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Small divisors for quasiperiodic linear equations

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INTER'ACTIONS EN MATHÉMATIQUES

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Diophantine numbers

- A vector is said **rationally independant** if

$$\forall m \in \mathbb{Z}^d \setminus \{0\}, \quad (m;\omega) \neq 0.$$

- A vector $\omega \in \mathbb{R}^d$ is said **diophantine**, denoted by $\omega \in DC(K,\tau)$ if

$$\forall m \in \mathbb{Z}^d, \quad |(m;\omega)| \geqslant \frac{K}{|m|^\tau}.$$

Rk. *These sets are « large » for the Lebesgue measure.*

We will take $\boxed{\omega \in DC(K_\omega,\tau)}$

Application. Ergodization time.

Quasi-periodic functions

- A function f is **quasiperiodic** if $\exists\, \omega \in \mathbb{R}^d$ and $F \in \mathcal{C}^0(\mathbb{T}^d, \mathbb{R}^n)$ such that

$$f(t) = F(t\omega) = F(t\omega_1, t\omega_2, \ldots, t\omega_d).$$

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Quasi-periodic functions

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$$f(t) = F(t\omega) = F(t\omega_1, t\omega_2, \ldots, t\omega_d).$$
- We consider the skew-product
$$\mathcal{S} \left\{ \begin{array}{rcl} X'(t) & = & F(\rho)X(t) \\ \rho' & = & \omega \end{array} \right. \quad \text{with} \quad F \in \mathcal{C}^a_r\left(\mathbb{T}^d, \mathcal{M}_p(\mathbb{R})\right).$$
- **Example.** The SCHRÖDINGER equation with quasiperiodic potential.

Remark. X is the fondamental matrix.

Reducibility

- $\mathcal{S}_1$ and $\mathcal{S}_2$ are conjugated if $\exists\, Y \in \mathcal{C}^a_r\left(\mathbb{T}^d, G\right)$ such that

$$\partial_\omega Y := (\nabla Y;\omega) = F_1 Y - Y F_2 \qquad (\mathcal{R})$$

- A system is said **reducible** if it is conjugated to a constant system

$$\partial_\omega Y = FY - YA, \qquad A \in \mathfrak{g}.$$

- If two systems are conjugated : $X_1(t) = Y(t\omega)X_2(t)$,
- Reducibility implies FLOQUET solutions.

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$\Rightarrow$ **All the systems are reducible ?**

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- If two systems are conjugated : $X_1(t) = Y(t\omega) X_2(t)$,
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$\Rightarrow$ **All the systems are reducible ?**

$\Rightarrow$ **What can be said in the perturbative case $A + \varepsilon F(t\omega)$?**

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Local picture

$$\mathcal{S}\left\{\begin{array}{lllr} X' & = & (\lambda A + F)X \quad \text{with} & |F|_r := \sup_{|\Im m(z)|\leqslant r} |F(z)| \\ \rho' & = & \omega & \omega \in DC(K_\omega, \tau) \end{array}\right.$$

We can check in many cases (when the group is compact)

- There is $\varepsilon_0(r, \omega, \|A\|_2, p)$ such that if $|F|_r \leqslant \varepsilon_0$:

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We can check in many cases (when the group is compact)

- There is $\varepsilon_0(r, \omega, \|A\|_2, p)$ such that if $|F|_r \leqslant \varepsilon_0$:

  1. Almost all the system are reducible,
  2. For every λ, there is $\mathcal{G}_\delta$-dense of non reducible systems.

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The Newton-K.A.M scheme

What does mean K.A.M ?

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Principle in the perturbative case

- Step 0. We start with the system

$$\partial_t X_1(t) = (A_1 + F_1(\varphi))X_1(t), \quad \varphi' = \omega, \quad F_1 \text{ « small »}.$$

- Step 1. Find a conjugate Y_1 such that $X_2 = Y_1 X_1$ and

$$\partial_t X_2(t) = (A_2 + F_2(\varphi))X_2(t), \qquad \|F_2\| \ll \|F_1\|.$$

- Step j. And so on to cancel the perturbation...

Which implies

- Solve $\partial_\omega Y_{j+1} = (A_{j+1} + F_{j+1})Y_{j+1} - Y_{j+1}(A_j + F_j)$.
- The convergence of the product $\prod_{j=1}^{n} Y_j \Rightarrow Y_j \sim I_p$.

