Fixed points of endomorphisms in Artin and preGarside monoids

Oussama Ajbal

Université de Caen

Lyon 2014

Plan

1. Preliminaries
   - Artin monoids
   - Norm
   - preGarside monoids
2. Fixed points and periodic points
   - Fixed points submonoid
   - Periodic points submonoid
   - The case of Artin monoids

Preliminaries

Artin monoids

Definition

Given a finite set $S \neq \emptyset$, a Coxeter matrix is a symmetric matrix $(m_{ab})_{a,b\in S}$ with entries in $\{1,2,\cdots,\infty\}$, such that $m_{aa} = 1$ and $m_{ab} \geq 2$, for $a \neq b$.

The associated Coxeter group is the group with presentation

$$W = \langle S \mid (ab)^{m_{ab}} = 1;\ m_{ab} \neq \infty \rangle.$$

The corresponding Artin monoid is the monoid with presentation

$$M = \langle S \mid [a,b\rangle^{m_{ab}} = [b,a\rangle^{m_{ab}};\ m_{ab} \neq \infty \rangle^{+},$$

where $[a,b\rangle^{m}$ denotes the alternating product $aba\cdots$ containing m terms.

Exemple

$S = \{s_1,\ldots,s_{n-1}\}$, $m_{i,i+1} = 3$ and $m_{ij} = 2$ pour $j > i+1$ $\rightsquigarrow$ $M = B_n^{+}$ the braid monoid with n strands.

Preliminaries

Artin monoids

Definition

Given a finite set $S \neq \emptyset$, a Coxeter matrix is a symmetric matrix $(m_{ab})_{a,b\in S}$ with entries in $\{1,2,\cdots,\infty\}$, such that $m_{aa}=1$ and $m_{ab}\geq 2$, for $a\neq b$.
The associated Coxeter group is the group with presentation

$$W = \langle S \mid (ab)^{m_{ab}} = 1;\ m_{ab}\neq\infty\rangle.$$

The corresponding Artin monoid is the monoid with presentation

$$M = \langle S \mid [a,b\rangle^{m_{ab}} = [b,a\rangle^{m_{ab}};\ m_{ab}\neq\infty\rangle^{+},$$

where $[a,b\rangle^{m}$ denotes the alternating product $aba\cdots$ containing m terms.

Exemple

$S=\{s_1,\ldots,s_{n-1}\}$, $m_{i,i+1}=3$ and $m_{ij}=2$ pour $j>i+1$ $\rightsquigarrow$ $M=B_n^+$ the braid monoid with n strands.

Preliminaries

Artin monoids

Definition

Given a finite set $S \neq \emptyset$, a Coxeter matrix is a symmetric matrix $(m_{ab})_{a,b\in S}$ with entries in $\{1,2,\cdots,\infty\}$, such that $m_{aa}=1$ and $m_{ab}\geq 2$, for $a\neq b$.
The associated Coxeter group is the group with presentation

$$W = \langle S \mid (ab)^{m_{ab}} = 1;\ m_{ab} \neq \infty \rangle.$$

The corresponding Artin monoid is the monoid with presentation

$$M = \langle S \mid [a,b\rangle^{m_{ab}} = [b,a\rangle^{m_{ab}};\ m_{ab} \neq \infty \rangle^{+},$$

where $[a,b\rangle^{m}$ denotes the alternating product $aba\cdots$ containing m terms.

Exemple

$S=\{s_1,\ldots,s_{n-1}\}$, $m_{i,i+1}=3$ and $m_{ij}=2$ pour $j>i+1$ $\rightsquigarrow$ $M=B_n^+$ the braid monoid with n strands.

Preliminaries

Artin monoids

Definition

Given a finite set $S \neq \emptyset$, a Coxeter matrix is a symmetric matrix $(m_{ab})_{a,b \in S}$ with entries in $\{1, 2, \cdots, \infty\}$, such that $m_{aa} = 1$ and $m_{ab} \geq 2$, for $a \neq b$.
The associated Coxeter group is the group with presentation

$$W = \langle S \mid (ab)^{m_{ab}} = 1;\ m_{ab} \neq \infty \rangle.$$

The corresponding Artin monoid is the monoid with presentation

$$M = \langle S \mid [a, b\rangle^{m_{ab}} = [b, a\rangle^{m_{ab}};\ m_{ab} \neq \infty \rangle^{+},$$

where $[a, b\rangle^{m}$ denotes the alternating product $aba\cdots$ containing m terms.

Exemple

$S = \{s_1, \ldots, s_{n-1}\}$, $m_{i,i+1} = 3$ *and* $m_{ij} = 2$ *pour* $j > i + 1 \rightsquigarrow M = B_n^+$ *the braid monoid with n strands.*

Preliminaries

For $\varphi \in \mathrm{End}(M)$, denote

- $\mathrm{Fix}(\varphi) = \{x \in M \mid \varphi(x) = x\}$ the submonoid of fixed points;
- $\mathrm{Per}(\varphi) = \bigcup_{n \geq 1} \mathrm{Fix}(\varphi^n)$ the submonoid of periodic points.

Theorem (Silva, Rodaro 2012)

Let M be a right angled Artin monoid, and $\varphi \in \mathrm{End}(M)$. Then the submonoids $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are finitely generated.

Lemma (J. Michel 1998, J. Crisp 1999)

Let M be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. If $\varphi|_S$ is a permutation, then $\mathrm{Fix}(\varphi)$ is an Artin monoid.

Theorem

Let M be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Then the submonoids $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are also Artin monoids.

Preliminaries

For $\varphi \in \mathrm{End}(M)$, denote

- $\mathrm{Fix}(\varphi) = \{x \in M \mid \varphi(x) = x\}$ the submonoid of fixed points;
- $\mathrm{Per}(\varphi) = \bigcup_{n \geq 1} \mathrm{Fix}(\varphi^n)$ the submonoid of periodic points.

