Actions of diagonalizable group schemes

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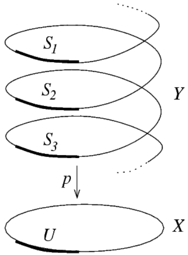
Coverings in Topology

A morphism of topological spaces (or differential manifolds) $f : Y \longrightarrow X$ is a covering map if there exists a discrete space F such that every point $x \in X$ has an open neighborhood U with $f^{-1}(U) \simeq U \times F$.



If G is a finite group acting freely on a topoogical space (or differential manifold) we get a covering map $Y \longrightarrow Y/G$. Locally we have

$$Y \simeq Y/G \times G.$$

Such maps are called G-torsors.

The theory of topological covering spaces is very rich (homotopy and fundamental group, Galois correspondence, inverse problems...).

Coverings in algebraic geometry

The concept of fibre bundle is of little use in algebraic geometry because the Zariski topology is too coarse (open sets are too big).

Still there is a notion of algebraic coverings : the finite étale morphisms. They behave in many ways like topological coverings.

If G is a finite algebraic group acting freely on an algebraic variety Y, we get an algebraic covering $Y \longrightarrow Y/G$. Locally (for the étale topology) we have $Y \simeq Y/G \times G$. Such maps are called G-torsors.

There exists a Galois theory for algebraic coverings, similar to the topological one (fundamental group, universal cover, Galois correspondence...)

Ramified coverings

We are interested in morphisms that are almost covering maps : the ramified coverings. These are morphisms $f : Y \longrightarrow X$ for which there exists a dense open subset $V \subset X$ such that $f_{|f^{-1}(V)}$ is a covering.

If G is a finite group acting freely on a dense open subset of a space Y we get a ramified covering $Y \longrightarrow Y/G$.

Riemann-Hurwitz formula

If $f : Y \longrightarrow X$ is a ramified covering of projective algebraic curves then the Riemann-Hurwitz formula relates the ramification locus with the genus of Y and X :

$$2\,g(Y) - 2 = \deg(f)\,(2\,g(X) - 2) + \deg(R_f)$$

where R_f is the ramification divisor of f.

The ramification indices $\deg(R_f)_P$ can be computed using number theory (cf J.P-Serre, *Local Fields*)

Riemann-Hurwitz formula

The RH formula comes from the exact sequence of differentials

$$0 \longrightarrow f^*\Omega^1_{X/k} \longrightarrow \Omega^1_{Y/k} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

valid for every ramified covering of algebraic k-varieties

$$f : Y \longrightarrow X.$$

Riemann-Hurwitz formula

Fact

f is unramified $\Leftrightarrow \Omega^1_{Y/k} \simeq f^*\Omega^1_{X/k} \Leftrightarrow \Omega^1_{Y/X} = 0.$

$\Omega^1_{Y/X}$ measures how far f is from being a covering. It completely determines R_f : we have $\Omega^1_{Y/X} = \mathcal{O}_X(R_f)$.

This applies in particular to finite algebraic group actions.

If char$(k) = p > 0$ the class of algebraic k-groups is not well-behaved. One has to work within the more general context of *group schemes*.

Contrary to algebraic groups, they can be non-smooth.

Group scheme actions

A k-group scheme is a k-scheme G such that, for every k- scheme T, the set $G(T) := \mathrm{Hom}_k(T, G)$ has a group structure.

An action of G on a k-scheme Y is a map $G \times Y \longrightarrow Y$ such that, for every k-scheme T, the induced map $G(T) \times Y(T) \longrightarrow Y(T)$ is a group action.

One can define the quotient of Y by G : it is a k-scheme Y/G such that, for every k-scheme T, $(Y/G)(T) = Y(T)/G(T)$.

One can also define free (resp. transitive, resp....) actions in the same way.

p-roots of unity in char. p

Suppose $\text{char}(k) = p > 0$. Consider the Frobenius morphism

$$\begin{array}{ccc} \mathbb{G}_{m,k} & \longrightarrow & \mathbb{G}_{m,k} \\ x & \mapsto & x^p \end{array}$$

Let $\mu_{p,k} = \text{Spec}(k[t]/t^p - 1)$ be its kernel. It has only one point.

Yet it is non-trivial !

Ramified μ_p-covers

As in the above cases, we define a μ_p-cover to be a morphism of k-schemes $Y \longrightarrow X$ such that

- μ_p acts on Y
- $Y/\mu_p \simeq X$
- The action in free on a dense open subset of Y.

These morphisms are generically ramified (they are "really not étale"), even when the action is free everywhere on Y.

Our previous notion of ramification is not appropriate anymore.

RH formula for μ_p-covers

We still want to relate the locus of points where the action is not free to the geometry of $Y \longrightarrow X$. We have to find substitutes for both Ω^1 and the ramification divisor.

We choose to replace :

- $\Omega^1_{Y/X}$ by Grothendieck's dualizing sheaf

$$\omega_{Y/X} := f^! \mathcal{O}_X = \mathrm{Hom}_{\mathcal{O}_Y}(f_* \mathcal{O}_Y, \mathcal{O}_X)$$

- R_f by the "fixed point divisor" : the divisor defined by the codimension 1 part of the fixed point scheme. The latter is a scheme Y^{μ_p} such that, for every k-scheme T, $Y^{\mu_p}(T)$ is the subset of $Y(T)$ fixed by $\mu_p(T)$.

RH formula for μ_p-covers

We can now state a kind of RH formula for μ_p-covers :

Theorem (Z.)

$$\omega_{X/Y} = \mathcal{O}_X((p-1)R_f)$$

where R_f is the fixed point divisor.

Actually, the same is true for a larger class of groups : the diagonalisable ones. These are the k-group schemes whose structure sheaves are of the form $k[M]$ for some finite p-group M.

Example

Let μ_p act on $\mathbb{P}^1_k$ by multiplication. As k-schemes, $\mathbb{P}^1_k/\mu_p \simeq \mathbb{P}^1_k$.

- The quotient map $f : \mathbb{P}^1_k \longrightarrow \mathbb{P}^1_k$ is the absolute Frobenius. It has degree p.
- The fixed point divisor is $R_f = [0] + [\infty]$. It has degree 2.
- The dualizing sheaf is $\omega_f = \mathcal{O}_{\mathbb{P}^1_k}(2p - 2)$

One checks the "Riemann-Hurwitz formula"

$$-2 = p \times (-2) + 2p - 2$$