\|c\|_\infty \| \frac{b}{e_1-e_2} \|_\infty \Big) \frac{C(T)}{|\omega|} \| V \|_\infty \tag{2.111}$$

where $C(T) = O(\exp(kT))$ for $k = \|a\|_\infty + \|d\|_\infty$. Hence for a sufficiently large ω, equation (2.111) imply that the operator $\mathcal{S} \circ \mathcal{R}$ is a contraction. Moreover equations (2.108) and (2.111) imply that $\mathcal{S} \circ \mathcal{R}$ preserves a ball B_ω with a radius of order $O(\frac{C(T)}{|\omega|})$. And we conclude the proof with a fixed point theorem in Banach spaces.□

Lemma *For the differential system with constant coefficient on* $(0,T)$

$$\begin{cases} u' + (a+i\omega e_1)u + bv = 0 \\ v' + (d+i\omega e_2)v + cu = 0 \end{cases} \tag{2.112}$$

with $e_1 \neq e_2$ *and* $T = o(\omega)$, *we have an approximation of the resolvent matrix:*

$$R(\omega,x) = \begin{pmatrix} \exp-(a+i\omega e_1)x & 0 \\ 0 & \exp-(d+i\omega e_2)x \end{pmatrix} (Id + Q(\omega,x)) \tag{2.113}$$

with

$$\| Q(\omega,.) \|_\infty = O(\frac{T}{\omega})$$

Proof. Since $e_1 \neq e_2$, when ω is large, the matrix is diagonalizable and the resolvent matrix has the form

$$R(x) = \begin{pmatrix} \exp-\lambda_1 x & 0 \\ 0 & \exp-\lambda_2 x \end{pmatrix} \tag{2.114}$$

where λ_i are the eigenvalues of the constant matrix. An easy calculation leads to the following expansions:

$$\begin{aligned} \lambda_1 &= i\omega e_1 + a + O\Big(\frac{1}{\omega}\Big), \\ \lambda_2 &= i\omega e_2 + d + O\Big(\frac{1}{\omega}\Big) \end{aligned} \tag{2.115}$$

We factor R to obtain

$$R(x) = \begin{pmatrix} \exp-(a+i\omega e_1)x & 0 \\ 0 & \exp-(d+i\omega d_2)x \end{pmatrix} \begin{pmatrix} \exp(O(\frac{1}{\omega})x) & 0 \\ 0 & \exp(O(\frac{1}{\omega})x)) \end{pmatrix} \tag{2.116}$$

And this completes the proof.□

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Chapitre 3

Phénomène de roll-waves pour les équations de Saint Venant avec fond périodique et dans le p-système avec terme source

3.1 Introduction

Dans ce chapitre, on étudie un écoulement de faible profondeur sur une pente avec un fond périodique de petite amplitude. On peut le décrire par les équations de Saint Venant adimensionnées

$$\begin{aligned} &h_t + (hu)_x = 0, \\ &u_t + uu_x + Gh_x = -G\epsilon b'(x) + \frac{1}{2}(1 - \frac{u^2}{h}) \end{aligned} \tag{3.1}$$

où $G = \frac{1}{F^2}$ et F désigne le nombre de Froude. Le paramètre ϵ est considéré proche de 0. La fonction b est 1-périodique en espace.
On étudie également un système plus simple ayant une structure proche du système de Saint Venant : le p-système avec un terme source

$$\begin{aligned} &u_t + v_x = 0, \\ &v_t + (p(u))_x = \epsilon b'(x) + f(u) - v \end{aligned} \tag{3.2}$$

où les fonctions p et f sont régulières et leurs premières dérivées sont globalement bornées.
Dans les deux cas, on cherche à mettre en évidence le phénomène de roll-waves.
Pour le modèle de Saint Venant (3.1), les expérimentations et les simulations numériques menées à l'Institut de Mécanique des Fluides de Toulouse

mettent en évidence l'existence de roll-waves avec une sélection de la longueur d'onde correspondant au double de la période du fond [12], [22] (voir également [17],[18], [19] pour une description du dispositif expérimental). Le but de ce chapitre est de tenter de donner une explication à ce phénomène. L'étude menée ici possède deux parties.
Dans un premier temps, on suit une démarche similaire à celle introduite par Dressler. On montre d'abord l'existence d'une solution stationnaire 1-périodique en espace. Ensuite on étudie la stabilité linéaire de cette solution stationnaire pour des perturbations n-périodiques et on montre l'existence d'un Froude critique $F_c = F(\epsilon, n)$ pour lequel le flot devient instable dans l'espace des fonctions n-périodiques. On donnera un développement limité de F_c au voisinage de $\epsilon = 0$. On montre que le Froude critique est maximal pour $n = 2$ et les perturbations de période $n = 2$ sont les moins stables. La question est alors de savoir quel type d'instabilité on observe lorsque $F > F_c$ (i.e. la solution stationnaire est instable). On s'intéresse ainsi aux instabilités de petite amplitude. Après avoir écrit ces systèmes avec les invariants de Riemann, on dérivera une équation de Burgers avec un terme source. Cette équation possède des solutions roll-waves dont la vitesse d'onde est donnée par

$$c'(t) = c_0 + \epsilon c_1(t) + O(\epsilon^2),$$

où c_1 est périodique et $c_0 = \sqrt{p'(u_0)}$. Soit $r = v + \phi(u)$ et $s = v - \phi(u)$, le p-système s'écrit

$$\begin{aligned} r_t + \sqrt{p' \circ \phi^{-1}(\frac{r-s}{2})} r_x &= \epsilon b'(x) + f \circ \phi^{-1}(\frac{r-s}{2}) - \frac{r+s}{2}, \\ s_t - \sqrt{p' \circ \phi^{-1}(\frac{r-s}{2})} s_x &= \epsilon b'(x) + f \circ \phi^{-1}(\frac{r-s}{2}) - \frac{r+s}{2}. \end{aligned} \tag{3.3}$$

On démontre le résultat suivant.

Théorème 6 *Il existe une solution entropique (r, s) des équations (3.3) de type roll-waves. On a*

$$\begin{aligned} r(x,t) &= r_0 + \epsilon R(\tfrac{x-c(t)}{\epsilon}, t) + O(\epsilon^2), \\ s(x,t) &= s_0 + \epsilon S(t) + O(\epsilon^2). \end{aligned} \tag{3.4}$$

où R est solution de

$$R_t + \frac{p''(u_0)}{4p'(u_0)}(R - \frac{R(1,t) + R(-1,t)}{2}) R_\xi = AR - BS(t) + b'(c_0 t). \tag{3.5}$$

3.2 Instabilités dans les équations de Saint Venant : mise en évidence d'un nombre de Froude critique

3.2.1 Existence d'une solution stationnaire périodique

On recherche une solution stationnaire (H, U) 1-périodique du système de Saint Venant. Elle vérifie le système

$$\begin{aligned} &(HU)' = 0, \\ &UU' + GH' = -G\epsilon b'(x) + \tfrac{1}{2}(1 - \tfrac{U^2}{H}). \end{aligned} \tag{3.6}$$

On normalise le débit $HU = 1$ et en injectant dans la deuxième équation de (3.6), on obtient

$$H' = \frac{1 - 2G\epsilon b'(x)}{2(GH^3 - 1)}(H(x)^3 - H_1(x, \epsilon)^3) \tag{3.7}$$

où $H_1(x, \epsilon) = (1 - 2G\epsilon b'(x))^{-\frac{1}{3}}$. On peut écrire $H_1(x, \epsilon) = 1 + \epsilon h_1(x) + \epsilon^2 h_2(x, \epsilon)$ où h_1 et h_2 sont 1-périodiques. On recherche H sous la forme $H(x) = 1 + \epsilon h(x)$. La fonction h vérifie

$$h' = \frac{3}{2(G-1)}(h + h_1(x)) + \epsilon f(h, x, \epsilon). \tag{3.8}$$

où

$$\begin{aligned} \epsilon f(h, x, \epsilon) = &\frac{1 - 2G\epsilon b'(x)}{2(G(1 + \epsilon h(x))^3 - 1)}((1 + \epsilon h(x))^3 - H_1(x, \epsilon)^3) - \\ &- \frac{3}{2(G-1)}(h + h_1(x)). \end{aligned} \tag{3.9}$$

En utilisant une méthode de variation de la constante, on obtient

$$\begin{aligned} h(x) = &\exp(\frac{3}{2(G-1)}x)h(0) + \\ &+ \int_0^x \exp \frac{3}{2(G-1)}(x - y)h_1(y)dy + \\ &+ \int_0^x \exp \frac{3}{2(G-1)}(x - y)\epsilon f(h(y), y, \epsilon)dy. \end{aligned} \tag{3.10}$$

On choisit $h(0)$ de telle sorte que $h(1) = h(0)$. La fonction h s'écrit alors $h(x) = h_0(x) + T(h, \epsilon)(x)$ où h_0 est la solution 1-périodique de l'équation principale

$$h' = \frac{3}{2(G-1)}(h + h_1(x)). \tag{3.11}$$

et

$$T(h,\epsilon)(x) = \exp(\frac{3}{2(G-1)}x)\frac{\int_0^1 \exp\frac{3}{2(G-1)}(1-y)\epsilon f(h(y),y,\epsilon)dy}{1-\exp(\frac{3}{2(G-1)})} + \\ + \int_0^x \exp\frac{3}{2(G-1)}(x-y)\epsilon f(h(y),y,\epsilon)dy. \tag{3.12}$$

On travaille sur $\mathbb{X} = \{h \text{ continues, } 1-\text{périodiques}/\|h\|_\infty \leq 2\|h_0\|_\infty\}$. On a $f(h(x),x,\epsilon) = f(h(x),x+1,\epsilon)$. Donc $T(h,\epsilon)(x+1) = T(h,\epsilon)(x)$ pour tout $h \in \mathbb{X}$. De plus, on peut montrer que $\|T(h,\epsilon)\|_\infty \leq K\epsilon$ et $\|T(.,\epsilon)\|_{Lip} \leq K\epsilon$. Donc T preserve l'espace de Banach $\mathbb{X}$ et pour ϵ suffisamment petit, T est contractant. En appliquant le théorème d'inversion globale, on a donc l'existence de h 1-périodique telle que $h(x) = h_0(x) + T(h,\epsilon)(x)$. On a ainsi l'existence d'une solution stationnaire 1-périodique $(H, U = \frac{1}{H})$ et $H(x) = 1 + \epsilon h_0(x) + 0(\epsilon^2)$. On analyse maintenant la stabilité linéaire d'une telle solution.

3.2.2 Analyse de stabilité linéaire

Dans le but de mener des calculs explicites, on choisit $b'(x) = \cos(2\pi x)$. Dans ce cas, la solution stationnaire H s'écrit

$$H(x) = 1 + \epsilon h_0(x) + 0(\epsilon^2) \tag{3.13}$$

et on a

$$h_0(x) = \frac{a}{a^2+b^2}\sin(2\pi x) + \frac{b}{a^2+b^2}\cos(2\pi x)$$

avec $a = 2\pi$ et $b = -\frac{3}{2(G-1)}$. On linéarise le système (3.1) au voisinage de la solution $(H, U = \frac{1}{H})$. Le système linéarisé s'écrit

$$\begin{array}{l} h_t + (hU + Hu)_x = 0, \\ u_t + (Uu)_x = -Gh_x + \frac{1}{2}(\frac{U^2}{H^2}h - \frac{2u}{H^2}). \end{array} \tag{3.14}$$

On se place dans l'espace des fonctions n-périodiques en espace et on recherche des solutions à croissance exponentielle en temps de la forme $u(x,t) = \exp(\lambda t)\underline{u}(x)$, $h(x,t) = \exp(\lambda t)\underline{h}(x)$. On obtient alors le problème spectral suivant

Trouver $\lambda \in \mathbb{C}$ et $(\underline{h}, \underline{u})$ n-périodiques telle que

$$\begin{array}{l} (\underline{h}U + H\underline{u})' + \lambda\underline{h} = 0, \\ (U\underline{u} + G\underline{h})' + \lambda\underline{u} = \frac{1}{2}(\frac{U^2}{H^2}\underline{h} - \frac{2u}{H^2}). \end{array} \tag{3.15}$$

En utilisant le développement limité (3.13), le système (3.15) s'écrit sous forme matricielle

$$\begin{pmatrix} \underline{h}' \\ \underline{u}' \end{pmatrix} + (A_0(\lambda, G) + \epsilon h_0(x) A_1(\lambda, G, x) + O(\epsilon^2)) \begin{pmatrix} \underline{h} \\ \underline{u} \end{pmatrix} = 0 \tag{3.16}$$

où

$$A_0(\lambda, G) = \frac{1}{G-1} \begin{pmatrix} -\lambda - \frac{1}{2} & \lambda + 1 \\ G\lambda + \frac{1}{2} & -\lambda - 1 \end{pmatrix}$$

et

$$A_1(\lambda, G, x) = \frac{1}{G-1} \begin{pmatrix} \lambda + \frac{3}{2} + \frac{h_0'}{h_0} & \lambda - 1 - 2\frac{h_0'}{h_0} \\ -(\frac{5}{2} + G\frac{h_0'}{h_0}) & \lambda + 3 + (G+1)\frac{h_0'}{h_0} \end{pmatrix} - \\ -\frac{G+2}{G-1} A_0(\lambda, G).$$

En appliquant la théorie de Floquet, les vecteurs formant une base de solution de (3.16) sont de la forme $(\underline{h}, \underline{u}) = e^{-\mu x} Z(x)$ où Z est une fonction 1 périodique. On obtient donc des solutions 1 périodiques si tous les coefficients $\mu(\lambda, G, \epsilon)$ sont égaux à $2i\pi k,\ k \in \mathbb{Z}$. On calcule un développement à l'ordre 1 de $\mu(\lambda, G, \epsilon) = \mu_0(\lambda, G) + \epsilon \mu_1(\lambda, G) + O(\epsilon^2)$. On recherche une solution de (3.16) sous la forme

$$(\underline{h}, \underline{v}) = e^{-(\mu_0 + \epsilon\mu_1 + O(\epsilon^2))x} (Z_0 + \epsilon Z_1(x) + O(\epsilon^2)) \tag{3.17}$$

où Z_0 est un vecteur constant (pour $\epsilon = 0$, on doit retrouver les solutions d'un système à coefficient constants) et Z_1 est 1-périodique. On fixe Z_1 en choisissant $Z_1(0) = 0$. On injecte (3.17) dans (3.16) et on identifie les termes d'ordre $O(\epsilon^0)$ et $O(\epsilon^1)$. A l'ordre 0, on trouve

$$\Big(A_0(\lambda, G) - \mu_0(\lambda, G)\Big) Z_0 = 0. \tag{3.18}$$

Donc μ_0 est valeur propre de A_0. On fixe μ_0 et Z_0 un vecteur propre associé. A l'ordre 1, on obtient

$$\frac{dZ_1}{dx} + (A_0 - \mu_0) Z_1 = -(h_0(x) A_1(x) - \mu_1) Z_0, \tag{3.19}$$

qui s'intègre en

$$Z_1(x) = -\int_0^x \exp - \Big((A_0 - \mu_0)(x - y)\Big) \Big(h_0(y) A_1(y) - \mu_1\Big) dy Z_0.$$

La fonction Z_1 est 1-périodique si et seulement si

$$\int_0^1 \exp\Big((A_0-\mu_0)y\Big)\Big(h_0(y)A_1(y)-\mu_1\Big)dyZ_0 = 0. \tag{3.20}$$

Comme $\exp\Big((A_0-\mu_0)y\Big)Z_0 = Z_0$ pour tout y, on obtient

$$\mu_1 = \frac{\Big\langle \int_0^1 \exp\Big((A_0-\mu_0)y\Big)h_0(y)A_1(y)dyZ_0/Z_0\Big\rangle}{\Big\langle Z_0/Z_0\Big\rangle} \tag{3.21}$$

où $\Big\langle ./. \Big\rangle$ désigne le produit scalaire. On a donc obtenu une relation de dispersion du type

$$\mu_0(\lambda, G) + \epsilon\mu_1(\lambda, G) = 2i\pi k,\ k \in \mathbb{Z}.$$

Le calcul est complètement identique lorsqu'on se place dans l'espace des fonctions n-périodiques en espaces : il suffit de remplacer 1 par n. On veut mettre en évidence l'existence d'un nombre de Froude critique F_c, pour lequel la solution stationnaire H devient instable. On se place à la limite d'instabilité $\lambda = i\omega$. En séparant partie réelle et partie imaginaire $\mu_0 = R_0 + iI_0$ $\mu_1 = R_1 + iI_1$, on peut écrire la relation de dispersion sous la forme

$$\begin{aligned} f_1(\omega, G, n, \epsilon) &= R_0(\omega, G, n) - \epsilon R_1(\omega, G, n) + O(\epsilon^2) = 0 \\ f_2(\omega, G, n, \epsilon) &= (I_0(\omega, G, n) - 2k\pi) - \epsilon I_1(\omega, G, n) + O(\epsilon^2) = 0 \end{aligned} \tag{3.22}$$

On fixe n et $\epsilon \approx 0$, et on veut montrer l'existence de $\omega_c(n, \epsilon), G_c(n, \epsilon)$ solution de (3.22). On commence par traiter le problème pour $\epsilon = 0$ et on résout.

$$\begin{aligned} R_0(\omega, G, n) &= 0 \\ I_0(\omega, G, n) &= 2k\pi \end{aligned} \tag{3.23}$$

On peut montrer que $G_c(0, n) = \frac{1}{4}$ soit $F_c0, n = 2$ et $\omega_c(0, n) = k*(\frac{3\pi}{n}),\ k \geq 1$ sont les solutions du problème (3.23). On s'intéresse maintenant au problème complet (3.22) pour $\epsilon > 0$ suffisamment petit. Dans les trois cas traités, la matrice jacobienne

$$\begin{pmatrix} \frac{\partial R_0}{\partial \omega}(\omega_c^0, G_c^0, n) & \frac{\partial R_0}{\partial G}(\omega_c^0, G_c^0, n) \\ \frac{\partial I_0}{\partial \omega}(\omega_c^0, G_c^0, n) & \frac{\partial I_0}{\partial G}(\omega_c^0, G_c^0, n) \end{pmatrix}$$

est inversible. En appliquant le théorème des fonctions implicites, on a donc l'existence de $\omega_c(n, \epsilon), G_c(n, \epsilon)$ solution de (3.22) pour ϵ petit. On peut calculer un développement limité de F_c. On a réalisé numériquement le calcul

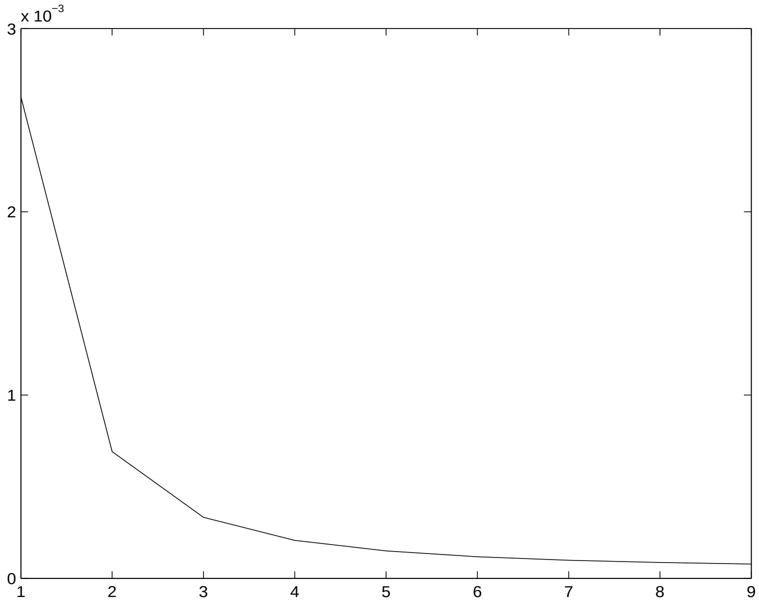

FIG. 3.1 – Graphe de $\frac{\partial F}{\partial \epsilon}(0, n)$ pour $n = 1, .., 9$

de $\frac{\partial F}{\partial \epsilon}(0, n)$ pour $n = 1, .., 9$. Le résultat est présenté dans la figure 3.1.

Pour ϵ suffisamment petit, on observe que le nombre de Froude critique $F_c(\epsilon, n)$ décroît avec n et les perturbations de plus grandes longueurs d'onde sont alors les plus instables. Si ce calcul a permis de mettre en évidence l'existence d'un nombre de Froude critique pour lequel le flot stationnaire devient instable comme pour le cas des canaux à fond plat, il n'explique cependant pas la formation préférentielle de roll-waves 2-périodiques : elles sont donc moins stables. On analyse donc le système lorsque la solution stationnaire est instable et on commence par les instabilités de petite amplitude.

3.3 Existence de roll-waves dans le p-système

3.3.1 Formulation du problème

On s'intéresse au p-système avec terme source

$$\begin{aligned} &u_t + v_x = 0, \\ &v_t + (p(u))_x = \epsilon b'(x) + f(u) - v, \end{aligned} \tag{3.24}$$

où les fonctions p et f sont régulières et leurs premières dérivées sont globalement bornées. La fonction b est régulière et 1-périodique.

Pour construire des roll-waves, on s'inspire de la méthode utilisée pour étudier la stabilité linéaire des chocs ([16], [20]), des transitions de phases ([1], [2]) dans différents systèmes de lois de conservations et qu'on a adapté dans le chapitre 2 pour l'analyse de stabilité des roll-waves. On travaille donc dans l'espace des fonctions C^1 par morceaux avec des discontinuités se propageant avec une vitesse presque constante. Au moyen d'un changement de variable $(\xi, t) = (x - c(t), t)$, on recherche des solutions C^1 par morceaux, L-périodiques en espace ($L \in \mathbb{N}^*$), les discontinuités étant fixées aux points $\{iL, i \in \mathbb{Z}\}$ et vérifiant les conditions de Rankine-Hugoniot :

$$\begin{aligned} c'(t) &= \frac{[v]}{[u]}, \\ c'(t) &= \frac{[p(u)]}{[v]}. \end{aligned} \tag{3.25}$$

On se place sur $(0, L)$ et on étudie le système

$$\begin{aligned} &u_t + (v - c'(t)u)_\xi = 0, \\ &v_t + (p(u) - c'(t)v)_\xi = \epsilon b'(\xi + c(t)) + f(u) - v. \end{aligned} \tag{3.26}$$

Le problème étant invariant par des translations $\xi \to \xi + L$, on obtient les roll-waves en translatant la solution obtenue sur $(0, L)$. Les conditions de Rankine-Hugoniot deviennent alors

$$\begin{aligned} c'(t) &= \tfrac{v(L,t)-v(0,t)}{u(L,t)-u(0,t)}, \\ c'(t) &= \tfrac{p(u(L,t))-p(u(0,t))}{v(L,t)-v(0,t)}. \end{aligned} \tag{3.27}$$

L'existence de roll-waves se réduit alors au problème suivant.
Trouver $(u, v) \in C^1(]0, L[\times \mathbb{T})$ *solutions classiques de*

$$\begin{aligned} &u_t + (v - \frac{v(L,t) - v(0,t)}{u(L,t) - u(0,t)} u)_\xi = 0, \\ &v_t + (p(u) - \frac{v(L,t) - v(0,t)}{u(L,t) - u(0,t)} v)_\xi = \epsilon b'(\xi + c(t)) + f(u) - v. \end{aligned}$$

tel que les valeurs aux bords vérifient

$$\frac{p(u(L,t)) - p(u(0,t))}{v(L,t) - v(0,t)} = \frac{v(L,t) - v(0,t)}{u(L,t) - u(0,t)}. \tag{3.28}$$

3.3.2 Roll-waves de petite amplitude

On diagonalise le p-système (3.24) en introduisant les invariants de Riemann $r = v + \phi(u)$ et $s = v - \phi(u)$. La fonction ϕ vérifie $\phi'(x) = \sqrt{p'(x)}$. Le système (3.24) s'écrit alors

$$\begin{aligned} r_t + \sqrt{p' \circ \phi^{-1}(\frac{r-s}{2})} r_x &= \epsilon b'(x) + f \circ \phi^{-1}(\frac{r-s}{2}) - \frac{r+s}{2}, \\ s_t - \sqrt{p' \circ \phi^{-1}(\frac{r-s}{2})} s_x &= \epsilon b'(x) + f \circ \phi^{-1}(\frac{r-s}{2}) - \frac{r+s}{2}. \end{aligned} \tag{3.29}$$

On s'inspire pour la recherche des solutions de type roll-waves (cas des solutions de petite amplitude) de la dérivation des équations de Burgers à partir du p-système par une méthode de relaxation [10]. Il existe de nombreuses études sur la stabilité des solutions d'ondes progressives pour les systèmes de relaxation [13]. L'originalité de ce travail tient au fait qu'on étudie des instabilités présentant des chocs pour un système de relaxation avec un forçage spatial.
On recherche des solutions autour de l'état constant (r_0, s_0) (cet état vérifiant $\frac{r_0+s_0}{2} = f \circ \phi^{-1}(\frac{r_0-s_0}{2})$) sous la forme $r = r_0 + \epsilon R(\xi, t)$ et $s = s_0 + \epsilon S(\xi, t)$ où $\xi = \frac{x-c(t)}{\epsilon}$, R et S sont des fonctions régulières en espace pour $\xi \in]-1, 1[$ et périodiques en temps (voir la section précédente). La vitesse c est définie au moyen des conditions de Rankine-Hugoniot. Dans ce cas le p-système s'écrit

$$\begin{aligned} R_t + \frac{1}{\epsilon}(\sqrt{p'(u_0)} + \epsilon \frac{p''(u_0)}{4p'(u_0)}(R-S) - c'(t) + \epsilon^2 P(R,S,\epsilon))R_\xi = \\ = \frac{1}{2}(\frac{f'(u_0)}{\sqrt{p'(u_0)}} - 1)R - \frac{1}{2}(\frac{f'(u_0)}{\sqrt{p'(u_0)}} + 1)S + b'(\epsilon\xi + c(t)) + \\ + \epsilon Q(R,S,\epsilon), \end{aligned} \tag{3.30}$$

$$\begin{aligned} S_t - \frac{1}{\epsilon}(\sqrt{p'(u_0)} + \epsilon \frac{p''(u_0)}{4p'(u_0)}(R-S) + c'(t) + \epsilon^2 P(R,S,\epsilon))S_\xi = \\ = \frac{1}{2}(\frac{f'(u_0)}{\sqrt{p'(u_0)}} - 1)R - \frac{1}{2}(\frac{f'(u_0)}{\sqrt{p'(u_0)}} + 1)S + b'(\epsilon\xi + c(t)) + \\ + \epsilon Q(R,S,\epsilon). \end{aligned} \tag{3.31}$$

Les fonctions P et Q sont régulières et vérifient $P(R,S,\epsilon) = O(|R|^2 + |S|^2)$, $Q(R,S,\epsilon) = O(|R|^2 + |S|^2)$.
Comme dans le premier chapitre, le petit paramètre ϵ induit une dynamique lente et une dynamique rapide. Le système (3.30,3.31) est écrit avec les

variables lentes. Lorsque $\epsilon = 0$, on obtient une "variété" lente invariante $S(t,\xi) = S(t)$ et la dynamique est donnée par l'équation de Burgers

$$R_t + \frac{p''(u_0)}{4p'(u_0)}(R - \frac{R(1,t)+R(-1,t)}{2})R_\xi = AR - BS + b'(c(t)). \tag{3.32}$$

On va suivre la démarche de Fenichel [7], [11](à ceci près qu'on est en dimension infinie) : on démontre dans un premier temps à partir de l'équation (3.31) l'existence d'un "graphe" invariant. On calcule alors le flot sur ce "graphe" et on dérive une équation de Burgers "modifiée" à partir de l'équation (3.30). On montre ensuite l'existence de solutions roll-waves.
Les conditions de Rankine Hugoniot donnent

$$c'(t)^2 = \frac{[p\circ\phi^{-1}(\frac{r_0-s_0+\epsilon(R-S)}{2})]_{-1}^{1}}{[\phi^{-1}(\frac{r_0-s_0+\epsilon(R-S)}{2})]_{-1}^{1}}.$$

Après un développement limité, on obtient

$$c'(t) = \sqrt{p'(u_0)} + \epsilon\frac{p''(u_0)}{4p'(u_0)}(\frac{R(1,t)+R(-1,t)}{2} - \frac{S(1,t)+S(-1,t)}{2}) + O(\epsilon^2). \tag{3.33}$$

De même, la condition aux bords

$$\frac{[p\circ\phi^{-1}(\frac{r_0-s_0+\epsilon(R-S)}{2})]_{-1}^{1}}{[\frac{r_0+s_0+\epsilon(R+S)}{2}]_{-1}^{1}} = \frac{[\frac{r_0+s_0+\epsilon(R+S)}{2}]_{-1}^{1}}{[\phi^{-1}(\frac{r_0-s_0+\epsilon(R-S)}{2})]_{-1}^{1}}$$

s'écrit

$$S(1,t) = S(-1,t) + \epsilon\mathcal{RH}(S(-1,t), R(1,t), R(-1,t), \epsilon), \tag{3.34}$$

et $\mathcal{RH}(x,y,z,,\epsilon) = O(x^2+y^2+z^2)$.

On calcule dans un premier temps un graphe $(R, S(R))$ invariant. On fixe la fonction R et on s'intéresse à l'équation (3.31) avec la condition aux bords (3.34). On remplace c' par son développement limité dans (3.31) et on obtient l'équation

$$S_t - \frac{1}{\epsilon}(2c_0 + \epsilon Q(R,S,t,\epsilon))S_\xi = AR - BS + b'(\epsilon\xi + c_0 t + \epsilon h(t,\epsilon)) + \epsilon P(R,S,\epsilon), \tag{3.35}$$

où $c_0 = \sqrt{p'(u_0)}$, $A = \frac{1}{2}(\frac{f'(u_0)}{\sqrt{p'(u_0)}} - 1)$ et $B = \frac{1}{2}(\frac{f'(u_0)}{\sqrt{p'(u_0)}} + 1)$. La fonction h est entièrement déterminée par $R(\pm1,t)$ et $S(\pm1,t)$.
En introduisant le rescaling, $\tau = \sqrt{p'(u_0)}t$, on peut supposer $c_0 = 1$. On démontre alors le lemme suivant.

Lemme 1 *Soit s_1, s_2 solutions 1-périodiques des équations*

$$\begin{aligned} \frac{ds_1}{dt} - Bs_1 &= -b'(t) \\ \frac{ds_2}{dt} + Bs_2 &= \frac{A}{2}\int_{-1}^{1} R(u,t)du. \end{aligned} \tag{3.36}$$

Alors il existe une solution $S^\epsilon \in C^1((-1,1)\times\mathbb{T})$, 1-périodique en temps tel que $S^\epsilon(x,t) = s_1(t)+s_2(t)+O(\epsilon)$ et vérifiant les conditions aux bords (3.34).

Preuve. Pour traiter les conditions aux bords, on introduit la fonction auxiliaire $\overline{S}$ définie par

$$\begin{aligned} \overline{S}(x,t) &= S(x,t) + \\ &+ \epsilon\frac{1-x}{2}\mathcal{RH}(S(-1,t),R(1,t),R(-1,t),\epsilon). \end{aligned} \tag{3.37}$$

La condition aux bords (3.34) est alors remplacée par une condition périodique $\overline{S}(1,t) = \overline{S}(-1,t)$. On réecrit alors l'équation (3.35) avec $\overline{S}$. On obtient

$$\overline{S}_t - \frac{2}{\epsilon}(1+\epsilon\overline{Q}(R,\overline{S},t,\epsilon))\overline{S}_\xi = AR - B\overline{S} + b'(\epsilon\xi + t + \epsilon\overline{h}(t,\epsilon)) + \epsilon\overline{P}(R,\overline{S},\epsilon). \tag{3.38}$$

Dans le but de simplifier les notations, on enlève les barres. Pour résoudre l'équation (3.38), on introduit les caractéristiques

$$\frac{dX}{dt} = -\frac{2}{\epsilon} - Q(R,S,t,\epsilon)(X(t))$$

avec $X(0) = \xi$. On définit également la caractéristique inverse

$$\frac{dY}{dt} = +\frac{2}{\epsilon} + Q(R,S,-t,\epsilon)(Y(t))$$

et $Y(0) = \xi$. On démontre la dépendance lipschitz vis à vis de S. On fait le calcul sur les caractéristiques directes, le calcul sur les caractéristiques inverses étant identique. On a

$$\begin{aligned} \frac{d\overline{X}}{dt} - \frac{dX}{dt} &= Q(R,\overline{S},t,\epsilon)(\overline{X}(t)) - Q(R,S,t,\epsilon)(\overline{X}(t)) + \\ &+ Q(R,S,t,\epsilon)(\overline{X}(t)) - Q(R,S,t,\epsilon)(\overline{X}(t)). \end{aligned}$$

On travaille sur l'ensemble des fonctions R, S tel que

$$\begin{aligned} &\sup_{t\in\mathbb{T}} \|R(.,t)\|_{C^1(-1,1)} \le M, \\ &\sup_{t\in\mathbb{T}} \|S(.,t)\|_{C^1(-1,1)} \le M. \end{aligned}$$

On obtient alors

$$\|\frac{d\overline{X}}{dt} - \frac{dX}{dt}\| \leq K(\|S - \overline{S}\|_\infty + \|X - \overline{X}\|_\infty).$$

En appliquant le lemme de Gronwall, on montre que

$$\|X - \overline{X}\|_\infty \leq K\|S - \overline{S}\|_\infty.$$

On fait le changement de variable $s(\xi, t) = S(X(t, \xi), t)$ dans (3.38). La fonction s vérifie

$$\frac{ds}{dt} + Bs = b'(\epsilon X(t, \eta) + t + \epsilon h(t, \epsilon)) + AR(X(t, \eta), t) + \epsilon P(R, S, \epsilon)(X(t, \eta)) \tag{3.39}$$

Au moyen d'une méthode de variation de la constante, on obtient

$$\begin{aligned} s(\eta, t) = A & \int_0^t R(X(u, \eta), u) \exp B(u - t) du + \\ & \int_0^t b'(\epsilon X(u, \eta) + u + \epsilon h(u, \epsilon)) \exp B(u - t) du + \\ & + \epsilon \int_0^t P(R, s, \epsilon)(X(u, \eta), u) \exp B(u - t) du, \end{aligned} \tag{3.40}$$

La fonction S est alors donnée par

$$\begin{aligned} S(\xi, t) = & S_0(Y(t, \xi)) \exp(-Bt) + A \int_0^t R(X(u, Y(t, \xi)), u) \exp B(u - t) du + \\ & \int_0^t b'(\epsilon X(u, Y(t, \xi)) + u + \epsilon h(u, \epsilon)) \exp B(u - t) du + \\ & + \epsilon \int_0^t P(R, S, \epsilon)(X(u, Y(t, \xi)), u) \exp B(u - t) du, \end{aligned} \tag{3.41}$$

On va choisir la fonction S_0 de telle sorte que la solution S soit 1-périodique. Les équations étant invariantes par $t \to t + 1$, il suffit de choisir S_0 de telle sorte que $S(1, .) = S(0, .)$. La fonction S apparait comme le point fixe d'un opérateur agissant sur $C(\mathbb{T}, C^1(-1, 1))$. Montrons que cet opérateur est contractant. Considérons

$$T_1(S)(\xi, t) = \int_0^t P(R, S, \epsilon)(X(u, Y(t, \xi)), u) \exp B(u - t) du.$$

En supposant les dérivées de p et f borées sur un compact, on a

$$\|T_1(S) - T_1(\overline{S})\| \leq C(\|S - \overline{S}\|_\infty + \|X - \overline{X}\|_\infty + \|Y - \overline{Y}\|_\infty), \tag{3.42}$$

Donc pour ϵ suffisamment petit, on obtient

$$\epsilon\|T_1(S) - T_1(\overline{S})\| \leq \frac{1}{4}\|S - \overline{S}\|_\infty.$$

On étudie alors l'opérateur

$$\begin{aligned} T_2(S)(\xi, t) = {} & S_2(Y(t,\xi))\exp(-Bt) + \\ & + \int_0^t b'(\epsilon X(u, Y(t,\xi)) + u + \epsilon h(u,\epsilon))\exp B(u-t)du. \end{aligned}$$

où S_2 est choisit de telle sorte que $T_2(S)$ soit périodique. On peut écrire

$$\begin{aligned} T_2(S)(\xi, t) = {} & S_2(Y(t,\xi))\exp(-Bt) + \\ & + \int_0^t b'(2t-u)\exp B(u-t)du + \\ & + \overline{T_2}(S), \end{aligned}$$

où

$$\begin{aligned} \overline{T_2}(S) = \int_0^t & \Big(b'(\epsilon(X(u, Y(t,\xi)) + \frac{2(u-t)}{\epsilon}) + 2t - u + \epsilon h(u,\epsilon)) - \\ & - b'(2t-u)\Big) \times \exp B(u-t)du. \end{aligned}$$

On peut montrer que $X(u, Y(t,\xi)) = \frac{2(t-u)}{\epsilon} + M(S,\epsilon)$ et M est un opérateur lipschitz en la variable S. Comme pour l'opérateur ϵT_1 et en utilisant la dépendance lipschitz des caractéristiques vis à vis de S, on a

$$\|\overline{T_2}(S) - \overline{T_2}(\overline{S})\| \leq C\epsilon\|b''\|_\infty\|\overline{S} - S\|_\infty.$$

Donc pour ϵ suffisamment petit, l'opérateur $\overline{T_2}$ est $\frac{1}{4}$-lipschitz. On examine alors l'intégrale

$$\begin{aligned} I(b,t) &= \int_0^t b'(2t-u)\exp B(u-t)du. \\ &= \int_0^t \exp(-Bv)b'(t+v)dv \end{aligned}$$

Soit $n = [t]$, on peut montrer que

$$\begin{aligned} I(b,t) = {} & \sum_{k=0}^{n-1}\exp(-Bk)\int_0^1 \exp(-Bv)b'(t+v)dv + \\ & + \int_n^t \exp(-Bv)b'(t+v)dv. \end{aligned}$$

On peut encore écrire

$$I(b,t) = \frac{1}{1-\exp -B}\int_0^1 \exp(-Bv)b'(t+v)dv + \exp(-Bt)f(t).$$

On choisit alors S_2 de telle sorte que $S_2(y) = -f(\frac{\epsilon}{2}y)$. On montre alors que $S_2(X(t,\xi)) = -f(t) + \epsilon N(S,\epsilon)$ et N est lipschitz vis à vis de S. On note s_1 la fonction 1-périodique

$$s_1(t) = \frac{1}{1-\exp -B}\int_0^1 \exp(-Bv)b'(t+v)dv.$$

Pour résumer, on a donc

$$T_2(S) = s_1(t) + \epsilon\mathcal{N}_2(S,R,\epsilon)$$

avec

$$\|\epsilon\mathcal{N}_2(S,R,\epsilon) - \epsilon\mathcal{N}_2(\overline{S},R,\epsilon)\| \leq C\epsilon\|S-\overline{S}\|_\infty.$$

On termine par l'étude de l'opérateur

$$T_3(S) = S_3(Y(t,\xi))\exp(-Bt) + A\int_0^t \exp B(s-t)R(X(s,Y(t,\xi)),s)du$$

où on choisit S_3 de telle sorte que $T_3(S)$ soit périodique. Comme pour le cas précédent, en utilisant $X(s,Y(t,\xi)) = \frac{2(t-s)}{\epsilon} + M(S,\epsilon)$, on peut montrer que

$$\begin{aligned} T_3(S) = A\int_0^t \exp B(s-t)R(\frac{2(t-s)}{\epsilon},s)ds + \\ +S_3(X(t,\xi))\exp(-Bt) + \epsilon\overline{T_3}(S,\epsilon), \end{aligned}$$

et l'opérateur $\epsilon\overline{T_3}$ est lipschitz en la variable S, de constante de lipschitz d'ordre $O(\epsilon)$. On s'intéresse à l'intégrale

$$I(R,\epsilon) = A\int_0^t \exp B(s-t)R(\frac{2(t-s)}{\epsilon},s)ds.$$

On fait le changement de variable $v = \frac{2(t-s)}{\epsilon}$ dans l'intégrale : on obtient alors

$$I(R,\epsilon) = A\frac{\epsilon}{2}\int_0^{\frac{2t}{\epsilon}} \exp -\frac{\epsilon Bu}{2}R(u,t-\frac{\epsilon u}{2})du.$$

On peut écrire $I(R,\epsilon)$ sous la forme

$$I(R,\epsilon) = A\frac{\epsilon}{2}\int_0^{2n} \exp -\frac{\epsilon Bu}{2}R(u,t-\frac{\epsilon u}{2})du + O(\epsilon),$$

avec $n = [\frac{t}{\epsilon}]$. En utilisant la périodicité spatiale de R, on a

$$I(R,\epsilon) = \frac{A\epsilon}{2}\sum_{k=0}^{n-1}\exp(-\epsilon Bk)\int_0^2 \exp(-\epsilon\frac{Bu}{2})R(u,t-\epsilon k)du + O(\epsilon).$$

On reconnait alors une somme de Riemann et on obtient

$$I(R,\epsilon) = A\int_0^t \exp -Bu\frac{1}{2}\int_0^2 R(x,t-u)dxdu + O(\epsilon).$$

On peut alors écrire $I(R,\epsilon)$ sous la forme

$$I(R,\epsilon) = s_2(t) + \exp(-Bt)g(t,\epsilon) + O(\epsilon),$$

où s_2 désigne la solution périodique de

$$\frac{ds_2}{dt} + Bs_2 = \frac{A}{2}\int_{-1}^1 R(x,t)dx.$$

Comme dans le cas de l'opérateur T_2, on choisit $S_3(y) = -g(\frac{\epsilon y}{2},\epsilon)$. On montre alors que $S_3(X(t,\xi)) = -g(t,\epsilon) + \epsilon P(S,\epsilon)$ et P est lispschitz vis à vis de S. Pour résumer, on peut écrire

$$T_3(S) = s_2(t) + \epsilon\mathcal{N}_3(S,R,\epsilon)$$

avec

$$\|\epsilon\mathcal{N}_3(S,R,\epsilon) - \epsilon\mathcal{N}_3(\overline{S},R,\epsilon)\| \leq C\epsilon\|S-\overline{S}\|_\infty.$$

Finalement, on a montré que S solution de (3.31) avec les conditions aux bords (3.34) vérifiait :

$$S(t,\xi) = s_1(t) + s_2(t) + \epsilon\mathcal{N}(R,S,\epsilon), \tag{3.43}$$

où $\mathcal{N}$ est un opérateur lipschitz. On conclut alors la démonstration du lemme en appliquant le théorème d'inversion global sur $C(\mathbb{T}\times(-1,1))$. Finalement, on obtient

$$S(t,\xi) = s_1(t) + s_2(t) + \epsilon\mathcal{P}(R,\epsilon). \tag{3.44}$$

□

On a donc démontré l'existence d'un graphe invariant $(R,S(R))$. On va déterminer le flot sur ce graphe. On s'intéresse maintenant à l'équation (3.30). On rappelle le développement limité de $c'(t)$:

$$c'(t) = \sqrt{p'(u_0)} + \epsilon\frac{p''(u_0)}{4p'(u_0)}(\frac{R(1,t)+R(-1,t)}{2} - s(t)) + O(\epsilon^2), \tag{3.45}$$

où $s(t) = s_1(t)+s_2(t)$. On injecte alors (3.44) et (3.45) dans (3.30), l'équation s'écrit

$$R_t + \frac{p''(u_0)}{4p'(u_0)}(R - \frac{R(1,T)+R(-1,t)}{2} + \epsilon P(R,S(R),\epsilon))R_\xi = \\ = AR - Bs(t) + b'(t+\epsilon\xi) + \epsilon Q(R,S(R),\epsilon). \quad (3.46)$$

avec $Q(R,S(R)\epsilon) = O(R^2)$ et $P(R,S(R),\epsilon) = O(R^2)$. Pour $\epsilon = 0$, on obtient une équation de type Burgers. On démontre le lemme suivant.

