Collective Motion Through Singular Limits

Charlotte Perrin, Ewelina Zatorska,
Didier Bresch

Colloque Inter'Actions, 2014, Lyon

Laboratoire de Mathématiques
Université de Savoie
France

22 May 2014

Classical Navier-Stokes equations

- Incompressible NS equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p - \nu \Delta u = 0 \\ \mathrm{div}\, u = 0 \end{cases}$$

- Compressible NS equations without temperature

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t (\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p \\ \qquad\qquad - \mu \Delta u - (\lambda + \mu) \nabla \mathrm{div}\, u = 0 \end{cases}$$

with $p = p(\rho)$

- Generalization with non-constant viscosities

A Free Boundary Problem in fluid mechanics

$$
\begin{cases}
\partial_t \rho + \mathrm{div}(\rho u) = 0 \\
\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p_1 + \nabla p_2 \\
\qquad\qquad -\mu \Delta u - (\lambda + \mu)\nabla(\mathrm{div}\, u) = 0
\end{cases}
$$

with

$$
\begin{cases}
0 \leq \rho \leq 1 \\
p_2 \geq 0 \\
p_1 = p_1(\rho) \\
(1-\rho)p_2 = 0
\end{cases}
$$

$\Longrightarrow$ compressible/incompressible system

Remarks : $p_1 \equiv 0$ and $\mu \equiv \lambda \equiv 0$

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p_2 = 0 \\ (1-\rho)\, p_2 = 0 \end{cases}$$

- **A** : $0 \le \rho < 1 \rightarrow$ pressureless Euler equations
- **B** : $\rho = 1 \rightarrow$ incompressible Euler equations

See F. Berthelin, F. Bouchut (2003–2012) :
Pressureless model from sticky particules system.

Collective motions and congestion : A flock of sheep
Picture from L. Navoret PhD Thesis

How to get the Free Boudary System from Comp. NS eq ?

Collective motion system from singular PDEs

Idea : Play with singular pressure

Previous works by Lions and Masmoudi

- Approximate system

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div}\, u + \nabla p_\gamma(\rho) = 0 \end{cases}$$

where

$$p_\gamma(\rho) = a\rho^\gamma$$

with $a > 0$ a fixed constant and γ a parameter.

- $\gamma \to \infty \implies$

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla \pi - \mu \Delta u - (\lambda + \mu)\nabla(\mathrm{div}\, u) = 0 \\ 0 \leq \rho \leq 1 \\ \pi \geq 0 \\ (1 - \rho)\pi = 0 \end{cases}$$

Pressure term coming from collective motion

- New approximate system

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \mathrm{div}\, u \\ \qquad + \nabla p_\varepsilon(\rho) = 0 \end{cases}$$

where

$$p_\varepsilon(\rho) = \varepsilon \rho^\gamma H(\rho) \text{ when } 0 \leq \rho < 1 \text{ and } +\infty \text{ otherwise}$$

with $\lim_{s \to 1^-} H(s) = +\infty$.

γ is fixed, $\varepsilon \to 0$: Collective motion model ?

See P. Degond, J. Hua, L. Navoret (2013) :
formal derivation and numerical schemes

$$p_\varepsilon = \varepsilon \frac{\rho^\gamma}{(1-\rho)^\gamma}$$

Remark

- we have $\rho \leq 1$ thanks to the pressure term

  $\longrightarrow$ the pressure plays the role of a barrier

  See B. Maury (2012)

- See also E.Feireisl, H.Petzeltovà, E.Rocca, G.Schimperna (2010)

  $\longrightarrow$ phase-field model for two-phase compressible fluids

Energy estimates

$\Omega = (0, L)$

- $p(\rho) = a\rho^{\gamma_n}$

$$\frac{d}{dt}\int_0^L \left(\frac{1}{2}\rho|u|^2 + \frac{a}{\gamma_n - 1}\rho^{\gamma_n}\right) dx + \int_0^L \mu|\partial_x u|^2 dx = 0$$

- $p(\rho) = \varepsilon\rho^{\gamma} H(\rho)$ **with** $H(s) = \dfrac{1}{(1-s)^{\beta}}$

$$\frac{d}{dt}\int_0^L \left(\frac{1}{2}\rho|u|^2 + \Pi(\rho)\right) dx + \int_0^L \mu|\partial_x u|^2 dx = 0$$

where $\Pi(\rho) = \rho\int_0^{\rho} \frac{p(s)}{s^2} ds.$

ex : $p = \varepsilon\dfrac{\rho^2}{(1-\rho)} \longrightarrow \Pi = \varepsilon\rho\log(1-\rho)$

$\Longrightarrow$ no a priori uniform bound on p

How to get extra information ?

$\longrightarrow$ $p(\rho) \in L^1_t L^1_x$ uniformly

Sketch of proof

- we test the momentum eq by $\phi(t,x) = \psi(t) \int_0^x (\rho(t,s) - \bar{\rho})ds$

$$\int_0^T \psi(t) \int_0^L p(\rho) \left(\rho - \int_0^L \rho \, dy \right) dx\, dt = \int_0^T \int_0^L \partial_t(\rho u)\phi dx\, dt$$
$$- \int_0^T \int_0^L \rho u^2 \partial_x \phi dx\, dt + \mu \int_0^T \int_0^L \partial_x u \partial_x \phi dx\, dt$$

- the energy estimates allows to control the r.h.s
- we slip the l.h.s into two parts depending on the density

$\Longrightarrow \quad p(\rho) \rightharpoonup \pi$ with π a positive measure.