Theorem (Silva, Rodaro 2012)

Let M be a right angled Artin monoid, and $\varphi \in \mathrm{End}(M)$. Then the submonoids $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are finitely generated.

Lemma (J. Michel 1998, J. Crisp 1999)

Let M be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. If $\varphi|_S$ is a permutation, then $\mathrm{Fix}(\varphi)$ is an Artin monoid.

Theorem

Let M be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Then the submonoids $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are also Artin monoids.

Preliminaries

For $\varphi \in \mathrm{End}(M)$, denote

- $\mathrm{Fix}(\varphi) = \{x \in M \mid \varphi(x) = x\}$ the submonoid of fixed points;
- $\mathrm{Per}(\varphi) = \bigcup_{n\geq 1} \mathrm{Fix}(\varphi^n)$ the submonoid of periodic points.

Theorem (Silva, Rodaro 2012)

Let M be a right angled Artin monoid, and $\varphi \in \mathrm{End}(M)$. Then the submonoids $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are finitely generated.

Lemma (J. Michel 1998, J. Crisp 1999)

Let M be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. If $\varphi|_S$ is a permutation, then $\mathrm{Fix}(\varphi)$ is an Artin monoid.

Theorem

Let M be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Then the submonoids $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are also Artin monoids.

Preliminaries

For $\varphi \in \mathrm{End}(M)$, denote

- $\mathrm{Fix}(\varphi) = \{x \in M \mid \varphi(x) = x\}$ the submonoid of fixed points;
- $\mathrm{Per}(\varphi) = \bigcup_{n \geq 1} \mathrm{Fix}(\varphi^n)$ the submonoid of periodic points.

Theorem (Silva, Rodaro 2012)

Let M be a right angled Artin monoid, and $\varphi \in \mathrm{End}(M)$. Then the submonoids $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are finitely generated.

Lemma (J. Michel 1998, J. Crisp 1999)

Let M be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. If $\varphi|_S$ is a permutation, then $\mathrm{Fix}(\varphi)$ is an Artin monoid.

Theorem

Let M be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Then the submonoids $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are also Artin monoids.

Preliminaries

For $\varphi \in \mathrm{End}(M)$, denote

- $\mathrm{Fix}(\varphi) = \{x \in M \mid \varphi(x) = x\}$ the submonoid of fixed points;
- $\mathrm{Per}(\varphi) = \bigcup_{n \geq 1} \mathrm{Fix}(\varphi^n)$ the submonoid of periodic points.

Theorem (Silva, Rodaro 2012)

Let M be a right angled Artin monoid, and $\varphi \in \mathrm{End}(M)$. Then the submonoids $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are finitely generated.

Lemma (J. Michel 1998, J. Crisp 1999)

Let M be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. If $\varphi|_S$ is a permutation, then $\mathrm{Fix}(\varphi)$ is an Artin monoid.

Theorem

Let M be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Then the submonoids $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are also Artin monoids.

Preliminaries

Norm

- M is cancellative, if $\forall a, b, c, d \in M$, $cad = cbd \Rightarrow a = b$.
- If $a = bc$, we say that :
  - b left-divides a, or that a is a right-multiple of b, and denote $b \preceq a$.
  - c right-divides a, or that a is a left-mutiple of c, and denote $a \succeq c$.
- An atom in a monoid is an element $a \in M$ verifying $a = bc \Rightarrow b = 1$ ou $c = 1$ for all $b, c \in M$.
- M is atomic if there is a norm $\nu : M \to \mathbb{N}$, satisfying :
  - $\nu(a) > 0$ for $a \neq 1$,
  - $\nu(ab) \geq \nu(a) + \nu(b)$ for all $a, b \in M$.

Preliminaries

Norm

- M is cancellative, if $\forall a, b, c, d \in M$, $cad = cbd \Rightarrow a = b$.
- If $a = bc$, we say that :
  - b left-divides a, or that a is a right-multiple of b, and denote $b \preceq a$.
  - c right-divides a, or that a is a left-mutiple of c, and denote $a \succeq c$.
- An atom in a monoid is an element $a \in M$ verifying $a = bc \Rightarrow b = 1$ ou $c = 1$ for all $b, c \in M$.
- M is atomic if there is a norm $\nu : M \to \mathbb{N}$, satisfying :
  - $\nu(a) > 0$ for $a \neq 1$,
  - $\nu(ab) \geq \nu(a) + \nu(b)$ for all $a, b \in M$.

Preliminaries

Norm

- M is cancellative, if $\forall a, b, c, d \in M$, $cad = cbd \Rightarrow a = b$.
- If $a = bc$, we say that :
  - b left-divides a, or that a is a right-multiple of b, and denote $b \preceq a$.
  - c right-divides a, or that a is a left-mutiple of c, and denote $a \succeq c$.
- An atom in a monoid is an element $a \in M$ verifying $a = bc \Rightarrow b = 1$ ou $c = 1$ for all $b, c \in M$.
- M is atomic if there is a norm $\nu : M \to \mathbb{N}$, satisfying :
  - $\nu(a) > 0$ for $a \neq 1$,
  - $\nu(ab) \geq \nu(a) + \nu(b)$ for all $a, b \in M$.

Preliminaries

Norm

- M is cancellative, if $\forall a,b,c,d \in M$, $cad = cbd \;\Rightarrow\; a = b$.
- If $a = bc$, we say that :
  - b left-divides a, or that a is a right-multiple of b, and denote $b \preceq a$.
  - c right-divides a, or that a is a left-mutiple of c, and denote $a \succeq c$.
- An atom in a monoid is an element $a \in M$ verifying $a = bc \Rightarrow b = 1$ ou $c = 1$ for all $b, c \in M$.
- M is atomic if there is a norm $\nu : M \to \mathbb{N}$, satisfying :
  - $\nu(a) > 0$ for $a \neq 1$,
  - $\nu(ab) \geq \nu(a) + \nu(b)$ for all $a, b \in M$.