Lemme 2 *Il existe une fonction $R_p \in C(\mathbb{T}, C^1(-1,1))$ solution de type roll-waves de l'équation de Burgers modifiée*

$$R_t + \frac{p''(u_0)}{4p'(u_0)}(R - \frac{R(1,t)+R(-1,t)}{2})R_\xi = AR - Bs(t) + b'(t), \qquad (3.47)$$

Preuve. On recherche une solution R_p dans $C(\mathbb{T}, C^1]-1,1[)$ sous la forme $R_p(\xi,t) = H(\xi)+F(t)$ et $H(-1)+H(1) = 0$. On choisit H tel que $\frac{p''(u_0)}{4p'(u_0)}H'(\xi) = A$. On a donc $H(\xi) = \frac{4p'(u_0)A}{p''(u_0)}\xi$. La fonction F vérifie alors l'équation différentielle

$$F' = AF - Bs_1(t) - Bs_2(t) + b'(t), \\ = AF - Bs_2(t) - \frac{ds_1}{dt}. \qquad (3.48)$$

En exploitant le fait que s_2 vérifie $s_2'(t) = -Bs_2 + AF(t)$ et en dérivant l'équation (3.48), on obtient

$$F'' = (A-B)F' - \frac{d}{dt}(Bs_1(t) + \frac{ds_1}{dt}), \qquad (3.49)$$

qu'on peut écrire

$$F' + F = -Bs_1(t) - \frac{ds_1}{dt}. \qquad (3.50)$$

On choisit alors F solution de (3.50) 1-périodique. La solution $R_p(\xi,t) = H(\xi)+F(t)$ est définie pour $\xi \in]-1,1[$. On obtient la solution de type roll-waves en translatant le profil obtenu.□

On étudie le problème complet (3.30) pour $\epsilon \approx 0$. On démontre le résultat suivant.

Lemme 3 *Il existe une fonction $R \in C(\mathbb{T}, C^1(-1,1))$ solution de type roll-waves de l'équation*

$$R_t + \frac{p''(u_0)}{4p'(u_0)}(R - \frac{R(1,t)+R(-1,t)}{2} + \epsilon\mathcal{F}(R,\epsilon))R_\xi = \\ = AR - Bs(t) + b'(t) + \epsilon\mathcal{G}(R,\xi,\epsilon), \qquad (3.51)$$

où

$$\mathcal{F}(R,\epsilon) = P(R, S(R), \epsilon) + \frac{\mathcal{P}(R,\epsilon)(1,t) + \mathcal{P}(R,\epsilon)(-1,t)}{2},$$

et

$$\mathcal{G}(R,\xi,\epsilon) = Q(R, S(R), \epsilon) + \frac{b'(\epsilon\xi + T) - b'(t)}{\epsilon}.$$

La fonction R s'écrit $R(\xi,t) = R_p(\xi,t) + O(\epsilon)$.

Preuve. On recherche la solution R de (3.51) sous la forme

$$R(\xi,t) = R_p(\xi,t) + \epsilon u$$

avec $u \in C^1(\mathbb{T}, C^1(-1,1))$. La fonction u vérifie

$$\begin{aligned} u_t + A(\xi + \epsilon\mathcal{P}_1(R,u,\epsilon))u_\xi = &-A\mathcal{F}(R+\epsilon u,\epsilon) + \mathcal{G}(R+\epsilon u,\xi,\epsilon) + \\ &+A\frac{u(-1,t)+u(1,t)}{2} - Bs(u)(t), \end{aligned} \tag{3.52}$$

où

$$\mathcal{P}_1(R,u,\epsilon) = \frac{4p'(u_0)}{p''(u_0)A}\left(\mathcal{F}(R+\epsilon u,\epsilon) + u - \frac{u(-1,t)+u(1,t)}{2}\right)$$

et $s(u)$ désigne la solution périodique de

$$\frac{ds}{dt} + Bs = \frac{A}{2}\int_{-1}^{1} u(x,t)dx$$

On écrit alors l'équation (3.52) de manière plus condensée

$$\begin{aligned} u_t + A(\xi + \epsilon\mathcal{K}^\epsilon(u))u_\xi = &\mathcal{L}^\epsilon(R+\epsilon u) + \\ &+A\frac{u(-1,t)+u(1,t)}{2} - Bs(u)(t), \end{aligned} \tag{3.53}$$

La fonction u apparait alors comme un point fixe de l'opérateur $\mathcal{H}$ défini par

$$\begin{aligned} \mathcal{H} : C(\mathbb{T}, C^1(-1,1)) &\to C(\mathbb{T}, C^1(-1,1)) \\ v &\to \mathcal{H}(v) = u, \end{aligned}$$

où $u \in C(\mathbb{T}, C^1(-1,1))$ est solution de

$$\begin{aligned} u_t + A(\xi + \epsilon\mathcal{K}^\epsilon(v))u_\xi = &\mathcal{L}^\epsilon(R+\epsilon v) + \\ &+A\frac{u(-1,t)+u(1,t)}{2} - Bs(u)(t), \end{aligned} \tag{3.54}$$

On écrit alors (3.54) comme une équation d'évolution dans $\mathbb{X} = C^1(-1,1)$. On a

$$u_t + \mathbb{L}_v^\epsilon u = \mathcal{L}^\epsilon(R + \epsilon v)(t,.) \tag{3.55}$$

où $\mathbb{L}_v^\epsilon$ désigne l'opérateur linéaire défini sur $\mathbb{X}$

$$\mathbb{L}_v^\epsilon u = Bs(u) - A\frac{u(1,t) + u(-1,t)}{2} + A(\xi + \epsilon\mathcal{K}^\epsilon(v))u_\xi.$$

On étudie le semi-groupe $\mathcal{S}(t) = \exp -t\mathbb{L}_v^\epsilon$ généré par l'opérateur borné $\mathbb{L}_v^\epsilon$ sur $\mathbb{X}$. On étudie donc le problème

$$\begin{aligned} u_t + \mathbb{L}_v^\epsilon u = 0,\\ u(0,x) = u_0(x). \end{aligned} \tag{3.56}$$

On pose $w = u_\xi$, la fonction w vérifie

$$w_t + A(\xi + \epsilon\mathcal{K}^\epsilon(v))w_\xi + A(1 + \epsilon\mathcal{K}^{\epsilon'}(v)v_\xi)w = 0. \tag{3.57}$$

On introduit alors les caractéristiques

$$\frac{dX}{dt} = A(X(t) + \epsilon\mathcal{K}^\epsilon(v)(X(t))),$$

et les caractéristiques inverses

$$\frac{dY}{dt} = -A(Y(t) + \epsilon\mathcal{K}^\epsilon(v)(Y(t))),$$

tel que $X(0) = Y(0) = \xi$. Par une méthode de variation de la constante, on obtient

$$Y(t) = \exp -At\xi - A\epsilon\int_0^t \exp -A(t-s)\mathcal{K}^\epsilon(v)(Y(s))ds.$$

On peut alors montrer que $Y(t,\xi) = \exp(-At)\xi(1 + O(\epsilon))$. De même on a $X(t,\xi) = \exp(At)\xi(1 + O(\epsilon))$. La fonction w est donnée par

$$w(\xi,t) = w_0(Y(t,\xi))\exp -\left(A\int_0^t (1 + \epsilon\mathcal{K}^{\epsilon'}(v)v_\xi(X(u,Y(t,\xi))))du\right) \tag{3.58}$$

On a donc l'estimation

$$\|u_x(t,.)\|_\infty \leq \|u_0\|_{\mathbb{X}} \exp -A(1 - \epsilon C(\|v\|_{\mathbb{X}}))t \tag{3.59}$$

On peut également montrer

$$\|u(t,.) - < u(t,.) >\|_\infty \leq \|u_0\|_{\mathbb{X}} \exp -A(1 - \epsilon C(\|v\|_{\mathbb{X}}))t, \tag{3.60}$$

où $< u >= \frac{1}{2}\int_{-1}^{1} u(x)dx$. On prend la moyenne spatiale sur l'équation

$$u_t + \mathbb{L}_v^\epsilon u = 0.$$

On obtient

$$\begin{aligned} &\frac{d < u >}{dt} - A < u >= -A\epsilon < \mathcal{K}^\epsilon(v)u_\xi > -Bs(u), \\ &\frac{ds(u)}{dt} + Bs(u) = A < u > . \end{aligned} \tag{3.61}$$

On soustrait les deux équations :

$$\frac{ds(u)}{dt} = \frac{d < u >}{dt} + A\epsilon < \mathcal{K}^\epsilon(v)u_\xi > \tag{3.62}$$

On a donc

$$\begin{aligned} &s(u) =< u > -A\epsilon \int_t^{+\infty} < \mathcal{K}^\epsilon(v)u_\xi(., s) > ds, \\ &s(u) =< u > +O(\epsilon \exp(-At)). \end{aligned} \tag{3.63}$$

On obtient donc finalement

$$\frac{d < u >}{dt} + (B - A) < u >= O(\epsilon \exp(-At))$$

et $B - A = 1$. On a donc

$$< u(t) >= \epsilon K(u_0, v) + O(\exp -t).$$

Finalement on obtient l'estimation

$$\|u(t, .) - \epsilon K(u_0, v)\|_\infty \leq \|u_0\|_{\mathbb{X}} \exp -\alpha(\epsilon, u_0, v)t, \tag{3.64}$$

avec

$$\alpha(\epsilon, u_0, v) = \min(A(1 - \epsilon C(\|v\|_{\mathbb{X}})), 1) > 0$$

pour ϵ suffisamment petit. On calcule alors l'opérateur $\mathcal{H}$ en utilisant la formule de Duhamel sur l'équation

$$u_t + \mathbb{L}_v^\epsilon u = \mathcal{L}^\epsilon(R + \epsilon v)(t, .). \tag{3.65}$$

On obtient alors

$$u(., t) = \exp -t\mathbb{L}_v u_0 + \int_0^t \exp -(t - s)\mathbb{L}_v \mathcal{L}^\epsilon(R + \epsilon v)(s, .)ds = \mathcal{H}(v). \tag{3.66}$$

Comme pour la démonstration du lemme 1, on choisit u_0 (fonction de v) de telle sorte que $\mathcal{H}(v)$ soit périodique. Grâce aux estimations (3.64) et (3.59), on montre que

$$\sup_{\mathbb{T}} \|\mathcal{H}(v_1) - \mathcal{H}(v_2)\|_{\mathbb{X}} \leq K\epsilon \sup_{\mathbb{T}} \|v_1 - v_2\|_{\mathbb{X}}. \tag{3.67}$$

Il existe donc un point fixe $u \in C(\mathbb{T}, C^1(-1,1))$ de l'opérateur $\mathcal{H}$ et le lemme est démontré. $\square$

Dans les coordonnées initiales, la solution s'écrit

$$\begin{array}{l} r(x,t) = r_0 + \frac{p''(u_0}{4p'(u_0)}(x - \sqrt{p'(u_0)}t + \epsilon h(t)) + O(\epsilon^2) \\ s(x,t) = s_0 + \epsilon\big(s_1(t) + s_2(t)\big) + O(\epsilon^2) \end{array} \tag{3.68}$$

où la fonction h est périodique et définie par $h(t) = F(t) - s_1(t) - s_2(t)$. On a donc une solution de type roll-wave dont la vitesse d'onde est oscillante autour de la valeur moyenne $c_0 = \sqrt{p'(u_0)}$. Ce phénomène est très similaire aux ondes pulsatoires observées dans les systèmes de réaction/diffusion non homogènes [4].

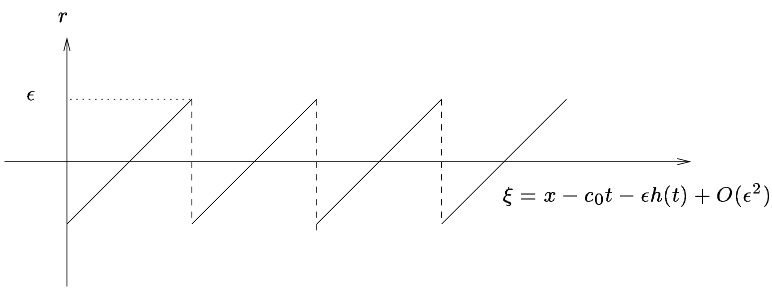

FIG. 3.2 – Allure des roll-waves dans le p-système

3.4 Roll-waves dans le système de Saint Venant avec fond périodique

Dans cette section, on étudie le système de Saint Venant avec fond périodiquement modulé.

$$\begin{array}{l} h_t + (hu)_x = 0, \\ u_t + uu_x + Gh_x = -G\epsilon b'(x) + \dfrac{1}{2}(1 - \dfrac{u^2}{h}) \end{array} \tag{3.69}$$

où $G = \frac{1}{F^2}$ et F désigne le nombre de Froude. Dans un premier temps, on suit la démarche de la section précédente et on dérive (formellement) une équation de type Burgers.

3.4.1 Roll-waves de petite amplitude

On diagonalise le système de Saint Venant en introduisant les invariants de Riemann $r = u + 2\sqrt{gh}$ et $s = u - 2\sqrt{gh}$. Le système (3.1) s'écrit alors

$$
\begin{aligned}
r_t + (\frac{3}{4}r + \frac{1}{4}s)r_x &= \epsilon b'(x) + \frac{1}{2}(1 - (\frac{r+s}{r-s})^2),\\
s_t + (\frac{3}{4}s + \frac{1}{4}r)s_x &= \epsilon b'(x) + \frac{1}{2}(1 - (\frac{r+s}{r-s})^2).
\end{aligned}
\tag{3.70}
$$

On recherche alors des solutions autour de l'état stationnaire $r_0 = 1 + \frac{2}{F}$ et $s_0 = 1 - \frac{2}{F}$ sous la forme $r = r_0 + \epsilon R(\xi, t)$ et $s = s_0 + \epsilon S(\xi, t)$ où $\xi = \frac{x - c(t)}{\epsilon}$. Les fonctions R et S vérifient

$$
\begin{aligned}
R_t \,+\, &\frac{1}{\epsilon}(1 + \frac{1}{F} + \epsilon\frac{S}{4} - c' + \epsilon\frac{3R}{4})R_\xi =\\
&= \frac{1}{2}\left((\frac{F}{2} - 1)R - (\frac{F}{2} + 1)S\right) +\\
&\quad + b'(\epsilon\xi + c(t)) + \epsilon M(R, S, \epsilon),
\end{aligned}
\tag{3.71}
$$

$$
\begin{aligned}
S_t \,+\, &\frac{1}{\epsilon}(1 - \frac{1}{F} - c' + \epsilon(\frac{3S}{4} + \frac{R}{4}))S_\xi =\\
&= \frac{1}{2}\left((\frac{F}{2} - 1)R - (\frac{F}{2} + 1)S\right) +\\
&\quad + b'(\epsilon\xi + c(t)) + \epsilon M(R, S, \epsilon).
\end{aligned}
\tag{3.72}
$$

Les conditions de Rankine Hugoniot sont données par

$$
\begin{aligned}
&[hu(T,t) - hu(0,t)] = c'(t)[h(T,t) - h(0,t)],\\
&[(G\frac{h^2}{2} + hu^2)(T,t) - (G\frac{h^2}{2} + hu^2)(0,t)] = c'(t)[hu(T,t) - hu(0,t)].
\end{aligned}
\tag{3.73}
$$

Un développement limité des conditions (3.73) donne

$$
\begin{aligned}
c'(t) \,=\, &1 + \frac{1}{F} + \epsilon\frac{3}{4}\frac{R(1,t) + R(-1,t)}{2} +\\
&+ \epsilon\frac{1}{4}\frac{S(1,t) + S(-1,t)}{2} + O(\epsilon^2)
\end{aligned}
\tag{3.74}
$$

et la condition aux bords

$$
S(1,t) = S(-1,t) + \epsilon\mathcal{RH}(R(1,t), R(-1,t), S(-1,t), \epsilon).
\tag{3.75}
$$

On fixe R. En remplaçant (3.74) dans (3.72), on obtient

$$
S_t - \frac{2}{F\epsilon}(1 + O(\epsilon))S_\xi = AR - BS + b'((1 + \frac{1}{F})t + \epsilon\xi) + O(\epsilon),
\tag{3.76}
$$

où $A = \frac{1}{2}(\frac{F}{2} - 1)$ et $B = \frac{1}{2}(\frac{F}{2} + 1)$. On peut montrer l'existence d'une solution $S(R)$ de (3.72) tel que $S(R) = s(t) + O(\epsilon)$. On injecte dans l'équation (3.71) et on obtient

$$R_t + \frac{3}{4}(R - \frac{R(1,t) + R(-1,t)}{2} + O(\epsilon))R_\xi = AR - Bs(t) + O(\epsilon). \quad (3.77)$$

Les équations possèdent la même structure que celles venant du p-système et on peut démontrer un résultat d'existence de solutions de type roll-waves avec une vitesse de propagation oscillant autour d'une vitesse moyenne.

3.5 Conclusion et perspectives

Dans ce chapitre, on a tenté d'expliquer l'apparition de roll-waves dans les écoulements de shallow water avec fond périodique et la sélection d'une longueur d'onde. D'une part, on a mis en évidence le fait que les perturbations périodiques de période double sont les moins stables. D'autre part pour un scaling donné, on a démontré l'existence de roll-waves de petite amplitude. Cela n'explique pourtant pas le phénomène de sélection de la longueur d'onde. Cependant, on conjecture que l'attracteur global pour notre système est constitué des solutions de type roll-waves et qu'il existe un scaling (du type $F = F_c + O(\epsilon)$) pour lequel on observerait ce type de phénomène. Les techniques qu'on a développées dans les chapitres 2 et 3 jointes à l'analyse du système dans une échelle correcte doit permettre de conclure à l'apparition préférentielle de roll-waves de période 2.

Pour conclure cette première partie, on peut indiquer quelques perspectives de recherche (outre celles présentées précédemment) pour continuer ce travail.

Dans le premier chapitre, on rappelle qu'on a montré l'existence de roll-waves $O(\epsilon^{\frac{1}{3}})$ proches des roll-waves construites par Dressler pour des paramètres $O(\epsilon)$ proche des paramètres de Dressler. Cette construction reste cependant très fragile et on peut se demander si il n'y aurait pas un moyen "plus naturel" de les obtenir. Il existe des modèles de Saint Venant prenant en compte une dérivée d'ordre 3 (et pas la viscosité) : dans ce cas on peut montrer l'existence de roll-waves continues de type cnoidale mais on perd l'aspect choc des roll-waves. Un modèle prenant en compte ces deux phénomènes devrait permettre d'obtenir plus naturellement les roll-waves et une dérivation rigoureuse des équations de Saint Venant est donc ici très importante.

Dans le deuxième chapitre, on a étudié la stabilité linéaire des roll-waves discontinues de Dressler. On peut se demander si cette étude peut nous permettre de donner des indications sur la stabilité linéaire des roll-waves visqueuses obtenues dans le premier chapitre. Le point crucial est de faire le lien entre le problème spectral lié à la stabilité des roll-waves discontinues et le problème lié à la stabilité linéaire des roll-waves visqueuses. Il existe déjà quelques travaux établissant le lien entre la stabilité des profils visqueux et la stabilité des chocs pour les systèmes de lois de conservation [8],[24],[3]. En distinguant une dynamique lente et une dynamique rapide, on doit pouvoir faire ce lien dans le cas des roll-waves. Enfin, une autre direction possible serait de savoir si la stabilité linéaire des roll-waves discontinues implique leur stabilité non linéaire.

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Chapitre 4

Existence de breathers dans les réseaux de Fermi Pasta Ulam diatomiques

Dans ce chapitre, on démontre l'existence de breathers (oscillations périodiques et localisées en espace) dans les réseaux Fermi Pasta Ulam (FPU) diatomiques avec un rapport de masse arbitraire. On complète ici les preuves d'existence de Livi Spicci Mac Kay obtenues pour les grands rapports de masse. On formule le problème au moyen d'un mapping dans un espace de fonctions périodique et on analyse celui ci par une réduction à une variété centrale discrète en espace. Les solutions ondes progressives du système linéarisé possède des fréquences dans deux bandes : une bande acoustique et une bande optique correspondant à des fréquences supérieures. Pour des fréquences proches des bords des bandes, toutes les solutions de petites amplitudes du système complet sont sur une variété de dimension finie, ce qui réduit localement le problème à l'étude d'un mapping en dimension finie. Pour certaines valeurs des paramètres, le mapping possède des orbites homoclines à 0 correspondant à des solutions breathers. Lorsque le potentiel d'interaction FPU vérifie une condition potentiel dur, on trouve des breathers de fréquence légèrement supérieur à celles de la bande optique et dans le gap entre les deux bandes juste au dessus de la bande acoustique. Lorsque le potentiel d'interaction vérifie une hypothèse de potentiel mou, on démontre l'existence de breathers dans le gap entre les bandes, pour des fréquences légèrement inférieures à celles de la bande optique.

Breathers on diatomic Fermi-Pasta-Ulam lattices

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Abstract

We prove the existence of breathers (spatially localized and time-periodic oscillations) in diatomic Fermi-Pasta-Ulam (FPU) chains with arbitrary mass ratio. This completes an existence result by Livi, Spicci and MacKay valid for large mass ratio. The problem is formulated as a mapping in a loop space and analyzed via a discrete spatial centre manifold reduction. Spatially peiriodic travelling wave solutions of the linearized system have frequencies in a higher "optic" band or a lower "acoustic" band. For frequencies close to band edges, all small amplitude solutions of the nonlinear system lie on a finite-dimensional centre manifold, which reduces the problem locally to the study of a finite-dimensional mapping. For good parameter values, the map admits homoclinic orbits to 0 corresponding to discrete breathers. When the FPU interaction potential satisfies a hardening condition, we find breathers with frequencies slightly above the optic band, or in the gap slightly above the acoustic band. For a potential satisfying the opposite softening condition, we obtain breathers with frequencies in the gap slightly below the optic band.

Keywords: diatomic Fermi-Pasta-Ulam lattice; discrete breather; centre manifold reduction; reversible map

4.1 Introduction

A Fermi-Pasta-Ulam (FPU) chain consists in a one-dimensional chain of masses connected by anharmonic springs. In this paper we consider an infinite diatomic FPU chain with two alternating different masses m_1, m_2 ($m_1 < m_2$). This system is described by the set of equations

$$m_n \frac{d^2}{dt^2} x_n = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z}, \tag{4.1}$$

where $m_{2n+1} = m_1$, $m_{2n} = m_2$, x_n represents mass displacements from their reference position and V denotes a smooth interaction potential with $V'(0) = 0$. For the sake of simplicity we assume V analytic in the neighbourhood of 0.

This paper deals with the existence of *discrete breathers* (DB), i.e. time-periodic solutions of (4.1) with spatially localized oscillations. More precisely, a solution of (4.1) is called discrete breather if $x_n(t+T) = x_n(t)$ for some $T > 0$ and if there exist constants $c_\pm \in \mathbb{R}$ such that $\lim_{n\to\pm\infty} \| x_n - c_\pm \|_{L^\infty(0,T)} = 0$. We also require that $\frac{dx_n}{dt}$ is not identically 0 (this eliminates equilibria and in particular simple shifts).

Discrete breather solutions are sustained by many nonlinear lattices and appear in different physical models, concerning e.g. DNA denaturation ([29]), dynamical properties of crystals ([33]) or Josephson junction arrays ([11]). We refer the reader to [12], [18], [30], [32] for reviews on the subject.

In diatomic FPU chains, DB have been considered in connection with the vibrational dynamics of ionic crystals. Their existence has been extensively studied numerically, see e.g. [26], [2], [13], [8], [14], [15]. These works explore various types of mass ratio ($\frac{m_2}{m_1}$ large or not) and interaction potentials (hard or soft ones). They are based on direct numerical integrations or rotating-wave approximations in which only the first frequency components are kept in the time dependence.

Mathematical results concerning DB in diatomic FPU chains have been obtained in several contexts.

On the one hand, Arioli et al ([3], [4], [5]) have used variational techniques. They have considered a class of potentials V superquadratic at infinity, corresponding to interaction forces anti-restoring at small amplitudes ($V''(0) < 0$) and restoring at large amplitudes ($x\,V'(x) > 0$ for $|x|$ large enough). In this case, the linearization of (4.1) at $x_n = 0$ does not possess plane wave solutions. The authors have shown that DB of period T exist provided T is large enough. Moreover, if one modifies the above assumptions by assuming $V''(0) = 0$ (purely restoring forces), then there exist DB of any period $T > 0$ ([3]). Note that their results apply to more general lattices with $m_{n+p} = m_n$ in (4.1).

On the other hand, Livi, Spicci and MacKay ([27]) have studied the more classical situation when nearest neighbours interactions act locally as restoring forces ($\kappa = V''(0) > 0$). In this case, the linearization of (4.1) at

$x_n = 0$ has plane wave solutions with frequencies ω in two separated bands, namely an "optic band" ($\omega^2 \in [\frac{2\kappa}{m_1}, 2\kappa(\frac{1}{m_1} + \frac{1}{m_2})]$) and an "acoustic band" ($\omega^2 \in [0, \frac{2\kappa}{m_2}]$). Livi et al consider hard interaction potentials having the form

$$V(x) = k_2\, x^2 + k_4\, x^4,$$

where k_2, $k_4 > 0$. For a large mass ratio m_2/m_1, they prove the existence of DB with frequencies above the optic band by continuation from the limit $m_2/m_1 = +\infty$. In this uncoupled limit (also called anticontinuous limit), heavy masses do not move and trivial DB can be constructed by exciting only one light mass and keeping the others at rest. Note that numerical continuation results up to $m_1 = m_2$ have been obtained in [8].

In this paper we assume $\kappa > 0$ and prove the existence of small amplitude DB for arbitrary mass ratio m_2/m_1. Moreover, we both treat the case of hard and soft interaction potentials. When V satisfies the hardening condition $\frac{\kappa}{2}\, V^{(4)}(0) - (V^{(3)}(0))^2 > 0$, we find DB with frequencies slightly above the optic band, or in the gap slightly above the acoustic band. When V satisfies the opposite softening condition $\frac{\kappa}{2}\, V^{(4)}(0) - (V^{(3)}(0))^2 < 0$, we obtain DB with frequencies in the gap slightly below the optic band. It is interesting to note that these conditions are independent of the mass ratio $\frac{m_2}{m_1}$. We state these results more precisely in the following theorem.

Theorem 4 *Fix values of* $B = \frac{\kappa}{2}\, V^{(4)}(0) - (V^{(3)}(0))^2$ *and* m_1, m_2 *with* $m_2 > m_1$. *Problem (4.1) has the following families of small amplitude discrete breather solutions, parametrized by their frequency* ω.

i) For $B < 0$, $\omega \approx \omega_c = \left(\frac{2\kappa}{m_1}\right)^{1/2}$ *and* $\omega < \omega_c$, *there exist DB solutions* x_n^1, x_n^2 *having the symmetries*

$$x^1_{-n}(t) = -x^1_{n-2}(t + \frac{\pi}{\omega}), \quad x^2_{-n}(t) = -x^2_n(t).$$

ii) For $B > 0$, $\omega \approx \omega_c = \left[2\kappa(\frac{1}{m_1} + \frac{1}{m_2})\right]^{1/2}$ *and* $\omega > \omega_c$, *there exist DB solutions* x_n^1, x_n^2 *having the symmetries*

$$x^1_{-n}(t) = -x^1_{n-2}(t + \frac{\pi}{\omega}), \quad x^2_{-n}(t) = -x^2_n(t + \frac{\pi}{\omega}).$$

iii) If $B > 0$, $\frac{m_2}{m_1} \in (k^2, k(k+2))$ *(for some integer* $k \geq 1$*),* $\omega \approx \omega_c = \left(\frac{2\kappa}{m_2}\right)^{1/2}$ *and* $\omega > \omega_c$, *there exist DB solutions* x_n^1, x_n^2 *having the symmetries*

$$x^1_{-n}(t) = -x^1_{n-2}(t), \quad x^2_{-n}(t) = -x^2_n(t + \frac{\pi}{\omega}).$$

In each case these solutions have the form

$$x_n^i(t) = d_n + X_n^i(t)$$

where X_n^i *has* 0 *time-average and* $\|X_n^i\|_{L^\infty}$ *decays exponentially as* $n \to \pm\infty$*. The stationary term* d_n *satisfies* $d_n = O(|\omega - \omega_c|)$ *for any fixed* n*,* $\lim_{n\to\pm\infty} d_n = O(|\omega - \omega_c|^{1/2})$ *and has a kink shape if* $V^{(3)}(0) \neq 0$*. The oscillatory parts* X_n^i *have the following form. In case i)*

$$X_{2n}^i(t) = O(|\omega - \omega_c|), \quad X_{2n-1}^i(t) = A_n \cos(\omega t) + O(|\omega - \omega_c|), \tag{4.2}$$

in case ii)

$$X_{2n}^i(t) = -\frac{m_1}{m_2} A_n \cos(\omega t) + O(|\omega-\omega_c|), \quad X_{2n-1}^i(t) = A_n \cos(\omega t) + O(|\omega-\omega_c|), \tag{4.3}$$

in case iii)

$$X_{2n}^i(t) = A_n \cos(\omega t) + O(|\omega - \omega_c|), \quad X_{2n-1}^i(t) = O(|\omega - \omega_c|), \tag{4.4}$$

where $0 < A_n \leq C\,|\omega - \omega_c|^{1/2}\,\nu^{-|n|}$*,* $\nu = 1 + O(|\omega - \omega_c|^{1/2}) > 1$*.*

These results are consistent with the numerical works previously mentioned. Note that an additional condition on $\frac{m_2}{m_1}$ is required in the case when ω lies slightly above the acoustic band (case iii)). This condition can be interpreted as a non-resonance condition with plane waves. More precisely, if this condition is not fulfilled (i.e. if $\frac{m_2}{m_1} \in [k(k+2), (k+1)^2]$, $k \geq 1$) and ω is sufficiently close to $(\frac{2\kappa}{m_2})^{1/2}$, then $(k+1)\,\omega$ belongs to the optic band.

Our proof also provides explicit expressions of DB solutions at leading order in $\omega - \omega_c$. Their oscillatory parts decay exponentially at infinity and can be viewed as spatial modulations of linear standing waves with frequencies $\omega = \omega_c$. Oscillatory parts are $O(|\omega - \omega_c|^{1/2})$ and superpose to $O(|\omega - \omega_c|)$ static distorsions of the lattice.

The proof of theorem 4 is based on the fact that one can rewrite (4.1) as an ill-posed recurrence relation on a loop space and analyze small amplitude solutions using an adapted centre manifold reduction theorem. This method was initially introduced for studying discrete breathers in monoatomic FPU lattices ([23]) and has been extended to more general discrete systems in [24].

To be more precise, V' is locally invertible ($V''(0) > 0$) and thus one can reformulate (4.1) using the rescaled force variable $f_n(t) = V'(x_n - x_{n-1})(t/\omega)$. This yields the equations

$$\omega^2\, m_1 m_2\, \frac{d^2}{dt^2}(V')^{-1}(f_n) = m_{n-1}\, f_{n+1} - (m_1 + m_2)\, f_n + m_n\, f_{n-1}, \quad n \in \mathbb{Z} \tag{4.5}$$

(recall $m_{2n+1} = m_1$, $m_{2n} = m_2$), where ω is viewed as a bifurcation parameter. Setting $Y_n = (f_{2n}, f_{2n-1})$, problem (4.5) can be rewritten as a mapping on a loop space

$$Y_{n+1} = F_\omega(Y_n) \quad \text{in } \mathbb{X}, \tag{4.6}$$

where equation (4.6) holds in a Sobolev space $\mathbb{X}$ of 2π-periodic functions of t and Y_n belongs to the domain $\mathbb{D} \subset \mathbb{X}$ of F_ω for all $n \in \mathbb{Z}$. The map F_ω is reversible due to the invariance $f_n \to f_{-n-1}$ in (4.5). It has a fixed point $Y = 0$ corresponding to the lattice at rest and the linearized operator $L_\omega = DF_\omega(0)$ is unbounded in $\mathbb{X}$ (hence the recurrence relation (4.6) is ill-posed). Discrete breathers correspond to solutions of (4.6) homoclinic to 0, i.e. satisfying $\lim_{n\to\pm\infty} \| Y_n \|_{\mathbb{D}} = 0$.

When ω takes one of the critical values ω_c given in theorem 4, L_ω has a double eigenvalue $+1$ or -1 and the remaining part of the spectrum lies away from the unit circle. This spectral separation allows us to apply a centre manifold theorem ([24]), which ensures that all small amplitude solutions of (4.6) with $\omega \approx \omega_c$ lie on a 2-dimensional centre manifold invariant under F_ω. This property reduces the local study of (4.6) to that of a two-dimensional reversible mapping, which admits homoclinic orbits to 0 for good parameter values. These orbits correspond to DB solutions of (4.6).

Note that the centre manifold theorem used in the present paper is a discrete analogue of centre manifold theorems for ill-posed differential equations in Banach spaces ([25], [28], [35]). These techniques have been used recently for studying travelling waves in different types of infinite one-dimensional lattices (Iooss and Kirchgässner [22], Iooss [21]). These problems take the form of scalar advance-delay reversible differential equations (written in the moving frame), where the continuous space coordinate plays the role of time. Note that other techniques have been used for studying different kinds of travelling waves in FPU lattices (see [16], [17], [20] and references therein).

In the present case when $V''(0) > 0$ (and with additional convexity assumptions), variational techniques should also work in principle and allow us to find finite amplitude discrete breathers (see [7] in the case of monoatomic FPU chains). In comparison, centre manifold theory is only local but provides more precise results concerning the shape of solutions, and works for general potentials.

Another advantage of centre manifold theory is that it does not fix a priori the spatial behaviour of solutions, since *all* small amplitude solutions with near-critical frequencies lie on a centre manifold. As a result, our analysis is not limited to DB and provides other types of time-periodic solutions. For good parameter values, we prove the existence of "dark breathers" (see [1] and references therein), which connect at infinity two spatially periodic solutions and have a smaller amplitude at the centre (these solutions are not spatially localized). Althought this point is not treated in the present paper, a more extended analysis of the reduced mapping would also provide spatially periodic and quasi-periodic solutions.

An interesting generalization of this work would be to analyze case iii) of theorem 4 when $\frac{m_2}{m_1}$ belongs to the "forbidden" bands $[k(k+2),(k+1)^2]$ $(k \geq 1)$. In this case, the spectrum of L_{ω_c} on the unit circle consists in a double eigenvalue -1 and a pair of simple imaginary eigenvalues (the centre manifold is 4-dimensional). It would be interesting to know if there still exist DB in this parameter regime. A possibility would be that DB do not exist in this case and are replaced by "nanopterons" having small oscillatory tails.

The outline of the paper is as follows. In section 4.2 we formulate (4.1) as a mapping in an adapted function space, and we determine the spectrum of the linearized operator in section 4.3 (depending on the parameters ω, m_1/m_2). Next we perform a centre manifold reduction when the frequency of solutions is close to one of the critical frequencies given in theorem 4. This is done in section 4.4, where we compute the (two-dimensional) centre manifolds and the reversible reduced mappings in normal form (some computations are detailed in appendix 4.6.2). The reduction is an application of the general result [24], except for the treatment of reversibility. In our case the reversibility symmetry is an unbounded operator in $\mathbb{D}$ and thus we have to redo a part of the proof (section 4.4.2 and appendix 4.6.2). We study small amplitude bifurcating orbits in section 4.5 (reversibility is important at this stage). We describe the corresponding lattice vibrations in section 4.6 and compare our results with previous numerical works.

4.2 Formulation of the mathematical problem

In this section we formulate the FPU system for time-periodic solutions as a mapping in an adapted function space.

We first rescale equation (4.1) for decreasing the number of parameters. Setting $x_n(t) = \tilde{x}_n((V''(0)/m_2)^{1/2}\, t)$, $V = V''(0)\, \tilde{V}$ in (4.1) and then dropping the tildes for convenience, we get

$$\begin{aligned} \frac{d^2}{dt^2} x_{2n} &= V'(x_{2n+1} - x_{2n}) - V'(x_{2n} - x_{2n-1}), \\ m\frac{d^2}{dt^2} x_{2n+1} &= V'(x_{2n+2} - x_{2n+1}) - V'(x_{2n+1} - x_{2n}), \end{aligned} \tag{4.7}$$

where $m = m_1/m_2 \in (0,1)$, $V'(0) = 0$ and $V''(0) = 1$. We look for solutions

of (4.7) with frequency ω. To cut off the invariance of (4.7) under translations, we use the rescaled force variable $y_n = V'(x_n - x_{n-1})(\frac{t}{\omega})$. Note that y_n is 2π-periodic in time. By integrating (4.7) one observes that the time average of y_n is independent of n. Since we are interested in spatially localized solutions, we fix $\int_0^{2\pi} y_n(t)dt = 0$ in the sequel.
Using variables y_n, problem (4.7) leads to the new system:

$$\begin{aligned} m\omega^2 \frac{d^2}{dt^2}(W(y_{2n+1})) &= y_{2n+2} - (m+1)y_{2n+1} + my_{2n}, \\ m\omega^2 \frac{d^2}{dt^2}(W(y_{2n})) &= my_{2n+1} - (m+1)y_{2n} + y_{2n-1}, \end{aligned} \tag{4.8}$$

where $W(y) = (V')^{-1}(y)$ is the local inverse of V' and satisfies $W(0) = 0$, $W'(0) = 1$.
Breather solutions of (4.7) with frequency ω correspond to 2π-periodic solutions of (4.8) satisfying $\lim_{|n|\to\infty} \|y_n\|_{L^\infty} = 0$.

