Parameterized Kovacic's algorithm

Thomas Dreyfus

University Paris 7

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Abstract

To the parameterized equation:

$$\partial_X^2 Y(X,t) = r(X,t)Y(X,t),$$

we associate a Galois group G that measures the algebraic and differentials relations between the solutions. We want to explain how to compute G.

- We consider: $\partial_X^2 Y(X) = r(X)Y(X),$ where $r(X) \in \mathbb{C}(X)$.
- We associate an algebraic subgroup H of $\mathrm{SL}_2(\mathbb{C})$ to this equation, which measure the algebraic relations between the solutions. We call H the differential Galois group.

Kovacic's algorithm uses the classification of the algebraic subgroups of $SL_2(\mathbb{C})$ to find the Liouvillian solutions, which are the solutions that involve exponentials, indefinite integrals and solutions of polynomial equations. There are four possibilities.

1. H is conjugated to a subgroup of $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \text{ where } a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$ and $\exists f(X) \in \mathbb{C}(X)$ such that $e^{\int_0^X f(u)du}$ is solution.
2. H is conjugated to a subgroup of $D_\infty = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cup \begin{pmatrix} 0 & b^{-1} \\ -b & 0 \end{pmatrix}, \text{ where } a, b \in \mathbb{C}^* \right\}$ and $\exists f(X) \notin \mathbb{C}(X)$, algebraic over $\mathbb{C}(X)$ of degree two, such that $e^{\int_0^X f(u)du}$ is solution.
3. H is finite and all the solutions are algebraic over $\mathbb{C}(X)$.
4. $H = \mathrm{SL}_2(\mathbb{C})$ and there are no Liouvillian solutions.

Let C be a differential field equipped with n commuting derivations: $\partial_1, \ldots, \partial_n$ and let $\Delta = \{\partial_1, \ldots, \partial_n\}$. We will assume that C is an universal (Δ)-field with characteristic 0.
We consider the parameterized equation

$$\begin{pmatrix} \partial_X Y(X) \\ \partial_X^2 Y(X) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r(X) & 0 \end{pmatrix} \begin{pmatrix} Y(X) \\ \partial_X Y(X) \end{pmatrix}, \text{ with } r(X) \in C(X). \quad (1)$$

The derivations in Δ could be seen as derivations with respect to the parameters.

$$\begin{pmatrix} \partial_X Y(X) \\ \partial_X^2 Y(X) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r(X) & 0 \end{pmatrix} \begin{pmatrix} Y(X) \\ \partial_X Y(X) \end{pmatrix}, \text{ with } r(X) \in C(X). \quad (1)$$

- A parameterized Picard-Vessiot extension for (1) is a (∂_X, Δ)-differential field extension generated by the entries of an invertible solution matrix.
- This extension for (1) exists and is unic.

Let U be an invertible solution matrix of (1) and let $\widetilde{C(X)}|C(X)$ denotes the parameterized Picard-Vessiot extension.

- The parameterized differential Galois group G, is the group of (∂_X, Δ)-differential field automorphism of $\widetilde{C(X)}$ letting $C(X)$ invariant.
- The image of $\{U^{-1}\varphi(U), \varphi \in G\}$, is a linear differential algebraic subgroup of $\mathrm{SL}_2(\mathrm{C})$: this is the zero of a set of (Δ)-differential polynomials in 4 variables.
- The Zariski closure of G is H, the classical differential Galois group.

$$\begin{pmatrix} \partial_X Y(X) \\ \partial_X^2 Y(X) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r(X) & 0 \end{pmatrix} \begin{pmatrix} Y(X) \\ \partial_X Y(X) \end{pmatrix}, \text{ with } r(X) \in C(X). \quad (1)$$

- The original Kovacic's algorithm can be used for equations having rational coefficient in an algebraic closed field.
- C algebraically closed $\Rightarrow$ we apply Kovacic's algorithm to compute H.
- In each of the four cases of the algorithm, we have to find G, which is Zariski dense in H.