Preliminaries

Norm

- M is cancellative, if $\forall a, b, c, d \in M$, $cad = cbd \Rightarrow a = b$.
- If $a = bc$, we say that :
  - b left-divides a, or that a is a right-multiple of b, and denote $b \preceq a$.
  - c right-divides a, or that a is a left-mutiple of c, and denote $a \succeq c$.
- An atom in a monoid is an element $a \in M$ verifying $a = bc \Rightarrow b = 1$ ou $c = 1$ for all $b, c \in M$.
- M is atomic if there is a norm $\nu : M \to \mathbb{N}$, satisfying :
  - $\nu(a) > 0$ for $a \neq 1$,
  - $\nu(ab) \geq \nu(a) + \nu(b)$ for all $a, b \in M$.

Preliminaries

Norm

- M is cancellative, if $\forall a,b,c,d \in M$, $cad = cbd \;\Rightarrow\; a = b$.
- If $a = bc$, we say that :
  - b left-divides a, or that a is a right-multiple of b, and denote $b \preceq a$.
  - c right-divides a, or that a is a left-mutiple of c, and denote $a \succeq c$.
- An atom in a monoid is an element $a \in M$ verifying $a = bc \Rightarrow b = 1$ ou $c = 1$ for all $b,c \in M$.
- M is atomic if there is a norm $\nu : M \to \mathbb{N}$, satisfying :
  - $\nu(a) > 0$ for $a \neq 1$,
  - $\nu(ab) \geq \nu(a) + \nu(b)$ for all $a,b \in M$.

Preliminaries

preGarside monoids

Definition

A monoid M is said to be preGarside if

(a) *it is atomic, cancellative, and finitely generated;*

(b_L) *for all $a, b \in M$, if $\{c \in M \mid a \preceq c \text{ and } b \preceq c\} \neq \emptyset$, then it has a least element, denoted by $a \vee_L b$ or $a \vee b$;*

(b_R) *for all $a, b \in M$, if $\{c \in M \mid c \succeq a \text{ and } c \succeq b\} \neq \emptyset$, then it has a least element, denoted by $a \vee_R b$.*

Preliminaries

preGarside monoids

Definition

A monoid M is said to be preGarside if

(a) *it is atomic, cancellative, and finitely generated;*

(b_L) *for all $a, b \in M$, if $\{c \in M \mid a \preceq c \text{ and } b \preceq c\} \neq \emptyset$, then it has a least element, denoted by $a \vee_L b$ or $a \vee b$;*

(b_R) *for all $a, b \in M$, if $\{c \in M \mid c \succeq a \text{ and } c \succeq b\} \neq \emptyset$, then it has a least element, denoted by $a \vee_R b$.*

Preliminaries

preGarside monoids

Definition

A monoid M is said to be preGarside if

- (a) *it is atomic, cancellative, and finitely generated;*
- (b_L) *for all $a, b \in M$, if $\{c \in M \mid a \preceq c \text{ and } b \preceq c\} \neq \emptyset$, then it has a least element, denoted by $a \vee_L b$ or $a \vee b$;*
- (b_R) *for all $a, b \in M$, if $\{c \in M \mid c \succeq a \text{ and } c \succeq b\} \neq \emptyset$, then it has a least element, denoted by $a \vee_R b$.*

Fixed points and periodic points

The morphism π

- Let $S \neq \emptyset$. Denote by S^* the set of finite words over S.
- If $T \subseteq S$, the forgetting morphism $\pi_T^* : S^* \to T^*$ is defined by

$$\pi_T^*(t) = \begin{cases} t & \text{if } t \in T \\ \varepsilon & \text{if } t \in S \setminus T \end{cases}.$$

- When it is well-defined, denote $\pi_T : M \to M$ the morphism induced by π_T^*.

Example

- For $M = \langle s, t \mid sts = tst \rangle^+$, with $S = \{s, t\}$ and $T = \{t\}$, we have $\pi_T^*(sts) = t$ and $\pi_T^*(tst) = tt$. But $t \neq t^2$ in M, so π_T is not well-defined.
- For $M = \langle s, t \mid stst = tsts \rangle^+$ with similar S and T, the morphism π_T is well-defined, since $\pi_T(stst) = \pi_T(tsts) = t^2$.

Fixed points and periodic points

The morphism π

- Let $S \neq \emptyset$. Denote by S^* the set of finite words over S.
- If $T \subseteq S$, the forgetting morphism $\pi_T^* : S^* \to T^*$ is defined by

$$\pi_T^*(t) = \begin{cases} t & \text{if } t \in T \\ \varepsilon & \text{if } t \in S \setminus T \end{cases} .$$

- When it is well-defined, denote $\pi_T : M \to M$ the morphism induced by π_T^*.

Example

- For $M = \langle s, t \mid sts = tst \rangle^+$, with $S = \{s, t\}$ and $T = \{t\}$, we have $\pi_T^*(sts) = t$ and $\pi_T^*(tst) = tt$. But $t \neq t^2$ in M, so π_T is not well-defined.
- For $M = \langle s, t \mid stst = tsts \rangle^+$ with similar S and T, the morphism π_T is well-defined, since $\pi_T(stst) = \pi_T(tsts) = t^2$.

Fixed points and periodic points

The morphism π

- Let $S \neq \emptyset$. Denote by S^* the set of finite words over S.
- If $T \subseteq S$, the forgetting morphism $\pi_T^* : S^* \to T^*$ is defined by

$$\pi_T^*(t) = \begin{cases} t & \text{if } t \in T \\ \varepsilon & \text{if } t \in S \setminus T \end{cases}.$$

- When it is well-defined, denote $\pi_T : M \to M$ the morphism induced by π_T^*.