We now formulate (4.8) as a first order recurrence relation in a loop space. For this purpose, we define $u_n = y_{2n}$, $v_n = y_{2n-1}$ and $Y_n = (u_n, v_n)$. Problem (4.8) can be rewritten

$$Y_{n+1} = F_{m,\omega}(Y_n) \tag{4.9}$$

with

$$F_{m,\omega}(u,v) = \begin{pmatrix} B(\frac{1}{m}(B(u) - v)) - mu \\ \frac{1}{m}(B(u) - v) \end{pmatrix} \tag{4.10}$$

and $B(u) = m\omega^2 \frac{d^2}{dt^2} W(u) + (m+1)u$. Now (m, ω) play the role of bifurcation parameters.

We now define appropriate function spaces on which $F_{m,\omega}$ is acting. The system (4.8) is invariant under the time reversibility $t \to -t$. Thus we restrict our attention to even functions of t. Moreover, this restriction divides by two the dimension of the centre manifold. We introduce the spaces $H^n_\#$ $(n \geq 0)$ defined by

$$H^n_\# = \{\, y \in H^n(\mathbb{R}/2\pi\mathbb{Z}) \,/\, y \text{ even}, \int_0^{2\pi} y\, dt = 0 \,\},$$

where H^n denotes the classical Sobolev space (with $H^0(\mathbb{R}/2\pi\mathbb{Z}) = L^2(\mathbb{R}/2\pi\mathbb{Z})$). We look for $(u_n, v_n) \in \mathbb{D} = H^4_\# \times H^2_\#$. The recurrence relation (4.9) holds in $\mathbb{X} = H^0_\# \times H^2_\#$. The operator $F_{m,\omega} : \mathbb{D} \to \mathbb{X}$ is analytic in a neighbourhood $\mathcal{U}$ of $Y = 0$ in $\mathbb{D}$. Note that the fixed point $Y = 0$ of $F_{m,\omega}$ corresponds to the lattice at rest.

We now examine the symmetry properties of equation (4.9). On the one hand, equation (4.9) is invariant under the symmetry $TY = Y(\cdot + \pi)$.

Moreover, if y_n is a solution of (4.8) then $\tilde{y}_n = y_{-n-1}$ also satisfies (4.8). As a consequence, if (u_n, v_n) is a solution of (4.9) then (v_{-n}, u_{-n}) is also a solution, i.e. equation (4.9) is reversible with respect to the symmetry $R(u,v) = (v,u)$. Note that this statement is only formal at the present stage since R does not map $\mathbb{D}$ into itself (we shall make this point rigorous in section 4.4.2). Reversibility is characterized by the property that for all $Y \in \mathcal{U}$ such that $RY \in \mathcal{U}$ and $R\,F_{m,\omega}(RY) \in \mathcal{U}$, one has $(F_{m,\omega} \circ R)^2 Y = Y$.

4.3 Spectral properties of the linearized operator

The linearized operator $L_{m,\omega} = DF_{m,\omega}(0)$ reads

$$L_{m,\omega}(u,v) = \begin{pmatrix} A(\frac{1}{m}(A\,u - v)) - m\,u \\ \frac{1}{m}(A\,u - v) \end{pmatrix},$$

where

$$A\,u = m\omega^2 \frac{d^2 u}{dt^2} + (m+1)u. \tag{4.11}$$

The operator $L_{m,\omega} : \mathbb{D} \subset \mathbb{X} \to \mathbb{X}$ is unbounded in $\mathbb{X}$ (of domain $\mathbb{D}$) and closed. Its spectrum is invariant under $z \to z^{-1}$ (due to reversibility) and $z \to \bar{z}$. The following lemma states the spectral properties of $L_{m,\omega}$ in more detail.

Lemma 7 *For all $m \in (0,1)$ and $\omega > 0$, the spectrum of $L_{m,\omega}$ is unbounded, discrete and can be written $\sigma(L_{m,\omega}) = \{0\} \cup \Sigma_{m,\omega}$, where 0 belongs to the essential spectrum and $\Sigma_{m,\omega}$ consists in non-zero eigenvalues. The set $\Sigma_{m,\omega}$ is contained in the union of the real axis and the unit circle, and invariant under $z \to \bar{z}$, $z \to z^{-1}$. The eigenvalues form sequences $(z_k)_{k\geq 1}$ and $(z_k^{-1})_{k\geq 1}$ (with $|z_k| \geq 1$ and $\mathrm{Im}\, z_k \geq 0$) determined by the equation*

$$\frac{1}{2}\,(z_k + z_k^{-1}) = 1 + k^2\omega^2\,(\frac{m}{2}k^2\omega^2 - m - 1), \quad k \geq 1. \tag{4.12}$$

Proof. For $z \in \mathbb{C}$ and $(f,g) \in \mathbb{X}$ we consider the spectral problem

$$(L_{m,\omega} - zI)\,(u,v) = (f,g), \quad (u,v) \in \mathbb{D}, \tag{4.13}$$

which reads

$$A(\frac{1}{m}(A\,u - v)) - (m+z)\,u = f, \quad \frac{1}{m}(A\,u - v) - z\,v = g.$$

The system is simplified by setting $w = \frac{1}{m}(A\,u - v)$ (the operator $(u,v) \mapsto (u,w)$ is bounded and invertible in $\mathbb{D}$). This yields

$$A\,w - (m+z)\,u = f, \quad (1+m\,z)\,w - zA\,u = g. \tag{4.14}$$

We first examine the case when $z = 0$. One obtains

$$m\,u = A\,g - f, \quad w = g,$$

hence $u \notin H^4_{\#}$ and thus $z = 0$ belongs to the essential spectrum.

We now consider the case when $z \neq 0$. Problem (4.14) can be written

$$A\,w - (m+z)\,u = f, \tag{4.15}$$
$$-A\,u + (m+z^{-1})\,w = z^{-1}\,g. \tag{4.16}$$

For $z \neq -m^{-1}$, equation (4.16) can be solved with respect to w, which yields

$$A^2\,u - (m+z)(m+z^{-1})\,u = (m+z^{-1})\,f - z^{-1}A\,g, \tag{4.17}$$
$$w = \frac{1}{1+m\,z}(zA\,u + g). \tag{4.18}$$

We now solve equation (4.17). The operator A^2 is unbounded in $H^0_{\#}$ (of domain $H^4_{\#}$), closed with a compact resolvent. It follows that (4.17) has a unique solution $(u,w) \in \mathbb{D}$ if and only if $(m+z)(m+z^{-1})$ is not an eigenvalue of A^2. Moreover, if $(m+z)(m+z^{-1})$ is an eigenvalue of A^2 then z is an eigenvalue of $L_{m,\omega}$. This occurs when z satisfies the dispersion relation

$$(m+z)(m+z^{-1}) = (m+1-m\,k^2\omega^2)^2, \quad k \in \mathbb{N}^* \tag{4.19}$$

which can be written

$$\frac{1}{2}\,(z+z^{-1}) = 1 + k^2\omega^2\,(\frac{m}{2}k^2\omega^2 - m - 1). \tag{4.20}$$

There remains to investigate the case when $z = -m^{-1}$. In this case, the system (4.15)-(4.16) reads

$$A\,w = f + (m - m^{-1})\,u, \tag{4.21}$$
$$A\,u = m\,g. \tag{4.22}$$

Note that $A : H^n_{\#} \to H^{n-2}_{\#}$ is invertible if and only if $\frac{m+1}{m\,\omega^2} \neq k^2$ for all $k \in \mathbb{N}^*$ (see (4.11) for the definition of A). When this is the case, (4.21)-(4.22) has a unique solution $(u,w) \in \mathbb{D}$ and $z = -m^{-1}$ belongs to the resolvent set of

$L_{m,\omega}$. Lastly, we consider the case when $\frac{m+1}{m\,\omega^2} = k^2$ $(k \in \mathbb{N}^*)$. In this case, one has $A\,u = \frac{m+1}{k^2}\,(\frac{d^2u}{dt^2} + k^2\,u)$. Equations (4.21) and (4.22) yield the solvability conditions

$$\int_0^{2\pi} u\,\cos(kt)\,dt = \frac{m}{1-m}\int_0^{2\pi} f\,\cos(kt)\,dt, \tag{4.23}$$

$$\int_0^{2\pi} g\,\cos(kt)\,dt = 0. \tag{4.24}$$

When condition (4.24) is satisfied, equations (4.22) and (4.23) determine $u \in H^4_{\#}$ uniquely. Then equation (4.21) has an infinite number of solutions $w \in H^2_{\#}$, defined up to an additive term $C\,\cos(kt)$. This shows that $z = -m^{-1}$ is an eigenvalue of $L_{m,\omega}$ when $\frac{m+1}{m\,\omega^2} = k^2$ $(k \in \mathbb{N}^*)$. For these parameter values, we note that $z = -m^{-1}$ satisfies the dispersion relation (4.19) (or equivalently (4.20)).

It follows from the above analysis that $L_{m,\omega}$ has essential spectrum at $z = 0$, while its eigenvalues are determined by the dispersion relation (4.20). Equation (4.20) shows that $z + z^{-1} \in \mathbb{R}$ for all eigenvalue z, hence the spectrum of $L_{m,\omega}$ is contained in the union of the real axis and the unit circle. Moreover, equation (4.20) shows that the spectrum is invariant under $z \to \bar{z}$, $z \to z^{-1}$ and is unbounded (one of the solutions of (4.20) tends to infinity as $k \to +\infty$). This completes the proof. □

We now study the variations of the spectrum of $L_{m,\omega}$ as we vary the parameter $(\omega^2, m) \in S$, $S = (0, +\infty) \times (0, 1)$. Equation (4.12) can be written

$$z_k^2 - 2(\,1 + k^2\omega^2\,(\frac{m}{2}k^2\omega^2 - m - 1)\,)\,z_k + 1 = 0. \tag{4.25}$$

The discriminant of equation (4.25) is negative if and only if

$$-2 \leq k^2\omega^2\,(\frac{m}{2}k^2\omega^2 - m - 1) \leq 0,$$

or equivalently

$$\frac{2}{m} \leq k^2\omega^2 \leq 2(1 + \frac{1}{m}) \quad \text{or} \quad 0 < k^2\omega^2 \leq 2.$$

In this case, the eigenvalues z_k, z_k^{-1} belong to the unit circle. Note that the intervals $[0, 2]$ and $[\frac{2}{m}, 2(1+\frac{1}{m})]$ are often referred as acoustic and optic band respectively.

The eigenvalues z_k, z_k^{-1} belong to the positive real axis when $k^2\omega^2 \geq 2(1 + \frac{1}{m})$ and collide at $z = +1$ when $k^2\omega^2 = 2(1 + \frac{1}{m})$. Note that all the eigenvalues belong to the positive real axis (and lie outside the unit circle) when $\omega^2 > 2(1 + \frac{1}{m})$.

The eigenvalues z_k, z_k^{-1} belong to the negative real axis when $2 \leq k^2\omega^2 \leq \frac{2}{m}$ and collide at $z = -1$ when $k^2\omega^2 = \frac{2}{m}$ and $k^2\omega^2 = 2$. One can see that z_k does not vary monotonically for $\omega^2 \in [\frac{2}{k^2}, \frac{2}{k^2\, m}]$, and reaches its minimum $z_k = -1/m$ at the mid value $\omega^2 = \frac{m+1}{k^2\, m}$. This frequency corresponds to the case when A is non-invertible.

The above analysis leads us to consider the following curves in the parameter space S

$$
\begin{aligned}
\Gamma_k^+ &: \omega^2 = \frac{2}{k^2}\,(1 + \frac{1}{m}),\\
\Gamma_k^- &: \omega^2 = \frac{2}{k^2\, m},\\
\Gamma_k^a &: \omega^2 = \frac{2}{k^2}.
\end{aligned}
$$

The infinite collection of curves Γ_k^+, Γ_k^-, Γ_k^a ($k \geq 1$) divides S in different regions corresponding to different numbers of eigenvalues on the unit circle (these curves are depicted in figure 4.1 for $k = 1, 2$).

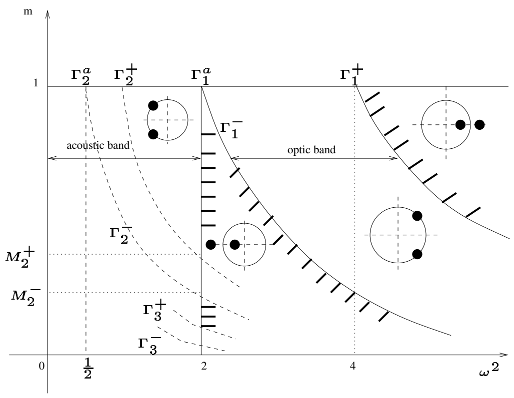


Figure 4.1: Spectrum of $L_{m,\omega}$ near the unit circle in some parameter regions. In the shaded regions, z_1, z_1^{-1} are the only eigenvalues close to the unit circle and the fixed point $Y = 0$ is hyperbolic.

Note that in our parameter space the anticontinuous limit developped in [27] consists in following a curve $\omega^2 = \frac{\Omega^2}{m}$, where $\Omega > \sqrt{2}$ is fixed and $m \to 0$. This is due to the fact that the system (4.1) is rescaled differently in this reference (the FPU chain consists in masses 1 and $M = m_2/m_1 > 1$).

The set of possible bifurcation scenarios is broad. With the aim of finding discrete breathers as homoclinic orbits to $Y = 0$, we shall restrict ourselves to the situation when z_1, z_1^{-1} are the only eigenvalues close to the unit circle and the fixed point $Y = 0$ is hyperbolic. The corresponding regions in the parameter space S are depicted in figure 4.1 (shaded regions). One of these regions is located in the neighbourhood of Γ_1^+, at the right side (i.e. above the top of the optic band). An other region is in the neighbourhood of Γ_1^-, at the left side (i.e. below the bottom of the optic band). The situation is more complicated when (ω^2, m) is close to Γ_1^a and at the right side (i.e. above the top of the acoustic band), since the curves Γ_1^a and $\Gamma_k^{\pm}$ $(k \geq 2)$ intersect at the point $(\omega^2, m) = (2, M_k^{\pm})$ with

$$M_k^+ = \frac{1}{k^2 - 1}, \quad M_k^- = \frac{1}{k^2}. \tag{4.26}$$

We shall only consider the case when $m \in (M_{k+1}^+, M_k^-)$ $(k \geq 1)$ is fixed and $\omega^2 \approx 2$, since z_1, z_1^{-1} are the only eigenvalues close to the unit circle. In the case when $m \in (M_k^-, M_k^+)$ $(k \geq 2)$ is fixed and $\omega^2 \approx 2$, there are two additional eigenvalues z_k, z_k^{-1} on the unit circle and the bifurcation problem is more complicated. Note that these eigenvalues indicate that $k\,\omega$ belongs to the optic band, and consequently resonance phenomena may occur in these parameter regimes. In the sequel, we shall consider the following line segments in the parameter space S

$$\Gamma_1^k \, : \, \omega^2 = 2, \; m \in (M_{k+1}^+, M_k^-).$$

We note that equation (4.8) has the invariance $(y_n, \omega) \to (y_n(kt), \frac{\omega}{k})$ (for any integer $k \geq 2$), and consequently the existence of a given solution y_n for $(\omega^2, m) \approx \Gamma_1^{\pm,a}$ implies the existence of an other solution $\tilde{y}_n = y_n(kt)$ for $(\omega^2, m) \approx \Gamma_k^{\pm,a}$. This new solution is artificial since y_n and $\tilde{y}_n$ both correspond to the same solution of (4.7). However, this remark would be useful for studying bifurcations near $\Gamma_k^{\pm,a}$ involving several modes.

Although we shall not investigate these cases in the present paper, we note the presence of simultaneous or multiple collisions of eigenvalues for particular values of (ω^2, m) (codimension two bifurcations). This phenomenon may yield the existence of new interesting types of solutions when the mass ratio is close to such critical values. For $(\omega^2, m) = (2, M_k^+)$, the central part of the spectrum is composed of two double eigenvalues $z_1 = -1$, $z_k = +1$, and for $(\omega^2, m) = (2, M_k^-)$ it is composed of one quadruple eigenvalue -1 $(z_1^{\pm 1} = z_k^{\pm 1} = -1)$. Moreover, the curves Γ_k^- and Γ_{k+1}^+ $(k \geq 3)$ cross at $(\omega^2, m) = (\frac{2}{2k+1}, m_k)$, with $m_k = (2k+1)/k^2$ (then $z_k = -1$, $z_{k+1} = +1$ are double eigenvalues, and a finite number of other eigenvalues lie on the unit circle). One can even observe a triple crossing of $\Gamma_{(p^2-1)/2}^-$, $\Gamma_{(p^2+1)/2}^+$ and Γ_p^a when $p \geq 3$ is odd (then $z = -1$, $z = +1$ are respectively quadruple and double eigenvalues, and a finite number of other eigenvalues lie on the unit circle).

Note also that a study of the codimension two bifurcation $(\omega^2, m) \approx (2, 1)$ should be of interest. Indeed, we shall see in section 4.6 that small amplitude breathers exist for $(\omega^2, m) \approx (2, 1)$ if one chooses good parameter values in the domain between Γ_1^a and Γ_1^-. This domain vanishes as $m \to 1$, and for $m = 1$ the limiting frequency $\omega = \sqrt{2}$ lies inside the acoustic band.

The following lemma describes the centre space $\mathbb{X}_c$ (invariant subspace under $L_{m,\omega}$ associated with the eigenvalues on the unit circle) when parameters lie on the curves $\Gamma_1^+, \Gamma_1^-, \Gamma_1^k$.

Lemma 8 *If $(\omega^2, m) \in \Gamma_1^+$, $z = +1$ is a double non-semi-simple eigenvalue of $L_{m,\omega}$. If $(\omega^2, m) \in \Gamma_1^-$ or $(\omega^2, m) \in \Gamma_1^k$ ($k \geq 1$), $z = -1$ is a double non-semi-simple eigenvalue of $L_{m,\omega}$. Moreover, the centre space $\mathbb{X}_c$ is the same for all these parameter values and is spanned by the vectors $V_u = (\cos t, 0)$ and $V_v = (0, \cos t)$. The spectral projection Π_c on $\mathbb{X}_c$ reads $\Pi_c(u, v) = (\pi_c u, \pi_c v)$, where $\pi_c u = \frac{1}{\pi} \int_0^{2\pi} u(t) \cos t \, dt \, \cos t$.*

Proof. The operators $L_{m,\omega}$ and Π_c commute since π_c commutes with $\frac{d^2}{dt^2}$. We now define $\mathbb{X}_c = \Pi_c \mathbb{X} = \text{Span}\,\{V_u, V_v\}$, $\Pi_h = I_{\mathbb{X}} - \Pi_c$, $\mathbb{X}_h = \Pi_h \mathbb{X}$, $\mathbb{D}_h = \Pi_h \mathbb{D}$, $L_c = L_{m,\omega|X_c}$ and $L_h = L_{m,\omega|X_h}$. Going back to lemma 7 and replacing $\mathbb{X}$, $\mathbb{D}$ by $\mathbb{X}_h$, $\mathbb{D}_h$, one obtains $\sigma(L_h) = \sigma(L_{m,\omega}) \setminus \{1\}$ for $(\omega^2, m) \in \Gamma_1^+$ and $\sigma(L_h) = \sigma(L_{m,\omega}) \setminus \{-1\}$ for $(\omega^2, m) \in \Gamma_1^-$ or $(\omega^2, m) \in \Gamma_1^k$. Moreover, $\sigma(L_h)$ is purely hyperbolic. Now we have in the basis $\{V_u, V_v\}$

$$L_c = \begin{pmatrix} \gamma & -\beta \\ \beta & -m^{-1} \end{pmatrix}, \tag{4.27}$$

where $\beta = 1 + m^{-1} - \omega^2$ and $\gamma = m\,(\beta^2 - 1)$. Consequently, $z = -1$ (respectively $z = +1$) is a double non-semi-simple eigenvalue of L_c for $(\omega^2, m) \in \Gamma_1^-$ or $(\omega^2, m) \in \Gamma_1^k$ (respectively $(\omega^2, m) \in \Gamma_1^+$).$\Box$

4.4 Centre manifold reduction

In this section we locally reduce the infinite-dimensional mapping (4.9) to a finite-dimensional one on a centre manifold, for parameter values close to the bifurcation curves Γ_1^k ($k \geq 1$), Γ_1^+, Γ_1^-. Section 4.4.1 states the reduction theorem, and we show in section 4.4.2 that the reduced mapping inherits the symmetries of the infinite-dimensional system. At this step one has to deal with a technical difficulty, due to the fact that the reversibility symmetry is an unbounded operator in $\mathbb{D}$. In section 4.4.3 we compute the principal terms in the Taylor expansions of the centre manifold and the reduced mapping, which is written in a simpler normal form in section 4.4.4.

4.4.1 Reduction theorem

We now consider the bifurcations at a double eigenvalue ± 1 occuring on the curves Γ_1^k ($k \geq 1$), Γ_1^+, Γ_1^-. We set $\omega^2 = \omega_c^2 + \mu$, where $\mu \approx 0$ is a small bifurcation parameter and ω_c^2, m are as follows

$$\begin{array}{lll} \text{for} & (\omega_c^2, m) \in \Gamma_1^+ \,:\, m \in (0,1), & \omega_c^2 = 2(1+\frac{1}{m}), \\ \text{for} & (\omega_c^2, m) \in \Gamma_1^- \,:\, m \in (0,1), & \omega_c^2 = \frac{2}{m}, \\ \text{for} & (\omega_c^2, m) \in \Gamma_1^k \,:\, m \in (M_{k+1}^+, M_k^-), & \omega_c^2 = 2. \end{array} \tag{4.28}$$

The recurrence (4.9) can be written in the form:

$$Y_{n+1} = LY_n + \mathcal{N}(Y_n, \mu) \tag{4.29}$$

with $L = DF_{m,\omega_c}(0)$ and $\mathcal{N}(Y,\mu) = F_{m,\omega}(Y) - LY = O(\|Y\|_{\mathbb{D}}^2 + |\mu|\|Y\|_{\mathbb{D}})$. We now use the notation $\alpha = 1/m$. Setting $Y = (u,v)$ and $Qu = (\omega_c^2 \frac{d^2}{dt^2} + (1+\alpha))u$, one has:

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha}(Q^2-1)u - Qv \\ Qu - \alpha v \end{pmatrix}$$

and $\mathcal{N}$ is given by:

$$\mathcal{N}(Y,\mu) = \begin{pmatrix} \frac{1}{\alpha}(Q\,N(u,\mu) + N(Qu - \alpha v + N(u,\mu),\mu)) \\ N(u,\mu) \end{pmatrix}$$

with

$$N(u,\mu) = \frac{d^2}{dt^2}(\mu W(u) + \omega_c^2(W(u) - u)). \tag{4.30}$$

Note that $W = (V')^{-1}$ can be expanded as

$$W(y) = y + W_2 y^2 + W_3 y^3 + O(y^4),$$

with

$$W_2 = -\frac{1}{2} V^{(3)}(0), \quad W_3 = \frac{1}{2}(V^{(3)}(0))^2 - \frac{1}{6} V^{(4)}(0). \tag{4.31}$$

Thus we have

$$N(u,\mu) = \frac{d^2}{dt^2}(\mu u + \omega_c^2 W_2 u^2 + W_2 \mu u^2 + \omega_c^2 W_3 u^3) + \text{h.o.t.} \tag{4.32}$$

The part of the spectrum of L lying on the unit circle consists in a double eigenvalue $+1$ or -1, which is isolated from the rest of the spectrum. Consequently, the centre manifold theorem applies (see theorem 1 in [24]). Therefore one can reduce locally the study of (4.29) to that of a two-dimensional map on the generalized eigenspace $\mathbb{X}_c$. In the sequel, we denote by Π_c the spectral projection on $\mathbb{X}_c$ (see lemma 8) and define $\Pi_h = Id - \Pi_c$, $\mathbb{X}_h = \Pi_h \mathbb{X}$, $L_h = L_{|\mathbb{X}_h}$, $L_c = L_{|\mathbb{X}_c}$, $Y^c = \Pi_c Y$, $Y^h = \Pi_h Y$. A direct application of the centre manifold theorem gives the following result:

Theorem 5 *Let us fix m and ω_c as in (4.28) and $k \geq 4$. There exist neighbourhoods Ω, Λ of 0 in $\mathbb{D}$, $\mathbb{R}$ respectively and a map $\Psi \in C^k(\mathbb{X}_c \times \Lambda; \mathbb{D}_h)$ with $\Psi(0,\mu) = 0$ and $D\Psi(0,0) = 0$ such that for any $\mu \in \Lambda$, the manifold*

$$\mathcal{M}_\mu = \{Y \in \mathbb{D} / Y = Y^c + \Psi(Y^c, \mu), Y^c \in \mathbb{X}_c\}$$

has the following properties.
(i) $\mathcal{M}_\mu$ is locally invariant under $L + \mathcal{N}(.,\mu)$, i.e. if $Y \in \mathcal{M}_\mu \cap \Omega$ then $LY + \mathcal{N}(Y,\mu) \in \mathcal{M}_\mu$.
(ii) If Y_n is a solution of (4.29) such that $Y_n \in \Omega$ for all $n \in \mathbb{Z}$, then $Y_n \in \mathcal{M}_\mu$ for all $n \in \mathbb{Z}$ and Y_n^c satisfies the recurrence relation

$$\forall n \in \mathbb{Z}, Y_{n+1}^c = f(Y_n^c, \mu) \tag{4.33}$$

with $f(Y,\mu) = L_c Y + \Pi_c \mathcal{N}(Y + \Psi(Y,\mu), \mu)$.
(iii) Conversely, if Y_n^c is a solution of (4.33) such that $Y_n^c \in \Omega$ for all $n \in \mathbb{Z}$, then $Y_n = Y_n^c + \Psi(Y_n^c, \mu)$ satisfies (4.29).

Note that the validity domain of the reduction (i.e. the size of the neighbourhood $\Omega \times \Lambda$) depends on the spectral gap between the unit circle and the hyperbolic part of the spectrum (see [24]). More precisely, this neighbourhood shrinks as other eigenvalues approach the unit circle, which is the case when $\omega_c^2 = 2$ and $m \to M_k^{\pm}$ $(k \geq 2)$. It also shrinks as $m \to 0$ since this limit is singular in (4.9).

The situation is different as $m \to 1$ on Γ_1^+, Γ_1^- and Γ_1^a (the hyperbolic part of the spectrum remains at a nonzero distance). In these limits, the validity domain of the reduction is uniform with respect to m. However, the frequency domains in which different classes of bifurcating solutions of the reduced mapping (4.33) exist may vanish as $m \to 1$. Indeed, the reduced mapping structure changes drastically at $(\omega^2, m) = (2,1)$ since the double eigenvalue -1 becomes semi-simple (see (4.27)).

4.4.2 Symmetries

We have seen in section 4.2 that the mapping (4.29) is equivariant with respect to the symmetry $TY = Y(\cdot + \pi)$ and reversible with respect to $R(u,v) = (v,u)$. We now have to check that the centre manifold reduction preserves these symmetries. This means that the centre manifold is invariant under T and R, and that the reduced mapping is equivariant under T and reversible under R. As we shall see, these properties simplify the computation of the centre manifold and play a fundamental role in the reduced mapping dynamics.

We now state the main result of this section.

Theorem 6 *The maps f and Ψ in theorem 5 satisfy*

$$\forall Y \in \mathbb{X}_c \cap \Omega, \quad R\Psi(Y,\mu) = \Psi(RY,\mu), \quad (f(\cdot,\mu)\circ R)^2 = Id, \tag{4.34}$$

$$\forall Y \in \mathbb{X}_c \cap \Omega, \quad T\Psi(Y,\mu) = \Psi(TY,\mu), \quad f(TY,\mu) = Tf(Y,\mu). \tag{4.35}$$

The main difficulty is related to the reversibility symmetry R. Indeed, the proof of theorem 6 requires to modify (4.29) by a cut-off preserving reversibility. The existence of a suitable cut-off is not as automatic for maps as for vector fields, due to the fact that a reversible vector field F is conjugate to $-F$ whereas a reversible map is conjugate to its inverse. This difficulty has been treated in [24] for a class of problems in which R is bounded in $\mathbb{D}$.

However, these results are not directly applicable here since R is unbounded in $\mathbb{D}$. More precisely, $R : \mathbb{D}_R \subset \mathbb{D} \to \mathbb{D}$ is closed and the domain $\mathbb{D}_R$ of R is $\mathbb{D}_R = H^4_\# \times H^4_\#$. Consequently we have to adapt the proof of reversibility preservation given in [24] (sections 4 and 5.1). The idea consists in using a cut-off function on the finite-dimensional centre space $\mathbb{X}_c$ (instead of $\mathbb{D}$) since R is bounded in $\mathbb{X}_c$. We shall only point out which steps are modified and refer to [24] for the complete proof.

The modification of (4.29) by a cut-off consists in replacing $\mathcal{N}(.,\mu)$ by a map $\mathcal{N}_\epsilon(.,\mu)$ equal to $\mathcal{N}(.,\mu)$ in a small neighbourhood of 0, globally bounded and lipschitzian on $\mathbb{X}_c$ (with small Lipschitz constant). This comes from the necessity of using the contraction mapping theorem in a space of sequences with possibly unbounded central parts (see section 3.1 in [24]). One first proves a global centre manifold reduction result for the truncated problem

$$Y_{n+1} = LY_n + \mathcal{N}_\epsilon(Y_n,\mu), \tag{4.36}$$

which gives a local centre manifold reduction result for the original problem (4.29). In order to prove that the centre manifold reduction preserves reversibility, one has to construct $\mathcal{N}_\epsilon$ such that the mapping (4.36) is also reversible. This means that one has to modify the nonlinear terms in (4.8) in such a way that the invariance $y_n \to y_{-n-1}$ is preserved. In what follows, we introduce an adapted cut-off of (4.8).

Setting $\omega^2 = \omega_c^2 + \mu$ and $\alpha = m^{-1}$ in (4.8), one obtains

$$\begin{aligned} \omega_c^2 \frac{d^2}{dt^2}(y_{2n+1}) + N(y_{2n+1},\mu) &= \alpha y_{2n+2} - (1+\alpha)y_{2n+1} + y_{2n}, \\ \omega_c^2 \frac{d^2}{dt^2}(y_{2n}) + N(y_{2n},\mu) &= y_{2n+1} - (1+\alpha)y_{2n} + \alpha y_{2n-1}, \end{aligned} \tag{4.37}$$

where $N(y,\mu) = \frac{d^2}{dt^2} g(y,\mu)$ and $g(y,\mu) = \mu W(y) + \omega_c^2(W(y) - y)$. We now consider the truncated problem

$$
\begin{aligned}
&\omega_c^2\, \frac{d^2}{dt^2}(y_{2n+1}) + N_\epsilon(y_{2n+1},\mu) = \alpha y_{2n+2} - (1+\alpha)y_{2n+1} + y_{2n},\\
&\omega_c^2\, \frac{d^2}{dt^2}(y_{2n}) + N_\epsilon(y_{2n},\mu) = y_{2n+1} - (1+\alpha)y_{2n} + \alpha y_{2n-1},
\end{aligned}
\tag{4.38}
$$

where $N_\epsilon(y,\mu) = \frac{d^2}{dt^2} g_\epsilon(y,\mu)$, $g_\epsilon(y,\mu) = g(y,\mu)\,\chi(\epsilon^{-1}\,|(y,\cos t)|)$, $(y,\cos t) = \int_0^{2\pi} y(t)\cos t\,dt$ and $\chi \in C^\infty([0,+\infty),[0,1])$ is a cut-off function satisfying $\chi(x) = 1$ for $x \in [0,1]$ and $\chi(x) = 0$ for $x \geq 2$. One can check that the invariance $y_n \to y_{-n-1}$ is preserved in (4.38).

Setting $Y_n = (u_n, v_n) = (y_{2n}, y_{2n-1})$, problem (4.38) takes the form (4.36) with

$$
L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha}(Q^2-1)u - Qv \\ Qu - \alpha v \end{pmatrix},
$$

$$
Qu = (\omega_c^2 \frac{d^2}{dt^2} + (1+\alpha))u,
$$

$$
\mathcal{N}_\epsilon(Y,\mu) = \begin{pmatrix} \frac{1}{\alpha}(Q\,N_\epsilon(u,\mu) + N_\epsilon(Qu - \alpha v + N_\epsilon(u,\mu),\mu)) \\ N_\epsilon(u,\mu) \end{pmatrix}.
$$

The cut-off in (4.38) is only performed on the (finite-dimensional) centre space. Indeed, since R is unbounded in $\mathbb{D}$ it is not possible to use a cut-off function on the whole of $\mathbb{D}$ (as it is done in [24]). This procedure will slightly modify the subsequent choice of spaces.

We now detail some important properties of $\mathcal{N}_\epsilon$. Let us consider the strip

$$
B^h_\epsilon = \{\, Y \in \mathbb{D} \,/\, \|Y^h\|_{\mathbb{D}} \leq \epsilon \,\}.
$$

One can find $\epsilon_0 > 0$ such that $\mathcal{N}_\epsilon \in C^k_b(B^h_{\epsilon_0} \times (-\epsilon_0,\epsilon_0), \mathbb{X})$. Moreover, one obtains using lemma 17 of appendix 4.6.2

$$
\|\mathcal{N}_\epsilon\|_{C^0_b(B^h_\epsilon \times (-\epsilon,\epsilon),\mathbb{X})} = O(\epsilon^2), \quad \|D_Y\mathcal{N}_\epsilon\|_{C^0_b(B^h_\epsilon \times (-\epsilon,\epsilon),\mathcal{L}(\mathbb{D},\mathbb{X}))} = O(\epsilon) \tag{4.39}
$$

as $\epsilon \to 0^+$. In addition, one has $\mathcal{N}_\epsilon(.,\mu) = \mathcal{N}(.,\mu)$ in a ball B_ϵ having the form

$$
B_\epsilon = \{\, Y \in \mathbb{D} \,/\, \|Y\|_{\mathbb{D}} < C\,\epsilon \,\},
$$

and thus problems (4.29) and (4.36) are locally identical.

We now examine more closely the reversibility symmetry in (4.36). One can check that if $u \in \mathbb{D}_R$, $R\,u \in B^h_{\epsilon_0}$, $(L + \mathcal{N}_\epsilon(.,\mu)) \circ R\,u \in \mathbb{D}_R$ and $R\,(L + \mathcal{N}_\epsilon(.,\mu)) \circ R\,u \in B^h_{\epsilon_0}$, then

$$
((L + \mathcal{N}_\epsilon(.,\mu)) \circ R)^2 u = u. \tag{4.40}
$$

This property is due to the invariance $y_n \to y_{-n-1}$ in (4.38) and characterizes the fact that (4.36) is reversible with respect to R.

In the case when R is bounded in $\mathbb{D}$, property (4.40) immediately implies that for any solution Y_n of (4.36), $Z_n = RY_{-n}$ is also a solution. The situation is more complicated in our case since R is unbounded in $\mathbb{D}$ and Y_{-n} does not *a priori* belong to $\mathbb{D}_R$.

However, one can show that the solutions of (4.36) have more regularity than the regularity of $\mathbb{D}$ and it turns out that $Y_{-n} \in \mathbb{D}_R$. More precisely, the following result is proved in appendix 4.6.2 (see lemma 21).

Lemma 9 *Fix $p \geq 2$. There exist $\epsilon_0, \gamma > 0$ (depending on p) such that for all $\epsilon < \epsilon_0$, $\mu \in [-\epsilon, \epsilon]$, any solution of (4.36) such that $Y_n \in B^h_\epsilon$ for all $n \in \mathbb{Z}$ satisfies $Y_n \in H^{p+2}_\# \times H^p_\#$ and $\| Y^h_n \|_{H^{p+2}_\# \times H^p_\#} \leq \gamma\epsilon$ for all $n \in \mathbb{Z}$.*

For proving lemma 9 one rewrites (4.36) as an evolutionnary system

$$\begin{aligned} &\frac{d^2}{dt^2}(\omega_c^2\, y_{2n+1} + g_\epsilon(y_{2n+1}, \mu)) = \alpha y_{2n+2} - (1+\alpha) y_{2n+1} + y_{2n}, \\ &\frac{d^2}{dt^2}(\omega_c^2\, y_{2n} + g_\epsilon(y_{2n}, \mu)) = y_{2n+1} - (1+\alpha) y_{2n} + \alpha y_{2n-1}, \end{aligned} \tag{4.41}$$

where the right side belongs to $H^i_\# \times H^i_\#$ (with $i = 2$), and one proceeds by induction on i.

Now property (4.40) and lemma 9 yield the following result.

Lemma 10 *If Y_n is a solution of (4.36), then $Z_n = RY_{-n}$ is also a solution.*

Proof. According to lemma 9 we have $Y_n \in H^6_\# \times H^4_\# \subset \mathbb{D}_R$, with the estimate $\| Y^h_n \|_{H^6_\# \times H^4_\#} \leq \gamma\epsilon$. Consequently, we have Z_n, $RZ_n \in \mathbb{D}_R$ and Z_n, $RZ_n \in B^h_{\epsilon_0}$ for ϵ small enough. Setting $n \to -n$ in (4.36) yields

$$(L + \mathcal{N}_\epsilon(., \mu))(RZ_n) = RZ_{n-1} \in \mathbb{D}_R,$$

with $R\,(L + \mathcal{N}_\epsilon(., \mu))(RZ_n) \in B^h_{\epsilon_0}$. Consequently, property (4.40) implies

$$Z_n = ((L + \mathcal{N}_\epsilon(., \mu))R)^2 Z_n = (L + \mathcal{N}_\epsilon(., \mu)) Z_{n-1},$$

which completes the proof.

□

We now give a *global* centre manifold reduction result (theorem 7 below) which preserves the reversible character of the truncated problem (4.36). The proof is completely parallel to [24] (section 4), except one has to make some obvious changes of spaces (since the cut-off is performed on $\mathbb{X}_c$ instead of $\mathbb{D}$). In appendix 4.6.2 we recall the principal steps of the proof and detail the modifications specific to our case.

We first introduce a suitable space of sequences for $Y = (Y_n)_{n\in\mathbb{Z}}$. Given a Banach space E and $\nu \in (0,1]$, we define the Banach space

$$B_\nu(E) = \{ \, Y \,/\, Y_n \in E, \|Y\|_{B_\nu(E)} < +\infty \, \},$$

where $\|Y\|_{B_\nu(E)} = \operatorname{Sup}_{n\in\mathbb{Z}} \; \nu^{|n|} \, \|Y_n\|_E$ (note that $B_1(E) = \ell_\infty(E)$). Now we look for Y in the set

$$B_\nu^\epsilon(\mathbb{D}) = \{ \, Y \in B_\nu(\mathbb{D}) \,/\, Y^h \in B_1(\mathbb{D}_h), \|Y^h\|_{B_1(\mathbb{D}_h)} \leq \epsilon \, \}$$

(sequences $Y \in B_\nu^\epsilon(\mathbb{D})$ can have an unbounded central part). The set $B_\nu^\epsilon(\mathbb{D})$ is a closed (convex) subset of $B_\nu(\mathbb{D})$ and consequently a complete metric space for the distance $\mathrm{d}(Y,Z) = \|Y - Z\|_{B_\nu(\mathbb{D})}$.

One can prove the following global reduction result using (4.39), the spectral properties of L and lemma 10 (see appendix 4.6.2).

Theorem 7 *There exists $r \in (0,1)$ such that for $r < \zeta < \nu^k < \nu < 1$, for ϵ small enough, for all $\mu \in [-\epsilon, \epsilon]$ and $x \in \mathbb{X}_c$, the problem*

$$Y_{n+1} = LY_n + \mathcal{N}_\epsilon(Y_n, \mu), \quad Y \in B_\nu^\epsilon(\mathbb{D}), \quad Y_0^c = x, \tag{4.42}$$

has a unique solution $Y_n = \phi_n^\epsilon(x,\mu)$ with

$$\phi^\epsilon \in C^0(\mathbb{X}_c \times [-\epsilon,\epsilon]\,,\, B_\nu^\epsilon(\mathbb{D})) \,\cap\, C^k(\mathbb{X}_c \times [-\epsilon,\epsilon]\,,\, B_\zeta(\mathbb{D})).$$

Moreover one has $Y_n^h = \psi_\epsilon(Y_n^c, \mu)$, where $\psi_\epsilon = \Pi_h\, \phi_0^\epsilon \in C^k(\mathbb{X}_c \times [-\epsilon,\epsilon], \mathbb{D}_h)$ and $\|\psi_\epsilon\|_{C_b^0(\mathbb{X}_c \times [-\epsilon,\epsilon], \mathbb{D}_h)} = O(\epsilon^2)$. The central part of Y_n satisfies the reduced recurrence relation

$$Y_{n+1}^c = f_\epsilon(Y_n^c, \mu) \quad \forall n \in \mathbb{Z}, \tag{4.43}$$

where

$$f_\epsilon(x,\mu) = L_c\, x + \Pi_c \mathcal{N}_\epsilon(x + \psi_\epsilon(x,\mu), \mu).$$

We have in addition $R\psi_\epsilon(.,\mu) = \psi_\epsilon(.,\mu) \circ R$ and $(f_\epsilon(.,\mu) \circ R)^2 = I$.