Using a result of Seidenberg, a finite (Δ)-differential field extension of $\mathbb{Q}$ generated by elements of C can be interpreted as a subfield of $(\mathcal{M}_D, \partial_{t_1}, \ldots, \partial_{t_n})$ of meromorphic functions on D, a poly-disk of $\mathbb{C}^n$. Then:

- The parameterized equation can be seen as an equation with coefficients in $\mathcal{M}_D(X)$.
- We will compute G as a subgroup of $\mathrm{SL}_2(\mathcal{M}_\mathrm{D})$.
- The Liouvillian solutions found are defined over the algebraic closure of $\mathcal{M}_D$.

Case 1

Let $t = (t_1, \dots, t_n)$. $\exists f(X,t) \in \mathcal{M}_D(X)$ such that:

$$g(X,t) = e^{\int_0^X f(u,t)du} \text{ is solution.}$$

If $g(X,t) \in \mathcal{M}_D(X)$, we can compute explicitly another solution, $g(X,t)\int_{u=0}^{X} g(u,t)^{-2}du$ and G.

In the other case, there exists $M \subset \mathrm{GL}_1(\mathcal{M}_D)$, and $P_1, \dots, P_k$ linear differential polynomials in coefficients in $\mathcal{M}_D$ such that

$$G \simeq \left\{ \begin{pmatrix} m(t) & a(t) \\ 0 & m(t)^{-1} \end{pmatrix}, \text{ where } m(t) \in M, P_i(a(t)) = 0\,, \forall i \right\}.$$

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Example

Let us consider $\partial_X^2 Y(X,t) = \frac{t}{X^2} Y(X,t)$. We have two Liouvillian solutions:

$$f_1(X,t) = \sqrt{X} X^{\frac{\sqrt{1+4t}}{2}} \textit{ and } f_2(X,t) = \sqrt{X} X^{-\frac{\sqrt{1+4t}}{2}}.$$

$$\begin{aligned} G &\simeq \left\{ \begin{pmatrix} be^{a(\sqrt{1+4t})} & 0 \\ 0 & b^{-1}e^{-a(\sqrt{1+4t})} \end{pmatrix}, \text{where } a \in \mathbb{C}\,, b \in \mathbb{C}^* \right\} \\ &\simeq \left\{ \begin{pmatrix} \alpha(t) & 0 \\ 0 & \alpha^{-1}(t) \end{pmatrix}, \text{where } \partial_t \left(\frac{\sqrt{1+4t}\,\partial_t \alpha(t)}{\alpha(t)} \right) = 0 \right\}. \end{aligned}$$

Case 2

In this case, there are two Liouvillian solutions of the form:

$$g_i(X,t) = e^{\int_0^X f_i(u,t)du}, i \in \{1,2\}$$

where $f_i(X,t)^2 + a(X,t)f_i(X,t) + b(X,t) = 0$.
Assume $n = 1$. $\exists \widetilde{P_1}, \ldots, \widetilde{P_k}$ linear differential polynomials with coefficients in $\mathcal{M}_D$ such that:

$$G \simeq \left\{ \begin{pmatrix} a(t) & 0 \\ 0 & a^{-1}(t) \end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1}(t) \\ -b(t) & 0 \end{pmatrix} \right\},$$

where $\widetilde{P_i}\left(\frac{\partial_t a(t)}{a(t)}\right) = \widetilde{P_i}\left(\frac{\partial_t b(t)}{b(t)}\right) = 0$ for all i.