Example

- For $M = \langle s, t \mid sts = tst \rangle^+$, with $S = \{s, t\}$ and $T = \{t\}$, we have $\pi_T^*(sts) = t$ and $\pi_T^*(tst) = tt$. But $t \neq t^2$ in M, so π_T is not well-defined.
- For $M = \langle s, t \mid stst = tsts \rangle^+$ with similar S and T, the morphism π_T is well-defined, since $\pi_T(stst) = \pi_T(tsts) = t^2$.

Fixed points and periodic points

The morphism π

- Let $S \neq \emptyset$. Denote by S^* the set of finite words over S.
- If $T \subseteq S$, the forgetting morphism $\pi_T^* : S^* \to T^*$ is defined by

$$\pi_T^*(t) = \begin{cases} t & \text{if } t \in T \\ \varepsilon & \text{if } t \in S \setminus T \end{cases}.$$

- When it is well-defined, denote $\pi_T : M \to M$ the morphism induced by π_T^*.

Example

- For $M = \langle s, t \mid sts = tst \rangle^+$, with $S = \{s, t\}$ and $T = \{t\}$, we have $\pi_T^*(sts) = t$ and $\pi_T^*(tst) = tt$. But $t \neq t^2$ in M, so π_T is not well-defined.
- For $M = \langle s, t \mid stst = tsts \rangle^+$ with similar S and T, the morphism π_T is well-defined, since $\pi_T(stst) = \pi_T(tsts) = t^2$.

Fixed points and periodic points

The morphism π

- Let $S \neq \emptyset$. Denote by S^* the set of finite words over S.
- If $T \subseteq S$, the forgetting morphism $\pi_T^* : S^* \to T^*$ is defined by

$$\pi_T^*(t) = \begin{cases} t & \text{if } t \in T \\ \varepsilon & \text{if } t \in S \setminus T \end{cases}.$$

- When it is well-defined, denote $\pi_T : M \to M$ the morphism induced by π_T^*.

Example

- For $M = \langle s, t \mid sts = tst \rangle^+$, with $S = \{s, t\}$ and $T = \{t\}$, we have $\pi_T^*(sts) = t$ and $\pi_T^*(tst) = tt$. But $t \neq t^2$ in M, so π_T is not well-defined.
- For $M = \langle s, t \mid stst = tsts \rangle^+$ with similar S and T, the morphism π_T is well-defined, since $\pi_T(stst) = \pi_T(tsts) = t^2$.

Fixed points submonoid

Let M be a preGarside monoid, with an additive and homogeneous norm ν. Let $\varphi \in \mathrm{End}(M)$, and set $S = \mathcal{S}(M)$.

Define

$$n_\varphi = \begin{cases} \max\{k \in \mathbb{N}^* \mid \exists s \in S \text{ with } \varphi^k(s) = 1 \text{ and } \varphi^{k-1}(s) \neq 1\} & \text{if } 1 \in \varphi(S) \\ 1 & \text{if } 1 \notin \varphi(S) \end{cases}.$$

Set

- $S_0 = S \cap \mathrm{Per}(\varphi)$
- $S_1 = S \cap (\varphi^{n_\varphi})^{-1}\{1\}$
- $S_2 = S \setminus S_1$
- $\pi := \pi_{S_2}$, when it is well-defined.

Fixed points submonoid

Let M be a preGarside monoid, with an additive and homogeneous norm ν. Let $\varphi \in \mathrm{End}(M)$, and set $S = \mathcal{S}(M)$.

Define

$$n_\varphi = \begin{cases} \max\{k \in \mathbb{N}^* \mid \exists s \in S \text{ with } \varphi^k(s) = 1 \text{ and } \varphi^{k-1}(s) \neq 1\} & \text{if } 1 \in \varphi(S) \\ 1 & \text{if } 1 \notin \varphi(S) \end{cases}.$$

Set

- $S_0 = S \cap \mathrm{Per}(\varphi)$
- $S_1 = S \cap (\varphi^{n_\varphi})^{-1}\{1\}$
- $S_2 = S \setminus S_1$
- $\pi := \pi_{S_2}$, when it is well-defined.

Fixed points submonoid

Let M be a preGarside monoid, with an additive and homogeneous norm ν. Let $\varphi \in \mathrm{End}(M)$, and set $S = \mathcal{S}(M)$.
Define

$$n_\varphi = \begin{cases} \max\{k \in \mathbb{N}^* \mid \exists s \in S \text{ with } \varphi^k(s) = 1 \text{ and } \varphi^{k-1}(s) \neq 1\} & \text{if } 1 \in \varphi(S) \\ 1 & \text{if } 1 \notin \varphi(S) \end{cases}.$$

Set

- $S_0 = S \cap \mathrm{Per}(\varphi)$
- $S_1 = S \cap (\varphi^{n_\varphi})^{-1}\{1\}$
- $S_2 = S \setminus S_1$
- $\pi := \pi_{S_2}$, when it is well-defined.

Fixed points submonoid

Let M be a preGarside monoid, with an additive and homogeneous norm ν. Let $\varphi \in \mathrm{End}(M)$, and set $S = \mathcal{S}(M)$.
Define

$$n_\varphi = \begin{cases} \max\{k \in \mathbb{N}^* \mid \exists s \in S \text{ with } \varphi^k(s) = 1 \text{ and } \varphi^{k-1}(s) \neq 1\} & \text{if } 1 \in \varphi(S) \\ 1 & \text{if } 1 \notin \varphi(S) \end{cases}.$$

Set

- $S_0 = S \cap \mathrm{Per}(\varphi)$
- $S_1 = S \cap (\varphi^{n_\varphi})^{-1}\{1\}$
- $S_2 = S \setminus S_1$
- $\pi := \pi_{S_2}$, when it is well-defined.