We now turn back to theorems 5 and 6, where Ω denotes the ball of radius $\epsilon/2$ in $\mathbb{D}$, Λ the interval $(-\epsilon,\epsilon)$ and $\Psi = \psi_\epsilon$. Note that $f(x,\mu) = f_\epsilon(x,\mu)$ for all $x \in \mathbb{X}_c \cap \Omega$ since $\|x + \Psi(x,\mu)\|_{\mathbb{D}} < \epsilon$ (one has $\|\psi_\epsilon\|_{C_b^0(\mathbb{X}_c \times (-\epsilon,\epsilon), \mathbb{D}_h)} = O(\epsilon^2)$). It follows immediately from theorem 7 that the maps f and Ψ in theorem 5 satisfy

$$\forall Y \in \mathbb{X}_c \cap \Omega, \quad R\Psi(Y,\mu) = \Psi(RY,\mu), \quad (f(\cdot,\mu) \circ R)^2 = Id. \tag{4.44}$$

Consequently, we have shown that reversibility is preserved throughout the reduction.

There remains to check that the reduction preserves the equivariance under $TY = Y(\cdot + \pi)$. Since T commutes with $L + \mathcal{N}_\epsilon(., \mu)$, it follows that the maps $\psi_\epsilon(., \mu)$ and $f_\epsilon(., \mu)$ in theorem 7 commute with T (see [24] p.39), which completes the proof of theorem 6.

4.4.3 Centre manifold computation

In this section we compute the Taylor expansions of the function Ψ and the reduced map f. For calculating Ψ, we project (4.29) on $\mathbb{X}_h$ and $\mathbb{X}_c$:

$$Y_{n+1}^h = L_h Y_n^h + \Pi_h \mathcal{N}(Y_n^c + Y_n^h, \mu),$$

$$Y_{n+1}^c = L_c Y_n^c + \Pi_c \mathcal{N}(Y_n^c + Y_n^h, \mu).$$

Choosing $Y_n \in \mathcal{M}_\mu \cap \Omega$ and using property (i) of theorem 1, we find:

$$\Psi(Y_{n+1}^c, \mu) = L_h \Psi(Y_n^c, \mu) + \Pi_h \mathcal{N}(Y_n^c + \Psi(Y_n^c, \mu), \mu), \tag{4.45}$$

$$Y_{n+1}^c = L_c Y_n^c + \Pi_c \mathcal{N}(Y_n^c + \Psi(Y_n^c, \mu), \mu) = f(Y_n^c, \mu). \tag{4.46}$$

Replacing (4.46) into (4.45) yields:

$$\begin{aligned} \Psi(L_c Y_n^c + \Pi_c \mathcal{N}(Y_n^c + \Psi(Y_n^c, \mu), \mu), \mu) = {} & L_h \Psi(Y_n^c, \mu) \\ & + \Pi_h \mathcal{N}(Y_n^c + \Psi(Y_n^c, \mu), \mu). \end{aligned} \tag{4.47}$$

The Taylor expansion of Ψ at $(Y^c, \mu) = 0$ can be computed by identifying the terms of equal orders in (Y_n^c, μ) obtained by expanding (4.47). We identify $\mathbb{X}_c$ with $\mathbb{R}^2$ by setting:

$$Y^c = \begin{pmatrix} a \\ b \end{pmatrix} \cos(t).$$

The symmetry properties of Ψ (theorem 6) imply that

$$\begin{aligned} T\Psi(a, b, \mu) &= \Psi(-a, -b, \mu), \\ R\Psi(a, b, \mu) &= \Psi(b, a, \mu). \end{aligned} \tag{4.48}$$

Consequently, the Taylor expansion of Ψ has the form:

$$\Psi(a, b, \mu) = \begin{pmatrix} \Psi_{011} a\mu + \Psi_{101} b\mu + \Psi_{020} a^2 + \Psi_{110} ab + \Psi_{200} b^2 \\ \Psi_{101} a\mu + \Psi_{011} b\mu + \Psi_{200} a^2 + \Psi_{110} ab + \Psi_{020} b^2 \end{pmatrix} + \text{h.o.t.} \tag{4.49}$$

where $\Psi_{pqr} \in H_\#^4$. Due to the invariance under T, we have also:

$$\begin{aligned} &\Psi_{011}, \Psi_{101} \in \langle \cos((2k+1)t)/k \in \mathbb{Z} \rangle = \mathbb{V}_{odd}, \\ &\Psi_{020}, \Psi_{110}, \Psi_{200} \in \langle \cos(2kt)/k \in \mathbb{Z} \rangle = \mathbb{V}_{even}. \end{aligned} \tag{4.50}$$

Setting

$$Y_n^c = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos(t),$$

in (4.47), expanding (4.47) in powers of (a_n, b_n, μ) and identifying quadratic terms leads to a linear system for the corresponding coefficients Ψ_{pqr} (calculations are given in the appendix). Solving this system yields:

$$\Psi_{011} = \Psi_{101} = 0$$

and $\Psi_{020}, \Psi_{200}, \Psi_{110}$ have the general form:

$$\begin{aligned} \Psi_{020} &= p(m) W_2 \cos(2t), \\ \Psi_{200} &= q(m) W_2 \cos(2t), \\ \Psi_{110} &= r(m) W_2 \cos(2t), \end{aligned} \tag{4.51}$$

with p, q, r depending on the considered value of ω_c^2 in (4.28). For $\omega_c^2 = 2(1+\alpha)$ we find:

$$\begin{aligned} p(m) &= -\frac{1}{16}\frac{8m-1}{m}, \\ q(m) &= \frac{1}{16m}, \\ r(m) &= \frac{1}{8m}. \end{aligned} \tag{4.52}$$

For $\omega_c^2 = 2\alpha$ we get:

$$\begin{aligned} p(m) &= -\frac{1}{16}\frac{7m^2-34m+3}{m(m-3)}, \\ q(m) &= \frac{1}{16}\frac{m^2-6m-3}{m(m-3)}, \\ r(m) &= \frac{1}{8}\frac{m^2+2m-3}{m(m-3)}. \end{aligned} \tag{4.53}$$

Considering the case $\omega_c^2 = 2$ yields:

$$\begin{aligned} p(m) &= -\frac{1}{16}\frac{24m^3+5m^2-6m+1}{m^2(3m-1)}, \\ q(m) &= \frac{1}{16}\frac{3m^2+6m-1}{m^2(3m-1)}, \\ r(m) &= \frac{1}{8}\frac{5m^2-6m+1}{m^2(3m-1)}. \end{aligned} \tag{4.54}$$

When $\omega_c^2 = 2$, note that Ψ is not defined for $m = 1/3 = M_2^+$ (see the comments following theorem 5).

The Taylor expansion of f is obtained by injecting (4.51) in equation (4.46) One obtains (see appendix 4.6.2)

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} F_1(a_n, b_n, \mu) \\ F_2(a_n, b_n, \mu) \end{pmatrix} = f(a_n, b_n, \mu) \tag{4.55}$$

with

$$\begin{aligned} \alpha\, F_1(a, b, \mu) &= (\beta^2 - 1)a - \alpha\beta b - 2\beta a\mu + \alpha b\mu \\ &\quad -\omega_c^2\, \alpha\, (\beta_4 a^3 + \beta_5 a^2 b + \beta_6 a b^2 + \beta_7 b^3) \\ &\quad +o(\|(a,b)\|\mu + \|(a,b)\|^3), \end{aligned}$$

$$\begin{aligned} F_2(a, b, \mu) &= \beta a - \alpha b - a\mu \\ &\quad -\omega_c^2(\beta_1 a^3 + \beta_2 a b^2 + \beta_3 a^2 b) \\ &\quad +o(\|(a,b)\|\mu + \|(a,b)\|^3), \end{aligned}$$

$$\begin{aligned} \beta_1 &= p(m)W_2^2 + \tfrac{3}{4}W_3, \\ \beta_2 &= q(m)W_2^2, \\ \beta_3 &= r(m)W_2^2, \end{aligned} \tag{4.56}$$

$$\begin{aligned} \alpha\beta_4 &= \beta\beta_1 + \beta^3\beta_1 + \beta\gamma^2\beta_2 + \beta^2\gamma\beta_3, \\ \alpha\beta_5 &= -3\beta^2\alpha\beta_1 - (\alpha\gamma^2 + 2\gamma\beta^2)\beta_2 + (\beta - \beta^3 - 2\alpha\beta\gamma)\beta_3, \\ \alpha\beta_6 &= 3\beta\alpha^2\beta_1 + (\beta + \beta^3 + 2\alpha\beta\gamma)\beta_2 + (2\beta^2\alpha + \alpha^2\gamma)\beta_3, \\ \beta_7 &= -\alpha^2\beta_1 - \beta^2\beta_2 - \alpha\beta\beta_3, \end{aligned} \tag{4.57}$$

and $\beta = 1 + \alpha - \omega_c^2$, $\gamma = \frac{\beta^2 - 1}{\alpha}$.
The reduced map f has the structure:

$$f(a, b, \mu) = L_c \begin{pmatrix} a \\ b \end{pmatrix} + \mu f_{11} \begin{pmatrix} a \\ b \end{pmatrix} + f_{30}(a, b) + o(\|(a,b)\|\mu + \|(a,b)\|^3) \tag{4.58}$$

where

$$L_c = \begin{pmatrix} \gamma & -\beta \\ \beta & -\alpha \end{pmatrix}, \tag{4.59}$$

$$f_{11} = \begin{pmatrix} -2\frac{\beta}{\alpha} & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.60}$$

$$f_{30} = \begin{pmatrix} P_3 \\ Q_3 \end{pmatrix}, \tag{4.61}$$

and P_3, Q_3 are homogeneous cubic polynomials in (a, b). The reduced map f inherits the symmetry properties of (4.29), i.e.

$$f(-a, -b, \mu) = -f(a, b, \mu), \tag{4.62}$$

$$(f(\cdot, \mu) \circ R)^2 = Id, \text{ or equivalently } f^{-1}(., \mu) = R \circ f(., \mu) \circ R. \tag{4.63}$$

4.4.4 Normal form computation

In this section, we write the reduced mapping (4.55) in *normal form*, i.e. we perform a change of variables which only keeps its essential terms. This greatly simplifies the recurrence relation, which takes the form

$$A_{n+1} \pm 2A_n + A_{n-1} = c\mu A_n + dA_n^3 + \text{h.o.t.},$$

the sign $\pm$ depending whether the bifurcation occurs at a double eigenvalue -1 or $+1$. Moreover, one can choose the change of variables such that the reversibility symmetry is transformed into an involution (nonlinear symmetry) which remains linear up to higher order terms. These results are detailed in the following lemma.

Lemma 11 *There exists a C^{k-2} local diffeomorphism h_μ defined on a neighbourhood of $(a,b)=0$ which transforms (4.55) into the following mapping:*

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \tag{4.64}$$

where $(A_n, B_n) = h_\mu(a_n, b_n)$ and

$$G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} + g_\mu(A_n, B_n) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.65}$$

with a $+$ sign in (4.65) for $(\omega^2, m) \in \Gamma_1^+$ and a $-$ sign for $(\omega^2, m) \in \Gamma_1^-$ or Γ_1^p $(p \geq 1)$,

$$g_\mu(A_n, B_n) = -\frac{\beta}{\alpha}(2\mu A_n - \frac{B\omega_c^2}{8}A_n^3) + o(\|(A_n,B_n)\|^3 + \mu\|(A_n,B_n)\|), \tag{4.66}$$

with

$$B = \frac{1}{2}V^{(4)}(0) - (V^{(3)}(0))^2. \tag{4.67}$$

The coefficient β in (4.66) depends on the value of (ω_c^2, m) considered in (4.28) and one has:

$$\begin{array}{lll} \textit{for} & (\omega_c^2, m) \in \Gamma_1^+ : & \beta = -1-\alpha, \\ \textit{for} & (\omega_c^2, m) \in \Gamma_1^- : & \beta = 1-\alpha, \\ \textit{for} & (\omega_c^2, m) \in \Gamma_1^p : & \beta = \alpha - 1. \end{array} \tag{4.68}$$

The maps h_μ and G_μ commute with $-Id$. Moreover, the map G_μ is reversible with respect to the involution:

$$\mathcal{R}_\mu = h_\mu \circ R \circ h_\mu^{-1}, \tag{4.69}$$

which depends on the considered value of (ω_c^2, m).
For $(\omega_c^2, m) \in \Gamma_1^+$, *we note* $\mathcal{R}_\mu = \mathcal{R}_\mu^+$ *and one has:*

$$\mathcal{R}_\mu^+ \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + o(\|(A,B)\|^3 + \mu\|(A,B)\|). \tag{4.70}$$

For $(\omega_c^2, m) \in \Gamma_1^-$, *we note* $\mathcal{R}_\mu = \mathcal{R}_\mu^-$ *with:*

$$\mathcal{R}_\mu^- \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + o(\|(A,B)\|^3 + \mu\|(A,B)\|). \tag{4.71}$$

For $(\omega_c^2, m) \in \Gamma_1^p$, *we note* $\mathcal{R}_\mu = \mathcal{R}_\mu^a$ *with:*

$$\mathcal{R}_\mu^a \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + o(\|(A,B)\|^3 + \mu\|(A,B)\|). \tag{4.72}$$

The principal part of h_μ *is given by:*

$$\begin{aligned}
&\textit{for} \quad (\omega_c^2, m) \in \Gamma_1^+ \;:\; h_\mu = \begin{pmatrix} -(\alpha+2) & -\alpha \\ -2(1+\alpha) & -2(1+\alpha) \end{pmatrix} + o(\|(a,b)\|), \\
&\textit{for} \quad (\omega_c^2, m) \in \Gamma_1^- \;:\; h_\mu = \begin{pmatrix} \alpha-2 & \alpha \\ -2(1-\alpha) & -2(1-\alpha) \end{pmatrix} + o(\|(a,b)\|), \\
&\textit{for} \quad (\omega_c^2, m) \in \Gamma_1^p \;:\; h_\mu = \begin{pmatrix} \alpha & 2-\alpha \\ 2(\alpha-1) & 2(1-\alpha) \end{pmatrix} + o(\|(a,b)\|).
\end{aligned} \tag{4.73}$$

Proof. In the sequel, the $\pm$ sign has to be understood as a $+$ sign for $(\omega_c^2, m) \in \Gamma_1^+$ and a $-$ sign for $(\omega_c^2, m) \in \Gamma_1^-$ or Γ_1^p.
We look for a change of variables $(A_n, B_n) = h_\mu(a_n, b_n)$ having the form:

$$\begin{aligned}
&A_n = P(a_n, b_n, \mu), \\
&B_n = A_n - (\pm A_{n-1}) = P(a_n, b_n, \mu) - (\pm P \circ f^{-1}(a_n, b_n, \mu)),
\end{aligned} \tag{4.74}$$

which transforms (4.55) into (4.64). We choose P such that the term g_μ in (4.65) has the form:

$$g_\mu(A,B) = c\mu A + dA^3 + o(\mu\|(A,B)\| + \|(A,B)\|^3).$$

The mapping (4.64) can be written as a second order scalar recurrence of the form:

$$\begin{aligned}
A_{n+1} - 2(\pm A_n) + A_{n-1} = {} & c\mu A_n + dA_n^3 \\
& + o(\|(A_n, A_{n-1})\|^3 + \mu\|(A_n, A_{n-1})\|).
\end{aligned} \tag{4.75}$$

We choose P as a cubic polynomial in (a, b, μ) which preserves the symmetry $-Id$ of (4.55), hence $P(-a, -b, \mu) = -P(a, b, \mu)$. Note that the mapping

(4.64) is reversible with respect to the involution $\mathcal{R}_\mu = h_\mu \circ R \circ h_\mu^{-1}$. We write $P = M_1 + \mu M_{11} + Q$ with M_1, M_{11} linear and

$$Q(a,b) = \delta_1 a^3 + \delta_2 a^2 b + \delta_3 a b^2 + \delta_4 b^3.$$

Equation (4.75) yields:

$$\begin{aligned} &P \circ f(a_n, b_n, \mu) - 2(\pm P(a_n, b_n, \mu)) + P \circ f^{-1}(a_n, b_n, \mu) \\ &= c\mu P(a_n, b_n, \mu) + dP(a_n, b_n, \mu)^3 + \text{h.o.t}\,. \end{aligned} \tag{4.76}$$

Using property (4.63), equation (4.76) can be written:

$$\begin{aligned} &P \circ f(a_n, b_n, \mu) - 2(\pm P(a_n, b_n, \mu)) + P \circ R \circ f \circ R(a_n, b_n, \mu) \\ &= c\mu P(a_n, b_n, \mu) + dP(a_n, b_n, \mu)^3 + \text{h.o.t}\,. \end{aligned} \tag{4.77}$$

Moreover, we choose a change of variable which transforms the reversibility symmetry R into an involution $\mathcal{R}_\mu$ which has the simplest possible form. In the case when $(\omega_c^2, m) \in \Gamma_1^+$, equation (4.70) is equivalent to

$$P \circ R \circ f(a,b,\mu) = -P(a,b,\mu) + o(\|(a,b)\|^3 + \mu\|(a,b)\|) \tag{4.78}$$

(use equations (4.69), (4.74), (4.63)). For $(\omega_c^2, m) \in \Gamma_1^-$, equation (4.71) yields similarly

$$P \circ R \circ f(a,b,\mu) = P(a,b,\mu) + o(\|(a,b)\|^3 + \mu\|(a,b)\|). \tag{4.79}$$

For $(\omega_c^2, m) \in \Gamma_1^a$, equation (4.72) leads to

$$P \circ R \circ f(a,b,\mu) = -P(a,b,\mu) + o(\|(a,b)\|^3 + \mu\|(a,b)\|). \tag{4.80}$$

We now start by computing P and the normal form (4.75) for $(\omega_c^2, m) \in \Gamma_1^+$. Let us determine M_1, M_{11}, Q, c, d by identification in the Taylor expansions of equations (4.77) and (4.78).
The identification of the linear terms in (4.77) and (4.78) leads to:

$$M_1 \circ L_c - 2M_1 + M_1 \circ L_c^{-1} = 0, \tag{4.81}$$

$$M_1 \circ RL_c = -M_1. \tag{4.82}$$

We recall that

$$L_c = \begin{pmatrix} \gamma & -\beta \\ \beta & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha+2 & 1+\alpha \\ -1-\alpha & -\alpha \end{pmatrix}. \tag{4.83}$$

Since $L_c + L_c^{-1} = 2Id$, it follows that (4.81) is satisfied for all M_1. Let us note $M_1(x,y) = a_1 x + b_1 y$. Equation (4.82) gives the condition:

$$a_1 \alpha - b_1 \gamma = 0.$$

Consequently, M_1 is defined up to a multiplicative constant and we take $M_1(x,y) = -(\gamma x + \alpha y)$. Then the principal part of h_μ is given by:

$$h_\mu = \begin{pmatrix} -(\alpha+2) & -\alpha \\ -2(1+\alpha) & -2(1+\alpha) \end{pmatrix} + o(\|(a,b)\|) = M + o(\|(a,b)\|).$$

Note that M is invertible and thus h_μ defines a local diffeomorphism.
The identification of $O(\mu)$ linear terms in (4.77) and (4.78) leads to:

$$M_{11} \circ (L_c - 2Id + L_c^{-1}) + M_1 \circ (f_{11} + Rf_{11}R - cId) = 0, \tag{4.84}$$

$$M_{11} \circ (RL_c + Id) = -M_1 R f_{11}, \tag{4.85}$$

where f_{11} is defined in (4.60). We note that $f_{11} + Rf_{11}R = -2\frac{\beta}{\alpha}Id$. Consequently, equation (4.84) yields $c = -2\frac{\beta}{\alpha}$. Let us note $M_{11}(x,y) = a_{11}x + b_{11}y$. Then equation (4.85) is equivalent to:

$$-a_{11}\alpha + b_{11}\gamma = -\alpha,$$

hence one can fix $M_{11}(x,y) = x$ (others choices are possible).

The identification of the cubic terms in equations (4.77) and (4.78) leads to:

$$Q \circ L_c - 2Q + Q \circ L_c^{-1} = dM_1^3 - M_1 \circ (f_{30} + R \circ f_{30} \circ R), \tag{4.86}$$

$$Q \circ RL_c + Q = -M_1 \circ Rf_{30}, \tag{4.87}$$

where f_{30} is defined in (4.61). The identification of the powers $a^i b^j$ in (4.86) and (4.87) leads to a couple of linear systems:

$$A_1 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_1, A_2 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_2 \tag{4.88}$$

with

$$A_1 = \begin{pmatrix} \gamma^3 - \alpha^3 - 2 & \gamma^2\beta - \beta\alpha^2 & \gamma\beta^2 - \alpha\beta^2 & 0 \\ 3\beta\alpha^2 - 3\gamma^2\beta & \delta & 0 & 3\beta^2\gamma - 3\beta^2\alpha \\ 3\beta^2\gamma - 3\beta^2\alpha & 0 & \delta & 3\beta\alpha^2 - 3\gamma^2\beta \\ 0 & \gamma\beta^2 - \alpha\beta^2 & \gamma^2\beta - \beta\alpha^2 & \gamma^3 - \alpha^3 - 2 \end{pmatrix},$$

$\delta = \alpha^2\gamma + 2\alpha\beta^2 - \alpha\gamma^2 - 2\gamma\beta^2 - 2,$

$$V_1 = \begin{pmatrix} -d\gamma^3 - \omega_c^2(\gamma\beta_4 + \alpha(\beta_7 + \beta_1)) \\ -3d\alpha\gamma^2 - \omega_c^2(\gamma(\beta_5 + \beta_2) + \alpha(\beta_3 + \beta_6)) \\ -3d\alpha^2\gamma - \omega_c^2(\alpha(\beta_5 + \beta_2) + \gamma(\beta_3 + \beta_6)) \\ -d\alpha^3 - \omega_c^2(\alpha\beta_4 + \gamma(\beta_7 + \beta_1)) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \beta^3+1 & \gamma\beta^2 & \gamma^2\beta & \gamma^3 \\ -3\beta^2\alpha & 1-\beta^3-2\alpha\beta\gamma & -\alpha\gamma^2-2\gamma\beta^2 & -3\beta\gamma^2 \\ 3\beta\alpha^2 & \gamma\alpha^2+2\alpha\beta^2 & 1+\beta^3+2\alpha\beta\gamma & 3\gamma\beta^2 \\ -\alpha^3 & -\alpha^2\beta & -\beta^2\alpha & 1-\beta^3 \end{pmatrix}$$

and

$$V_2 = -\omega_c^2 \begin{pmatrix} \alpha\beta_4+\gamma\beta_1 \\ \alpha\beta_5+\gamma\beta_3 \\ \alpha\beta_6+\gamma\beta_2 \\ \alpha\beta_7 \end{pmatrix},$$

where the coefficients β_i are defined in (4.56) and (4.57). The couple of systems seems to be overdetermined but we shall see in the sequel that this is not the case for a suitable choice of d.
We now solve (4.88) and start with the linear system:

$$A_1 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_1 \tag{4.89}$$

which corresponds to the identification of cubic terms in (4.86). Since $\gamma = \alpha+2$ and $\beta = -1-\alpha$, A_1 has a simpler form

$$A_1 = \beta^2 \begin{pmatrix} 6 & -4 & 2 & 0 \\ 12 & -6 & 0 & 6 \\ 6 & 0 & -6 & 12 \\ 0 & 2 & -4 & 6 \end{pmatrix}.$$

We have $rank(A_1) = 2$. This yields the conditions of compatibility:

$$\det \begin{pmatrix} 0 & 2 & -d\alpha^3-\omega_c^2(\alpha\beta_4+\gamma(\beta_7+\beta_1)) \\ 6 & 0 & -3d\alpha^2\gamma-\omega_c^2(\alpha(\beta_5+\beta_2)+\gamma(\beta_3+\beta_6)) \\ 12 & -6 & -3d\alpha\gamma^2-\omega_c^2(\gamma(\beta_5+\beta_2)+\alpha(\beta_3+\beta_6)) \end{pmatrix} = 0 \tag{4.90}$$

and

$$\det \begin{pmatrix} 0 & 2 & -d\alpha^3-\omega_c^2(\alpha\beta_4+\gamma(\beta_7+\beta_1)) \\ 6 & 0 & -3d\alpha^2\gamma-\omega_c^2(\alpha(\beta_5+\beta_2)+\gamma(\beta_3+\beta_6)) \\ 6 & -4 & -d\gamma^3-\omega_c^2(\gamma\beta_4+\alpha(\beta_7+\beta_1)) \end{pmatrix} = 0. \tag{4.91}$$

Thanks to relations (4.57) (which are due to the reversibility of f) , conditions (4.90) and (4.91) are both satisfied for:

$$3d = -\frac{\omega_c^2(3\gamma(\beta_1+\beta_7)+3\alpha\beta_4+(\alpha-2\gamma)(\beta_3+\beta_6)+(\gamma-2\alpha)(\beta_2+\beta_5))}{(\alpha^3+\alpha\gamma^2-2\alpha^2\gamma)}. \tag{4.92}$$

Injecting equations (4.57) in (4.92) yields:

$$12\alpha d = -6\omega_c^2\beta(\beta_1+\beta_2-\beta_3).$$

Using (4.56) and (4.52) gives

$$d = \frac{\beta\omega_c^2}{8\alpha}B,$$

where B is given in (4.67). Consequently, we have computed the coefficients c, d in the normal form (4.75).
There remains to check the existence of a solution $(\delta_1, \delta_2, \delta_3, \delta_4)$ of (4.88). The system (4.89) is equivalent to:

$$\beta^2 \begin{pmatrix} 6 & 0 & -6 & 12 \\ 0 & 2 & -4 & 6 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} V_{13} \\ V_{14} \end{pmatrix}$$

where V_{13} and V_{14} are respectively the third and the fourth component of V_1.
We now study the second linear system in (4.88) corresponding to the identification of cubic terms in (4.87). We have $rank(A_2) = 2$. The compatibility conditions read:

$$\det \begin{pmatrix} \beta^3+1 & \gamma\beta^2 & -\omega_c^2(\alpha\beta_4+\gamma\beta_1) \\ -\alpha^3 & -\alpha^2\beta & -\omega_c^2(\alpha\beta_7) \\ -3\beta^2\alpha & 1-\beta^3-2\alpha\beta\gamma & -\omega_c^2(\alpha\beta_5+\gamma\beta_3) \end{pmatrix} = 0 \tag{4.93}$$

and

$$\det \begin{pmatrix} \beta^3+1 & \gamma\beta^2 & -\omega_c^2(\alpha\beta_4+\gamma\beta_1) \\ -\alpha^3 & -\alpha^2\beta & -\omega_c^2(\alpha\beta_7) \\ 3\beta\alpha^2 & \gamma\alpha^2+2\alpha\beta^2 & -\omega_c^2(\alpha\beta_6+\gamma\beta_2) \end{pmatrix} = 0. \tag{4.94}$$

Thanks to relations (4.57), conditions (4.93) and (4.94) are both satisfied and the second system is equivalent to:

$$\begin{pmatrix} \beta^3+1 & \gamma\beta^2 & \gamma^2\beta & \gamma^3 \\ -\alpha^3 & -\alpha^2\beta & -\beta^2\alpha & 1-\beta^3 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} V_{21} \\ V_{24} \end{pmatrix}.$$

Here we denote by V_{21} and V_{24} respectively the first and the fourth component of V_2.
As a conclusion, (4.88) is equivalent to:

$$d = \frac{\beta\omega_c^2}{8\alpha}B$$

and

$$A_3(\delta_1, \delta_2, \delta_3, \delta_4)^t = (V_{13}, V_{14}, V_{21}, V_{24})^t, \tag{4.95}$$

where

$$A_3 = \begin{pmatrix} 6\beta^2 & 0 & -6\beta^2 & 12\beta^2 \\ 0 & 2\beta^2 & -4\beta^2 & 6\beta^2 \\ \beta^3+1 & \gamma\beta^2 & \gamma^2\beta & \gamma^3 \\ -\alpha^3 & -\alpha^2\beta & -\beta^2\alpha & 1-\beta^3 \end{pmatrix}.$$

We have $rank(A_3) = 3$. The compatibility condition reads

$$\det \begin{pmatrix} 6\beta^2 & 0 & -6\beta^2 & V_{13} \\ 0 & 2\beta^2 & -4\beta^2 & V_{14} \\ \beta^3+1 & \gamma\beta^2 & \gamma^2\beta & V_{21} \\ -\alpha^3 & -\alpha^2\beta & -\beta^2\alpha & V_{24} \end{pmatrix} = 0.$$

Thanks to equations (4.57), this condition is again satisfied and thus we can find a (non unique) solution to (4.88). As a conclusion, we have proved lemma 11 in the case when $(\omega_c^2, m) \in \Gamma_1^+$.

We now consider the case when $(\omega_c^2, m) \in \Gamma_1^-$. Since the computations are similar to previous ones, we keep the same notations and only give the outline of the proof. We recall that $\omega_c^2 = 2\alpha$, $\beta = 1-\alpha$ and $\gamma = \alpha - 2$. The identification of the linear terms in (4.77) and (4.79) leads to $M_1(x, y) = \gamma x + \alpha y$. Then the principal part of h_μ is given by

$$h_\mu = \begin{pmatrix} \alpha - 2 & \alpha \\ -2(1-\alpha) & -2(1-\alpha) \end{pmatrix} + o(\|(a, b)\|) = M + o(\|(a, b)\|).$$

We note that M is invertible and h_μ defines a local diffeomorphism. The identification of $O(\mu)$ linear terms in (4.77) and (4.79) leads to $c = -2\frac{\beta}{\alpha}$ and $M_{11}(x, y) = x$ (this choice is non unique). The identification of cubic terms in (4.77) and (4.79) leads to:

$$A_1 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_1, A_2 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_2 \tag{4.96}$$

with

$$A_1 = \begin{pmatrix} \gamma^3 - \alpha^3 + 2 & \gamma^2\beta - \beta\alpha^2 & \gamma\beta^2 - \alpha\beta^2 & 0 \\ 3\beta\alpha^2 - 3\gamma^2\beta & \delta & 0 & 3\beta^2\gamma - 3\beta^2\alpha \\ 3\beta^2\gamma - 3\beta^2\alpha & 0 & \delta & 3\beta\alpha^2 - 3\gamma^2\beta \\ 0 & \gamma\beta^2 - \alpha\beta^2 & \gamma^2\beta - \beta\alpha^2 & \gamma^3 - \alpha^3 + 2 \end{pmatrix},$$

$\delta = \alpha^2\gamma + 2\alpha\beta^2 - \alpha\gamma^2 - 2\gamma\beta^2 + 2,$

$$V_1 = \begin{pmatrix} d\gamma^3 + \omega_c^2(\gamma\beta_4 + \alpha(\beta_7 + \beta_1)) \\ 3d\alpha\gamma^2 + \omega_c^2(\gamma(\beta_5 + \beta_2) + \alpha(\beta_3 + \beta_6)) \\ 3d\alpha^2\gamma + \omega_c^2(\alpha(\beta_5 + \beta_2) + \gamma(\beta_3 + \beta_6)) \\ d\alpha^3 + \omega_c^2(\alpha\beta_4 + \gamma(\beta_7 + \beta_1)) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \beta^3 - 1 & \gamma\beta^2 & \gamma^2\beta & \gamma^3 \\ -3\beta^2\alpha & -1 - \beta^3 - 2\alpha\beta\gamma & -\alpha\gamma^2 - 2\gamma\beta^2 & -3\beta\gamma^2 \\ 3\beta\alpha^2 & \gamma\alpha^2 + 2\alpha\beta^2 & -1 + \beta^3 + 2\alpha\beta\gamma & 3\gamma\beta^2 \\ -\alpha^3 & -\alpha^2\beta & -\beta^2\alpha & -1 - \beta^3 \end{pmatrix}$$

and

$$V_2 = \omega_c^2 \begin{pmatrix} \alpha\beta_4 + \gamma\beta_1 \\ \alpha\beta_5 + \gamma\beta_3 \\ \alpha\beta_6 + \gamma\beta_2 \\ \alpha\beta_7 \end{pmatrix},$$

where the coefficients β_i are defined in (4.56) and (4.57).
Let us study the first system in (4.96). Since $\beta = 1 - \alpha$ and $\gamma = \alpha - 2$, A_1 has a simpler form:

$$A_1 = \beta^2 \begin{pmatrix} -6 & 4 & -2 & 0 \\ -12 & 6 & 0 & -6 \\ -6 & 0 & 6 & -12 \\ 0 & -2 & 4 & -6 \end{pmatrix}.$$

The compatibility conditions have the same form (4.90) and (4.91) and we find:

$$d = -\frac{\beta\omega_c^2}{2\alpha}(\beta_1 + \beta_2 - \beta_3). \tag{4.97}$$

Injecting (4.56) and (4.53) into (4.97), we obtain $d = \frac{\beta\omega_c^2}{2\alpha}B$. Then it is lenghty but straightforward to check the existence of a solution $(\delta_1, \delta_2, \delta_3, \delta_4)$ of (4.96) (these calculations are similar to previous ones). This completes the proof of lemma 11 for $(\omega_c^2, m) \in \Gamma_1^-$.
We now give the main steps of the proof for $(\omega_c^2, m) \in \Gamma_1^p$. We recall that $\omega_c^2 = 2$, $\beta = \alpha - 1$ and $\gamma = \alpha + 2$.
The identification of the linear terms in (4.77) and (4.80) leads to $M_1(x, y) = \alpha x - \gamma y$. Then the principal part of h_μ is given by

$$h_\mu = \begin{pmatrix} \alpha & 2 - \alpha \\ 2(\alpha - 1) & 2(1 - \alpha) \end{pmatrix} + o(\|(a, b)\|) = M + o(\|(a, b)\|).$$

Note that M is invertible and thus h_μ defines a local diffeomorphism. The identification of $O(\mu)$ linear terms in (4.77) and (4.80) leads to $c = -2\frac{\beta}{\alpha}$ and

$M_{11}(x,y) = x$ (this choice is non unique). The identification of cubic terms in (4.77) and (4.80) leads to:

$$A_1 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_1, A_2 \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = V_2 \tag{4.98}$$

with

$$A_1 = \begin{pmatrix} \gamma^3 - \alpha^3 + 2 & \gamma^2\beta - \beta\alpha^2 & \gamma\beta^2 - \alpha\beta^2 & 0 \\ 3\beta\alpha^2 - 3\gamma^2\beta & \delta & 0 & 3\beta^2\gamma - 3\beta^2\alpha \\ 3\beta^2\gamma - 3\beta^2\alpha & 0 & \delta & 3\beta\alpha^2 - 3\gamma^2\beta \\ 0 & \gamma\beta^2 - \alpha\beta^2 & \gamma^2\beta - \beta\alpha^2 & \gamma^3 - \alpha^3 + 2 \end{pmatrix},$$

$\delta = \alpha^2\gamma + 2\alpha\beta^2 - \alpha\gamma^2 - 2\gamma\beta^2 + 2,$

$$V_1 = \begin{pmatrix} d\alpha^3 + \omega_c^2(\alpha\beta_4 - \gamma(\beta_7 + \beta_1)) \\ -3d\alpha^2\gamma + \omega_c^2(\alpha(\beta_5 + \beta_2) - \gamma(\beta_3 + \beta_6)) \\ 3d\alpha\gamma^2 + \omega_c^2(-\gamma(\beta_5 + \beta_2) + \alpha(\beta_3 + \beta_6)) \\ -d\gamma^3 + \omega_c^2(-\gamma\beta_4 + \alpha(\beta_7 + \beta_1)) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \beta^3 + 1 & \gamma\beta^2 & \gamma^2\beta & \gamma^3 \\ -3\beta^2\alpha & 1 - \beta^3 - 2\alpha\beta\gamma & -\alpha\gamma^2 - 2\gamma\beta^2 & -3\beta\gamma^2 \\ 3\beta\alpha^2 & \gamma\alpha^2 + 2\alpha\beta^2 & 1 + \beta^3 + 2\alpha\beta\gamma & 3\gamma\beta^2 \\ -\alpha^3 & -\alpha^2\beta & -\beta^2\alpha & 1 - \beta^3 \end{pmatrix}$$

and

$$V_2 = \omega_c^2 \begin{pmatrix} -\gamma\beta_4 + \alpha\beta_1 \\ -\gamma\beta_5 + \alpha\beta_3 \\ -\gamma\beta_6 + \alpha\beta_2 \\ -\gamma\beta_7 \end{pmatrix},$$

where the coefficients β_i are defined in (4.56) and (4.57).
We now study the first system in (4.98). Since $\beta = \alpha - 1$ and $\gamma = \alpha - 2$, A_1 has a simpler form:

$$A_1 = \beta^2 \begin{pmatrix} -6 & -4 & -2 & 0 \\ 12 & 6 & 0 & -6 \\ -6 & 0 & 6 & 12 \\ 0 & -2 & -4 & -6 \end{pmatrix}.$$

The compatibility conditions yield:

$$d = -\frac{\beta\omega_c^2}{2\alpha}(\beta_1 + \beta_2 + \beta_3)$$

(note that this expression is slightly different from (4.97)).
Using (4.56) and (4.54), we obtain $d = \frac{\beta\omega_c^2}{2\alpha} B$ and one checks as above that there exists a solution $(\delta_1, \delta_2, \delta_3, \delta_4)$ of (4.98). This complete the proof of lemma 11. $\square$

4.5 Study of the reduced mapping

In this section we study small amplitude bifurcating solutions of the reduced mapping (4.55). We focus our attention on homoclinic and heteroclinic solutions, which will be related in section 4.6 to breather and "dark breather" solutions of the FPU system. As we shall see, these results heavily rely on the reversibility of the reduced system. Note that one could complete our analysis by studying the existence of quasi-periodic orbits on invariant tori ([31]) and periodic orbits.

We shall not examine the question of *transverse* intersections of stable and unstable manifolds in homoclinic orbits. The case of a transverse intersection is generic and yields a rich variety of solutions, because there exists an invariant Cantor set on which some iterate of the map is topologically conjugate to a full shift on N symbols ([6]). Proving transverse intersections is particularly difficult in our context because the splitting size is beyond all orders in μ (hence Melnikov theory cannot be applied directly) and the map is a priori not analytic (see [24]). The lack of analyticity enables us to use recent techniques for proving exponentially small splitting of separatrices (see [19], [9], [10] and references therein).

For studying the reduced mapping we shall consider the normal form (4.64). A mapping having the same form as (4.64) has been studied in [24] (see section 6.2.3). The difference with respect to [24] is that higher order terms are present in the reversibility symmetry (see (4.70)-(4.72)) but this detail does not change the results. Consequently we shall refer to [24] for the study of the normal form (existence of homoclinic and heteroclinic orbits are obtained using reversibility and approximation by a flow).

Let us start with the case when $(\omega_c^2, m) \in \Gamma_1^+$.

Lemma 12 *Assume* $(\omega_c^2, m) \in \Gamma_1^+$ *and* $B = \frac{1}{2}V^{(4)}(0) - \left(V^{(3)}(0)\right)^2 \neq 0$. *For* $\mu = \omega^2 - \omega_c^2 = \omega^2 - 2(1 + \frac{1}{m}) \approx 0$, *the recurrence relation (4.55) has the following solutions.*
i) For $\mu > 0$ *and* $B > 0$, *(4.55) has at least two homoclinic solutions* (a_n^{1+}, b_n^{1+}), (a_n^{2+}, b_n^{2+}) *such that* $\lim_{n\to\pm\infty}(a_n^{i+}, b_n^{i+}) = 0$. *These solutions have the symmetries*

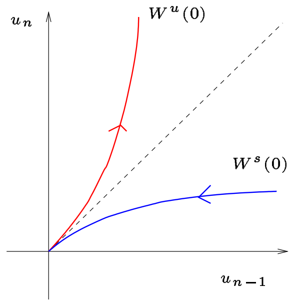
$$-R(a_{-n}^{1+}, b_{-n}^{1+}) = (a_n^{1+}, b_n^{1+}), \quad -R(a_{-n+1}^{2+}, b_{-n+1}^{2+}) = (a_n^{2+}, b_n^{2+})$$

and satisfy for some $C > 0 : 0 < b_n^{i+} \leq C\,\mu^{1/2}\,|z_1|^{-|n|}$, $-C\,\mu^{1/2}\,|z_1|^{-|n|} \leq a_n^{i+} < 0$ *and* $|a_n^{i+} + b_n^{i+}| \leq C\,\mu\,|z_1|^{-|n|}$ *(*$i = 1, 2$*), where* $|z_1| = 1 + O(\mu^{1/2}) > 1$.
ii) If μ *and* B *have the same sign, (4.55) has two symmetric fixed points* $\pm(a^*, -a^*)$ *with* $a^* = O(|\mu|^{1/2})$.
iii) For $\mu < 0$ *and* $B < 0$*, (4.55) has at least two heteroclinic solutions* (a_n^{3+}, b_n^{3+}), (a_n^{4+}, b_n^{4+}) *(with the other solutions* $-(a_n^{3+}, b_n^{3+})$, $-(a_n^{4+}, b_n^{4+})$*) such that* $\lim_{n\to\pm\infty}(a_n^{i+}, b_n^{i+}) = \pm(a^*, -a^*)$*. These solutions have the symmetries*

$$R\,(a_{-n}^{3+}, b_{-n}^{3+}) = (a_n^{3+}, b_n^{3+}), \qquad R\,(a_{-n+1}^{4+}, b_{-n+1}^{4+}) = (a_n^{4+}, b_n^{4+}).$$

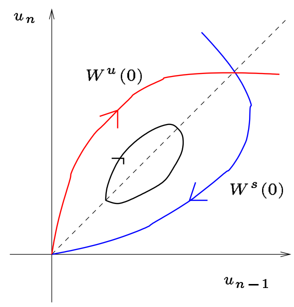
Moreover, (a_n^{3+}, b_n^{3+}), (a_n^{4+}, b_n^{4+}) *are* $O(|\mu|^{1/2})$ *as* $n \to \pm\infty$ *and* $O(|\mu|)$ *for bounded values of* n.