$\Longrightarrow \quad \rho p(\rho) \rightharpoonup \pi_1$ with π_1 a positive measure.

- Can we pass to the limit in the system ?
- Can we recover the constraint $(\rho - 1)\pi = 0$?

Sketch of proof $p = \varepsilon \frac{\rho^\gamma}{(1-\rho)^\beta}$, 1d case

- using the strong convergence of the density

$$\rho\pi = \pi_1$$

- to characterize the limit π we write

$$\underbrace{\varepsilon \frac{(\rho)^\gamma \rho}{(1-\rho)^\beta}}_{\rightharpoonup \pi_1} = \underbrace{-\varepsilon \frac{(\rho)^\gamma}{(1-\rho)^{\beta-1}}}_{\to 0} + \underbrace{\varepsilon \frac{(\rho)^\gamma}{(1-\rho)^\beta}}_{\rightharpoonup \pi}$$

$p = \varepsilon \frac{\rho^\gamma}{(1-\rho)^\beta}$, conclusion

the limit (ρ, u, π) satisfies

$$\begin{aligned}
\partial_t \rho + \partial_x(\rho u) &= 0 && \text{in } (0,T)\times(0,L) \\
\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x \pi - \mu \partial_x^2 u &= 0 && \text{in } \mathcal{D}'((0,T)\times(0,L)) \\
0 \le \rho &\le 1 && \text{in } (0,T)\times(0,L) \\
\pi &\ge 0 && \text{in } \mathcal{M}_+((0,T)\times(0,L)) \\
(1-\rho)\pi &= 0 && \text{in } \mathcal{D}'((0,T)\times(0,L))
\end{aligned}$$

Existence results

Singular pressure

Existence of solutions, ε fixed

- let $\varepsilon, \mu, \lambda$ be fixed positive constants and let

$$(u^0, \rho^0) \in H^1_0(0, L) \times H^1(0, L) \qquad \text{with} \quad 0 < \rho^0 < 1.$$

Then there exists a regular solution $(u^\varepsilon, \rho^\varepsilon)$ such that

$$\rho^\varepsilon \in L^\infty(0, T; H^1(0, L)) \cap H^1(0, T; L^2(0, L))$$

$$u^\varepsilon \in L^2(0, T; H^1_0(0, L)) \cap L^\infty(0, T; L^2(0, L))$$

uniformly with respect to ε and there exist constants c and $C(\varepsilon)$ s.t.

$$0 < c \leq \rho^\varepsilon \leq C(\varepsilon) < 1.$$

$\longrightarrow$ we pass to Lagrangian coordinates

Upper bound on the density

$$\begin{cases} \partial_\tau \rho - \rho^2 \partial_X u = 0 \\ \partial_\tau u - \mu \partial_X \left(\rho \partial_X u\right) + \varepsilon \partial_X p_2(\rho) = 0 \end{cases}$$

$\Longrightarrow$

$$\frac{d}{d\tau} \int_0^M \left(\frac{u^2}{2} + \varepsilon \int_0^\rho \frac{p_2(s)}{s^2} ds \right) dX + \mu \int_0^M \rho |\partial_X u|^2 dX = 0$$

$$\frac{d}{d\tau} \int_0^M \left(\frac{\mu}{2} \left(\frac{\partial_X \rho}{\rho} \right)^2 + \frac{u \partial_X \rho}{\rho} \right) dX + \varepsilon \int_0^M p_2'(\rho) \frac{(\partial_X \rho)^2}{\rho} dX = \int_0^M \rho (\partial_X u)^2 dX$$

$\Longrightarrow \quad \rho < 1$

Perspectives

- multi-d case
- degenerate viscosities

preuve $\pi_1 = \pi$

$$\varepsilon \frac{\rho^\gamma}{(1-\rho)^{\beta-1}} \longrightarrow 0 \text{ in } L^{\beta/\beta-1}(Q_T)$$

- $\beta = 1$

$$|\varepsilon \frac{\rho^\gamma}{(1-\rho)^{\beta-1}}| = \varepsilon\rho^\gamma \leq \varepsilon$$

- $\beta > 1$

$$\begin{aligned}\int_{Q_T}\left(\frac{\rho^\gamma}{(1-\rho)^{\beta-1}}\right)^{\frac{\beta}{\beta-1}} &\leq \varepsilon^{\frac{\beta}{\beta-1}}\int_{Q_T}\left(\frac{(\rho^\gamma)^{\frac{\beta}{\beta-1}}}{(1-\rho)^{\beta-1}}\right)^{\frac{\beta}{\beta-1}} \\ &= \varepsilon^{\frac{1}{\beta-1}}\varepsilon\int_{Q_T}\left(\frac{\rho^\gamma}{(1-\rho)^{\beta}}\right)^{\frac{\beta}{\beta-1}} \\ &\leq C\varepsilon^{\frac{1}{\beta-1}}\end{aligned}$$