Fixed points submonoid

Let M be a preGarside monoid, with an additive and homogeneous norm ν. Let $\varphi \in \mathrm{End}(M)$, and set $S = \mathcal{S}(M)$.
Define

$$n_\varphi = \begin{cases} \max\{k \in \mathbb{N}^* \mid \exists s \in S \text{ with } \varphi^k(s) = 1 \text{ and } \varphi^{k-1}(s) \neq 1\} & \text{if } 1 \in \varphi(S) \\ 1 & \text{if } 1 \notin \varphi(S) \end{cases}.$$

Set

- $S_0 = S \cap \mathrm{Per}(\varphi)$
- $S_1 = S \cap (\varphi^{n_\varphi})^{-1}\{1\}$
- $S_2 = S \setminus S_1$
- $\pi := \pi_{S_2}$, when it is well-defined.

Fixed points submonoid

Let M be a preGarside monoid, with an additive and homogeneous norm ν. Let $\varphi \in \mathrm{End}(M)$, and set $S = \mathcal{S}(M)$.
Define

$$n_\varphi = \begin{cases} \max\{k \in \mathbb{N}^* \mid \exists s \in S \text{ with } \varphi^k(s) = 1 \text{ and } \varphi^{k-1}(s) \neq 1\} & \text{if } 1 \in \varphi(S) \\ 1 & \text{if } 1 \notin \varphi(S) \end{cases}.$$

Set

- $S_0 = S \cap \mathrm{Per}(\varphi)$
- $S_1 = S \cap (\varphi^{n_\varphi})^{-1}\{1\}$
- $S_2 = S \setminus S_1$
- $\pi := \pi_{S_2}$, when it is well-defined.

Fixed points submonoid

Let M be a preGarside monoid, with an additive and homogeneous norm ν. Let $\varphi \in \mathrm{End}(M)$, and set $S = \mathcal{S}(M)$.

Define

$$n_\varphi = \begin{cases} \max\{k \in \mathbb{N}^* \mid \exists s \in S \text{ with } \varphi^k(s) = 1 \text{ and } \varphi^{k-1}(s) \neq 1\} & \text{if } 1 \in \varphi(S) \\ 1 & \text{if } 1 \notin \varphi(S) \end{cases}.$$

Set

- $S_0 = S \cap \mathrm{Per}(\varphi)$
- $S_1 = S \cap (\varphi^{n_\varphi})^{-1}\{1\}$
- $S_2 = S \setminus S_1$
- $\pi := \pi_{S_2}$, when it is well-defined.

Fixed points submonoid

Theorem

Theorem

If π is well-defined, then $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are preGarside monoids.

Proof : 1^{st} case : $\varphi|_S$ permutation ($\Leftrightarrow$ φ automorphism)

Lemma (D. Bessis, F. Digne, J. Michel 2002)

If $\varphi|_S$ is a permutation, then $\mathrm{Fix}(\varphi)$ is a preGarside monoid.

Exemple

- $M = \langle a_1, b_1 \mid a_1 b_1 a_1 = b_1^2 \rangle^+ \times \langle a_2, b_2 \mid a_2 b_2 a_2 = b_2^2 \rangle^+$ with $\varphi(a_1) = a_2$, $\varphi(a_2) = a_1$, $\varphi(b_1) = b_2$ and $\varphi(b_2) = b_1$. One has $\mathrm{Fix}(\varphi) = \langle a_1 a_2, b_1 b_2 \rangle^+$.

Theorem

If π is well-defined, then $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are preGarside monoids.

Proof : 1^{st} case : $\varphi|_S$ permutation ($\Leftrightarrow$ φ automorphism)

Lemma (D. Bessis, F. Digne, J. Michel 2002)

If $\varphi|_S$ is a permutation, then $\mathrm{Fix}(\varphi)$ is a preGarside monoid.

Exemple

- $M = \langle a_1, b_1 \mid a_1 b_1 a_1 = b_1^2 \rangle^+ \times \langle a_2, b_2 \mid a_2 b_2 a_2 = b_2^2 \rangle^+$ with $\varphi(a_1) = a_2$, $\varphi(a_2) = a_1$, $\varphi(b_1) = b_2$ and $\varphi(b_2) = b_1$. One has $\mathrm{Fix}(\varphi) = \langle a_1 a_2, b_1 b_2 \rangle^+$.

Fixed points submonoid

Theorem

Theorem

If π is well-defined, then $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are preGarside monoids.

Proof : 1^{st} case : $\varphi|_S$ permutation ($\Leftrightarrow$ φ automorphism)

Lemma (D. Bessis, F. Digne, J. Michel 2002)

If $\varphi|_S$ is a permutation, then $\mathrm{Fix}(\varphi)$ is a preGarside monoid.

Exemple

- $M = \langle a_1, b_1 \mid a_1 b_1 a_1 = b_1^2 \rangle^+ \times \langle a_2, b_2 \mid a_2 b_2 a_2 = b_2^2 \rangle^+$ with $\varphi(a_1) = a_2$, $\varphi(a_2) = a_1$, $\varphi(b_1) = b_2$ and $\varphi(b_2) = b_1$. One has $\mathrm{Fix}(\varphi) = \langle a_1 a_2, b_1 b_2 \rangle^+$.

Theorem

If π is well-defined, then $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ are preGarside monoids.

Proof : 1[st] case : $\varphi|_S$ permutation ($\Leftrightarrow \varphi$ automorphism)

Lemma (D. Bessis, F. Digne, J. Michel 2002)

If $\varphi|_S$ is a permutation, then $\mathrm{Fix}(\varphi)$ is a preGarside monoid.

Exemple

- $M = \langle a_1, b_1 \mid a_1 b_1 a_1 = b_1^2 \rangle^+ \times \langle a_2, b_2 \mid a_2 b_2 a_2 = b_2^2 \rangle^+$ with $\varphi(a_1) = a_2$, $\varphi(a_2) = a_1$, $\varphi(b_1) = b_2$ and $\varphi(b_2) = b_1$. One has $\mathrm{Fix}(\varphi) = \langle a_1 a_2, b_1 b_2 \rangle^+$.