Let us summarize the different situations that occur for the case $\mu > 0$, $B > 0$ the case $\mu B < 0$ and the case $\mu < 0$ and $B < 0$.



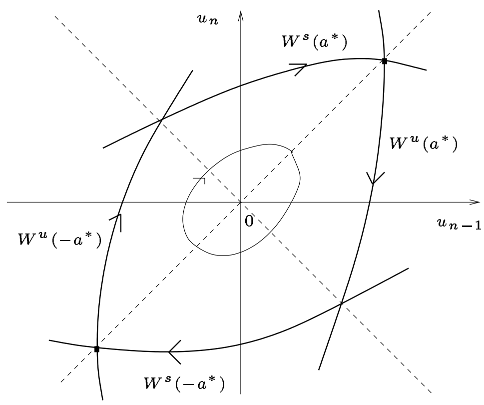
The case $B\mu < 0$

No homoclinic solutions



The case $\mu > 0$ and $B > 0$

Existence of homoclinic solutions



The case $B < 0$ and $\mu < 0$: Existence of heteroclinic solutions

Proof. Using lemma 11, we write (4.55) in the normal form

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \tag{4.99}$$

where

$$G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} + g_\mu(A_n, B_n) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.100}$$

$$g_\mu(A_n, B_n) = 2(m+1)\,\mu A_n - \frac{m}{4}(1+\alpha)^2\, B\, A_n^3 + o(\|(A_n, B_n)\|^3 + \mu\|(A_n, B_n)\|). \tag{4.101}$$

The map G_μ commutes with $-I$. Moreover, (4.99) is reversible with respect to involutions $\mathcal{R}_\mu^+$ and $-\mathcal{R}_\mu^+$ given by

$$\mathcal{R}_\mu^+ \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + o(\|(A, B)\|^3 + \mu\|(A, B)\|) \tag{4.102}$$

and $\mathcal{R}_\mu^+$ commutes with $-I$. Since the involutions $\pm\mathcal{R}_\mu^+$ are reversors of (4.99), it is a classical result that (4.99) is also reversible with respect to $\pm\mathcal{R}_\mu^+\, G_\mu^p$ ($p \in \mathbb{Z}$). For example, $\pm\mathcal{S}_\mu^+ = \pm\mathcal{R}_\mu^+\, G_\mu$ are also reversors. An orbit (A_n, B_n) is said to be reversible with respect to $\mathcal{R}_\mu^+$ if it has the symmetry $(A_{-n}, B_{-n}) = \mathcal{R}_\mu^+(A_n, B_n)$. One can check that any reversible orbit with respect to $\pm\mathcal{R}_\mu^+\, G_\mu^p$ is a shift of a reversible orbit under $\pm\mathcal{R}_\mu^+$ or $\pm\mathcal{S}_\mu^+$. Consequently, we only consider the reversors $\pm\mathcal{R}_\mu^+$ and $\pm\mathcal{S}_\mu^+$ in the sequel.

We now discuss the existence of small amplitude homoclinic and heteroclinic solutions of (4.99) for $\mu \approx 0$. The proof of their existence for the full system can be found e.g. in [24], section 6.2.3. These results heavily rely on the reversibility of (4.99). One also uses the fact that the map G_μ is almost conjuged (up to higher order terms) to the time one map of the integrable vector field

$$v'' = 2(m+1)\mu v - \frac{m}{4}(1+\alpha)^2 B v^3.$$

The difference with respect to [24] is that higher order terms are present in (4.102), but this detail does not change the results.

The fixed point $(A, B) = 0$ of (4.99) is hyperbolic for $\mu > 0$ and elliptic for $\mu < 0$. In the case when μ and B have the same sign, (4.99) has two other symmetric fixed points $(\pm A^*, 0)$ satisfying $\mathcal{R}_\mu^+(A^*, 0) = (-A^*, 0)$. These fixed points are elliptic for $\mu > 0$, $B > 0$, hyperbolic for $\mu < 0$, $B < 0$ and one has $A^* = (\frac{8\,\mu}{(1+\alpha)\,B})^{1/2} + O(|\mu|)$.

In the case when $\mu > 0$ and $B > 0$, (4.99) has reversible solutions (A_n^{1+}, B_n^{1+}), (A_n^{2+}, B_n^{2+}) homoclinic to $(A, B) = 0$ and satisfying

$$-\mathcal{R}_\mu^+(A_{-n}^{1+}, B_{-n}^{1+}) = (A_n^{1+}, B_n^{1+}), \quad -\mathcal{S}_\mu^+(A_{-n}^{2+}, B_{-n}^{2+}) = (A_n^{2+}, B_n^{2+})$$

(the same holds for the symmetric solutions $-(A_n^{1+}, B_n^{1+})$, $-(A_n^{2+}, B_n^{2+})$). Moreover, one has $0 < A_n^{i+} \leq C\,\mu^{1/2}\,|z_1|^{-|n|}$ and $|B_n^{i+}| \leq C\,\mu\,|z_1|^{-|n|}$ ($i = 1,2$), where $|z_1| = 1 + O(\mu^{1/2}) > 1$.

In the case when $B < 0$ and $\mu > 0$ ($\mu \approx 0$), the local stable and unstable manifolds of $(A,B) = 0$ do not intersect in a small neighbourhood of 0. Consequently, there exist no small amplitude homoclinic orbits to $(A,B) = 0$ in this parameter range.

In the case when $\mu < 0$ and $B < 0$, (4.99) has reversible heteroclinic solutions (A_n^{3+}, B_n^{3+}), (A_n^{4+}, B_n^{4+}) (with also $-(A_n^{3+}, B_n^{3+})$, $-(A_n^{4+}, B_n^{4+})$) connecting the hyperbolic fixed points $(\pm A^*, 0)$. They satisfy

$$\lim_{n\to\pm\infty} (A_n^{i+}, B_n^{i+}) = (\pm A^*, 0),$$

with

$$\mathcal{R}_\mu^+(A_{-n}^{3+}, B_{-n}^{3+}) = (A_n^{3+}, B_n^{3+}), \quad \mathcal{S}_\mu^+(A_{-n}^{4+}, B_{-n}^{4+}) = (A_n^{4+}, B_n^{4+}).$$

Moreover, B_n^{3+}, B_n^{4+} are $O(|\mu|)$, and A_n^{3+}, A_n^{4+} are $O(|\mu|^{1/2})$ as $n \to \pm\infty$, $O(|\mu|)$ for bounded values of n.

The above analysis of the normal form (4.64) allows us to describe small amplitude homoclinic and heteroclinic solutions of the original mapping (4.55), which completes the proof.

□

We now consider the case when $(\omega_c^2, m) \in \Gamma_1^-$.

Lemma 13 *Assume $(\omega_c^2, m) \in \Gamma_1^-$ and $B = \frac{1}{2}V^{(4)}(0) - \left(V^{(3)}(0)\right)^2 \neq 0$. For $\mu = \omega^2 - \omega_c^2 = \omega^2 - \frac{2}{m} \approx 0$, the recurrence relation (4.55) has the following solutions.*
i) For $\mu < 0$ and $B < 0$, (4.55) has at least two homoclinic solutions (a_n^{1-}, b_n^{1-}), (a_n^{2-}, b_n^{2-}) such that $\lim_{n\to\pm\infty} (a_n^{i-}, b_n^{i-}) = 0$. These solutions have the symmetries

$$-R(a_{-n}^{1-}, b_{-n}^{1-}) = (a_n^{1-}, b_n^{1-}), \quad R(a_{-n+1}^{2-}, b_{-n+1}^{2-}) = (a_n^{2-}, b_n^{2-})$$

and satisfy for some $C > 0$: $0 < b_n^{i-} \leq C\,|\mu|^{1/2}\,|z_1|^{-|n|}$, $-C\,|\mu|^{1/2}\,|z_1|^{-|n|} \leq a_n^{i-} < 0$ and $|a_n^{i-} + b_n^{i-}| \leq C\,|\mu|\,|z_1|^{-|n|}$ ($i = 1,2$), where $|z_1| = 1 + O(|\mu|^{1/2}) > 1$.
ii) If μ and B have the same sign, (4.55) has a period-2 orbit $(a_n^0, b_n^0) = (-1)^n\,(a^*, -a^*)$ with $a^* = O(|\mu|^{1/2})$.
iii) For $\mu > 0$ and $B > 0$, (4.55) has at least two heteroclinic solutions (a_n^{3-}, b_n^{3-}), (a_n^{4-}, b_n^{4-}) (with the other solutions $-(a_n^{3-}, b_n^{3-})$, $-(a_n^{4-}, b_n^{4-})$) such

that $\lim_{n\to\pm\infty} |\, (a_n^{i-}, b_n^{i-}) \mp (a_n^0, b_n^0) \,| = 0$. *These solutions have the symmetries*

$$R\,(a_{-n}^{3-}, b_{-n}^{3-}) = (a_n^{3-}, b_n^{3-}), \quad -R\,(a_{-n+1}^{4-}, b_{-n+1}^{4-}) = (a_n^{4-}, b_n^{4-}).$$

Moreover, (a_n^{3-}, b_n^{3-}), (a_n^{4-}, b_n^{4-}) *are* $O(\mu^{1/2})$ *as* $n \to \pm\infty$ *and* $O(\mu)$ *for bounded values of* n.

Proof. Using lemma 11, we write (4.55) in the normal form

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \tag{4.103}$$

where

$$G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} + g_\mu(A_n, B_n) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.104}$$

$$g_\mu(A_n, B_n) = 2(1-m)\,\mu A_n + \frac{1-\alpha}{4}\, B\, A_n^3 + o(\|(A_n, B_n)\|^3 + \mu\|(A_n, B_n)\|). \tag{4.105}$$

In order to recover the case of a double eigenvalue $+1$ considered above, one makes the change of variable $(\tilde{A}_n, \tilde{B}_n) = (-1)^n\,(A_n, B_n)$ (this yields an autonomous mapping since G_μ commutes with $-I$). We obtain

$$\begin{pmatrix} \tilde{A}_{n+1} \\ \tilde{B}_{n+1} \end{pmatrix} = \tilde{G}_\mu \begin{pmatrix} \tilde{A}_n \\ \tilde{B}_n \end{pmatrix}, \tag{4.106}$$

where $\tilde{G}_\mu = -G_\mu$ has the same structure as (4.100). The maps G_μ, $\tilde{G}_\mu$ commute with $-I$ and are reversible with respect to involutions $\pm\mathcal{R}_\mu^-$ having the form

$$\mathcal{R}_\mu^- \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + o(\|(A,B)\|^3 + \mu\|(A,B)\|). \tag{4.107}$$

We shall also consider the complementary reversors $\pm\mathcal{S}_\mu^- = \pm\mathcal{R}_\mu^-\, G_\mu$.

The description of small amplitude homoclinic and heteroclinic solutions of (4.106) has been done previously and yields the following results for the original mapping (4.103).

The fixed point $(A,B) = 0$ of (4.103) is hyperbolic for $\mu < 0$ and elliptic for $\mu > 0$. In the case when μ and B have the same sign, (4.103) has a period-2 orbit $(A_n^0, B_n^0) = (-1)^n(A^*, 0)$ with $\mathcal{R}_\mu^-(A^*, 0) = (-A^*, 0)$ and $A^* = (\frac{8m}{B}\,\mu)^{1/2} + O(|\mu|)$. This orbit is elliptic for $\mu < 0$, $B < 0$ and hyperbolic for $\mu > 0$, $B > 0$.

In the case when $\mu < 0$ and $B < 0$, (4.103) has reversible solutions (A_n^{1-}, B_n^{1-}), (A_n^{2-}, B_n^{2-}) homoclinic to $(A, B) = 0$ and satisfying

$$-\mathcal{R}_\mu^-(A_{-n}^{1-}, B_{-n}^{1-}) = (A_n^{1-}, B_n^{1-}), \quad \mathcal{S}_\mu^-(A_{-n}^{2-}, B_{-n}^{2-}) = (A_n^{2-}, B_n^{2-})$$

(the same holds for the symmetric solutions $-(A_n^{1-}, B_n^{1-})$, $-(A_n^{2-}, B_n^{2-})$). Moreover, one has $0 < (-1)^n A_n^{i-} \leq C\, \mu^{1/2} |z_1|^{-|n|}$ and $|B_n^{i-}| \leq C\, \mu\, |z_1|^{-|n|}$ $(i = 1, 2)$, where $|z_1| = 1 + O(\mu^{1/2}) > 1$.

In the case when $B > 0$ and $\mu < 0$ ($\mu \approx 0$), the stable and unstable manifolds of $(A, B) = 0$ do not intersect in a small neighbourhood of 0 and thus there exist no small amplitude homoclinic orbits to $(A, B) = 0$.

In the case when $\mu > 0$ and $B > 0$, (4.103) has reversible homoclinic solutions (A_n^{3-}, B_n^{3-}), (A_n^{4-}, B_n^{4-}) (with also $-(A_n^{3-}, B_n^{3-})$, $-(A_n^{4-}, B_n^{4-})$) connecting the period-2 orbit (A_n^0, B_n^0) with the shifted orbit (A_{n+1}^0, B_{n+1}^0). Note that these solutions can be considered as heteroclinic solutions connecting (A_n^0, B_n^0) with $-(A_n^0, B_n^0)$ (one has $(A_{n+1}^0, B_{n+1}^0) = -(A_n^0, B_n^0)$). They satisfy $\lim_{n\to\pm\infty} (-1)^n (A_n^{i-}, B_n^{i-}) = (\pm A^*, 0)$, with

$$\mathcal{R}_\mu^-(A_{-n}^{3-}, B_{-n}^{3-}) = (A_n^{3-}, B_n^{3-}), \quad -\mathcal{S}_\mu^-(A_{-n}^{4-}, B_{-n}^{4-}) = (A_n^{4-}, B_n^{4-}).$$

Moreover, B_n^{3-}, B_n^{4-} are $O(|\mu|)$, and A_n^{3-}, A_n^{4-} are $O(|\mu|^{1/2})$ as $n \to \pm\infty$, $O(|\mu|)$ for bounded values of n.

The above analysis of the normal form (4.64) provides small amplitude homoclinic and heteroclinic solutions of the original mapping (4.55), which completes the proof.

□

Lastly, we consider the case when $(\omega_c^2, m) \in \Gamma_1^p$ $(p \geq 1)$.

Lemma 14 *Assume* $(\omega_c^2, m) \in \Gamma_1^p$ $(p \geq 1)$ *and* $B = \frac{1}{2} V^{(4)}(0) - \left(V^{(3)}(0)\right)^2 \neq 0$. *For* $\mu = \omega^2 - \omega_c^2 = \omega^2 - 2 \approx 0$, *the recurrence relation (4.55) has the following solutions.*
i) For $\mu > 0$ *and* $B > 0$, *(4.55) has at least two homoclinic solutions* (a_n^{1a}, b_n^{1a}), (a_n^{2a}, b_n^{2a}) *such that* $\lim_{n\to\pm\infty} (a_n^{ia}, b_n^{ia}) = 0$. *These solutions have the symmetries*

$$R(a_{-n}^{1a}, b_{-n}^{1a}) = (a_n^{1a}, b_n^{1a}), \quad -R(a_{-n+1}^{2a}, b_{-n+1}^{2a}) = (a_n^{2a}, b_n^{2a})$$

and satisfy for some $C > 0 : 0 < a_n^{ia} \leq C\, \mu^{1/2} |z_1|^{-|n|}$, $0 < b_n^{ia} \leq C\, \mu^{1/2} |z_1|^{-|n|}$, *and* $|a_n^{ia} - b_n^{ia}| \leq C\, \mu\, |z_1|^{-|n|}$ $(i = 1, 2)$, *where* $|z_1| = 1 + O(\mu^{1/2}) > 1$.
ii) If μ *and* B *have the same sign, (4.55) has a period-2 orbit* $(a_n^0, b_n^0) = (-1)^n (a^*, a^*)$ *with* $a^* = O(|\mu|^{1/2})$.
iii) For $\mu < 0$ *and* $B < 0$, *(4.55) has at least two heteroclinic solutions* (a_n^{3a}, b_n^{3a}), (a_n^{4a}, b_n^{4a}) *(with the other solutions* $-(a_n^{3a}, b_n^{3a})$, $-(a_n^{4a}, b_n^{4a})$*) such*

that $\lim_{n\to\pm\infty} |\,(a_n^{ia}, b_n^{ia}) \mp (a_n^0, b_n^0)\,| = 0$. *These solutions have the symmetries*

$$-R\,(a_{-n}^{3a}, b_{-n}^{3a}) = (a_n^{3a}, b_n^{3a}), \quad R\,(a_{-n+1}^{4a}, b_{-n+1}^{4a}) = (a_n^{4a}, b_n^{4a}).$$

Moreover, (a_n^{3a}, b_n^{3a}), (a_n^{4a}, b_n^{4a}) *are* $O(|\mu|^{1/2})$ *as* $n \to \pm\infty$ *and* $O(|\mu|)$ *for bounded values of* n.

Proof. Using lemma 11, we write (4.55) in the normal form

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \tag{4.108}$$

where

$$G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} + g_\mu(A_n, B_n) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{4.109}$$

$$g_\mu(A_n, B_n) = 2(m-1)\,\mu A_n + \frac{1-m}{4}\,B\,A_n^3 + o(\|(A_n, B_n)\|^3 + \mu\|(A_n, B_n)\|). \tag{4.110}$$

The map G_μ commutes with $-I$ and is reversible with respect to involutions $\pm\mathcal{R}_\mu^a$ having the form

$$\mathcal{R}_\mu^a \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + o(\|(A, B)\|^3 + \mu\|(A, B)\|). \tag{4.111}$$

We shall also consider the complementary reversors $\pm\mathcal{S}_\mu^a = \pm\mathcal{R}_\mu^a\, G_\mu$.

The structure of (4.109) is the same as in (4.104) but coefficients of g_μ have opposite signs. Moreover, reversors (4.71), (4.72) have opposite principal parts. These variations induce several differences with the bifurcation results obtained for (4.104).

The fixed point $(A, B) = 0$ of (4.103) is hyperbolic for $\mu > 0$ and elliptic for $\mu < 0$. In the case when μ and B have the same sign, (4.108) has a period-2 orbit $(A_n^0, B_n^0) = (-1)^n (A^*, 0)$ with $\mathcal{R}_\mu^a(A^*, 0) = (A^*, 0)$ and $A^* = (\frac{8\mu}{B})^{1/2} + O(|\mu|)$. This orbit is elliptic for $\mu > 0$, $B > 0$ and hyperbolic for $\mu < 0$, $B < 0$.

In the case when $\mu > 0$ and $B > 0$, (4.103) has reversible solutions (A_n^{1a}, B_n^{1a}), (A_n^{2a}, B_n^{2a}) homoclinic to $(A, B) = 0$ and satisfying

$$\mathcal{R}_\mu^a(A_{-n}^{1a}, B_{-n}^{1a}) = (A_n^{1a}, B_n^{1a}), \quad -\mathcal{S}_\mu^a(A_{-n}^{2a}, B_{-n}^{2a}) = (A_n^{2a}, B_n^{2a})$$

(the same holds for the symmetric solutions $-(A_n^{1a}, B_n^{1a})$, $-(A_n^{2a}, B_n^{2a})$). Moreover, one has $0 < (-1)^n A_n^{ia} \le C\,\mu^{1/2}\,|z_1|^{-|n|}$ and $|B_n^{ia}| \le C\,\mu\,|z_1|^{-|n|}$ ($i = 1, 2$), where $|z_1| = 1 + O(\mu^{1/2}) > 1$.

In the case when $B < 0$ and $\mu > 0$ ($\mu \approx 0$), the stable and unstable manifolds of $(A, B) = 0$ do not intersect in a small neighbourhood of 0 and thus there exist no small amplitude homoclinic orbits to $(A, B) = 0$.

In the case when $\mu < 0$ and $B < 0$, (4.108) has reversible homoclinic solutions (A_n^{3a}, B_n^{3a}), (A_n^{4a}, B_n^{4a}) (with also $-(A_n^{3a}, B_n^{3a})$, $-(A_n^{4a}, B_n^{4a})$) connecting the period-2 orbit (A_n^0, B_n^0) with (A_{n+1}^0, B_{n+1}^0). These solutions can again be considered as heteroclinic solutions connecting (A_n^0, B_n^0) with $-(A_n^0, B_n^0)$, since $(A_{n+1}^0, B_{n+1}^0) = -(A_n^0, B_n^0)$. They satisfy $\lim_{n\to\pm\infty} (-1)^n (A_n^{ia}, B_n^{ia}) = (\pm A^*, 0)$, with

$$-\mathcal{R}_\mu^a(A_{-n}^{3a}, B_{-n}^{3a}) = (A_n^{3a}, B_n^{3a}), \quad \mathcal{S}_\mu^a(A_{-n}^{4a}, B_{-n}^{4a}) = (A_n^{4a}, B_n^{4a}).$$

Moreover, B_n^{3a}, B_n^{4a} are $O(|\mu|)$, and A_n^{3a}, A_n^{4a} are $O(|\mu|^{1/2})$ as $n \to \pm\infty$, $O(|\mu|)$ for bounded values of n.

This analysis of the normal form (4.64) provides small amplitude homoclinic and heteroclinic solutions of the original mapping (4.55), which completes the proof.

□

If $(\omega_c^2, m) \in \Gamma_1^+$ or $(\omega_c^2, m) \in \Gamma_1^p$, $B < 0$ and $\mu > 0$ ($\mu \approx 0$), note that the local stable and unstable manifolds of $(a, b) = 0$ do not intersect and thus (4.55) has no small amplitude homoclinic solution. In the same way, (4.55) has no small amplitude homoclinic solution to $(a, b) = 0$ for $(\omega_c^2, m) \in \Gamma_1^-$, $B > 0$ and $\mu < 0$ ($\mu \approx 0$).

4.6 Breathers and "dark" breathers

In this section, we deduce the existence of breathers and "dark" breathers from the reduced mapping properties and study their spatial geometry. Their symmetries are of interest since they seem strongly related to stability (see [26] and references therein). We have divided our analysis into two parts.

Section 4.6.1 describes the solutions of (4.9) corresponding to the homoclinic and heteroclinic orbits found in section 4.5 (breathers and "dark" breathers). It gives also a rather simple explanation of the fact that the coefficient B determining their existence is independent of m.

Section 4.6.2 describes the shape of DB using the original displacement variable x_n of system (4.7). These results are compared to different numerical studies ([8], [26], [14]).

4.6.1 Existence results

According to theorem 5, each solution (a_n, b_n) in lemma 12, 13 and 14 corresponds to a solution $Y_n = (u_n, v_n)$ of (4.9) for $\omega^2 = \mu + \omega_c^2$. This solution is given by

$$Y_n = (a_n, b_n)\, \cos t + \Psi(a_n, b_n, \mu), \tag{4.112}$$

where $\Psi \in C^k(\mathbb{R}^3; \mathbb{D}_h)$ has the symmetries

$$\begin{aligned} T\Psi(a, b, \mu) &= \Psi(-a, -b, \mu), \\ R\Psi(a, b, \mu) &= \Psi(b, a, \mu). \end{aligned} \tag{4.113}$$

We describe some of these solutions in theorems 8-10 below (these results follow directly from lemma 12-14 and equations (4.112),(4.113)). Solutions $Y_n^{1\pm}$, $Y_n^{2\pm}$, Y_n^{1a}, Y_n^{2a} below correspond to discrete breathers (they are time-periodic and spatially localized). They have a large extent $(\ln|z_1|)^{-1}$ ($|z_1| \approx 1$) but decay exponentially at infinity. Solutions $Y_n^{3\pm}$, $Y_n^{4\pm}$, Y_n^{3a}, Y_n^{4a} are sometimes referred as dark breathers, by analogy with dark solitons (see e.g. [1]). Indeed, oscillations have an amplitude $O(|\omega - \omega_c|)^{1/2}$ as $n \to \pm\infty$ and a smaller amplitude $O(|\omega - \omega_c|)$ in the centre.

Let us start with the case when $(\omega_c^2, m) \in \Gamma_1^+$.

Theorem 8 *Suppose* $B = \frac{1}{2} V^{(4)}(0) - \left(V^{(3)}(0)\right)^2 \neq 0$ *and* $m \in (0, 1)$. *For* $\omega \approx \omega_c = (2 + \frac{2}{m})^{1/2}$, *problem (4.9) has the following solutions with* $Y_n \in \mathbb{D}$ *for all* $n \in \mathbb{Z}$.
i) For $\omega > \omega_c$ *and* $B > 0$, *(4.9) has at least two homoclinic solutions* Y_n^{1+}, Y_n^{2+} *(and also* $T\,Y_n^{1+}$, $T\,Y_n^{2+}$*) such that* $\lim_{n\to\pm\infty} \|Y_n^{i+}\|_{\mathbb{D}} = 0$. *These solutions satisfy*

$$T\,R\,Y_{-n}^{1+} = Y_n^{1+}, \quad T\,R\,Y_{-n+1}^{2+} = Y_n^{2+},$$

or equivalently

$$T\,v_{-n}^{1+} = u_n^{1+}, \quad T\,v_{-n+1}^{2+} = u_n^{2+}.$$

They have the form

$$Y_n^{i+} = (-b_n^{i+}, b_n^{i+})\, \cos t + O(|\omega - \omega_c|),$$

where $0 < b_n^{i+} \leq C\,(\omega - \omega_c)^{1/2}\, |z_1|^{-|n|}$, $|z_1| = 1 + O((\omega - \omega_c)^{1/2}) > 1$.
ii) If $\omega - \omega_c$ *and* B *have the same sign, (4.9) has two symmetric fixed points* Y^{0+}, $TY^{0+} \in \mathbb{D}$. *They have the form* $Y^{0+} = (y^0, T\,y^0)$ *with* $y^0(t) = a^* \cos t + O(|\omega - \omega_c|)$ *(*$y^0 \in H^4_\#$*) and* $a^* = O(|\omega - \omega_c|^{1/2})$.
iii) For $\omega < \omega_c$ *and* $B < 0$, *(4.9) has at least two heteroclinic solutions* Y_n^{3+}, Y_n^{4+} *(and also* $T\,Y_n^{3+}$, $T\,Y_n^{4+}$*) such that* $\lim_{n\to-\infty} \|Y_n^{i+} - TY^{0+}\|_{\mathbb{D}} = 0$, $\lim_{n\to+\infty} \|Y_n^{i+} - Y^{0+}\|_{\mathbb{D}} = 0$. *These solutions satisfy*

$$R\,Y_{-n}^{3+} = Y_n^{3+}, \quad R\,Y_{-n+1}^{4+} = Y_n^{4+},$$

or equivalently

$$v^{3+}_{-n} = u^{3+}_n, \quad v^{4+}_{-n+1} = u^{4+}_n.$$

Moreover, $\|Y^{3+}_n\|_{\mathbb{D}}$, $\|Y^{4+}_n\|_{\mathbb{D}}$ *are* $O(|\omega - \omega_c|^{1/2})$ *as* $n \to \pm\infty$ *and* $O(|\omega - \omega_c|)$ *for bounded values of* n.

We now consider the case when $(\omega_c^2, m) \in \Gamma_1^-$.

Theorem 9 *Suppose* $B = \frac{1}{2} V^{(4)}(0) - \left(V^{(3)}(0)\right)^2 \neq 0$ *and* $m \in (0,1)$. *For* $\omega \approx \omega_c = (\frac{2}{m})^{1/2}$, *problem (4.9) has the following solutions with* $Y_n \in \mathbb{D}$ *for all* $n \in \mathbb{Z}$.

i) For $\omega < \omega_c$ *and* $B < 0$, *(4.9) has at least two homoclinic solutions* Y^{1-}_n, Y^{2-}_n *(and also* $T\,Y^{1-}_n$, $T\,Y^{2-}_n$*) such that* $\lim_{n\to\pm\infty} \|Y^{i-}_n\|_{\mathbb{D}} = 0$. *These solutions satisfy*

$$T\,R\,Y^{1-}_{-n} = Y^{1-}_n, \quad R\,Y^{2-}_{-n+1} = Y^{2-}_n,$$

or equivalently

$$T\,v^{1-}_{-n} = u^{1-}_n, \quad v^{2-}_{-n+1} = u^{2-}_n.$$

They have the form

$$Y^{i-}_n = (-b^{i-}_n, b^{i-}_n)\,\cos t + O(|\omega - \omega_c|),$$

where $0 < b^{i-}_n \leq C\,|\omega - \omega_c|^{1/2}\,|z_1|^{-|n|}$, $|z_1| = 1 + O(|\omega - \omega_c|^{1/2}) > 1$.

ii) If $\omega - \omega_c$ *and* B *have the same sign, (4.9) has a solution* Y^{0-}_n *being* 2-*periodic in* n. *It has the form* $Y^{0-}_n = (T^n\,y^0, T^{n+1}\,y^0)$ *with* $y^0(t) = a^*\,\cos t + O(|\omega - \omega_c|)$ $(y^0 \in H^4_{\#})$ *and* $a^* = O(|\omega - \omega_c|^{1/2})$.

iii) For $\omega > \omega_c$ *and* $B > 0$, *(4.9) has at least two heteroclinic solutions* Y^{3-}_n, Y^{4-}_n *(and also* $T\,Y^{3-}_n$, $T\,Y^{4-}_n$*) such that* $\lim_{n\to-\infty} \|Y^{i-}_n - TY^{0-}_n\|_{\mathbb{D}} = 0$, $\lim_{n\to+\infty} \|Y^{i-}_n - Y^{0-}_n\|_{\mathbb{D}} = 0$. *These solutions satisfy*

$$R\,Y^{3-}_{-n} = Y^{3-}_n, \quad T\,R\,Y^{4-}_{-n+1} = Y^{4-}_n,$$

or equivalently

$$v^{3-}_{-n} = u^{3-}_n, \quad T\,v^{4-}_{-n+1} = u^{4-}_n.$$

Moreover, $\|Y^{3-}_n\|_{\mathbb{D}}$, $\|Y^{4-}_n\|_{\mathbb{D}}$ *are* $O(|\omega - \omega_c|^{1/2})$ *as* $n \to \pm\infty$ *and* $O(|\omega - \omega_c|)$ *for bounded values of* n.

Lastly, we consider the case when $(\omega_c^2, m) \in \Gamma_1^k$ $(k \geq 1)$.

Theorem 10 *Suppose* $B = \frac{1}{2} V^{(4)}(0) - \left(V^{(3)}(0)\right)^2 \neq 0$ *and* $m \in (\frac{1}{k(k+2)}, \frac{1}{k^2})$ $(k \geq 1)$. *For* $\omega \approx \omega_c = \sqrt{2}$, *problem (4.9) has the following solutions with* $Y_n \in \mathbb{D}$ *for all* $n \in \mathbb{Z}$.

i) For $\omega > \omega_c$ and $B > 0$, (4.9) has at least two homoclinic solutions Y_n^{1a}, Y_n^{2a} (and also $T\,Y_n^{1a}$, $T\,Y_n^{2a}$) such that $\lim_{n\to\pm\infty} \|Y_n^{ia}\|_{\mathbb{D}} = 0$. These solutions satisfy

$$R\,Y_{-n}^{1a} = Y_n^{1a}, \quad T\,R\,Y_{-n+1}^{2a} = Y_n^{2a},$$

or equivalently

$$v_{-n}^{1a} = u_n^{1a}, \quad T\,v_{-n+1}^{2a} = u_n^{2a}.$$

They have the form

$$Y_n^{ia} = (b_n^{ia}, b_n^{ia})\,\cos t + O(|\omega - \omega_c|),$$

where $0 < b_n^{ia} \leq C\,|\omega - \omega_c|^{1/2}\,|z_1|^{-|n|}$, $|z_1| = 1 + O(|\omega - \omega_c|^{1/2}) > 1$.
ii) If $\omega - \omega_c$ and B have the same sign, (4.9) has a solution Y_n^{0a} being 2-periodic in n. It has the form $Y_n^{0a} = T^n\,(y^0, y^0)$ with $y^0(t) = a^*\,\cos t + O(|\omega - \omega_c|)$ ($y^0 \in H^4_\#$) and $a^* = O(|\omega - \omega_c|^{1/2})$.
iii) For $\omega < \omega_c$ and $B < 0$, (4.9) has at least two heteroclinic solutions Y_n^{3a}, Y_n^{4a} (and also $T\,Y_n^{3a}$, $T\,Y_n^{4a}$) such that $\lim_{n\to-\infty} \|Y_n^{ia} - TY_n^{0a}\|_{\mathbb{D}} = 0$, $\lim_{n\to+\infty} \|Y_n^{ia} - Y_n^{0a}\|_{\mathbb{D}} = 0$. These solutions satisfy

$$T\,R\,Y_{-n}^{3a} = Y_n^{3a}, \quad R\,Y_{-n+1}^{4a} = Y_n^{4a},$$

or equivalently

$$T\,v_{-n}^{3a} = u_n^{3a}, \quad v_{-n+1}^{4a} = u_n^{4a}.$$

Moreover, $\|Y_n^{3a}\|_{\mathbb{D}}$, $\|Y_n^{4a}\|_{\mathbb{D}}$ are $O(|\omega - \omega_c|^{1/2})$ as $n \to \pm\infty$ and $O(|\omega - \omega_c|)$ for bounded values of n.

In addition, note that for $B < 0$ and $(\omega_c^2, m) \in \Gamma_1^+$ or Γ_1^k, there exists no small amplitude discrete breather solution $Y_n \in \mathbb{D}$ with $\omega > \omega_c$ and $\omega \approx \omega_c$ (since (4.55) has no small amplitude solution homoclinic to 0). In the same way, for $B > 0$ and $(\omega_c^2, m) \in \Gamma_1^-$ there exists no small amplitude breather $Y_n \in \mathbb{D}$ with $\omega < \omega_c$ and $\omega \approx \omega_c$.
We shall also mention that the validity domain of the reduction depends on the spectral gap between the unit circle and the hyperbolic part of the spectrum. This domain shrinks as other eigenvalues approach the unit circle which does happen here when $m \to M_k^\pm$ ($k \geq 2$).
We now discuss the a priori surprising fact that coefficient B in the DB existence conditions is independent of m. Let us first investigate the case when $(\omega_c^2, m) \in \Gamma_1^+$. We have a bifurcation at a double eigenvalue $+1$ and the analysis of the reduced mapping shows that the existence of small amplitude homoclinic solutions $Y_n = (u_n, v_n)$ is equivalent to the existence of a pair of bifurcating fixed points. Consequently, let us look for non trivial solutions

$(u_n, v_n) = (u, v)$ of (4.9), where u, v are time-periodic functions. The system (4.8) reads

$$
\begin{aligned}
m\omega^2 \frac{d^2}{dt^2}(W(v)) &= (m+1)(u-v), \\
m\omega^2 \frac{d^2}{dt^2}(W(u)) &= (m+1)(v-u).
\end{aligned}
\tag{4.114}
$$

We note $\omega^2 = \omega_c^2 + (1 + \frac{1}{m})\bar{\mu} = (1 + \frac{1}{m})(2 + \overline{\mu})$ where $\overline{\mu} \approx 0$. This yields

$$
\begin{aligned}
(2+\overline{\mu}) \frac{d^2}{dt^2}(W(v)) &= (u-v), \\
(2+\overline{\mu}) \frac{d^2}{dt^2}(W(u)) &= (v-u).
\end{aligned}
\tag{4.115}
$$

The condition $B > 0$ leading to a pair of time-periodic solutions bifurcating from $(u, v) = (0, 0)$ as $\overline{\mu} \approx 0^+$ could be obtained from (4.115) by a standard Lyapounov-Schmidt procedure. We shall not detail this point here but it is clear that the condition $B > 0$ is independent of the mass ratio (since (4.115) does not depend on m).

We now consider the case when $(\omega_c^2, m) \in \Gamma_1^-$. We have a bifurcation at a double eigenvalue -1 and the study of the reduced mapping proves that the existence of small amplitude discrete breathers is equivalent to the existence of a small amplitude period 2 orbit $Y_n^0 = (u_n, v_n) = (T^n y^0, T^{n+1} y^0)$ (see property ii) of theorem 9). Consequently, let us look for solutions of (4.9) with $u_{2n} = u$, $v_{2n} = v$, $u_{2n-1} = v$ and $v_{2n-1} = u$ where u, v are time-periodic functions. Let us note in this case $\omega^2 = \omega_c^2 + \frac{1}{m}\overline{\mu} = \frac{1}{m}(2 + \overline{\mu})$ $(\overline{\mu} \approx 0)$. Then system (4.8) again takes the form (4.115) independent of m. Consequently, the condition $B < 0$ for the existence of small amplitude DB is independent of the mass ration m.

We conclude with the case $(\omega_c^2, m) \in \Gamma_1^a$, which also corresponds to a bifurcation at a double eigenvalue -1. The existence of small amplitude homoclinic solutions is equivalent to the existence of a small amplitude solution $Y_n^0 = (u_n, v_n) = T^n (y^0, y^0)$ being 2 periodic in n (see property ii) of theorem 10). Consequently, let us look for solutions of (4.9) with $u_{2n} = u$, $v_{2n} = u$, $u_{2n-1} = v$ and $v_{2n-1} = v$ where u, v are time-periodic functions. Setting $\omega^2 = \omega_c^2 + \bar{\mu} = 2 + \overline{\mu}$ $(\overline{\mu} \approx 0)$, system (4.8) again takes the form (4.115) independent of m.

Note that we have not investigated all the possible breather solutions for a given bifurcation. As we precised in section 4.5 for the reduced map, the intersection of stable and unstable manifolds is generically transverse. In this case, the reduced map admits an infinity of homoclinic orbits to 0, each one corresponding to a different breather solution of the FPU system.

The next section examines the shape of the discrete breathers described in theorems 8-10, using the original displacement variable x_n. We shall focus our attention on breather solutions, but the case of dark breathers could be treated in the same way.

4.6.2 Discrete breathers geometry

The displacement variable x_n can be recovered using the following formula (integrate (4.7) and use the evenness of x_n)

$$M_n\, x_n(t) = M_n\, x_n(0) + \frac{1}{\omega^2}\int_0^{\omega t}\int_0^{\tau} (y_{n+1}(s) - y_n(s))\, ds\, d\tau, \qquad (4.116)$$

where $M_{2n} = 1$, $M_{2n+1} = m$ and $x_n(0)$ is obtained up to an arbitrary constant $x_0(0)$ by the case $t = 0$ of

$$\begin{array}{ll} x_n(t) = x_0(t) + \sum_{k=1}^{n} W(y_k(\omega t)), & n \geq 1, \\ x_n(t) = x_0(t) - \sum_{k=n}^{-1} W(y_{k+1}(\omega t)), & n \leq -1. \end{array} \qquad (4.117)$$

Given a solution Y_n of (4.9), formula (4.116)-(4.117) determine x_n up to an additive constant c, due to the invariance $x_n \to x_n + c$ in (4.7).

In the sequel, $x_n^{1\pm}, x_n^{2\pm}$ denote displacement variables associated with the homoclinic solutions $Y_n^{1\pm}, Y_n^{2\pm}$, and similarly x_n^{1a}, x_n^{2a} correspond to the homoclinic solutions Y_n^{1a}, Y_n^{2a}. Displacements are described in the following lemma.