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How to solve $(\mathcal{R})$?

$$A_2 + F_2 = \partial_\omega Y_1 Y_1^{-1} + \mathrm{Ad}(Y_1)(A_1 + F_1)$$

$$\Downarrow \qquad Y_1 = e^Y \simeq I_p + Y$$

$$A_2 = \partial_\omega Y(I_p - Y) + (\mathrm{id} + \mathrm{ad}(Y))(A_1 + F_1)$$

$$\Downarrow$$

$$\partial_\omega Y - [A_1, Y] = A_2 - A_1 - F_1$$

$$\Downarrow \qquad A_2 = A_1 + \widehat{F_1}(0)$$

$$\partial_\omega Y = [A_1, Y] + F_1 - \widehat{F_1}(0)$$

Small divisors problem

The simplified linearized equation is : $y \in \mathbb{T} \mapsto \mathbb{C}$

$$\partial_\omega y = \alpha y + f \quad \text{with} \quad \alpha \in \mathrm{Sp}(\mathrm{ad}_{A_j}).$$

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- This equation can be solved in Fourier series, $n \in \mathbb{Z}^d$

$$\widehat{y}(n) = \frac{\widehat{f}(n)}{\mathrm{i}\,(n;\omega) - \alpha}.$$

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- This equation can be solved in Fourier series, $n \in \mathbb{Z}^d$

$$\widehat{y}(n) = \frac{\widehat{f}(n)}{\mathrm{i}\,(n;\omega) - \alpha}.$$

- We need the diophantine conditions :

$$|\mathrm{i}\,(n;\omega) - \alpha| \geqslant \frac{K}{|n|^\tau}$$

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Diophantine case

Proposition (Resolution of the linearized equation)**.**

Let $A \in \mathfrak{g}$ and $F \in \mathcal{C}_r^a\left(\mathbb{T}^d, \mathfrak{g}\right)$. We suppose that

$$|\mathrm{i}(\omega; n) - \alpha| \geqslant K \cdot |n|^{-\tau}.$$

Then there is $Y \in \mathcal{C}_s^a\left(\mathbb{T}^d, \mathfrak{g}\right)$ solution of

$$\partial_\omega Y = \mathrm{ad}_A(Y) + F - \widehat{F}(0)$$

$$|Y|_s \leqslant c \frac{|F|_r}{K(r-s)^{2\tau}} \qquad \text{with} \qquad s < r$$

Remark. *The conjugate is given by*

$$A + \varepsilon F \xrightarrow{Y_1} A_1 + \varepsilon^2 F_1 \qquad \textit{with} \qquad Y_1 = e^Y \sim I_p + O(\varepsilon).$$

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How to avoid small divisors ?

- By a choice of the matrix A

$$\lambda - \mu \notin \mathrm{i}\,\mathbb{R}^* \qquad \text{with} \qquad \lambda, \mu \in \mathrm{Sp}(A).$$

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- By withdrawing a small set of parameter

$$\Omega_{j+1} = \Omega_j \setminus \{\omega \,:\, \mathrm{ad}_{A_j} \not\subset DC_\omega(K_j, \tau)\}.$$

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- By a choice of the matrix A

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$$\Omega_{j+1} = \Omega_j \setminus \{\omega \,:\, \mathrm{ad}_{A_j} \not\subset DC_\omega(K_j, \tau)\}.$$

- By conjugating far from identity

$$Y_j \not\sim I_p + O(\varepsilon_j) \qquad \text{and} \qquad \|F_{j+1}\| \ll \|F_j\|.$$

- . . .

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Examples of reducibility results

Let $A \in \mathfrak{g}$ and $F \in \mathcal{C}_r^a(\mathbb{T}^d, \mathfrak{g})$.

Theorem (If we choose A)**.**

If the real part of the eigenvalues of A are different and if $\varepsilon \leqslant \varepsilon^*$ then the system is $\mathcal{C}_{r/2}^a$-reducible.

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$\Rightarrow$ **If G is compact ? $\mathfrak{g} = \mathfrak{so}_p$ or $\mathfrak{u}_p$?**

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Theorem (If we choose ω).

There are ε_* and a CANTOR set $\Pi_\varepsilon \subset [0;1]^d$ of large measure

$$\mathrm{Leb}_d\left([0;1]^d \setminus \Pi_\varepsilon\right) \leqslant {\varepsilon_*}^{\mathrm{C^{te}}}$$

such that if $\omega \in \Pi_\varepsilon$ and $\varepsilon \leqslant \varepsilon_*$ then the system is reducible.

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The Schrödinger equation

$$\begin{cases} u = u(t,x) & t \in \mathbb{R},\ x \in \mathbb{T} \\ \mathrm{i}\,\partial_t u = \Delta u + \varepsilon P(t\omega, x) u \end{cases}$$

We study the growth of the SOBOLEV norm

$$\psi : \mathbb{T}^m \mapsto \mathbb{C}, \qquad \|\psi\|_{H^s(\mathbb{T}^m)} = \left(\sum_{n\in\mathbb{Z}^m} \langle n\rangle^s |\hat{\psi}(n)|^2 \right)^{\frac{1}{2}}$$

with $\langle n\rangle = (1+|n|^2)^{1/2}$. Then we get the system on $\ell^2(\mathbb{Z})$

$$\mathcal{S} \begin{cases} X' = (\mathrm{i}\,D + F(\rho))X \\ \rho' = \omega \end{cases}$$

with F TOEPLTIZ and D diagonal.

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- What can be said in the Lie group ?
- Infinite dimension for others systems ?
- K.A.M and the stability of the solar system.
- ...

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Thank you for your attention

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