Example

Let us consider $\partial_X^2 Y(X,t) = \left(\frac{t}{X} - \frac{3}{16X^2}\right) Y(X,t)$. We have two Liouvillian solutions:

$$f_1(X,t) = (X)^{1/4} e^{2(tX)^{1/2}} \text{ and } f_2(X,t) = (X)^{1/4} e^{-2(tX)^{1/2}}.$$

In that basis:

$$G \simeq \left\{ \begin{pmatrix} a(t) & 0 \\ 0 & a^{-1}(t) \end{pmatrix} \bigcup \begin{pmatrix} 0 & b^{-1}(t) \\ -b(t) & 0 \end{pmatrix}, \text{ where } a(t), b(t) \in \mathbb{C}^* \right\}.$$

Case 3

In this case where H is finite, and H=G.

Case 4

In this case, G is Zariski dense in $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$. There exists $\mathbb{D}$, a vectorial space spanned by the derivations such that $G \simeq \mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}})$, where

$$\mathcal{M}_D^{\mathbb{D}} := \{f(t) \in \mathcal{M}_D | \forall \partial \in \mathbb{D}, \partial f(t) = 0\}.$$

Proposition (1 $\Leftrightarrow$ 2, Cassidy/Singer, 2 $\Leftrightarrow$ 3 $\Leftrightarrow$ 4, D)

We have the following equivalences:

1. *G is conjugated to $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}})$ over $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$.*
2.
3.
4.

Proposition ($1 \Leftrightarrow 2$, Cassidy/Singer, $2 \Leftrightarrow 3 \Leftrightarrow 4$, D)

We have the following equivalences:

1. G *is conjugated to* $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}})$ *over* $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$.
2. $\exists \partial_1, \ldots, \partial_k$, *commutating basis of* $\mathbb{D}$, $\exists A_1(X,t), \ldots, A_k(X,t) \in \mathrm{GL}_2(\mathcal{M}_{\mathrm{D}})$ *such that* $\forall 0 \leq i, j \leq k$:

$$\partial_j A_i(X,t) - \partial_i A_j(X,t) = A_j(X,t)A_i(X,t) - A_i(X,t)A_j(X,t),$$

where $A_0(X,t) = \begin{pmatrix} 0 & 1 \\ r(X,t) & 0 \end{pmatrix}$ *and* $\partial_0 = \partial_X$.

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Proposition (1 $\Leftrightarrow$ 2, Cassidy/Singer, 2 $\Leftrightarrow$ 3 $\Leftrightarrow$ 4, D)

We have the following equivalences:

1. G *is conjugated to* $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}})$ *over* $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$.
2.
3. *For all* $\partial' \in \mathbb{D}$, *there exists* $A_1(X,t) \in \mathrm{GL}_2(\mathcal{M}_{\mathrm{D}})$ *such that*
$$\partial' A_0(X,t) - \partial_X A_1(X,t) = A_0(X,t)A_1(X,t) - A_1(X,t)A_0(X,t).$$
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Proposition ($1 \Leftrightarrow 2$, Cassidy/Singer, $2 \Leftrightarrow 3 \Leftrightarrow 4$, D)

We have the following equivalences:

1. G *is conjugated to* $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}}^{\mathbb{D}})$ *over* $\mathrm{SL}_2(\mathcal{M}_{\mathrm{D}})$.
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3.
4. *For all* $\partial' \in \mathbb{D}$*, the following parameterized differential equation has a solution in* $\mathcal{M}_D(X)$*:*

$$\frac{\partial_X^3 b(X,t)}{2} = 2\partial_X b(X,t) r(X,t) + b(X,t)\partial_X r(X,t) - \partial' r(X,t).$$

Example

Let us consider $\partial_X^2 Y(X,t) = \left(X^{2n+1} + \sum_{i=0}^{2n} t_i X^i \right) Y(X,t)$.

We find: $G \simeq \mathrm{SL}_2\left(\mathcal{M}_{\mathrm{D}}^{\partial_{t'}}\right)$, where

$$\partial_{t'} = (2n+1)\partial_{t_{2n}} + \sum_{i=0}^{2n-1} (i+1) t_{i+1} \partial_{t_i}.$$