Lemma 15 *Solutions $Y_n^{i\pm}$, Y_n^{ia} $(i = 1, 2)$ of (4.9) provided by theorems 8, 9, 10 (property i)) correspond via formula (4.116)-(4.117) to DB solutions of (4.7) $x_n^{i\pm}$, x_n^{ia} having the form*

$$x_n^i(t) = c + d_n + X_n^i(t)$$

where X_n^i is time-periodic (with frequency ω), with 0 time-average and $\|X_n^i\|_{L^\infty}$ decays exponentially as $n \to \pm\infty$. The stationary term d_n satisfies $d_n = O(|\mu|)$ for any fixed n and $\lim_{n\to\pm\infty} d_n = O(|\mu|^{1/2})$. It has a kink shape if $V^{(3)}(0) \neq 0$ (decreasing if $V^{(3)}(0) > 0$ and increasing if $V^{(3)}(0) < 0$). One has $d_n = 0$ in the special case when V is even. The constant $c \in \mathbb{R}$ is arbitrary. The oscillatory parts X_n^i have the form

$$X_{2n}^{i+}(t) = -\frac{2b_n^{i+}}{\omega^2}\cos(\omega t) + O(|\mu|), \quad X_{2n-1}^{i+}(t) = \frac{2b_n^{i+}}{m\omega^2}\cos(\omega t) + O(|\mu|), \qquad (4.118)$$

$$X_{2n}^{i-}(t) = O(|\mu|), \quad X_{2n-1}^{i-}(t) = \frac{2b_n^{i-}}{m\omega^2}\cos(\omega t) + O(|\mu|), \tag{4.119}$$

$$X_{2n}^{ia}(t) = \frac{2b_n^{ia}}{\omega^2}\cos(\omega t) + O(|\mu|), \quad X_{2n-1}^{ia}(t) = O(|\mu|), \tag{4.120}$$

where $b_n^i = O(|\mu|^{1/2})$ *are the homoclinic solutions of the reduced mapping described in lemma 12, 13, 14 (property i)).*

Proof. We recall that $Y_n = (u_n, v_n) = (y_{2n}, y_{2n-1})$ with in addition $y_n = V'(x_n - x_{n-1})(\frac{t}{\omega})$. Equation (4.7) yields

$$\begin{aligned} X_{2n}''(t) &= v_{n+1}(\omega t) - u_n(\omega t), \\ mX_{2n-1}''(t) &= u_n(\omega t) - v_n(\omega t). \end{aligned} \tag{4.121}$$

Let us consider the case when $(\omega_c^2, m) \in \Gamma_1^+$ (we drop the $i+$ index in the notations). One obtains using (4.121) and theorem 8 (property i))

$$\begin{aligned} X_{2n}''(t) &= (b_n + b_{n+1})\,\cos(\omega t) + O(|\mu|), \\ mX_{2n-1}''(t) &= -2b_n\,\cos(\omega t) + O(|\mu|). \end{aligned} \tag{4.122}$$

The orbits (a_n, b_n) in lemma 12 (property i)) satisfy

$$(a_{n+1}, b_{n+1}) = L_c\,(a_n, b_n) + O(|\mu|^{\frac{3}{2}}), \tag{4.123}$$

where

$$L_c = \begin{pmatrix} \alpha+2 & 1+\alpha \\ -1-\alpha & -\alpha \end{pmatrix}$$

$(\alpha = 1/m)$ and $a_n = -b_n + O(|\mu|)$. This implies that

$$b_{n+1} = b_n + O(|\mu|). \tag{4.124}$$

One obtains equation (4.118) by inserting (4.124) in (4.122) and integrating twice.

We now turn to the case when $(\omega_c^2, m) \in \Gamma_1^-$. Using (4.121) and theorem 9 (property i)) leads again to equations (4.122) and (4.123), with

$$L_c = \begin{pmatrix} \alpha-2 & \alpha-1 \\ 1-\alpha & -\alpha \end{pmatrix}$$

and $a_n = -b_n + O(|\mu|)$ (see lemma 13, property i)). This implies that

$$b_{n+1} = -b_n + O(|\mu|) \tag{4.125}$$

and equation (4.119) follows by inserting (4.125) in (4.122) and integrating twice.

We now consider the case when $(\omega_c^2, m) \in \Gamma_1^a$. One obtains using (4.121) and theorem 10 (property i))

$$\begin{aligned} &X''_{2n}(t) = (-b_n + b_{n+1}) \cos(\omega t) + O(|\mu|), \\ &mX''_{2n-1}(t) = O(|\mu|). \end{aligned} \tag{4.126}$$

In equation (4.123) we have

$$L_c = \begin{pmatrix} \alpha - 2 & 1 - \alpha \\ \alpha - 1 & -\alpha \end{pmatrix}$$

and $a_n = b_n + O(|\mu|)$ (see lemma 14, property i)). This implies that

$$b_{n+1} = -b_n + O(|\mu|) \tag{4.127}$$

and equation (4.120) follows by inserting (4.127) in (4.126) and integrating twice.

We now consider the steady part of x_n. According to equation (4.117) one has $\overline{x_n} = c + d_n$ (the bar denotes time-average) where

$$\begin{aligned} d_n &= \textstyle\sum_{k=1}^{n} \overline{W(y_k(\omega t))}, \quad n \geq 1, \\ d_n &= -\textstyle\sum_{k=n+1}^{0} \overline{W(y_k(\omega t))}, \quad n \leq -1, \end{aligned} \tag{4.128}$$

$d_0 = 0$ and $c = \overline{x_0}$. We recall that $W(y_n) = y_n - \frac{1}{2}V^{(3)}(0)\, y_n^2 + O(|y_n^3|)$ and $\overline{y_n} = 0$. The estimates on d_n follow from the decay properties of y_n (theorems 8, 9, 10, property i)). In addition, if $V^{(3)}(0) \neq 0$ then $d_{n+1} - d_n$ has the sign of $-V^{(3)}(0)$ for μ small enough.

If V is even then W is odd and equation (4.9) is invariant under $-I$. It follows that the centre manifold is invariant under $-I$, and consequently $\overline{T\,Y_n} = -Y_n$ (since $T = -I$ on the centre space). This implies that $\overline{W(y_n(\omega t))} = 0$ and thus $d_n = 0$.

□

A suitable choice of the additive constant c yields the symmetric DB solutions of (4.7) listed in lemma 16. In what follows we note $\tilde{T}x = x(.+\pi/\omega)$.

Lemma 16 *One can choose a constant $c = O(|\mu|)$ in lemma 15 such that*

$$x_{-n}^{1+}(t) = -\tilde{T}\, x_{n-2}^{1+}(t), \quad x_{-n}^{2+}(t) = -T\, x_n^{2+}(t),$$

$$x_{-n}^{1-}(t) = -\tilde{T}\, x_{n-2}^{1-}(t), \quad x_{-n}^{2-}(t) = -x_n^{2-}(t),$$

$$x_{-n}^{1a}(t) = -x_{n-2}^{1a}(t), \quad x_{-n}^{2a}(t) = -\tilde{T}\, x_n^{2a}(t).$$

Proof. We prove the lemma for x_n^{1+}, the other cases being similar. To shorten notations we drop the 1+ index in the computations. We shall prove that $x_{-n} = -\tilde{T}\, x_{n-2}$ for all $n \in \mathbb{Z}$, for a suitable choice of the translation constant c. Note that it suffices to show this equality for $n \geq 0$ (the case $n \leq -1$ follows by symmetry).

According to theorem 8 (i), the homoclinic solution $Y_n = (u_n, v_n)$ has the symmetry

$$T\, v_{-n} = u_n.$$

Recalling $(u_n, v_n) = (y_{2n}, y_{2n-1})$, this means that

$$T\, y_{-n+1} = y_{n-2}. \tag{4.129}$$

Using (4.117), we have for $n \geq 1$

$$\begin{aligned} x_{-n}(t) &= -\textstyle\sum_{i=-n}^{-1} W(y_{i+1}(\omega t)) + x_0(t) \\ &= -\textstyle\sum_{i=1}^{n} W(y_{-i+1}(\omega t)) + x_0(t) \\ &= -\tilde{T}\, \textstyle\sum_{i=1}^{n} W(y_{i-2}(\omega t)) + x_0(t) \\ &= -\tilde{T}\, x_{n-2}(t) + x_0(t) + \tilde{T} x_{-2}(t). \end{aligned}$$

Now one observes that $x_0 + \tilde{T} x_{-2}$ is independent of t (differentiate twice, use (4.7) and (4.129)). Consequently, one can use the invariance $x_n \to x_n + c$ in (4.7) and choose the constant c such that $x_0 + \tilde{T} x_{-2} = 0$. In this case one has $c = O(|\mu|)$ (use lemma 15 and equation (4.124)) and one obtains

$$x_{-n} = -\tilde{T}\, x_{n-2} \tag{4.130}$$

for all $n \geq 0$.

□

Lemma 15 and 16 allow us to sketch the shape of the above discrete breather solutions. We plot their $\cos(\omega t)$ Fourier component in figures 4.2-4.7 (the error with respect to x_n is $O(|\mu|)$). These results are in agreement with previous numerical works (see e.g. [8], [26], [14]).

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Appendix

Global reduction preserving reversibility

This appendix is aimed at proving theorem 7 of section 4.4.2 (to which we refer for part of the notations). Our analysis is organized as follows.

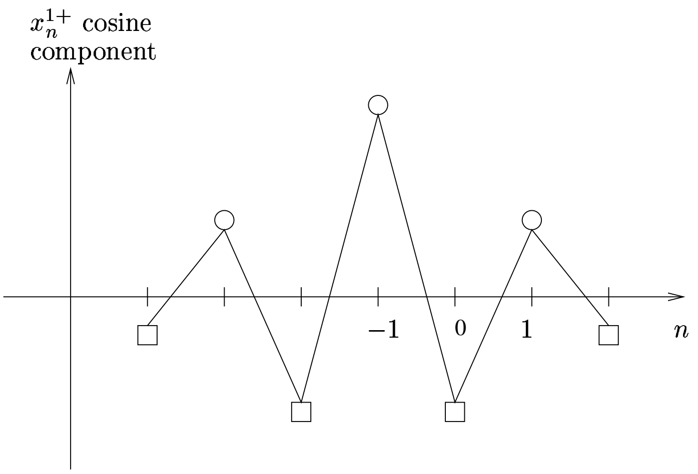

Figure 4.2: Sketch of the breather solution x_n^{1+}, as a function of n and at fixed t. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-x_{-n}^{1+}(t) = x_{n-2}^{1+}(t+\pi/\omega)$. Light masses displacements are larger and nearest neighbours are out of phase.

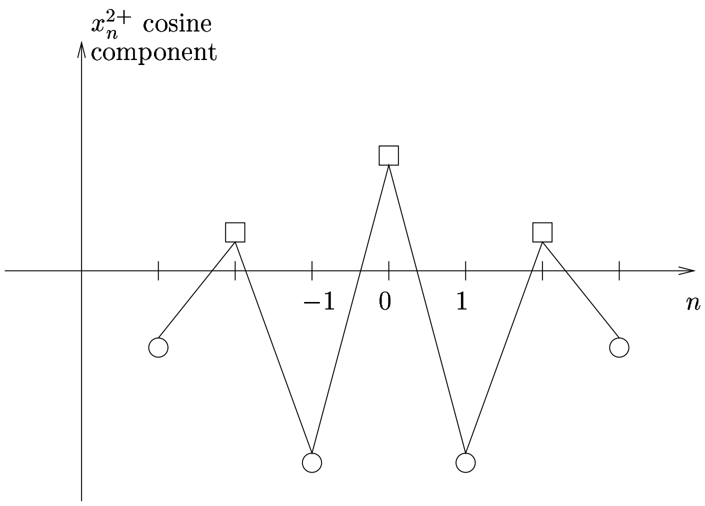

Figure 4.3: Sketch of the breather solution x_n^{2+}, as a function of n and at fixed t. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-x_{-n}^{2+}(t) = x_n^{2+}(t+\pi/\omega)$. Light masses displacements are larger and nearest neighbours are out of phase.

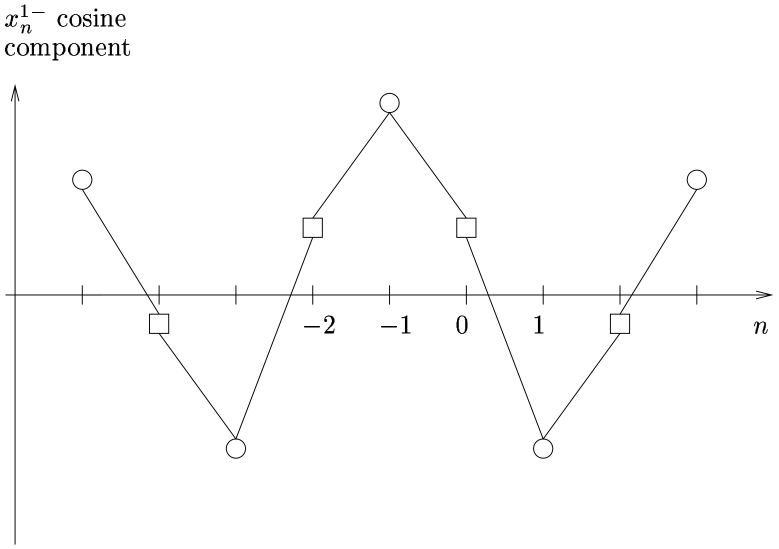

Figure 4.4: Sketch of the breather solution x_n^{1-}, as a function of n and at fixed t. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-x_{-n}^{1-}(t) = x_{n-2}^{1-}(t + \pi/\omega)$. Light masses are out of phase and have larger displacements than heavy masses.

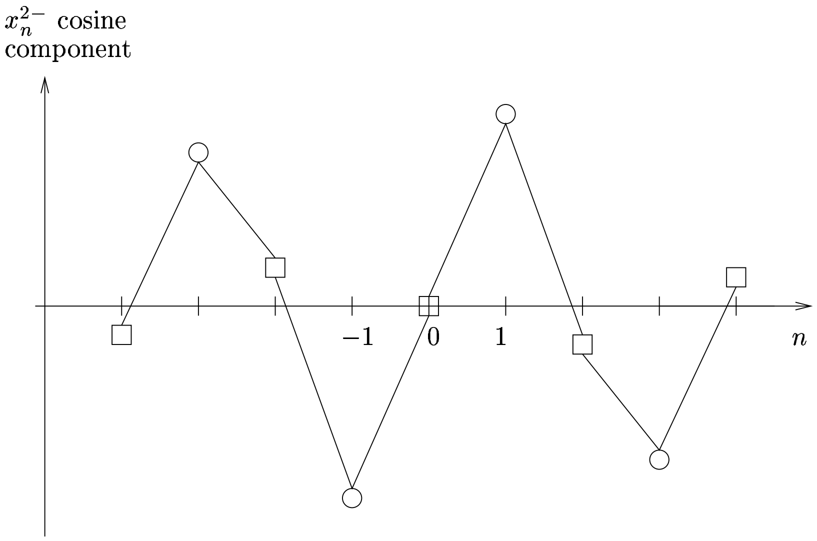

Figure 4.5: Sketch of the breather solution x_n^{2-}, as a function of n and at fixed t. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-x_{-n}^{2-}(t) = x_n^{2-}(t)$. Light masses are out of phase and have larger displacements than heavy masses.

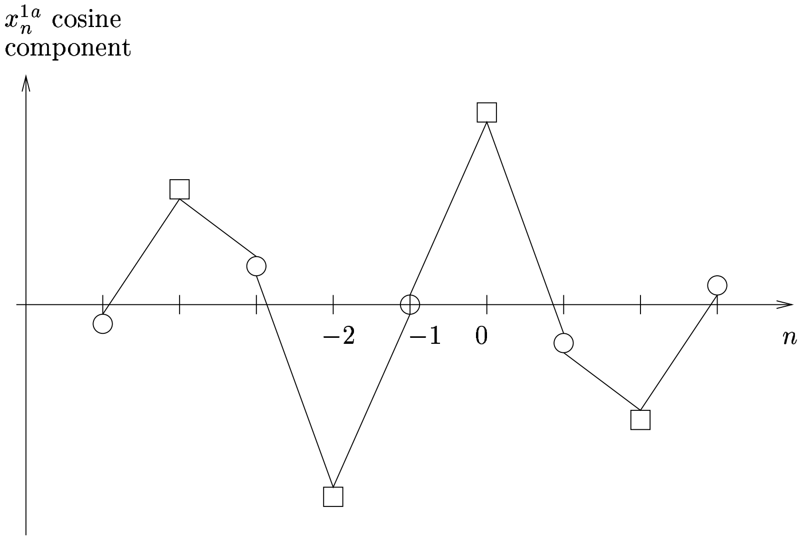

Figure 4.6: Sketch of the breather solution x_n^{1a}, as a function of n and at fixed t. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-x_{-n}^{1a}(t) = x_{n-2}^{1a}(t)$. Heavy masses are out of phase and have larger displacements than light masses.

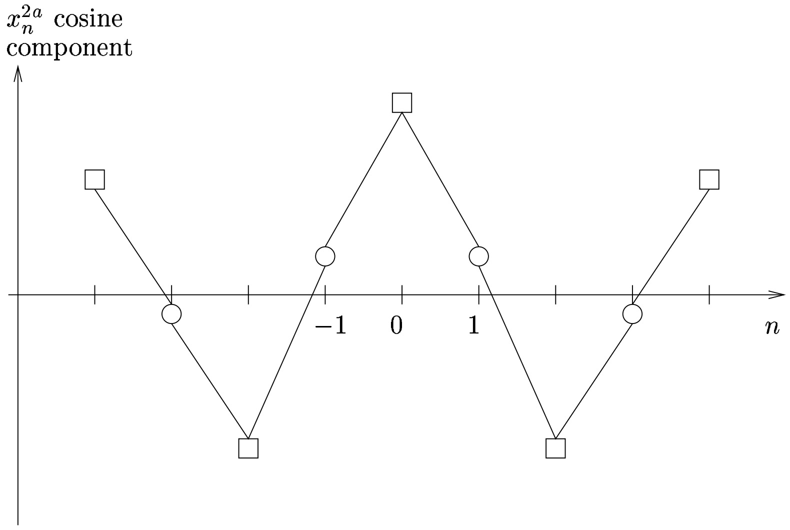

Figure 4.7: Sketch of the breather solution x_n^{2a}, as a function of n and at fixed t. Circles indicate the light atoms, while squares refer to heavy masses. One has the site-centred symmetry $-x_{-n}^{2a}(t) = x_n^{2a}(t + \pi/\omega)$. Heavy masses are out of phase and have larger displacements than light masses.

We first derive basic estimates on $\mathcal{N}_\epsilon$ and use them (in conjunction with the spectral properties of L) to obtain a global centre manifold reduction result for the truncated problem. This part is merely an adaptation of [24] (the difference is that the cut-off is performed on $\mathbb{X}_c$ instead of $\mathbb{D}$).

Next we show that our global reduction procedure preserves reversibility. The reversibility symmetry R is unbounded in $\mathbb{D}$ and thus we need a regularity result on each solution Y_n (lemma 9) for proving that RY_{-n} is also a solution. The remainder of the proof is identical to [24], section 5.1.

The truncated problem reads

$$Y_{n+1} = LY_n + \mathcal{N}_\epsilon(Y_n, \mu), \tag{4.131}$$

where

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha}(Q^2-1)u - Qv \\ Qu - \alpha v \end{pmatrix},$$

$$Qu = (\omega_c^2 \frac{d^2}{dt^2} + (1+\alpha))u,$$

$$\mathcal{N}_\epsilon(Y,\mu) = \begin{pmatrix} \frac{1}{\alpha}(Q\, N_\epsilon(u,\mu) + N_\epsilon(Qu - \alpha v + N_\epsilon(u,\mu),\mu)) \\ N_\epsilon(u,\mu) \end{pmatrix},$$

$N_\epsilon(y,\mu) = \frac{d^2}{dt^2} g_\epsilon(y,\mu)$ and $g_\epsilon(y,\mu) = g(y,\mu)\,\chi(\epsilon^{-1}\,|(y,\cos t)|)$. We recall that $g(y,\mu) = \mu W(y) + \omega_c^2(W(y) - y)$, $(y,\cos t) = \int_0^{2\pi} y(t)\cos t\, dt$ and $\chi \in C^\infty([0,+\infty),[0,1])$ is a cut-off function satisfying $\chi(x)=1$ for $x\in[0,1]$ and $\chi(x)=0$ for $x\geq 2$.

The first step consists in estimating $g_\epsilon(y,\mu)$. In the sequel we use the notations $\pi_c u = \frac{1}{\pi}\int_0^{2\pi} u(t)\cos t\, dt\, \cos t$ and $\pi_h = I - \pi_c$. Given $n\geq 1$, we denote by H^n_e the subspace of $H^n(\mathbb{R}/2\pi\mathbb{Z})$ consisting in even functions of t (equiped with the usual norm $\|\,\|_{H^n}$). We shall also consider

$$H^n_\# = \{\, y\in H^n_e \,/\, \int_0^{2\pi} y\, dt = 0\,\}, \quad H^n_\epsilon = \{\, y \in H^n_\# \,/\, \|\,\pi_h\, y\,\|_{H^n} \leq \epsilon\,\}.$$

The following estimates follow from the fact that $g(0,\mu)=0$, $\frac{\partial g}{\partial y}(0,0)=0$.

Lemma 17 *There exists $\epsilon_0>0$ such that for all $\epsilon<\epsilon_0$, $g_\epsilon : H^n_\epsilon \to H^n_e$ is C^∞. Moreover, there exists $C>0$ (depending on n) such that for all $y\in H^n_\epsilon$ and $\mu\in[-\epsilon,\epsilon]$ one has*

$$\|g_\epsilon(y,\mu)\|_{H^n} \leq C\,\epsilon^2,$$

$$\|D_y g_\epsilon(y,\mu)\|_{\mathcal{L}(H^n_\#, H^n_e)} \leq C\,\epsilon.$$

We are now ready to estimate $\mathcal{N}_\epsilon$ in the strip $B^h_\epsilon = \{ Y \in \mathbb{D} / \|Y^h\|_{\mathbb{D}} \leq \epsilon \}$. An application of lemma 17 yields as $\epsilon \to 0^+$

$$\|\mathcal{N}_\epsilon\|_{C^0_b(B^h_\epsilon \times (-\epsilon,\epsilon),\mathbb{X})} = O(\epsilon^2), \quad \|D_Y \mathcal{N}_\epsilon\|_{C^0_b(B^h_\epsilon \times (-\epsilon,\epsilon),\mathcal{L}(\mathbb{D},\mathbb{X}))} = O(\epsilon). \quad (4.132)$$

Our next step is to prove a global centre manifold reduction theorem for (4.131), where we look for Y_n in the following closed subspace of $B_\nu(\mathbb{D})$

$$B^\epsilon_\nu(\mathbb{D}) = \{ \, Y \in B_\nu(\mathbb{D}) \, / \, Y^h \in B_1(\mathbb{D}_h), \|Y^h\|_{B_1(\mathbb{D}_h)} \leq \epsilon \, \}.$$

The method consists in formulating (4.131) as a fixed point equation in $B^\epsilon_\nu(\mathbb{D})$ and then apply the contraction mapping theorem.

For this purpose, the following results on the projected affine equations are essential. They have been proved in [24] (section 3) and originate from the spectral properties of L.

Lemma 18 *The affine recurrence relation in $\mathbb{X}_c$*

$$Y^c_0 = a, \quad Y^c_{n+1} = L_c \, Y^c_n + f^c_n \quad \forall n \in \mathbb{Z} \quad (4.133)$$

has a unique solution

$$Y^c = L^n_c \, a + K_c \, f^c. \quad (4.134)$$

For all $\nu \in (0,1)$ one has $Y^c \in B_\nu(\mathbb{X}_c)$ and $K_c \in \mathcal{L}(B_\nu(\mathbb{X}_c))$.

Lemma 19 *There exists $r \in (0,1)$ such that for all $\nu \in (r,1]$, for any $f^h \in B_\nu(\mathbb{X}_h)$, the problem*

$$Y^h \in B_\nu(\mathbb{D}_h), \quad Y^h_{n+1} = L_h \, Y^h_n + f^h_n \quad \forall n \in \mathbb{Z} \quad (4.135)$$

has a unique solution $Y^h = K_h \, f^h$ with $K_h \in \mathcal{L}(B_\nu(\mathbb{X}_h), B_\nu(\mathbb{D}_h))$.

In lemma 19, the constant r must be chosen so that $\sigma(L_h)$ lies strictly inside the ball of radius r or outside the ball of radius r^{-1}.

Splitting (4.131) on $\mathbb{X}_c$, $\mathbb{X}_h$ and choosing $\nu \in (r,1)$ as in lemmas 18, 19, one obtains the equivalent problem

$$Y \in B^\epsilon_\nu(\mathbb{D}), \; Y = L^n_c \, Y^c_0 + (K_c \Pi_c + K_h \Pi_h) \, \mathcal{N}_\epsilon(Y,\mu). \quad (4.136)$$

We now solve (4.136) when ϵ is small enough and $(Y^c_0, \mu) \in \mathbb{X}_c \times [-\epsilon, \epsilon]$ is given. In (4.136) we view $\mathcal{N}_\epsilon(.,\mu)$ as a map from $B^\epsilon_\nu(\mathbb{D})$ into $B_\nu(\mathbb{X})$. Using (4.132) gives

$$\operatorname*{Sup}_{Y \in B^\epsilon_\nu(\mathbb{D}), |\mu| \leq \epsilon} \|\mathcal{N}_\epsilon(Y,\mu)\|_{B_1(\mathbb{X})} = O(\epsilon^2), \quad (4.137)$$

$$\operatorname*{Sup}_{Y\neq Z\in B_\nu^\epsilon(\mathbb{D}),|\mu|\leq\epsilon}\frac{\|\mathcal{N}_\epsilon(Y,\mu)-\mathcal{N}_\epsilon(Z,\mu)\|_{B_\nu(\mathbb{X})}}{\|Y-Z\|_{B_\nu(\mathbb{D})}}=O(\epsilon). \tag{4.138}$$

By combining (4.137)-(4.138) with lemmas 18 and 19, one finds $\epsilon_0(\nu)>0$ such that for all $\epsilon<\epsilon_0$ and $(x,\mu)\in\mathbb{X}_c\times[-\epsilon,\epsilon]$, the map

$$Y\mapsto F_\epsilon(Y,x,\mu)=L_c^n\,x+(K_c\Pi_c+K_h\Pi_h)\,\mathcal{N}_\epsilon(Y,\mu)$$

maps $B_\nu^\epsilon(\mathbb{D})$ into itself and is a contraction. Consequently, for a given $(Y_0^c,\mu)\in\mathbb{X}_c\times[-\epsilon,\epsilon]$ it follows from the contraction mapping theorem that (4.136) has a unique solution $Y=\phi^\epsilon(Y_0^c,\mu)$ with $\phi^\epsilon\in C^{0,1}(\mathbb{X}_c\times[-\epsilon,\epsilon],B_\nu^\epsilon(\mathbb{D}))$. Since for any fixed $p\in\mathbb{Z}$ the shifted sequence Y_{n+p} is also a solution, one has by uniqueness $Y_{n+p}=\phi_{n+p}^\epsilon(Y_0^c,\mu)=\phi_n^\epsilon(Y_p^c,\mu)$. Setting $n=0$ gives in particular

$$Y_p=\phi_p^\epsilon(Y_0^c,\mu)=\phi_0^\epsilon(Y_p^c,\mu)\quad\forall p\in\mathbb{Z}.$$

Consequently, Y is solution of (4.136) if and only if

$$Y_n=\phi_0^\epsilon(Y_n^c,\mu)\quad\forall n\in\mathbb{Z}. \tag{4.139}$$

This implies that

$$Y_n^h=\psi_\epsilon(Y_n^c,\mu), \tag{4.140}$$

where $\psi_\epsilon=\Pi_h\,\phi_0^\epsilon\in C_b^0(\mathbb{X}_c\times[-\epsilon,\epsilon],\mathbb{D}_h)$ (with $\psi_\epsilon=O(\epsilon^2)$). The next step is to prove that $\psi_\epsilon\in C^k(\mathbb{X}_c\times[-\epsilon,\epsilon]),\mathbb{D}_h)$. This property does not follow directly from the implicit function theorem since $\mathcal{N}_\epsilon(.,\mu)\,:\,B_\nu^\epsilon(\mathbb{D})\to B_\nu(\mathbb{X})$ is not differentiable (due to the possible divergence of $Y\in B_\nu^\epsilon(\mathbb{D})$). The C^k-regularity of ψ_ϵ is obtained in the same way as in [24], [35], [34], to which we refer for details. The proof is based on the fiber contraction theorem and the fact that $\mathcal{N}_\epsilon(.,\mu)\in C^k(B_\nu^\epsilon(\mathbb{D}),B_\zeta(\mathbb{X}))$ for $\zeta<\nu^k<\nu<1$.

Differentiability in $B_\nu^\epsilon(\mathbb{D})$ has to be understood as follows. Consider a Banach space E and an operator $N\,:\,B_\nu^\epsilon(\mathbb{D})\to E$. One has $N\in C^1(B_\nu^\epsilon(\mathbb{D}),E)$ if there exists a mapping $DN\in C^0(B_\nu^\epsilon(\mathbb{D}),\mathcal{L}(B_\nu(\mathbb{D}),E))$ such that for all $Y,Z\in B_\nu^\epsilon(\mathbb{D})$

$$\|\,N(Y)-N(Z)-DN(Z)(Y-Z)\,\|_E=o(\|\,Y-Z\,\|_{B_\nu(\mathbb{D})}),\quad \|\,Y-Z\,\|_{B_\nu(\mathbb{D})}\to 0.$$

Obviously one has $N\in C^k(B_\nu^\epsilon(\mathbb{D}),E)$ if $D^jN\in C^1(B_\nu^\epsilon(\mathbb{D}),\mathcal{L}^j(B_\nu(\mathbb{D}),E))$ for all $j=0,\ldots,k-1$.

Thanks to property (4.140), one obtains a (global) reduced recurrence relation by projecting (4.131) on $\mathbb{X}_c$. This yields

$$Y_{n+1}^c=f_\epsilon(Y_n^c,\mu)\quad\forall n\in\mathbb{Z}, \tag{4.141}$$

where

$$f_\epsilon(x,\mu) = L_c\, x + \Pi_c \mathcal{N}_\epsilon(x + \psi_\epsilon(x,\mu),\mu).$$

We now address the question of the reversibility of (4.141) and the invariance under R of the centre manifold. As we mentioned previously, one has to prove lemma 9 as a first step. For this purpose we need the following intermediate result.

Lemma 20 *There exist $\epsilon_0, \gamma > 0$ (depending on n) such that for all $\epsilon < \epsilon_0$, $\mu \in [-\epsilon, \epsilon]$ and $f \in H^n_\epsilon$, the problem*

$$\frac{d^2}{dt^2}(\omega_c^2 y + g_\epsilon(y,\mu)) = f, \quad y \in H^{n+2}_{\gamma\epsilon} \tag{4.142}$$

has a unique solution y.

Proof. We define $\pi_0\, u = u - \frac{1}{2\pi}\int_0^{2\pi} u(t)\, dt$. Applying π_0 to equation (4.142) and integrating twice yields

$$\omega_c^2 y + \pi_0\, g_\epsilon(y,\mu) = F, \tag{4.143}$$

where $\frac{d^2F}{dt^2} = f$ and $F \in H^{n+2}_\epsilon$ (this determines F uniquely). Consequently, we are led to solve the fixed point equation

$$y = U_{\epsilon,\mu}(y), \tag{4.144}$$

where $U_{\epsilon,\mu}(y) = \frac{1}{\omega_c^2}(F - \pi_0\, g_\epsilon(y,\mu))$. Choosing $\gamma > \frac{1}{\omega_c^2}$ and ϵ small enough, it follows from lemma 17 that $U_{\epsilon,\mu}$ is a contraction on $H^{n+2}_{\gamma\epsilon}$. Therefore, according to the contraction mapping theorem (4.142) has a unique solution y in $H^{n+2}_{\gamma\epsilon}$.

□

We are now ready to prove lemma 9, which can be formulated in the following way.

Lemma 21 *Fix $p \geq 2$. There exist $\epsilon_0, \gamma > 0$ (depending on p) such that for all $\epsilon < \epsilon_0$ and $\mu \in [-\epsilon, \epsilon]$, any solution $Y \in B^\epsilon_\nu(\mathbb{D})$ of (4.131) satisfies $Y_n \in H^{p+2}_{\gamma\epsilon} \times H^p_{\gamma\epsilon}$.*

Proof. We prove this result by induction. Firstly we have $Y_n \in H^4_\epsilon \times H^2_\epsilon$. Secondly, assume $p > 2$ and $Y_n \in H^{k+2}_{\lambda\epsilon} \times H^k_{\lambda\epsilon}$ with $2 \leq k < p$. Setting $Y_n = (u_n, v_n)$, problem (4.131) yields

$$[\,Q + N_\epsilon(.,\mu)\,]\, v_{n+1} = \alpha u_{n+1} + u_n, \tag{4.145}$$

$$[\,Q + N_\epsilon(.,\mu)\,]\, u_n = \alpha v_n + v_{n+1}. \tag{4.146}$$

Equation (4.145) can be written

$$\frac{d^2}{dt^2}(\omega_c^2 v_{n+1} + g_\epsilon(v_{n+1},\mu)) = \alpha u_{n+1} + u_n - (1+\alpha)\, v_{n+1} \in H^k_{\lambda\,\epsilon}.$$

Consequently, lemma 20 ensures that $v_n \in H^{k+2}_{\gamma\,\lambda\,\epsilon}$. Turning to equation (4.146), we get

$$\frac{d^2}{dt^2}(\omega_c^2 u_n + g_\epsilon(u_n,\mu)) = \alpha v_n + v_{n+1} - (1+\alpha)\, u_n \in H^{k+2}_{\kappa\,\epsilon}$$

and lemma 20 yields $u_n \in H^{k+4}_{\gamma\,\kappa\,\epsilon}$. Consequently we have obtained $Y_n \in H^{k+4}_{c\,\epsilon} \times H^{k+2}_{c\,\epsilon}$, and the proof follows by induction.

□

From lemma 21 it follows that if Y_n is a solution of (4.131) then $R\,Y_{-n}$ is also a solution (see lemma 10 of section 4.4.2). Proceeding as in [24] (lemma 5 p.41), one obtains in this case $R\psi_\epsilon(.,\mu) = \psi_\epsilon(.,\mu)\circ R$ and $(f_\epsilon(.,\mu)\circ R)^2 = I$. This completes the proof of theorem 7.

Centre manifold computation

When μ is small, the centre manifold theorem 5 ensures that the solutions of (4.9) staying in the neighbourhood of $Y=0$ in $\mathbb{D}$ for any $n \in \mathbb{Z}$ have the form $Y_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos(t) + \Psi(a_n, b_n, \mu)$, where $\Psi \in C^k(\mathbb{R}^3; \mathbb{D}_h)$. The coordinates (a_n, b_n) satisfy a recurrence relation in $\mathbb{R}^2$:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} F_1(a_n,b_n,\mu) \\ F_2(a_n,b_n,\mu) \end{pmatrix} = f(a_n,b_n,\mu). \tag{4.147}$$

In this appendix we compute the leading order terms in the Taylor expansions of Ψ and the reduced map (4.147).

Computation of Ψ:
We first compute the Taylor expansion of Ψ in (a,b,μ) at $(a,b,\mu)=0$. Due to the invariance under R, it has the form:

$$\Psi(a,b,\mu) = \begin{pmatrix} \Psi_{011}a\mu + \Psi_{101}b\mu + \Psi_{020}a^2 + \Psi_{110}ab + \Psi_{200}b^2 \\ \Psi_{101}a\mu + \Psi_{011}b\mu + \Psi_{200}a^2 + \Psi_{110}ab + \Psi_{020}b^2 \end{pmatrix} + \text{h.o.t.} \tag{4.148}$$

The invariance under T implies that:

$$\begin{array}{l} T\Psi_{011} = -\Psi_{011}, T\Psi_{101} = -\Psi_{101}, \\ T\Psi_{020} = \Psi_{020}, T\Psi_{110} = \Psi_{110}, T\Psi_{200} = \Psi_{200}. \end{array} \tag{4.149}$$

Hence we have:

$$\begin{array}{l}\Psi_{011}, \Psi_{101} \in \langle \cos((2k+1)t)/k \in \mathbb{Z} \rangle = \mathbb{V}_{odd}, \\ \Psi_{020}, \Psi_{110}, \Psi_{200} \in \langle \cos(2kt)/k \in \mathbb{Z} \rangle = \mathbb{V}_{even}.\end{array} \tag{4.150}$$

Setting $Y_n = (u_n, v_n)$, one has:

$$\begin{array}{l}u_n = a_n \cos(t) + \Psi_{011} a_n \mu + \Psi_{101} b_n \mu + \Psi_{020} a_n^2 + \Psi_{110} a_n b_n + \Psi_{200} b_n^2 + \text{h.o.t}, \\ v_n = b_n \cos(t) + \Psi_{101} a_n \mu + \Psi_{011} b_n \mu + \Psi_{200} a_n^2 + \Psi_{110} a_n b_n + \Psi_{020} b_n^2 + \text{h.o.t.}\end{array} \tag{4.151}$$

Problem (4.9) can be written

$$\begin{array}{l}m\omega^2 \dfrac{d^2}{dt^2} W(v_{n+1}) = u_{n+1} - (m+1) v_{n+1} + m u_n, \\ m\omega^2 \dfrac{d^2}{dt^2} W(u_n) = m v_{n+1} - (m+1) u_n + v_n,\end{array} \tag{4.152}$$

or equivalently

$$\begin{array}{l}(\mu + \omega_c^2) \dfrac{d^2}{dt^2} W(v_{n+1}) = \alpha u_{n+1} - (1+\alpha) v_{n+1} + u_n, \\ (\mu + \omega_c^2) \dfrac{d^2}{dt^2} W(u_n) = v_{n+1} - (1+\alpha) u_n + \alpha v_n,\end{array} \tag{4.153}$$

with $\alpha = 1/m$. We calculate the Taylor expansion of Ψ as explained in section 4.4.3. We identify the quadratic terms $a_n\mu$, $b_n\mu$, a_n^2, b_n^2 and $a_n b_n$ in the Taylor expansion of (4.47). Due to the symmetries of Ψ (see (4.148)), we just need the second component of (4.47) to perform the identification. In the sequel we use the notations $\pi_c u = \frac{1}{\pi} \int_0^{2\pi} u(t) \cos t \, dt \, \cos t$ and $\pi_h = I - \pi_c$.

We first derive a system which allows to compute $(\Psi_{020}, \Psi_{200}, \Psi_{110})$. The identification at order a_n^2 gives:

$$\begin{aligned}\omega_c^2 \tfrac{d^2}{dt^2} \Psi_{020} + W_2 \omega_c^2 \tfrac{d^2}{dt^2} \pi_h(\cos^2(t)) = {} & \beta \tfrac{\beta^2-1}{\alpha} \Psi_{110} + (\tfrac{\beta}{\alpha})^2 \Psi_{200} + \\ & \beta^2 \Psi_{020} - (1+\alpha) \Psi_{020} + \alpha \Psi_{200},\end{aligned} \tag{4.154}$$

where W_2 is defined in (4.31) and $\beta = 1 + \alpha - \omega_c^2$. Moreover the identification at order b_n^2 yields:

$$\begin{aligned}\omega_c^2 \tfrac{d^2}{dt^2} \Psi_{200} = {} & \beta^2 \Psi_{200} + \alpha\beta \Psi_{110} \\ & + \alpha^2 \Psi_{020} - (1+\alpha) \Psi_{200} + \alpha \Psi_{020}.\end{aligned} \tag{4.155}$$

Identification at order $a_n b_n$ leads to:

$$\begin{aligned}-\omega_c^2 \tfrac{d^2}{dt^2} \Psi_{110} = {} & 2\beta \tfrac{\beta^2-1}{\alpha} \Psi_{200} + 2\beta^2 \Psi_{110} \\ & + 2\alpha\beta \Psi_{020}.\end{aligned} \tag{4.156}$$

By expanding Ψ_{200}, Ψ_{020}, Ψ_{110} in Fourier series in (4.154), (4.155) and (4.156), one shows that these functions are colinear to $\cos(2t)$. Let us note $\Psi_{200} = \psi_{200}\cos(2t)$ and define ψ_{110} and ψ_{020} in the same way. Equations (4.154), (4.155) and (4.156) yield:

$$D(\alpha, \omega_c^2) \begin{pmatrix} \psi_{200} \\ \psi_{020} \\ \psi_{110} \end{pmatrix} = b(W_2, \omega_c^2) \tag{4.157}$$

with

$$b(W_2, \omega_c^2) = \begin{pmatrix} -2W_2\omega_c^2 \\ 0 \\ 0 \end{pmatrix}$$

and

$$D(\alpha, \omega_c^2) = \begin{pmatrix} (\frac{\beta^2-1}{\alpha})^2 + \alpha & \beta^2 - (\beta - 3\omega_c^2) & \beta\frac{\beta^2-1}{\alpha} \\ \beta^2 - (\beta - 3\omega_c^2) & \alpha(\alpha+1) & \alpha\beta \\ \beta\frac{\beta^2-1}{\alpha} & \alpha\beta & \beta^2 - 2\omega_c^2 \end{pmatrix}. \tag{4.158}$$

Now we solve the linear system (4.157) for the different critical values listed in (4.28).
Starting with $\omega_c^2 = 2(1+\alpha)$, we find:

$$\begin{aligned} \Psi_{020} &= -\frac{1}{16}\frac{8m-1}{m}W_2\cos(2t), \\ \Psi_{200} &= \frac{1}{16m}W_2\cos(2t), \\ \Psi_{110} &= \frac{1}{8m}W_2\cos(2t). \end{aligned} \tag{4.159}$$

For $\omega_c^2 = 2\alpha$, one has:

$$\begin{aligned} \Psi_{020} &= -\frac{1}{16}\frac{7m^2-34m+3}{m(m-3)}W_2\cos(2t), \\ \Psi_{200} &= \frac{1}{16}\frac{m^2-6m-3}{m(m-3)}W_2\cos(2t), \\ \Psi_{110} &= \frac{1}{8}\frac{m^2+2m-3}{m(m-3)}W_2\cos(2t). \end{aligned} \tag{4.160}$$

Considering the case $\omega_c^2 = 2$, we obtain:

$$\begin{aligned} \Psi_{020} &= -\frac{1}{16}\frac{24m^3+5m^2-6m+1}{m^2(3m-1)}W_2\cos(2t), \\ \Psi_{200} &= \frac{1}{16}\frac{3m^2+6m-1}{m^2(3m-1)}W_2\cos(2t), \\ \Psi_{110} &= \frac{1}{8}\frac{5m^2-6m+1}{m^2(3m-1)}W_2\cos(2t). \end{aligned} \tag{4.161}$$

The computations of the other coefficients Ψ_{011} and Ψ_{101} are similar. Identification at order $a_n\mu$ and $b_n\mu$ leads to $\Psi_{011} = 0$ and $\Psi_{101} = 0$.