Fixed points submonoid

2^{nd} case : $1 \notin \varphi(S)$

Lemma

$\varphi(S_0) = S_0$, and $M_0 = \langle S_0 \rangle^+$ is preGarside, and $\nu|_{M_0}$ is additive and homogeneous.

$\varphi(S_0) = S_0 \quad \rightsquigarrow \quad \varphi_0 = \varphi|_{M_0} \in \mathrm{Aut}(M_0)$.
And we show that $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_0)$.

3^{rd} case : $1 \in \varphi(S)$

Lemma

$M_2 = \langle S_2 \rangle^+$ is preGarside, and $\nu|_{M_2}$ is additive and homogeneous.

Consider $\varphi_2 = (\pi \circ \varphi)|_{M_2} \in \mathrm{End}(M_2)$. We show that $1 \notin \varphi_2(S_2)$,
$\mathrm{Fix}(\varphi) = \varphi^{n_\varphi}(\mathrm{Fix}(\varphi_2))$, and $\mathrm{Fix}(\varphi) \sim \mathrm{Fix}(\varphi_2)$.

Fixed points submonoid

2^{nd} case : $1 \notin \varphi(S)$

Lemma

$\varphi(S_0) = S_0$, *and* $M_0 = \langle S_0 \rangle^+$ *is preGarside, and* $\nu|_{M_0}$ *is additive and homogeneous.*

$\varphi(S_0) = S_0 \quad \rightsquigarrow \quad \varphi_0 = \varphi|_{M_0} \in \mathrm{Aut}(M_0)$.
And we show that $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_0)$.

3^{rd} case : $1 \in \varphi(S)$

Lemma

$M_2 = \langle S_2 \rangle^+$ *is preGarside, and* $\nu|_{M_2}$ *is additive and homogeneous.*

Consider $\varphi_2 = (\pi \circ \varphi)|_{M_2} \in \mathrm{End}(M_2)$. We show that $1 \notin \varphi_2(S_2)$,
$\mathrm{Fix}(\varphi) = \varphi^{n_\varphi}(\mathrm{Fix}(\varphi_2))$, and $\mathrm{Fix}(\varphi) \sim \mathrm{Fix}(\varphi_2)$.

Fixed points submonoid

2^{nd} case : $1 \notin \varphi(S)$

Lemma

$\varphi(S_0) = S_0$, and $M_0 = \langle S_0 \rangle^+$ is preGarside, and $\nu|_{M_0}$ is additive and homogeneous.

$\varphi(S_0) = S_0 \quad \rightsquigarrow \quad \varphi_0 = \varphi|_{M_0} \in \mathrm{Aut}(M_0)$.
And we show that $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_0)$.

3^{rd} case : $1 \in \varphi(S)$

Lemma

$M_2 = \langle S_2 \rangle^+$ is preGarside, and $\nu|_{M_2}$ is additive and homogeneous.

Consider $\varphi_2 = (\pi \circ \varphi)|_{M_2} \in \mathrm{End}(M_2)$. We show that $1 \notin \varphi_2(S_2)$, $\mathrm{Fix}(\varphi) = \varphi^{n_\varphi}(\mathrm{Fix}(\varphi_2))$, and $\mathrm{Fix}(\varphi) \sim \mathrm{Fix}(\varphi_2)$.

Fixed points submonoid

2^{nd} case : $1 \notin \varphi(S)$

Lemma

$\varphi(S_0) = S_0$, and $M_0 = \langle S_0 \rangle^+$ is preGarside, and $\nu|_{M_0}$ is additive and homogeneous.

$\varphi(S_0) = S_0 \quad \rightsquigarrow \quad \varphi_0 = \varphi|_{M_0} \in \mathrm{Aut}(M_0)$.
And we show that $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_0)$.

3^{rd} case : $1 \in \varphi(S)$

Lemma

$M_2 = \langle S_2 \rangle^+$ is preGarside, and $\nu|_{M_2}$ is additive and homogeneous.

Consider $\varphi_2 = (\pi \circ \varphi)|_{M_2} \in \mathrm{End}(M_2)$. We show that $1 \notin \varphi_2(S_2)$, $\mathrm{Fix}(\varphi) = \varphi^{n_\varphi}(\mathrm{Fix}(\varphi_2))$, and $\mathrm{Fix}(\varphi) \sim \mathrm{Fix}(\varphi_2)$.

Fixed points submonoid

2^{nd} case : $1 \notin \varphi(S)$

Lemma

$\varphi(S_0) = S_0$, and $M_0 = \langle S_0 \rangle^+$ is preGarside, and $\nu|_{M_0}$ is additive and homogeneous.

$\varphi(S_0) = S_0 \quad \rightsquigarrow \quad \varphi_0 = \varphi|_{M_0} \in \mathrm{Aut}(M_0)$.
And we show that $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_0)$.

3^{rd} case : $1 \in \varphi(S)$

Lemma

$M_2 = \langle S_2 \rangle^+$ is preGarside, and $\nu|_{M_2}$ is additive and homogeneous.

Consider $\varphi_2 = (\pi \circ \varphi)|_{M_2} \in \mathrm{End}(M_2)$. We show that $1 \notin \varphi_2(S_2)$,
$\mathrm{Fix}(\varphi) = \varphi^{n_\varphi}(\mathrm{Fix}(\varphi_2))$, and $\mathrm{Fix}(\varphi) \sim \mathrm{Fix}(\varphi_2)$.

Fixed points submonoid

2^{nd} case : $1 \notin \varphi(S)$

Lemma

$\varphi(S_0) = S_0$, and $M_0 = \langle S_0 \rangle^+$ is preGarside, and $\nu|_{M_0}$ is additive and homogeneous.