As a conclusion we have achieved the centre manifold computation. To simplify the notations, we shall write

$$\begin{aligned} \Psi_{020} &= p(m)W_2\cos(2t), \\ \Psi_{200} &= q(m)W_2\cos(2t), \\ \Psi_{110} &= r(m)W_2\cos(2t). \end{aligned} \tag{4.162}$$

Reduced map computation:

Now we compute the leading order terms in the Taylor expansion of the reduced map (4.147). Projecting (4.153) on $\mathbb{X}_c$ yields (we use the fact that $\pi_c(\frac{d^2}{dt^2}) = -\pi_c$)

$$\begin{aligned} -(\mu+\omega_c^2)(v_{n+1}^c + \pi_c G(v_{n+1})) &= \alpha u_{n+1}^c - (1+\alpha)v_{n+1}^c + u_n^c, \\ -(\mu+\omega_c^2)(u_n^c + \pi_c G(u_n)) &= v_{n+1}^c - (1+\alpha)u_n^c + \alpha v_n^c, \end{aligned} \tag{4.163}$$

where $(u_n^c, v_n^c) = \Pi_c(u_n, v_n) = (\pi_c\, u_n, \pi_c\, v_n) = (a_n, b_n)\ \cos(t)$ and

$$G(y) = W(y) - y = W_2y^2 + W_3y^3 + O(y^4).$$

The recurrence relation (4.147) is obtained by inserting expressions (4.151) in equation (4.163). The second equation of (4.163) then gives:

$$b_{n+1} = \beta a_n - \alpha b_n - a_n\mu - (\mu+\omega_c^2)I\pi_c G(u_n)$$

where we note $I(a\cos(t)) = a$. To simplify the computation of $\pi_c G(u_n)$, we note that $\pi_c\mathbb{V}_{even} = 0$ (see definition (4.150)). Using expressions (4.151), (4.162) and the symmetry properties (4.150), one obtains:

$$\begin{aligned} G(u_n) = {} & (2W_2\Psi_{020}\cos(t) + W_3\cos^3(t))a_n^3 + 2W_2\Psi_{110}\cos(t)\, a_n^2 b_n \\ & + 2W_2\Psi_{200}\cos(t)\, a_n b_n^2 + k_n(t) + \text{h.o.t} \end{aligned}$$

with $k_n \in \mathbb{V}_{even}$. Finally, we have:

$$\begin{aligned} b_{n+1} &= F_2(a_n, b_n, \mu) \\ &= \beta a_n - \alpha b_n - a_n\mu - \omega_c^2(\beta_1 a_n^3 + \beta_2 a_n b_n^2 + \beta_3 a_n^2 b_n) + \text{h.o.t} \end{aligned} \tag{4.164}$$

with:

$$\begin{aligned} \beta_1 &= I\pi_c(2W_2\Psi_{020}\cos(t) + W_3\cos^3(t)) = p(m)W_2^2 + \frac{3}{4}W_3, \\ \beta_2 &= I\pi_c(2W_2\Psi_{110}\cos(t)) = q(m)W_2^2, \\ \beta_3 &= I\pi_c(2W_2\Psi_{200}\cos(t)) = r(m)W_2^2. \end{aligned} \tag{4.165}$$

In the sequel we note

$$F_2(a,b,\mu) = L_2(a,b) - a\mu + Q(a,b) + \text{h.o.t}\,, \tag{4.166}$$

where $L_2(a,b) = \beta a - \alpha b$ and $Q(a,b) = -\omega_c^2(\beta_1 a^3 + \beta_2 ab^2 + \beta_3 a^2 b)$.

We now expand the first equation $a_{n+1} = F_1(a_n, b_n, \mu)$ in (4.147). Let us note

$$F_1(a,b,\mu) = L_1(a,b) + \mu M_1(a,b) + P(a,b) + \text{h.o.t}\,, \tag{4.167}$$

where L_1, M_1 are linear forms and P is a homogeneous cubic polynomial. The first equation of (4.163) yields

$$\alpha a_{n+1} = \beta b_{n+1} - a_n - \mu b_{n+1} - (\mu + \omega_c^2) I \pi_c G(v_{n+1}). \tag{4.168}$$

Inserting (4.164) in equation (4.168), one finds

$$L_1(a,b) = \frac{\beta^2 - 1}{\alpha} a - \beta b, \quad M_1(a,b) = -2\frac{\beta}{\alpha} a + b.$$

Cubic terms of (4.167) could be obtained by combining equations (4.168), (4.164), (4.153) (v_{n+1} is expressed as a function of (u_n, v_n)) and (4.151). Here we proceed in a simpler way, using the reversibility of (4.147) under $R\,(a,b) = (b,a)$ (see theorem 6). One has

$$f \circ R \circ f \circ R = Id \tag{4.169}$$

and consequently

$$F_2(F_2(b,a,\mu), F_1(b,a,\mu)) = b. \tag{4.170}$$

Now we identify cubic terms in (4.170) and find

$$L_2(Q(b,a), P(b,a)) + Q(L_2(b,a), L_1(b,a)) = 0.$$

This yields

$$\alpha P(b,a) = \beta Q(b,a) + Q(L_2(b,a), L_1(b,a)). \tag{4.171}$$

One obtains consequently $P(a,b) = -\omega_c^2(\beta_4 a^3 + \beta_5 a^2 b + \beta_6 ab^2 + \beta_7 b^3)$, with

$$\begin{aligned}
\alpha\beta_4 &= \beta\beta_1 + \beta^3\beta_1 + \beta\gamma^2\beta_2 + \beta^2\gamma\beta_3, \\
\alpha\beta_5 &= -3\beta^2\alpha\beta_1 - (\alpha\gamma^2 + 2\gamma\beta^2)\beta_2 + (\beta - \beta^3 - 2\alpha\beta\gamma)\beta_3, \\
\alpha\beta_6 &= 3\beta\alpha^2\beta_1 + (\beta + \beta^3 + 2\alpha\beta\gamma)\beta_2 + (2\beta^2\alpha + \alpha^2\gamma)\beta_3, \\
\beta_7 &= -\alpha^2\beta_1 - \beta^2\beta_2 - \alpha\beta\beta_3
\end{aligned} \tag{4.172}$$

and $\gamma = \frac{\beta^2-1}{\alpha}$. As a conclusion, we find

$$\begin{aligned}
a_{n+1} &= F_1(a_n, b_n, \mu) \\
&= \gamma a_n - \beta b_n - 2\tfrac{\beta}{\alpha} a_n \mu + b_n \mu - \omega_c^2(\beta_4 a_n^3 + \beta_5 a_n^2 b_n + \beta_6 a_n b_n^2 + \beta_7 b_n^3) \\
&\quad + \text{h.o.t.}
\end{aligned} \tag{4.173}$$

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Chapitre 5

Existence de breathers dans les chaînes de spins

Dans ce chapitre, on étudie la dynamique d'une chaîne de spins interagissant avec leurs plus proches voisins et possédant une anisotropie locale ("on site") au voisinage de deux configurations bien particulières. Dans un premier temps, on se place au voisinage d'un état de base (i.e. une solution statique des équations du mouvement minimisant en plus une certaine fonctionnelle d'énergie) où tous les spins sont dans un plan (qu'on supposera être le plan XY), parallèles et pointés dans le même sens. On parle alors de réseaux ferromagnétiques avec une anisotropie "easy plane". On démontre rigoureusement l'existence de breathers "out of plane" : dans ce cas, un ou plusieurs spins ont un mouvement de précession autour de l'axe Z (appelé "hard axis") tandis que les autres spins restent proches d'un axe inclus dans le plan XY. Bien que les ondes progressives du système linéarisé aient leurs fréquences dans une bande acoustique, on peut encore utiliser la méthode de la limite anticontinue introduite par Aubry et MacKay, en restreignant les espaces fonctionnels. Cette démonstration est indépendante de la dimension du réseau et on la fait en dimension 1.

On étudie ensuite le mouvement des spins au voisinage d'un état de base où tous les spins sont alignés le long d'un même axe (qu'on supposera être l'axe Z) et dans le même sens : on parle de chaîne ferromagnétique de type easy axis. Dans ce cas, on démontre l'existence de breathers de petite amplitude (où les spins tournent autour du même axe avec un mouvement de précession dont l'amplitude est petite et décroît lorsqu'on s'éloigne du centre du réseau). On formule le problème au moyen d'un mapping dans un espace de fonctions périodiques et on analyse celui-ci par une réduction à une variété centrale. Les résultats ainsi obtenus complètent les démonstrations de Flach et Speight utilisant la méthode de la limite anticontinue.

Existence of breathers in classical ferromagnetic lattices

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Abstract

In this paper, we study the dynamics of classical spins interacting via the Heisenberg exchange in the presence of a single-ion anisotropy on spatial d-dimensional lattices. We focus on easy-plane ferromagnets and easy-axis ferromagnets. We present a rigorous proof for the existence of the so called "out of plane" breathers which have no analog in the continuum theory. In this configuration, one or several spins precess around an axis out of a plane where all the other spins lie. Such discrete breathers represent excitations with a tilted magnetization and possess an energy threshold. The travelling waves of the linearized system possess frequencies in an acoustic band but we shall see that with a suitable restriction of the functional spaces, the anticontinuous method introduced by MacKay and Aubry is still working. This analysis is independent of the lattice dimension and we make the proof in the one dimensional case. On the other hand, we prove the existence of small amplitude breathers precessing around an axis (so called "easy" axis) using centre manifold reduction for quasilinear mappings. In this case, the motion of precession is small and decreases to 0 far from the lattice centre.

5.1 Introduction

Discrete breathers are time periodic, spatially localized solutions in networks of coupled oscillators (including spins and rotors). They arise in very general systems due to the interplay between nonlinear and discrete effects [5]. Therefore there is a lot of applications in many area of physics in particular in condensed matter and biophysics. A rigorous proof for the existence of breathers has been given by MacKay and Aubry [13] for weakly coupled chains via the anticontinuous limit. They represent the breathers solutions as zeros of a suitable mapping. Then starting from the uncoupled case where trivial breathers exist (several oscillators are excited while the others are at rest), they continue these solutions using the implicit function theorem. Recently, James [8], [9] has proved the existence of small amplitude breathers in Fermi-Pasta-Ulam chains (where there is no "anticontinuous limit") using centre manifold reduction for quasilinear discrete systems. In this case, the equations of motion are written as a recurrence relation in a loop space to apply centre manifold techniques. It has been proved that in the monoatomic case, small amplitude breathers exist for a hardening potential [9]. In the diatomic case, we find small amplitude breathers for hardening or softening potential [10]. We shall also mention the proof of Aubry in the hardening case using a variational approach. From an experimental point of view, discrete breathers have been observed in Josephson arrays [17], arrays of weakly coupled waveguide [4], [7], low dimensional crystals [16] and biological systems [18].

In this paper, we study the classical equations of motion for a ferromagnetic spin chain with a single ion anisotropy. Due to the spatial periodicity, the lattices of interacting spins are ideal candidates to observe discrete breathers. Nonlinear waves in such systems have extensively been studied in particular the solitary wave property close to the continuum limit. However, neglecting discreteness effects may lead to lose important features of nonlinear wave dynamics. For example, since only high-symmetry continuous systems possess breather solutions, the area of potentially interesting system is drastically reduced. Several numerical studies of discrete breathers have been performed in the case of easy axis antiferromagnets and easy plane ferromagnets [12]. In recent papers, rigorous existence proofs have been presented for the easy axis ferromagnet with an isotropic exchange interaction or with a strongly anisotropic exchange interaction in the presence of a single ion anisotropy via the anticontinuous limit approach [6], [15].

In this article, we first consider an easy plane ferromagnet. Several numerical studies ([6], [11]) have been carried out on such sytems: they exhibit a new type of solutions, namely "out of plane" breathers where one or several spins are precessing around one axis (called "hard" axis) while all the others are precessing around another axis lying in the "easy" plane (which we assume to be the XY plane). They have no analog in the continuum and possess an energy threshold. There exists a frequency threshold upon which breathers are static. Morevover, the linear stability of such breathers is strongly related to their geometry. The aim of this paper is to give a rigorous proof for the existence of "out of plane" discrete breathers and their static analogues in the anticontinuous limit. The travelling waves solutions of the linearized sytem have frequencies in an acoustic band but we shall see that the anticontinuous method is still working after a suitable restriction of the functional spaces.

Then we consider the spins dynamic in an easy axis ferromagnet. The ground

state corresponds to all spins parallels to the same axis (we assume this "easy" axis to be the Z axis). The purpose of this study is to give a rigorous proof for the existence of breathers in a larger range of parameter values than those considered in [6], [15]. We will consider small amplitude breathers close to the ground state. The equations of motion can be written as a recurrence relation in a loop space and applying centre manifold reduction for quasilinear mappings [8], we are able to describe *all* small amplitude periodic solutions close to the ground state.
The paper is organized as follows. In the next section we present the equations of motion in a classical spin chain. Section 5.3 is devoted to the rigorous existence proof of "out of plane" breathers and their static analogues in easy plane ferromagnets. In section 5.4, we focus on easy axis ferromagnets and prove the existence of small amplitude breathers close to the ground state.

5.2 Equations of motions

We consider a lattice of classical spins described by the "hamiltonian" with Heisenberg XYZ exchange interaction and single-ion anisotropy

$$E = -\frac{1}{2}\sum_n (J_x x_{n+1} x_n + J_y y_{n+1} y_n + J_z z_{n+1} z_n) - D\sum_n z_n^2. \tag{5.1}$$

This "hamiltonian" is more precisely a magnetic energy which is a conserved quantity. Here x_n, y_n, z_n are the n-th spin components (n labels lattice sites) that satisfy the normalization condition

$$x_n^2 + y_n^2 + z_n^2 = 1 \ , \ n \in \mathbb{Z}. \tag{5.2}$$

The constants J_x, J_y, J_z are the exchange integrals and D is the on-site anisotropy constant.
The equations of motions for the spin components in the one-dimensional spin chain with nearest-neighbour interactions are the well-known Landau-Lifshitz equations

$$\begin{pmatrix} \dot{x}_n \\ \dot{y}_n \\ \dot{z}_n \end{pmatrix} = -\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} \times \left(\frac{\partial E}{\partial x_n}, \frac{\partial E}{\partial y_n}, \frac{\partial E}{\partial z_n}\right)^t \tag{5.3}$$

which reads

$$\begin{aligned} \dot{x}_n &= \tfrac{1}{2}[J_y z_n(y_{n-1} + y_{n+1}) - J_z y_n(z_{n-1} + z_{n+1})] - 2D y_n z_n, \\ \dot{y}_n &= \tfrac{1}{2}[J_z x_n(z_{n-1} + z_{n+1}) - J_x z_n(x_{n-1} + x_{n+1})] + 2D x_n z_n, \\ \dot{z}_n &= \tfrac{1}{2}[J_x y_n(x_{n-1} + x_{n+1}) - J_y x_n(y_{n-1} + y_{n+1})] \ , \ n \in \mathbb{Z}. \end{aligned} \tag{5.4}$$

In this paper, we consider ferromagnetic spin chains: $J_x > 0$, $J_y > 0$ and $J_z > 0$ (the case of antiferromagnets $J_x, J_y, J_z < 0$ is not treated here).
We study the dynamics in the neigbourhood of two different equilibria. On the one hand, we consider "easy axis" ferromagnets: this is a spin lattice such that the ground state (static solution of (5.4) which minimizes E) corresponds to $z_n = \pm 1, x_n = y_n = 0$. This can be achieved by introducing either a strong exchange anisotropy $J_x, J_y << J_z$ or an on-site anisotropy term $D > 0$ with $\max(J_x, J_y, J_z) << D$.
On the other hand, we study the chain dynamic in an "easy plane" ferromagnet where the ground state corresponds to all spins lying in the XY plane (easy plane) and parallels. For example we can choose $x_n = 1, y_n = z_n = 0$. In the sequel, we take $J_x = J_y = J_z$ and $D < 0$ such that $\max(J_x, J_y, J_z) << |D|$. In this case, the isotropy $J_x = J_y$ implies that this ground state is degenerated.
We will keep the term "easy axis" anisotropy when $D > 0$ and "easy plane" anisotropy when $D < 0$ even if the hypothesis $\max(J_x, J_y, J_z) << |D|$ is not satisfied.

5.3 "Out of plane" breathers in easy plane ferromagnets

In this section, we present the equations of motion in an easy plane ferromagnet for an isotropic exchange interaction with an "easy plane" type on-site anisotropy ($D < 0$) and carry out a spectral analysis around the ground state. Then we prove the existence of "out of plane" breathers with the anticontinuous limit method. In "out of plane" breathers, one or several spins are precessing around the "hard" axis (we assume it is the Z axis) while all the others lie in the "easy" plane (we assume here it is the XY plane).

5.3.1 Equations of motion and linear spectrum analysis

In the sequel, we will consider an easy plane ferromagnet where the ground state is given by $x_n = 1, y_n = z_n = 0$. Let us choose

$$J_x = J_y = J_z = 2J$$

and an on site anisotropy $D < 0$ such that $\max(J_x, J_y, J_z) << |D|$. Changing D into $-D$, the conserved magnetic energy (5.1) has now the form

$$E = -J \sum_n (x_{n+1}x_n + y_{n+1}y_n + z_{n+1}z_n) + D \sum_n z_n^2. \tag{5.5}$$

The equations of motion are given by the Landau-Lifshitz equations (see the previous section):

$$\begin{array}{l} \dot{x}_n = J[z_n(y_{n-1} + y_{n+1}) - y_n(z_{n-1} + z_{n+1})] + 2Dy_nz_n, \\ \dot{y}_n = J[x_n(z_{n-1} + z_{n+1}) - z_n(x_{n-1} + x_{n+1})] - 2Dx_nz_n, \\ \dot{z}_n = J[y_n(x_{n-1} + x_{n+1}) - x_n(y_{n-1} + y_{n+1})] \ , \ n \in \mathbb{Z}. \end{array} \tag{5.6}$$

It is also convenient to introduce the stereographic coordinates:

$$\xi_n = \frac{x_n + i\, y_n}{1 + z_n}.$$

The Landau-Lifschitz equations (5.6) now read

$$\begin{array}{rl} i\dot{\xi}_n = & J[\dfrac{\xi_{n-1} - \xi_n^2\xi_{n-1}^*}{1 + |\xi_{n-1}|^2} + \dfrac{\xi_{n+1} - \xi_n^2\xi_{n+1}^*}{1 + |\xi_{n+1}|^2} \\ & -2\xi_n(\dfrac{1 - |\xi_{n-1}|^2}{1 + |\xi_{n-1}|^2} + \dfrac{1 - |\xi_{n+1}|^2}{1 + |\xi_{n+1}|^2})] \\ & +2D\xi_n(\dfrac{1 - |\xi_n|^2}{1 + |\xi_n|^2}), \ n \in \mathbb{Z}. \end{array} \tag{5.7}$$

We study the ferromagnet case $J > 0$ (if $J < 0$ this an antiferromagnet). The ground state of the lattice corresponds to spins lying in the XY plane (easy plane). This ground state is degenerated and spins can be oriented arbitrary (but parallels) in the XY plane. Let us assume the ground state to be the following spin distribution

$$x_n = 1 \ , \ y_n = z_n = 0 \ , \ n \in \mathbb{Z}. \tag{5.8}$$

We linearize the equations (5.6) in the neighbourhood of the ground state. Then looking for travelling waves solutions $\delta z_n = \delta_z \sin(qn - \Omega(q)t)$, $\delta y_n = \delta_y \cos(qn - \Omega(q)t)$ and $\delta x_n = 0$, we find the following dispersion laws

$$\begin{array}{l} \Omega^2(q) = 4\left(J^2(1 - \cos(q))^2 + 2JD(1 - \cos(q))\right), \\ \Omega^2(0) = 0 \ , \ \Omega^2(\pi) = 8J(2J + D). \end{array} \tag{5.9}$$

Thus travelling waves of the linearized system have frequencies in an acoustic band similarly to Fermi-Pasta-Ulam lattices. The anticontinuous method requires in general a nonresonance condition with the phonon band. Since the spectrum is gapless, this condition is in general never satisfied. But the system still possesses periodic solutions with spins precessing around the hard axis (which is not the case in FPU lattices). Then for $J = 0$, we can construct breathers where one or several spins are precessing around the hard axis while the others are at rest. We shall see that we can avoid the eigenvalue 0 by choosing a suitable functional space and then continue breather solutions for $J \neq 0$.

5.3.2 Existence of "out of plane" breathers

In the sequel, we present a rigorous proof for the existence of "out of plane" breathers. Following MacKay and Aubry [13], we apply the method based on the anticontinuum limit of our system. It consists in decoupling the lattice sites and exciting only one or several sites while the other spins are at rest. Then, the persistence of the localized solution is shown for small values of the coupling. We focus here on breathers with one excited site. For $J = 0$, let us take the following lattice site configuration

$$\begin{array}{l} x_0(t) = a_0 \cos(\omega t)\ ,\ y_0(t) = a_0 \sin(\omega t)\ ,\ z(t) = z_0, \\ x_n = 1\ ,\ y_n = z_n = 0 \text{ if } n \neq 0, \end{array} \tag{5.10}$$

where $a_0 = \sqrt{1 - z_0^2}$ and $\omega = 2Dz_0$. It is easily seen that this distribution is a solution of (5.6) when $J = 0$. Now, we state the main theorem of this paper.

Theorem 11 *Suppose $z_0 \in (0,1)$. For $\omega \neq 0$ and $D > 0$, the periodic orbit of the equations of motions (5.6) at $J = 0$ given by the spin precession distribution (5.10) has a locally unique continuation as a periodic orbit of the equations (5.6) with the same frequency ω for a sufficiently small J.*

Proof. We are looking for periodic solutions with frequency $\omega = 2Dz_0$. The rescaling in time $\tau = \omega t$ yields

$$\omega\, x_n' = J[z_n(y_{n-1} + y_{n+1}) - y_n(z_{n-1} + z_{n+1})] + 2Dy_n z_n, \tag{5.11}$$

$$\omega\, y_n' = J[x_n(z_{n-1} + z_{n+1}) - z_n(x_{n-1} + x_{n+1})] - 2Dx_n z_n, \tag{5.12}$$

$$\omega\, z_n' = J[y_n(x_{n-1} + x_{n+1}) - x_n(y_{n-1} + y_{n+1})], \tag{5.13}$$

where $x' = \frac{dx}{d\tau}$ and x_n, y_n, z_n are now 2π-periodic functions. Let us introduce the following functional spaces

$$X_e^n = \{\, y \in H^n(\mathbb{R}/2\pi\mathbb{Z}) \,/\, y \text{ even}\},$$

where H^n denotes the classical Sobolev space ($H^0(\mathbb{R}/2\pi\mathbb{Z}) = L^2(\mathbb{R}/2\pi\mathbb{Z})$) and

$$X_o^n = \{\, y \in H^n(\mathbb{R}/2\pi\mathbb{Z}) \,/\, y \text{ odd}\}.$$

Let us assume that $x_n(-t) = x_n(t)$, $y_n(-t) = -y_n(t)$ and $z_n(-t) = z_n(t)$. We shall see later that the choice of the odd parity for y_n is important. We are looking for $x_n, z_n \in X_e^1$ and $y_n \in X_o^1$ which are close to (5.10). Thus for $n \neq 0$, we want $x_n(t) = \sqrt{1 - y_n(t)^2 - z_n(t)^2}$. The set of equations (5.6) is redundant and we just keep equations (5.12) and (5.13) to describe the motions of the n-th spin. For $n = 0$, we choose $z_0 = \sqrt{1 - x_0^2 - y_0^2}$ and we keep equations (5.11) and (5.12) to describe the motion of the central spin. We thus are looking for solutions satisfying $\sup_{t\in(0,2\pi)} |y_n|^2 + |z_n|^2 < 1$ for $n \neq 0$ and $\sup_{t\in(0,2\pi)} |x_0|^2 + |y_0|^2 < 1$. Now following the approach of Flach et al. [6] for the proof of breather in easy axis ferromagnet, let us introduce the mapping $F : \mathcal{O} \to l^2(X_o^0) \times l^2(X_e^0)$ defined by

$$F(z, J) = w$$

with

$$z = \{(x_0, y_0), \{z_n, y_n\}_{n\in\mathbb{Z}^*}\}, \; w = \{a_n, b_n\}_{n\in\mathbb{Z}}$$

where

$$\begin{aligned} a_n &= \omega z_n' - J[y_n(x_{n-1} + x_{n+1}) - x_n(y_{n-1} + y_{n+1})], \\ b_n &= \omega y_n' + 2Dx_n z_n - J[x_n(z_{n-1} + z_{n+1}) - z_n(x_{n-1} + x_{n+1})], \end{aligned} \tag{5.14}$$

for $n \neq 0$ and

$$\begin{aligned} a_0 &= \omega x_0' - 2Dy_0 z_0 - J[z_0(y_{-1} + y_1) - y_0(z_{-1} + z_{+1})], \\ b_0 &= \omega y_0' + 2Dx_0 z_0 - J[x_0(z_{-1} + z_{+1}) - z_0(x_{-1} + x_{+1})], \end{aligned} \tag{5.15}$$

with the notations $x_n = \sqrt{1 - y_n^2 - z_n^2}$ (for $n \neq 0$) and $z_0 = \sqrt{1 - x_0^2 - y_0^2}$. The set $\mathcal{O}$ is an open subset of $l^2(X_e^1) \times l^2(X_o^1)$ defined by

$$\mathcal{O} = \{(u_n, v_n)_{n\in\mathbb{N}} \in l^2(X_e^1) \times l^2(X_o^1) \,/\, \sup_{t\in(0,2\pi)} |u_n(t)|^2 + |v_n(t)|^2 < 1\}.$$

We can see that F is well defined on $\mathcal{O}$. The symmetric solutions of equations of motions (5.6) with the "out of plane" geometry are in one-to-one correspondence with zeros of F and we can see that for $J = 0$, the spin-precession configuration (5.10) is a zero of F which lies in $\mathcal{O}$. Using the implicit function theorem, we prove this solution to have a locally unique continuation $z(J)$ for sufficiently small J provided its derivative with respect to z is invertible with a bounded inverse at $J = 0$. The derivative $\mathcal{D}F(z(0), 0) : l^2(X_e^1) \times l^2(X_o^1) \to l^2(X_o^0) \times l^2(X_e^0)$ is given by:

$$\begin{aligned} \mathcal{D}F(z(0), 0)(u, v)_n^1 &= \omega u_n', \\ \mathcal{D}F(z(0), 0)(u, v)_n^2 &= \omega v_n' + 2Du_n, \end{aligned} \tag{5.16}$$

when $n \neq 0$ and

$$\begin{aligned} \mathcal{D}F(z(0), 0)(u, v)_0^1 &= \omega(u_0' - [(1 + \alpha s^2(t))v_0 + \alpha s(t)c(t)u_0]), \\ \mathcal{D}F(z(0), 0)(u, v)_0^2 &= \omega(v_0' + [(1 + \alpha c^2(t))u_0 + \alpha s(t)c(t)v_0]), \end{aligned} \tag{5.17}$$

with $c(t) = \cos(t)$ and $s(t) = \sin(t)$. Now let us choose $(f_n, g_n)_{n\in\mathbb{Z}} \in l^2(X_o^0) \times l^2(X_e^0)$. We want to prove that there exists a unique $(u_n, v_n)_{n\in\mathbb{Z}} \in l^2(X_e^1) \times l^2(X_o^1)$ such that

$$DF(z(0), 0)(u, v)_n = (f_n, g_n)^t.$$

Thus when $n \neq 0$, we have to solve the system

$$\begin{aligned} &\omega u_n' = f_n, \\ &\omega v_n' + 2Du_n = g_n. \end{aligned} \tag{5.18}$$

It is easily seen that in general, v_n will be defined up to a constant and thus it is non unique and $\mathcal{D}F(z(0), 0)$ is non invertible. The fact that $v_n(-t) = -v_n(t)$ imposes that the constant is fixed to 0 and then system (5.18) will have a unique solution. We expand u and v into Fourier series taking into account their parity:

$$u_n(t) = \sum_{k=0}^{+\infty} u_n^k c(kt)\ ,\ v_n(t) = \sum_{k=1}^{+\infty} v_n^k s(kt),$$

$$f_n(t) = \sum_{k=1}^{+\infty} f_n^k s(kt)\ ,\ v_n(t) = \sum_{k=0}^{+\infty} g_n^k c(kt).$$

This yields the system of equations

$$\begin{aligned} &-k\omega u_n^k = f_n^k, \\ &k\omega v_n^k + 2Du_n^k = g_n^k,\ k \geq 0. \end{aligned} \tag{5.19}$$

The system (5.19) is invertible if $\omega \neq 0$ and $D \neq 0$. Moreover, it is easily seen that $u_n \in X_e^1$ and $v_n \in X_o^1$ and we have the estimate

$$\|u_n\|_{H^1} + \|v_n\|_{H^1} \leq C(\omega, D)(\|f_n\|_{L^2} + \|v_n\|_{L^2}), \tag{5.20}$$

where C is a continuous function on $(0, +\infty)^2$. Now let us consider the case when $n = 0$. We are going to use the stereographic coordinates and linearize the equations around $\xi_0(t) = \Psi_0 \exp(i\omega t)$. Let us consider the perturbations $\xi_n(t) = (\Psi_0 + w(t)) \exp(i\omega t)$ which is not a restriction. It does not change the functional spaces since $T : X_e + i\, X_o \to X_e + i\, X_o$, defined by $T(w)(t) = \exp(i\omega t) w(t)$ is an isometry. The derivative $\mathcal{D}F(z(0), 0)$ for $n = 0$ reads

$$\mathcal{D}F(z(0), 0)(u, v)_0 = \omega(iw' - \frac{2D|\Psi_0|^2}{(1 + |\Psi_0|^2)^2}(w + w^*)), \tag{5.21}$$

where $w = u + i\, v$. Let us denote $f = \frac{f_0}{\omega}$ and $g = \frac{g_0}{\omega}$. We have to find a unique solution $(x, y) = (u_0, v_0) \in X_e^1 \times X_o^1$ satisfying the system

$$\begin{aligned} &x' = f, \\ &y' + \tfrac{4D|\Psi_0|^2}{(1+|\Psi_0|^2)^2} x = -g. \end{aligned} \tag{5.22}$$

We expand x and y into Fourier series and under the condition that

$$\frac{4D|\Psi_0|^2}{(1+|\Psi_0|^2)^2} \neq 0$$

and $\omega \neq 0$ (which is equivalent to $z_0 \neq 0, 1$), we can find a unique solution $(x, y) \in X_e^1 \times X_o^1$ to system (5.22). Moreover we have the estimate

$$\|x\|_H^1 + \|y\|_H^1 \leq K(\omega, z_0)(\|f\|_{L^2} + \|g\|_{L^2}), \tag{5.23}$$

where K is a continuous function of ω, z_0 with $\omega > 0$ and $z_0 \in (0,1)$. Thus we have proved that $\mathcal{D}F(z(0), 0)$ is invertible and estimates (5.23) and (5.20) show that this inverse is bounded. Then the implicit function theorem does apply and $z(0)$ has a locally unique continuation as a periodic orbit of the equations (5.6), $z(J) \in \mathcal{O}$. Since $z(J) \in \mathcal{O}$, the assumptions $\sup_{t\in(0,2\pi)} |y_n|^2 + |z_n|^2 < 1$ and $\sup_{t\in(0,2\pi)} |x_0|^2 + |y_0|^2 < 1$ are automatically satisfied. Moreover, since $(y_n, z_n) \in l^2(X_o^1) \times l^2(X_e^1)$, we have

$$\lim_{n\to\pm\infty} \|y_n\|_{H^1} = \lim_{n\to\pm\infty} \|z_n\|_{H^1} = 0.$$

Then the periodic solution $z(J)$ is spatially localized and far from the center, spins precess around an axis lying close to the easy plane and this completes the proof of the theorem.□

Since the continuation occurs in l^2 spaces, it has weak localization built in. We can improve this statement to exponential spatial localization (that is there exist $K, \lambda > 1$ such that $\sup_{t\in(0,2\pi)}(|y_n(t)| + |z_n(t)|) \leq K\lambda^{-|n|}$) by applying some results by Baesens and MacKay [3]. Flach et al. [6] have computed an optimal $\lambda = \lambda_{opt}$ by linearizing the equations of motion (5.6) around the ground state (5.8) in the breather tails and

$$\lambda_{opt} = 1 + \frac{D}{J} + ((1+\frac{D}{J})^2 - 1)^2. \tag{5.24}$$

This is an easy computation to extend this result to lattices in higher dimensions and multibreathers with several spins precessing around the hard axis Z. We can see that for $\omega > 2D$ and $J = 0$, we have only static solutions of out of plane type. For $J \neq 0$, this threshold still exists [11]. We present here an existence proof of such static solutions.

Proposition 11 *For $D > 0$ and $J \neq 0$ but sufficiently small, there exist static solutions (x_n, y_n, z_n) such that*

$$\lim_{n\to\pm\infty} y_n = \lim_{n\to\infty} z_n = 0,$$

and $z_0 = 1 + O(J)$.

Proof. We suppose $y_n = 0$ and $x_n = \cos(\theta_n)$, $z_n = \sin(\theta_n)$. They are static solutions of (5.6) iff

$$P(\{\theta_n\}, J)_n = D\sin(2\theta_n) - J[\sin(\theta_{n+1} - \theta_n) + \sin(\theta_{n-1} - \theta_n)] = 0 \quad (5.25)$$

Clearly, $P(., J)$ map $l^2(\mathbb{Z})$ into itself and is C^1. We will use the implicit function theorem. Let us take $\theta_n^0 = 0$ if $n \neq 0$ and $\theta_0^0 = \frac{\pi}{2}$: this is a trivial out of plane static solution for $J = 0$. Moreover $\mathcal{D}P(\theta^0, 0)(h)_n = 2Dh_n$ if $n \neq 0$ and $\mathcal{D}P(\theta^0, 0)(h)_0 = -2Dh_0$. Thus $\mathcal{D}P(\theta^0, 0)$ is invertible provided $D \neq 0$. The implicit function theorem does apply and this completes the proof.□

Once again, we can prove that these solutions are exponentially localized. Now let us consider the case of larger J. The existence of a "out of plane" static solution is equivalent to the existence of a solution of (5.25) homoclinic to 0. When $r = \frac{J}{D} \to \infty$, we are looking for small amplitude homoclinic solutions which is equivalent to the existence of a small amplitude fixed point. Since the only fixed points solutions of (5.25) are $\theta_n = k\frac{\pi}{2}$, $k \in \mathbb{Z}$, we do not find small amplitude homoclinic solutions to 0. Thus there exists a critical ratio r_c such that for $r > r_c$ there are no "out of plane" static solutions. This is coherent with the threshold observed by Zolotaryuk et al. [11] for the existence of out of plane breathers (if we consider that a static solution is a particular breather).

5.4 Existence of small amplitude breather in easy axis ferromagnets

In this section, we give a rigorous proof for the existence of small amplitude breathers in easy axis ferromagnet close to the ground state $z_n = 1$. The underlying method is a centre manifold reduction for quasilinear mappings [8] recently applied to the existence of small amplitude breathers in FPU chains (monoatomic [9], diatomic [10]). The set of equations (5.4) can be written as a recurrence relation in a loop space and we apply the set of theorems proved in [8]. We first make a spectral study of the linearized operator and compute the centre subspace (the generalized eigenspace corresponding to the eigenvalues lying on the unit circle). Then we apply the centre manifold reduction for quasilinear mappings and calculate the Taylor expansion of the centre manifold. Then projecting the equations of motions on the centre subspace, we get the *reduced form* of the equations which is a mapping in finite dimension. Then we perform a change of variable which leads to the simpler *normal form*. This only keeps the essential terms. We then make the proof for the existence of homoclinic orbits to 0 which correspond to discrete breather solutions of (5.4). Using a reversibility property of the whole system, we can prove the persistence of these homoclinic solutions.

5.4.1 Formulation of the mathematical problem

We recall that the equations of motion for the spin components in the one-dimensional spin chain with nearest neighbor interactions are the Landau-Lifshitz equations

$$\dot{x}_n = \frac{1}{2}(J_y z_n(y_{n+1} + y_{n-1}) - J_z y_n(z_{n+1} + z_{n-1})) - 2D y_n z_n, \tag{5.26}$$

$$\dot{y}_n = \frac{1}{2}(J_z x_n(z_{n+1} + z_{n-1}) - J_x z_n(x_{n+1} + x_{n-1})) + 2D x_n z_n, \tag{5.27}$$

$$\dot{z}_n = \frac{1}{2}(J_x y_n(x_{n+1} + x_{n-1}) - J_y x_n(y_{n+1} + y_{n-1})). \tag{5.28}$$

We wish to study analytically the existence of "small amplitude" breathers close to the ground state $z_n = 1$ in a large range of values for J_x, J_y, D and extend the results obtained via the anticontinuous limit ($\frac{\max(J_x,J_y)}{J_z} \to 0$). Thus we write $z_n = \sqrt{1 - x_n^2 - y_n^2}$ and the set of equations (5.26), (5.27), (5.28) is redundant: we only keep (5.26), (5.27). We look for solutions (x_n, y_n) of (5.26), (5.27) in the form $x_n(t) = \overline{x_n}(J_z\omega t)$ and $y_n(t) = \overline{y_n}(J_z\omega t)$ where $\overline{x_n}, \overline{y_n}$ is 2π-periodic and the normalized frequency ω is treated as a bifurcation parameter. Dropping the lines, the system (5.26), (5.27) now reads

$$\begin{aligned} \omega\dot{x}_n &= \tfrac{1}{2}(\beta z_n(y_{n+1} + y_{n-1}) - y_n(z_{n+1} + z_{n-1})) - 2d y_n z_n, \\ \omega\dot{y}_n &= \tfrac{1}{2}(x_n(z_{n+1} + z_{n-1}) - \alpha z_n(x_{n+1} + x_{n-1})) + 2d x_n z_n, \end{aligned} \tag{5.29}$$

where $\alpha = \frac{J_x}{J_z}$, $\beta = \frac{J_y}{J_z}$ and $d = \frac{D}{J_z}$. Now, following James approach [8],[9], let us write the system (5.29) in the form of a recurrence relation in a loop space

$$Y_{n+1} = F(Y_n, \omega, \alpha, \beta, d), \tag{5.30}$$

where $Y_n = (x_{n-1}, x_n, y_{n-1}, y_n)$ is a 2π-periodic function of t. Discrete breathers solutions of (5.29) correspond to solutions of (5.30) homoclinic to 0. We now define appropriate function spaces on which F is acting. We look for

$$(x_{n-1}, x_n, y_{n-1}, y_n) \in \mathbb{D}$$

where

$$\begin{aligned} \mathbb{D} = \{&(x_1, x_2, x_3, x_4) \in (\mathbb{H}^1)^4, x_1, x_2 \; even, x_3, x_4 \; odd, \\ &x_i(t+\pi) = -x_i(t)\;, i = 1..4\} \end{aligned}$$

and $\mathbb{H}^n$ denotes the classical Sobolev space $H^n(\mathbb{R}/2\pi\mathbb{Z})$. The recurrence relation (5.30) holds in

$$\mathbb{X} = \{(x_1, x_2, x_3, x_4) \in (\mathbb{H}^0)^4, x_1, x_2 \; even, x_3, x_4 \; odd, \\ x_i(t+\pi) = -x_i(t) \; , i = 1..4\}.$$

The operator $F : \mathbb{D} \to \mathbb{X}$ is smooth in a neighbourhhood $\mathcal{U}$ of $Y = 0$ in $\mathbb{D}$. Note that the fixed point $Y = 0$ corresponds to the lattice at rest. We now examine the symmetry properties of equations (5.30). On the one hand, equation (5.30) is invariant under the symmetry $TY = Y(.+\pi)$ i.e. F commutes with T. Moreover, if (x_n, y_n) is solution of (5.29) then (x_{-n}, y_{-n}) is also a solution. This implies that F is reversible with respect to the symmetry $R(x_1, x_2, x_3, x_4) = (x_2, x_1, x_4, x_3)$ i.e. if $Y \in \mathcal{U}$ and $F(RY) \in \mathcal{U}$, one has $(F \circ R)^2 Y = Y$. We can see also that if (x_n, y_n) is solution of (5.29) then $(-x_n, -y_n)$ is also a solution: thus F commutes with $-Id$. We now study the spectrum of the linearized operator at $Y = 0$.