$\varphi(S_0) = S_0 \quad \rightsquigarrow \quad \varphi_0 = \varphi|_{M_0} \in \mathrm{Aut}(M_0)$.
And we show that $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_0)$.

3^{rd} case : $1 \in \varphi(S)$

Lemma

$M_2 = \langle S_2 \rangle^+$ is preGarside, and $\nu|_{M_2}$ is additive and homogeneous.

Consider $\varphi_2 = (\pi \circ \varphi)|_{M_2} \in \mathrm{End}(M_2)$. We show that $1 \notin \varphi_2(S_2)$,
$\mathrm{Fix}(\varphi) = \varphi^{n_\varphi}(\mathrm{Fix}(\varphi_2))$, and $\mathrm{Fix}(\varphi) \sim \mathrm{Fix}(\varphi_2)$.

Periodic points submonoid

Lemma

If π is well-defined, then we have $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi^{pn})$.

Proposition

If π is well-defined, then $\mathrm{Per}(\varphi)$ *is a preGarside monoid.*

Preuve : $S_1(\varphi) = S_1(\varphi^{pn})$ $\rightsquigarrow$ we apply the theorem.

Periodic points submonoid

Lemma

If π *is well-defined, then we have* $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi^{pn})$.

Proposition

If π *is well-defined, then* $\mathrm{Per}(\varphi)$ *is a preGarside monoid.*

Preuve : $S_1(\varphi) = S_1(\varphi^{pn}) \rightsquigarrow$ we apply the theorem.

Periodic points submonoid

Lemma

If π is well-defined, then we have $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi^{pn})$.

Proposition

If π is well-defined, then $\mathrm{Per}(\varphi)$ *is a preGarside monoid.*

Preuve : $S_1(\varphi) = S_1(\varphi^{pn})$ $\rightsquigarrow$ we apply the theorem.

The case of Artin monoids

Let $M = \langle S \mid [a,b\rangle^{m_{ab}} = [b,a\rangle^{m_{ab}};\ m_{ab} \neq \infty\rangle^+$ be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Artin monoids are preGarside (Brieskorn, Saito 1972).

Proposition

The morphism π is well-defined.

Preuve : It suffices to check that $\pi([a,b\rangle^{m_{ab}}) = \pi([b,a\rangle^{m_{ab}})$ for all $m_{ab} \neq \infty$. If m_{ab} is even, or if a and b are both in S_1 or in S_2, the equality holds. Suppose $m_{ab} = 2k+1$, and $a \in S_1$. Then $\varphi^n([a,b\rangle^{m_{ab}}) = (\varphi^n(b))^k$ and $\varphi^n([b,a\rangle^{m_{ab}}) = (\varphi^n(b))^{k+1}$. Thus, by cancellation, $\varphi^n(b) = 1$, and $b \in S_1$.

The case of Artin monoids

Let $M = \langle S \mid [a,b\rangle^{m_{ab}} = [b,a\rangle^{m_{ab}};\ m_{ab} \neq \infty \rangle^{+}$ be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Artin monoids are preGarside (Brieskorn, Saito 1972).

Proposition

The morphism π is well-defined.

Preuve : It suffices to check that $\pi([a,b\rangle^{m_{ab}}) = \pi([b,a\rangle^{m_{ab}})$ for all $m_{ab} \neq \infty$. If m_{ab} is even, or if a and b are both in S_1 or in S_2, the equality holds. Suppose $m_{ab} = 2k+1$, and $a \in S_1$. Then $\varphi^n([a,b\rangle^{m_{ab}}) = (\varphi^n(b))^k$ and $\varphi^n([b,a\rangle^{m_{ab}}) = (\varphi^n(b))^{k+1}$. Thus, by cancellation, $\varphi^n(b) = 1$, and $b \in S_1$.

The case of Artin monoids

Let $M = \langle S \mid [a,b\rangle^{m_{ab}} = [b,a\rangle^{m_{ab}};\ m_{ab} \neq \infty \rangle^+$ be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Artin monoids are preGarside (Brieskorn, Saito 1972).

Proposition

The morphism π is well-defined.

Preuve : It suffices to check that $\pi([a,b\rangle^{m_{ab}}) = \pi([b,a\rangle^{m_{ab}})$ for all $m_{ab} \neq \infty$. If m_{ab} is even, or if a and b are both in S_1 or in S_2, the equality holds. Suppose $m_{ab} = 2k+1$, and $a \in S_1$. Then $\varphi^n([a,b\rangle^{m_{ab}}) = (\varphi^n(b))^k$ and $\varphi^n([b,a\rangle^{m_{ab}}) = (\varphi^n(b))^{k+1}$. Thus, by cancellation, $\varphi^n(b) = 1$, and $b \in S_1$.

The case of Artin monoids

Let $M = \langle S \mid [a,b\rangle^{m_{ab}} = [b,a\rangle^{m_{ab}};\ m_{ab} \neq \infty\rangle^+$ be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Artin monoids are preGarside (Brieskorn, Saito 1972).

Proposition

The morphism π is well-defined.

Preuve : It suffices to check that $\pi([a,b\rangle^{m_{ab}}) = \pi([b,a\rangle^{m_{ab}})$ for all $m_{ab} \neq \infty$. If m_{ab} is even, or if a and b are both in S_1 or in S_2, the equality holds. Suppose $m_{ab} = 2k+1$, and $a \in S_1$. Then $\varphi^n([a,b\rangle^{m_{ab}}) = (\varphi^n(b))^k$ and $\varphi^n([b,a\rangle^{m_{ab}}) = (\varphi^n(b))^{k+1}$. Thus, by cancellation, $\varphi^n(b) = 1$, and $b \in S_1$.

The case of Artin monoids

Let $M = \langle S \mid [a,b\rangle^{m_{ab}} = [b,a\rangle^{m_{ab}};\ m_{ab} \neq \infty\rangle^+$ be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Artin monoids are preGarside (Brieskorn, Saito 1972).

Proposition

The morphism π is well-defined.

Preuve : It suffices to check that $\pi([a,b\rangle^{m_{ab}}) = \pi([b,a\rangle^{m_{ab}})$ for all $m_{ab} \neq \infty$. If m_{ab} is even, or if a and b are both in S_1 or in S_2, the equality holds. Suppose $m_{ab} = 2k+1$, and $a \in S_1$. Then $\varphi^n([a,b\rangle^{m_{ab}}) = (\varphi^n(b))^k$ and $\varphi^n([b,a\rangle^{m_{ab}}) = (\varphi^n(b))^{k+1}$. Thus, by cancellation, $\varphi^n(b) = 1$, and $b \in S_1$.

The case of Artin monoids

Let $M = \langle S \mid [a,b\rangle^{m_{ab}} = [b,a\rangle^{m_{ab}};\ m_{ab} \neq \infty\rangle^+$ be an Artin monoid, and $\varphi \in \mathrm{End}(M)$. Artin monoids are preGarside (Brieskorn, Saito 1972).

Proposition

The morphism π is well-defined.

Preuve : It suffices to check that $\pi([a,b\rangle^{m_{ab}}) = \pi([b,a\rangle^{m_{ab}})$ for all $m_{ab} \neq \infty$. If m_{ab} is even, or if a and b are both in S_1 or in S_2, the equality holds. Suppose $m_{ab} = 2k+1$, and $a \in S_1$. Then $\varphi^n([a,b\rangle^{m_{ab}}) = (\varphi^n(b))^k$ and $\varphi^n([b,a\rangle^{m_{ab}}) = (\varphi^n(b))^{k+1}$. Thus, by cancellation, $\varphi^n(b) = 1$, and $b \in S_1$.

The case of Artin monoids

Theorem

Theorem

$\mathrm{Fix}(\varphi)$ *and* $\mathrm{Per}(\varphi)$ *are also Artin monoids.*

Exemples

- For $M = \langle s, t, u \mid st = ts,\ sus = usu,\ tut = utu\rangle^+$, with $\varphi(u) = u$, $\varphi(t) = s$ and $\varphi(s) = t$. One has $\mathrm{Fix}(\varphi) = \langle u, st\rangle^+$.
- Let $M = \langle a, b, c \mid abab = baba,\ ac = ca\rangle^+$, and $\varphi \in \mathrm{End}(M)$ such that $\varphi(a) = b$, $\varphi(b) = a$ and $\varphi(c) = 1$. We have $n = 1$, $S_1 = \{c\}$, $S_2 = \{a, b\}$, $M_2 = \langle a, b \mid abab = baba\rangle^+$ and $\varphi_2 \in \mathrm{End}(M_2)$, with $\varphi_2(a) = b$ and $\varphi_2(b) = a$. Then $\mathrm{Fix}(\varphi_2) = \langle abab\rangle^+$, and $\mathrm{Fix}(\varphi) = \varphi(\mathrm{Fix}(\varphi_2)) = \mathrm{Fix}(\varphi_2) = \langle abab\rangle^+$.

The case of Artin monoids

Theorem

Theorem

$\mathrm{Fix}(\varphi)$ *and* $\mathrm{Per}(\varphi)$ *are also Artin monoids.*

Exemples

- For $M = \langle s, t, u \mid st = ts,\ sus = usu,\ tut = utu \rangle^+$, with $\varphi(u) = u$, $\varphi(t) = s$ and $\varphi(s) = t$. One has $\mathrm{Fix}(\varphi) = \langle u, st \rangle^+$.
- Let $M = \langle a, b, c \mid abab = baba,\ ac = ca \rangle^+$, and $\varphi \in \mathrm{End}(M)$ such that $\varphi(a) = b$, $\varphi(b) = a$ and $\varphi(c) = 1$. We have $n = 1$, $S_1 = \{c\}$, $S_2 = \{a, b\}$, $M_2 = \langle a, b \mid abab = baba \rangle^+$ and $\varphi_2 \in \mathrm{End}(M_2)$, with $\varphi_2(a) = b$ and $\varphi_2(b) = a$. Then $\mathrm{Fix}(\varphi_2) = \langle abab \rangle^+$, and $\mathrm{Fix}(\varphi) = \varphi(\mathrm{Fix}(\varphi_2)) = \mathrm{Fix}(\varphi_2) = \langle abab \rangle^+$.

The case of Artin monoids

Theorem

$\mathrm{Fix}(\varphi)$ *and* $\mathrm{Per}(\varphi)$ *are also Artin monoids.*

Exemples

- For $M = \langle s, t, u \mid st = ts,\ sus = usu,\ tut = utu \rangle^+$, with $\varphi(u) = u$, $\varphi(t) = s$ and $\varphi(s) = t$. One has $\mathrm{Fix}(\varphi) = \langle u, st \rangle^+$.
- Let $M = \langle a, b, c \mid abab = baba,\ ac = ca \rangle^+$, and $\varphi \in \mathrm{End}(M)$ such that $\varphi(a) = b$, $\varphi(b) = a$ and $\varphi(c) = 1$. We have $n = 1$, $S_1 = \{c\}$, $S_2 = \{a, b\}$, $M_2 = \langle a, b \mid abab = baba \rangle^+$ and $\varphi_2 \in \mathrm{End}(M_2)$, with $\varphi_2(a) = b$ and $\varphi_2(b) = a$. Then $\mathrm{Fix}(\varphi_2) = \langle abab \rangle^+$, and $\mathrm{Fix}(\varphi) = \varphi(\mathrm{Fix}(\varphi_2)) = \mathrm{Fix}(\varphi_2) = \langle abab \rangle^+$.