5.4.2 Spectral properties of the linearized operator

The linearized operator $L_\omega = DF_\omega(0)$ reads

$$L_\omega \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{2}{\alpha}((1+2d)x_2 - \omega \frac{d}{dt} x_4) - x_1 \\ x_4 \\ \frac{2}{\beta}(\omega \frac{d}{dt} x_2 + (1+2d)x_4) - x_3 \end{pmatrix}. \tag{5.31}$$

The operator $L_\omega : \mathbb{D} \subset \mathbb{X} \to \mathbb{X}$ is unbounded in $\mathbb{X}$ (of domain $\mathbb{D}$) and closed. We now examine the spectral properties of L_ω.

Lemma 22 *For $\alpha > 0$, $\beta > 0$, $d \geq 0$ and $\omega > 0$, the spectrum of L_ω is unbounded, discrete and can be written $\sigma(L_\omega) = \{0\} \cup \Sigma_\omega$, where 0 belongs to the essential spectrum and Σ_ω consists in non-zero eigenvalues. The set Σ_ω is contained in the union of the real axis and the unit circle, and invariant under $z \to z^{-1}$ and $z \to \overline{z}$. The eigenvalues form sequences $(u_k)_{k\geq 1}$, $(u_k^{-1})_{k\geq 1}$, $(v_k)_{k\geq 1}$ and $(v_k^{-1})_{k\geq 1}$ (with $|u_k| \geq 1$, $|v_k| \geq 1$, $\Im u_k \geq 0$ and $\Im v_k \geq 0$) determined by the equations*

$$\begin{aligned} \frac{1}{2}(u_k + u_k^{-1}) &= \frac{1}{2\alpha\beta}[(1+2d)(\alpha+\beta) + \sqrt{(\alpha-\beta)^2(1+2d)^2 + 4\alpha\beta(p_k\omega)^2}], \\ \frac{1}{2}(v_k + v_k^{-1}) &= \frac{1}{2\alpha\beta}[(1+2d)(\alpha+\beta) - \sqrt{(\alpha-\beta)^2(1+2d)^2 + 4\alpha\beta(p_k\omega)^2}], \end{aligned} \tag{5.32}$$

where $p_k = 2k - 1$.

We now discuss the variation of the spectrum of L_ω as we vary the parameter $\omega \in (0, +\infty)$. Let us first consider $(u_k, u_k^{-1})_{k\geq 1}$. The eigenvalues $(u_k, u_k^{-1})_{k\geq 1}$ belong to the positive real axis when $p_k^2\omega^2 \geq \omega_1^2$ (with $\omega_1^2 = (\alpha - (1 + 2d))(\beta - (1 + 2d)))$ and collide at $z = 1$ when $p_k^2\omega^2 = \omega_1^2$. This situation occurs only if $\min(\alpha, \beta) > 1 + 2d$ otherwise the eigenvalues remain real and far from $z = 1$. Note that all these eigenvalues belong to the positive real axis (and lie outside the unit circle) when $\omega^2 > \omega_1^2$. For $p_k^2\omega^2 \leq \omega_1^2$, the eigenvalues u_k, u_k^{-1} belong to the unit circle. We now consider $(v_k, v_k^{-1})_{k\geq 1}$. The eigenvalues $(v_k, v_k^{-1})_{k\geq 1}$ belong to the negative real axis when $p_k^2\omega^2 \geq \omega_2^2$ (with $\omega_2^2 = (\alpha+(1+2d))(\beta+(1+2d)))$ and collide at $z = -1$ when $p_k^2\omega^2 = \omega_2^2$. All these eigenvalues belong to the negative real axis when $\omega^2 > \omega_2^2$. For $\omega_1^2 \leq p_k^2\omega^2 \leq \omega_2^2$, the eigenvalues v_k, v_k^{-1} belong to the unit circle and collide at $z = 1$ for $\omega^2 = \omega_1^2$. This last situation only occurs if $\max(\alpha, \beta) < 1 + 2d$. In this case for $p_k^2\omega^2 \leq \omega_1^2$ the eigenvalues v_k, v_k^{-1} lies on the positive real axis otherwise they remain on the unit circle. The set of possible bifurcation scenarios is broad. With the aim of finding discrete breathers as homclinic orbits to $Y = 0$, we shall restrict ourselves to the simplest situation when (u_1, u_1^{-1}) or (v_1, v_1^{-1}) are the only eigenvalues close to the unit circle and the fixed point $Y = 0$ is hyperbolic. This corresponds to two regions. The first one is situated in the neighbourhood of $\omega = \omega_2$ at the right hand side. The other one is situated in the neighbourhood of $\omega^2 = \omega_1^2$ on the left hand side when the parameters α, β, d satisfy the relations: $\max(\alpha, \beta) < 1 + 2d$ and $\omega_1^2 > \frac{\omega_2^2}{9}$. In the following lemma, we compute the centre space $\mathbb{X}_c$ i.e. the generalized eigenspace corresponding to the eigenvalues lying on the unit circle.

Lemma 23 *If $\omega = \omega_2$, $z = -1$ is a double non-semi-simple eigenvalue of L_ω. The centre space $\mathbb{X}_c$ is spanned by the vectors $U_2 = (\cos t, 0, \gamma \sin t, 0)$ and $V_2 = (0, \cos t, 0, \gamma \sin t)$ with $\gamma = \sqrt{\frac{\alpha+1+2d}{\beta+1+2d}}$. If $\omega = \omega_1$ and the parameters α, β, d satisfy the relations: $\max(\alpha, \beta) < 1 + 2d$, $\omega_1^2 > \frac{\omega_2^2}{9}$, $z = 1$ is a double non-semi-simple eigenvalue of L_ω. The centre space $\mathbb{X}_c$ is spanned by the vectors $U_1 = (\cos t, 0, \delta \sin t, 0)$ and $V_1 = (0, \cos t, 0, \delta \sin t)$ with $\delta = \sqrt{\frac{\alpha-1-2d}{\beta-1-2d}}$.*

Let us set $\omega^2 = \omega_c^2 + \mu$ where $\omega_c = \omega_1$ or $\omega_c = \omega_2$ and $\mu \approx 0$ is a bifurcation parameter. Problem (5.30) can be written $Y_{n+1} = L_{\omega_c} Y_n + \mathcal{N}(Y_n, \mu)$ where $\mathcal{N}(0, \mu) = 0$ and $D_Y\mathcal{N}(0, 0) = 0$.

5.4.3 Centre manifold and normal form computation

Using the centre manifold theorem for quasilinear mapping, one obtains for $\mu \approx 0$ the existence of a smooth local manifold $\mathcal{M}_\mu$ invariant under F_ω and the symmetries T, R and $-Id$ (see [8] for more details). This manifold can be written $\mathcal{M}_\mu = \{Y \in \mathbb{D}/Y = Y_c + \Psi(Y_c, \mu), Y_c \in \mathbb{X} \cap \Omega\}$ where $\Psi(., \mu) : \mathbb{X} \to (I - \Pi_c))\mathbb{D}$ is a smooth map satisfying $\Psi(Y, \mu) = O(\|Y\|^2 + |\mu|\|Y\|)$ and Ω is a small neighbourhood of 0 in $\mathbb{D}$. For $\mu \approx 0$, the centre manifold $\mathcal{M}_\mu$ contains all solutions Y_n staying in Ω for all $n \in \mathbb{Z}$. Their component $Y_n^c = \Pi_c Y_n$ is given by the two dimensional mapping

$$Y_{n+1}^c = f(Y_n^c, \mu), \tag{5.33}$$

where $f(Y^c, \mu) = L_{\omega_c} Y^c + \Pi_c \mathcal{N}(Y^c + \Psi(Y_c, \mu), \mu)$. The map $f : \mathbb{X}_c \to \mathbb{X}_c$ is smooth, reversible under R and commutes with T and $-Id$. Now we compute the Taylor expansion of Ψ in the case when $\omega_c = \omega_2$. Setting $Y_c = aU_2(t) + bV_2(t)$, we identify $\mathbb{X}_c$ to $\mathbb{R}^2$. The symmetry properties of Ψ implies that $\Psi(-a, -b, \mu) = -\Psi(a, b, \mu)$. Consequently the Taylor expansion of Ψ has the form

$$\Psi(a, b, \mu) = P_1(t)a\mu + P_2(t)b\mu + Q(t)(a, b) + h.o.t.,$$

where $P_i \in (I - \Pi_c)\mathbb{X}$ and $Q(t)$ is a cubic homogeneous polynom in a and b. The corrections introduced by the centre manifold are easily seen to be $o(\mu\|Y_c\|)$ and $o(\|Y_c\|^3)$. Since we are only interested in the terms of order $O(\mu\|Y_c\|)$ and $O(\|Y_c\|^3)$ in the Taylor expansion of f, we do not compute P_i and Q. Thus (5.33) reads

$$Y_{n+1}^c = L_{\omega_c} Y_n^c + \Pi_c \mathcal{N}(Y_n^c, \mu) + o(\mu\|Y_c\| + \|Y_c\|^3). \tag{5.34}$$

The same argument holds for the bifurcation $\omega_c = \omega_1$ and in both case, the Taylor expansion of the reduced mapping (5.33) is easily obtained.We first consider $\omega_c = \omega_1$. Let us denote $Y_n^c = a_n U_1(t) + b_n V_1(t)$, system (5.34) is equivalent to:

$$\begin{aligned} &a_{n+1} = b_n \\ &b_{n+1} - 2b_n + b_{n-1} = -\tfrac{2\delta}{\alpha} b_n + Q_1(b_n, b_{n-1}) + h.o.t, \end{aligned} \tag{5.35}$$

where Q_1 is a cubic homogeneous polynom. After a suitable change of variable $u_n = b_n + \beta b_n^3$ (which preserves the symmerties of (5.33)), the system (5.35) can be written under the normal form:

$$u_{n+1} - 2u_n + u_{n-1} = g_\mu^1(u_n, u_{n-1}), \tag{5.36}$$

where

$$\begin{aligned} g_\mu^1(u,v) &= -\frac{2\delta}{\alpha}\mu u + \frac{\delta^2+3}{8\alpha}(\alpha-(1+2d))u^3 \\ &\quad + O((|u|+|v|)((|u|+|v|)^2+|\mu|^2)^2). \end{aligned}$$

The same holds for $\omega_c = \omega_2$ and the normal form of (5.33) can be written (with the notation $Y_n^c = a_n U_2(t) + b_n V_2(t)$):

$$u_{n+1} + 2u_n + u_{n-1} = g_\mu^2(u_n, u_{n-1}), \tag{5.37}$$

where

$$\begin{aligned} g_\mu^2(u,v) &= -\frac{2\gamma}{\alpha}\mu u - \frac{\gamma^2+3}{8\alpha}(\alpha+(1+2d))u^3 \\ &\quad + O((|u|+|v|)((|u|+|v|)^2+|\mu|^2)^2). \end{aligned}$$

As a conclusion, the mapping (5.33) can be locally transformed into the simpler second scalar recurrence relation (5.36) and (5.37) via a simple change of variable $u_n = b_n + \beta b_n^3$ for a suitable β.

A particular case: $J_x = J_y = J$

Let us use the steregographic coordinates:

$$\xi_n = \frac{x_n + i\,y_n}{1+z_n}.$$

The Landau-Lifschitz equations (5.6) are now equivalent to

$$\begin{aligned} 2i\xi_n' = J[&\frac{\xi_{n-1} - \xi_n^2\xi_{n-1}^*}{1+|\xi_{n-1}|^2} + \frac{\xi_{n+1} - \xi_n^2\xi_{n+1}^*}{1+|\xi_{n+1}|^2} \\ &-J_z\xi_n(\frac{1-|\xi_{n-1}|^2}{1+|\xi_{n-1}|^2} + \frac{1-|\xi_{n+1}|^2}{1+|\xi_{n+1}|^2})] \\ &-4D\xi_n(\frac{1-|\xi_n|^2}{1+|\xi_n|^2}),\ n\in\mathbb{Z}. \end{aligned} \tag{5.38}$$

As in discrete nonlinear Schrodinger equations, we seach breather solutions under the form $\xi_n = \exp(i\omega t)\psi_n$ with $\psi_n \in \mathbb{R}$. The system (5.38) reads:

$$\begin{aligned} -2\omega\psi_n = J[&\frac{\psi_{n-1} - \psi_n^2\psi_{n-1}}{1+|\psi_{n-1}|^2} + \frac{\psi_{n+1} - \psi_n^2\psi_{n+1}}{1+|\psi_{n+1}|^2} \\ &-J_z\psi_n(\frac{1-|\psi_{n-1}|^2}{1+|\psi_{n-1}|^2} + \frac{1-|\psi_{n+1}|^2}{1+|\psi_{n+1}|^2})] \\ &-4D\psi_n(\frac{1-|\psi_n|^2}{1+|\psi_n|^2}),\ n\in\mathbb{Z}. \end{aligned} \tag{5.39}$$

Breathers solutions of (5.38) then correspond to $\psi_n \in \mathbb{R}$ solution of the recurrence relation (5.39) with $\lim_{n\to\pm\infty}\psi_n = 0$. In this case, the centre manifold is flat and (5.39) is the reduced form of (5.38).

5.4.4 Homoclinic solutions of the reduced mapping

We now consider the recurrence relation (5.36) for $\mu \leq 0$ and study the existence of small amplitude homoclinic orbits to $u_n = 0$ (these orbits correspond to small amplitude breather solutions of (5.29). Recall that

$$u_{n+1} - 2u_n + u_{n-1} = -\frac{2\delta}{\alpha}\mu u_n + \frac{\delta^2+3}{8\alpha}(\alpha - (1+2d))u_n^3 + h.o.t. \qquad (5.40)$$

Moe precisely, the normal form can be written

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix}, \qquad (5.41)$$

with

$$G_\mu \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} + g_\mu(A_n, B_n) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad (5.42)$$

and

$$\begin{aligned} g_\mu(A_n, B_n) = &\frac{\delta}{\alpha}\Big(-2\mu A_n - \frac{\delta^2+3}{8\delta}\big((1+2d)-\alpha\big)A_n^3\Big) + \\ &+ o(\|(A_n,B_n)\|^3 + |\mu|\|(A_n,B_n)\|), \end{aligned} \qquad (5.43)$$

The proof of the existence of such homoclinic solutions for the full system can be found e.g. in [8] section 6.2.3. This result heavily relies on the fact that (5.40) is reversible (due to the reversibility of f). One also uses the fact that G_μ is almost conjuged (up to high order terms) to the integrable vector field

$$v'' = \frac{\delta}{\alpha}\Big(-2\mu v - \frac{\delta^2+3}{8\delta}\big((1+2d)-\alpha\big)v^3\Big).$$

The fixed point $(A,B) = 0$ is hyperbolic if $\mu < 0$. Moreover, since $-\mu(\alpha - (1+2d)) < 0$, (5.42) has two symetric fixed points $(\pm A^*, 0)$ which are elliptic. We have $-\mu(\alpha-(1+2d)) < 0$: the local stable manifold $W^s(0)$ and unstable manifold $W^u(0)$ intersect and (5.42) has reversible homoclinic solution to $(A,B) = 0$ [8].

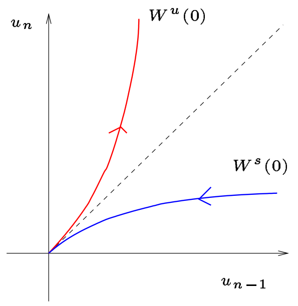
We now consider the recurrence relation (5.37) for $\mu \geq 0$. The equation of the normal form is

$$u_{n+1} + 2u_n + u_{n-1} = -\frac{2\gamma}{\alpha}\mu u_n - \frac{\gamma^2+3}{8\alpha}(\alpha + (1+2d))u_n^3 + h.o.t. \qquad (5.44)$$

We introduce the variable $U_n = (-1^n)u_n$. This yields

$$U_{n+1} - 2U_n + u_{n-1} = \frac{2\gamma}{\alpha}\mu U_n + \frac{\gamma^2+3}{8\alpha}(\alpha + (1+2d))U_n^3 + h.o.t. \qquad (5.45)$$

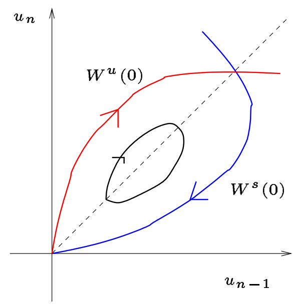
The analysis is similar to the previous case. Since $\mu(\alpha + (1 + 2d)) > 0$, the local stable and unstable manifolds $W^s(0)$ and $W^u(0)$ do not intersect in the neighbourhood of 0 and (5.29) admits no small amplitude breather solution in the corresponding parameter range.



$0 < \omega - \omega_2 << 1$

$\mu > 0,\ \alpha + 1 + 2d > 0$

No small amplitude breathers



$0 < \omega_1 - \omega << 1$

$\mu < 0,\ 1 + 2d - \alpha > 0$

Existence of small amplitude breathers

As a conclusion, we have proved the existence of small amplitude breathers in a parameter range where α, β, d satisfy the relations: $\max(\alpha, \beta) < 1 + 2d$ and $9\omega_1^2 > \omega_2^2$. This is equivalent to the condition:

$$0 < \beta < \frac{4(1+2d) - 5\alpha}{5(1+2d) - 4\alpha}(1 + 2d).$$

Thus we have proved the following proposition.

Theorem 12 *Let us note* $\alpha = \frac{J_x}{J_z}$, $\beta = \frac{J_y}{J_z}$ *and* $d = \frac{D}{J_z}$. *In the domain*

$$0 < \beta < \frac{4(1+2d) - 5\alpha}{5(1+2d) - 4\alpha}(1 + 2d)$$

and for $0 < \omega_c - \omega << 1$ *with* $\omega_c^2 = (\alpha - (1 + 2d))(\beta - (1 + 2d))$, *there exist small amplitude discrete breathers with a period* $\frac{2\pi}{J_z \omega}$ *solution of (5.29).*

The domain obtained contains a large neighbourhood of the point $(\alpha, \beta) = (0, 0)$ which corresponds to the anticontinuous limit and this analysis completes the anticontinuous approach carried out in [6]. We can also describe how spins precess around the easy axis: the projection in the XY plane of the trajectory is an ellipse (since $y_n = \delta b_n \sin t + h.o.t$ and $x_n = b_n \cos t + h.o.t.$).

5.5 Conclusion

We first have proved in this paper the existence of out of plane breathers and static solutions in easy plane ferromagnets using the anticontinuous limit. Then we have proved the existence of small amplitude breathers in a large range of parameter using centre manifold techniques. A natural question that arises is the linear stability of such breather solutions. In the easy axis case, it is proved using the Aubry's band theory that the one site breather is stable close to the anticontinuous limit. In the case of multibreathers, where several sites are excited, Archilla et al. [1] have recently proved stability and unstability results in Klein Gordon chains for two sites breathers and carried out the study for three sites breathers. The method is based on a degenerated Aubry's band theory. This approach should also applies in the easy axis case close to the anticontinuous limit to study the linear stability of multibreathers. In the case of easy plane ferromagnets, Zolotaryuk et al. [11] have investigated numerically the linear stability of "out of plane" breathers. The results obtained show a strong relation between the linear stability of these breathers and their geometry. For instance, the one site excited breather seems to be unstable (but the instability is not localized in [11]). On the other hand, the two sites breather with spins precessing in phase is stable and the two sites breather with spins precessing out of phase is unstable.
The equilibrium in the easy plane ferromagnet is degenerated and the mathematical framework introduced by MacKay et al. [14] and Aubry [2] do not apply. In the anticontinuous limit ($J_x = J_y = J_z = 2J = 0$), the eigenvalue 1 has a multiplicity greater than 2. The phase degeneracy and the time translation invariance fix the eigenvalue 1 with a multiplicity 2 when $J \neq 0$ but we have no control on the other eigenvalues. In particular, they can escape the unit circle. Thus the mathematical analysis of the linear stability of "out of plane" breathers is still an open question.

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Chapitre 6

Stabilité des breathers “out of plane” dans la limite anticontinue

Dans le chapitre précédent, on a démontré l'existence d'une famille de breathers ”out of plane” dans la limite $J \approx 0$ paramétrée par leur fréquence ω. Pour les breathers avec un site principal excité, il existe une fréquence critique ω_c, tel que pour $\omega \geq \omega_c$ le breather est une solution stationnaire des équations de Landau Lifshitz. On s'intéresse ici à la stabilité des breathers out of plane avec un site excité. On va montrer que ces breathers sont instables pour $\omega \approx \omega_c$ en étudiant la dynamique autour de la solution stationnaire. On démontre ici ce résultat pour un nombre fini mais arbitraire de spins. On ne peut appliquer les méthodes standards d'analyse de stabilité des breathers dans la limite anticontinue introduites par Aubry et MacKay [2]. En effet pour $J = 0$, le spectre du linéarisé possède un grand nombre de fois le multiplicateur de Floquet 1. L'invariance par translation de phase et l'autre pour l'amplitude implique la présence de la valeur propre 1 deux fois pour $J \neq 0$, les autres valeurs propres 1 pouvant bifurquer en des modes stables ou instables. On réalise donc une analyse numérique. La première analyse directe de la stabilité des breathers par le calcul direct des coefficients de Floquet dépendait de la taille du réseau étudié (voir [12] alors que les résultats standards sont indépendants de la taille du réseau. On commence par calculer numériquement la solution stationnaire de type “out of plane” puis on linéarise les équations de Landau Lifshitz au voisinage de cette solution.

6.1 Rappel des équations et calcul de la solution stationnaire

On va mener l'étude pour deux tailles de réseaux $2N = 50$ et $2N = 100$. On étudie le système de Landau Lifshtiz

$$\begin{aligned} x_n' &= J\left(z_n(y_{n-1} + y_{n+1}) - y_n(z_{n-1} + z_{n+1})\right) + 2Dy_n z_n, \\ y_n' &= J\left(x_n(z_{n-1} + z_{n+1}) - z_n(x_{n-1} + x_{n+1})\right) - 2Dx_n z_n, \\ z_n' &= J\left(y_n(x_{n-1} + x_{n+1}) - x_n(y_{n-1} + y_{n+1})\right), \ \forall n \in [1, N] \end{aligned} \tag{6.1}$$

avec les conditions aux bords libres $X_0 = X_1$ et $X_N = X_{2N+1}$ où $X_n = (x_n, y_n, z_n)$. On introduit également les coordonnées stéréographiques

$$\begin{aligned} i\xi_n' = J\Bigg(&\Big(\frac{\xi_{n-1} - \xi_n^2\overline{\xi}_{n-1}}{1 + |\xi_{n-1}|^2} + \frac{\xi_{n+1} - \xi_n^2\overline{\xi}_{n+1}}{1 + |\xi_{n+1}|^2}\Big) - \\ &- 2\xi_n\Big(\frac{1 - |\xi_{n-1}|^2}{1 + |\xi_{n-1}|^2} + \frac{1 - |\xi_{n+1}|^2}{1 + |\xi_{n+1}|^2}\Big)\Bigg) + 2D\xi_n\frac{1 - |\xi_n|^2}{1 + |\xi_n|^2}. \end{aligned} \tag{6.2}$$

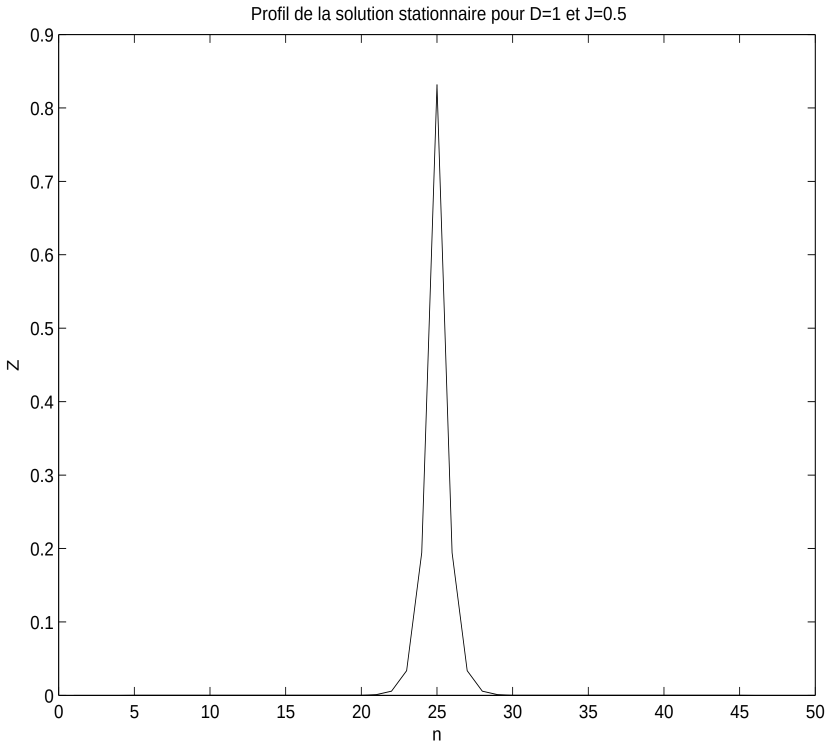
On cherche la solution stationnaire sous la forme $x_n^0 = \cos(\theta_n)$, $y_n^0 = 0$ et $z_n^0 = \sin(\theta_n)$. En coordonnées stéréographiques, elle s'écrit $\psi_n = \frac{x_n^0}{1+z_n^0}$. La suite θ_n vérifie le système d'équations

$$D\sin(2\theta_n) = J\left(\sin(\theta_{n+1} - \theta_n) + \sin(\theta_{n-1} - \theta_n)\right), \ \forall n \in [1, 2N] \tag{6.3}$$

avec $\theta_0 = \theta_1$ et $\theta_{2N} = \theta_{2N+1}$. On peut écrire ce système sous la forme $F(\theta, J) = 0$. On résout ce système par une méthode de Newton. Pour $J = 0$, on a une solution out of plane triviale $\theta_N = \frac{\pi}{2}$ et $\theta_n = 0$ pour $n \neq N$. On se sert de cette solution comme solution approchée du système pour J_1 petit différent de 0 par la méthode de Newton. Pour un $J_0 = kJ_1$ fixé, on itère k fois cette opération, la solution calculée après une itération servant de solution approchée pour l'itération suivante. On a les résultats suivants.

6.2 Etude de la stabilité linéaire de la solution stationnaire

On linéarise les équations (6.2) au voisinage de la solution stationnaire ψ_n. On écrit $\xi_n = \psi_n + \epsilon_n$ et on obtient le système linéarisé (avec condition



aux bords libres).

$$
\begin{aligned}
i\epsilon_n' = 2D &\left(\epsilon_n \frac{1-\psi_n^2}{1+\psi_n^2} - 2\frac{\psi_n^2}{(1+\psi_n^2)^2}(\epsilon_n+\overline{\epsilon}_n)\right) + \\
&+ J\Bigg(4\psi_n\Big(\frac{\psi_{n-1}}{(1+\psi_{n-1}^2)^2}(\epsilon_{n-1}+\overline{\epsilon}_{n-1}) + \frac{\psi_{n+1}}{(1+\psi_{n+1}^2)^2}(\epsilon_{n+1}+\overline{\epsilon}_{n+1})\Big) - \\
&- 2\epsilon_n\Big(\frac{1-\psi_{n-1}^2}{1+\psi_{n-1}^2} + \frac{1-\psi_{n+1}^2}{1+\psi_{n+1}^2}\Big) - \\
&- (1-\psi_n^2)\Big(\frac{\psi_{n-1}^2}{(1+\psi_{n-1}^2)^2}(\epsilon_{n-1}+\overline{\epsilon}_{n-1}) + \frac{\psi_{n+1}^2}{(1+\psi_{n+1}^2)^2}(\epsilon_{n+1}+\overline{\epsilon}_{n+1})\Big) - \\
&- 2\psi_n\Big(\frac{\psi_{n-1}}{1+\psi_{n-1}^2} + \frac{\psi_{n+1}}{1+\psi_{n+1}^2}\Big)\epsilon_n + \frac{\epsilon_{n-1}}{1+\psi_{n-1}^2} + \frac{\epsilon_{n+1}}{1+\psi_{n+1}^2} - \\
&- \psi_n^2\Big(\frac{\overline{\epsilon}_{n-1}}{1+\psi_{n-1}^2} + \frac{\overline{\epsilon}_{n+1}}{1+\psi_{n+1}^2}\Big)\Bigg) \qquad (6.4)
\end{aligned}
$$

On écrit alors $\epsilon_n = a_n + ib_n$ et on obtient le système

$$\frac{d}{dt}\begin{pmatrix} A \\ B \end{pmatrix} = M\begin{pmatrix} A \\ B \end{pmatrix}. \qquad (6.5)$$

Considérant la solution stationnaire comme une solution périodique, on calcule le spectre de $\exp T_c M$ où $T_c = \frac{2\pi}{\omega_c}$. On obtient les résultats suivants.

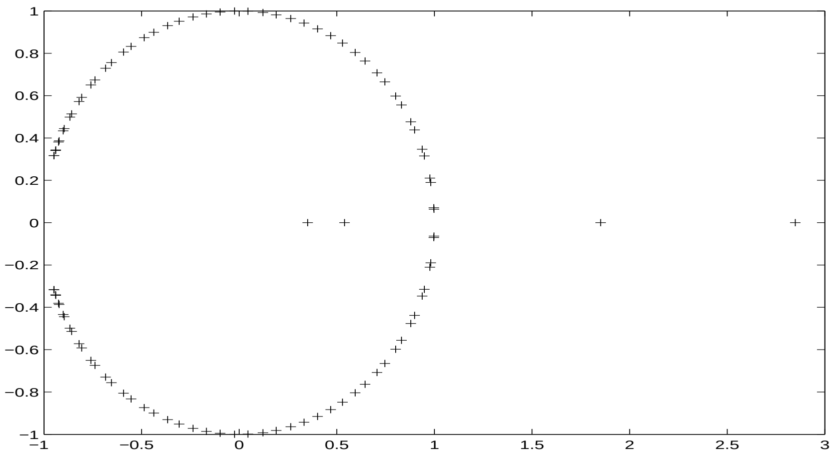
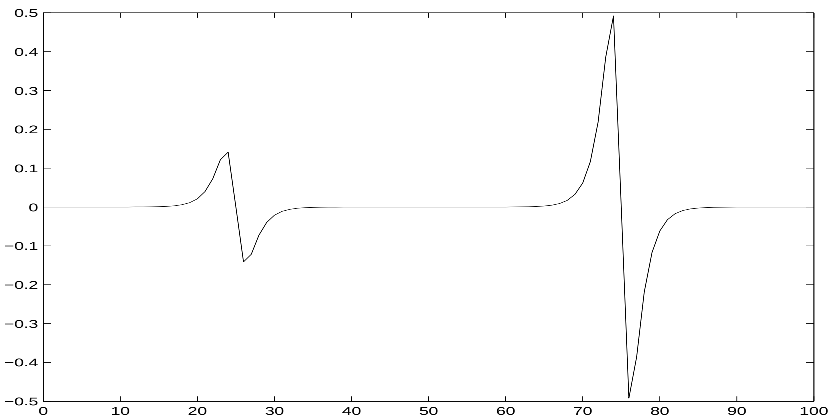


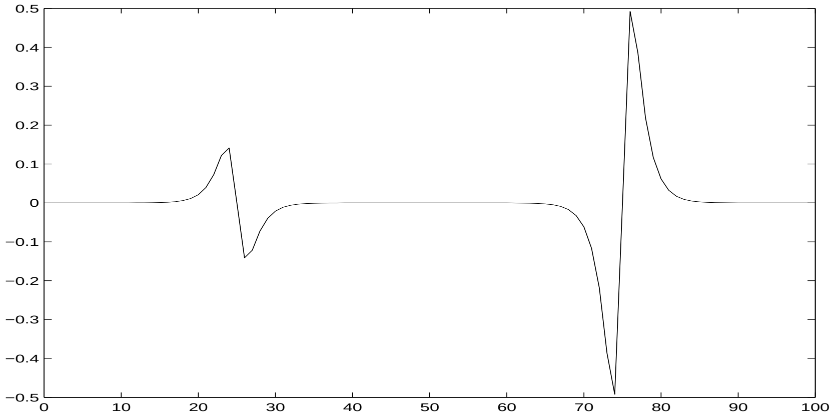
FIG. 6.1 – Spectre de $\exp T_c M$

6.3 Analyse de stabilité du système complet

On a l'existence de deux couples de valeurs propres instables $(\lambda, \frac{1}{\lambda}) \in \mathbb{R}$ pour le système linéaire pour $\omega = \omega_c$. On trace ici les graphes des vecteurs propres associés pour voir l'influence de ces instabilités sur le réseau.



Le premier type d'instabilité associé à un couple de valeur instable $(\lambda, \frac{1}{\lambda})$



tend à “tordre” le réseau tel que les spins n avec $n > N$ tournent autour de l'axe Z dans le sens des aiguilles d'une montre et les spins n avec $n < N$ dans le sens contraire (voir la figure).
Ce type d'instabilité a été observée numériquement par Zolotaryuk et al. Cependant, notre analyse est ici indépendante de la taille du réseau et les modes instables sont localisés. En effet, on a réalisé la simulation numérique pour $2N = 100$ et on retrouve les mêmes instabilités (voir figure 6.2).

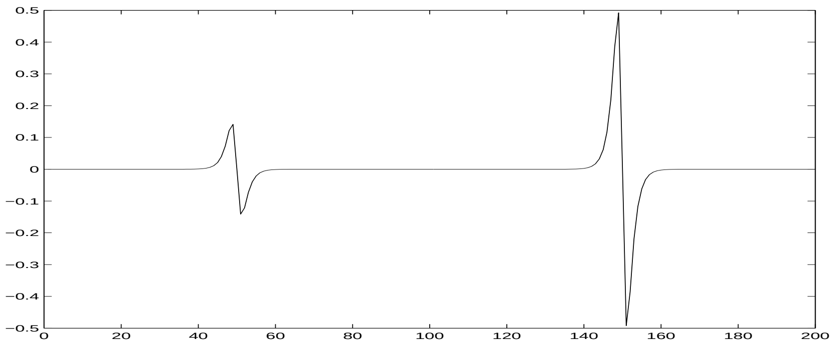


FIG. 6.2 – Exemple d'instabilité pour $2N = 100$

On examine maintenant la deuxième instabilité. Les vecteurs propres associés à l'autre paire $(\lambda, \frac{1}{\lambda})$ de valeurs propres instables sont décrit sur la figure 6.3.

La deuxième instabilité tend à faire aligner le réseau dans un des états de base soit l'état où tous les spins sont parallèles dans le plan XY ou l'état où tous les spins sont parallèles à l'axe Z et dans le même sens. Par continuité du spectre, ces modes persistent pour $\omega \approx \omega_c$ et $\omega < \omega_c$. On a ainsi prouvé

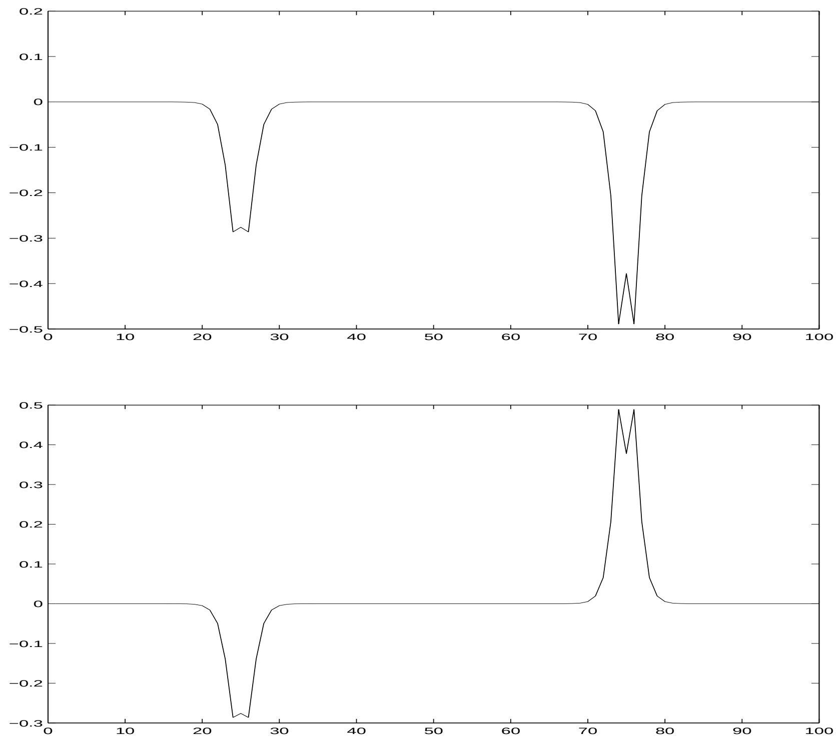


FIG. 6.3 – Profils du deuxième type d'instabilités pour $2N = 50$

numériquement et pour le système linéarisé l'instabilité des breathers "out of plane" avec un site principal excité. Cette méthode ne fonctionne pas pour des multibreathers car on a alors deux types de breathers (en phase et déphasé) qui ont la même limite stationnaire alors que ces deux types de breathers n'ont pas les mêmes propriétés de stabilité : l'un est stable et l'autre est instable (voir [12]).

6.4 Conclusion et perspectives

Dans ce dernier chapitre, on a fait une analyse de stabilité linéaire de breather dans la limite anticontinue. Les résultats "classiques" de stabilité linéaire sont obtenus pour des breathers obtenus par limite anticontinue et pour des réseaux de taille arbitraire mais finis. La méthode consiste à analyser la situation lorsque le couplage entre les particules est nul. Dans le cas des breathers de faibles amplitudes obtenues par réduction à une variété centrale, on ne peut pas faire la même approche et le problème est ouvert. Des simulations numériques menées sur la stabilité linéaire des breathers dans

FPU diatomiques mettent en avant le role essentiel joué par la géométrie des breathers. L'étude de la géométrie des breathers menées dans le chapitre 4 est donc un point important. Une piste possible est donc d'étudier d'un point de vue analytique le problème de la stabilité linéaire des breathers dans les chaines FPU diatomiques.
On s'est intéressé ici aux breathers dans des chaines $1D$. On peut se demander ce qui se passe en dimension 2 et dimension 3 et si les méthodes utilisées dans cette thèse peuvent s'appliquer. Sievers [13] a mené différentes simulations numériques pour des réseaux FPU diatomique $3D$ et mis en évidence le phénomène de breathers pour un cristal ionique. Cependant, Flach [5] a mis en évidence un phénomène de seuil pour l'existence de breathers $2D$ et $3D$. Ainsi, les oscillations type breathers ont une énergie et donc une amplitude minimale. Les techniques de réduction à une variété centrale ne peuvent donc pas s'appliquer pour rechercher des solutions breathers de petite amplitude. Les seules preuves connues d'existence pour les systèmes FPU sont valables pour des potentiels d'interaction homogènes : dans ce cas, on peut séparer les variables d'espace et de temps et les breathers apparaissent comme des points selle d'une certaine fonctionnelle [17] . Le problème de l'existence de breathers dans les réseaux FPU en deux et trois dimensions est donc ouvert.
Le phénomène de localisation de l'énergie dans des systèmes discrets a de nombreuses applications dans des domaines aussi divers que la physique de la matière condensée pour l'étude des oscillations dans des cristaux qu'en biologie pour tenter d'expliquer le phénomène de dénaturation de l'ADN [18] et on n'a abordé ici que des modèles très simplifiés. Cependant avant d'aborder des modèles et des structures plus complexes, on peut commencer par aborder la question de la localisation spatiale de l'énergie dans des molécules simples. Elles ont été observées expérimentalement dans des petites molécules comme SnD_4 où l'énergie est localisée sur une des liaisons [8]. Le problème n'a pas encore été abordé théoriquement et constitue une suite possible à ce travail, les questions essentielles portant sur les conditions d'existence de ces modes localisés et comment on peut les exciter.

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