MOMENT MAP IN DEFORMATION QUANTIZATION.

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Abstract.

We consider a Hamiltonian action of a group on a symplectic manifold with a given moment map. Provided with an equivariant connection on this manifold we show that the action in the deformed algebra is the same, namely by the Poisson bracket. We employ Fedosov's construction of deformation quantization.

1. Deformation quantization of a symplectic manifold.

Deformation quantization [2] is a formal way to deform a commutative product on the algebra of functions on a symplectic (or Poisson) manifold. Recently Kontsevich [6] proved through formality theorem that every Poisson structure can be quantized. We are concerned now with the question of the lifting the group action on the manifold to the algebra of deformed functions. We show how to construct a quantum moment map to such a deformed algebra on a symplectic manifold if the deformation is equivariant.
A noncommutative associative product on M, a $*$–product, is defined as a product on the algebra of series in a formal variable $\hbar$ with coefficients in $C^\infty(M)$. This noncommutative algebra $\mathbb{A}^\hbar$ should be a deformation of the algebra of functions on the manifold, $C^\infty(M)$. Let (M,ω) be a symplectic manifold. Then the symplectic form ω defines a Lie–algebra structure on $C^\infty(M)$, namely the Poisson structure. The Jacobi identity follows from closedness of the form ω. For $f,g \in C^\infty(M)$ let $\{f,g\} = (df)^\sharp(g)$, where $\sharp : \mathcal{T}^*M \to \mathcal{T}M$, defined by ω.

Definition 1. Deformation quantization of a symplectic manifold (M,ω) is an associative algebra $(\mathbb{A}^\hbar, *)$ over $\mathbb{C}[[\hbar]]$ with an isomorphism $\epsilon : \mathbb{A}^\hbar/\hbar\mathbb{A}^\hbar \to C^\infty(M)$ s.t.

1. $\mathbb{A}^\hbar$ is flat as a $\mathbb{C}[[\hbar]]$–module,
2. $\mathbb{A}^\hbar$ is separable and complete in $\hbar$–topology,
3. for any $f,g \in C^\infty(M)$
$$\epsilon\left(\frac{i}{\hbar}(\check{f} * \check{g} - \check{g} * \check{f})\right) = \{f,g\}$$
for $\check{f}$ and $\check{g}$ s.t. $\epsilon\{\check{f}\} = f$ and $\epsilon\{\check{g}\} = g$,
4. The structure of the product $*$ on $\mathbb{A}^\hbar/\hbar^n\mathbb{A}^\hbar$ for all $n \geq 0$ is given by bidifferential operators.

We introduce a notion of an F–manifold:

Definition 2. An *F–manifold* is the following triple (manifold, deformation of a symplectic form, a connection) :
$$(M, \Omega, \nabla),$$
where the class of Ω in $H^2(M)[[\hbar]]$ is called *characteristic class of deformation.* $\Omega = \omega + \hbar\alpha$, and $\alpha \in \Gamma(M, \Lambda^2 T^* M)[[\hbar]]$, a series in $\hbar$ with coefficients being closed 2–forms on the manifold.

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In a recent article [5] a similar object is called a Fedosov manifold, namely, a symplectic manifold together with a torsion free symplectic connection. Indeed, these three objects (M, ω, ∇) define the first three terms in the $*$–product:

- Classically, a manifold can be considered as an algebra of functions on it, that is M defines the structure of the commutative product.
- A symplectic form defines the Poisson structure and hence the term at $\hbar$.
- A connection defines the term at $\hbar^2$ (as follows from [8]).

It turns out that these three terms determine the higher terms. However, in general it is not necessary that the connection is symplectic, for any connection there exists a suitable way to construct a torsion free symplectic connection [9]. We formulate the theorem of deformation quantization as follows

Theorem 3. *An F–manifold (M, Ω, ∇) uniquely determines a $*$-product on the underlying manifold M.*

Fedosov's construction involves the Weyl algebra bundle $\mathcal{W}$. Fibre of $\mathcal{W}$ at a point $p \in M$ is a Weyl algebra of the cotangent space T_p^*M.

Definition 4. The Weyl algebra of any space E is the universal enveloping algebra of the Heisenberg algebra of E. It has the natural associative product $\circ$, called the Moyal product.

Heisenberg algebra structure of the space T_p^*M at a point $p \in M$ is given by $P \in \Gamma(M, \Lambda^2 TM)$, a bivector form on T^*M, a dual of the form ω at each point $P_p : \Lambda^2 T_p^*M \to \mathbb{C}$. For $x, y \in T_p^*M$ the Lie bracket is

$$[x, y] = 2\hbar P_p(x, y). \tag{1}$$

The Weyl algebra bundle is a bundle of graded associative algebras. Grading is assigned as follows: elements of T^*M are of degree 1, $\hbar$ is of degree 2. The Lie bracket (1) is then homogeneous.
All the operators acting on the cotangent bundle can be lifted to the Weyl algebra bundle, in particular, an operator of covariant derivative can be considered as an operator $\nabla : \Gamma(M, \mathcal{W}) \to \Omega^1(M, \mathcal{W})$. Locally ∇ can be expressed as a sum of the differential along the base $d : C^\infty(M) \to \Omega^1(M)$ lifted to the Weyl algebra bundle and the form $A \in \Omega^1(M, \mathcal{W})$, which acts by an inner automorphism:

$$\nabla s = ds + [A, s], \quad s \in \Gamma(M, \mathcal{W}).$$

The connection preserves the degree. There is also a natural operator δ which lowers the degree by 1, acting by multiplying elements from the base of $\Omega^1(M)$ and contracting corresponding elements from the Weyl algebra bundle of degree 1. This operator is a differential in the Koszul–type resolution:

$$0 \to \Gamma(M, \mathcal{W}) \xrightarrow{\delta} \Omega^1(M, \mathcal{W}) \xrightarrow{\delta} \Omega^2(M, \mathcal{W}) \xrightarrow{\delta} \dots . \tag{2}$$

It has zero cohomology since there is an operator δ^* which gives a homotopy to identity (δ^* kills elements from $\Omega^1(M)$ and puts instead corresponding elements of $\mathcal{W}$).
The statement of the Theorem 3 that an F–manifold uniquely determines a $*$-product on the underlying manifold M can be reformulated as follows (see Fedosov [3]) .

Theorem 5. *There exists square zero operator (Fedosov connection)*

$$D : \Gamma(M, \mathcal{W}) \to \Omega^1(M, \mathcal{W}),$$

such that

- *D is a derivation with respect to the Moyal product.*
- *D can be constructed inductively*

$$D = -\delta + \nabla + r, \quad \textit{where } r \textit{ lifts the degree.} \tag{3}$$

- *D is uniquely defined by additional condition*
$$\delta^* r = 0.$$
- *there is a one-to-one correspondence:*
$$Q : C^\infty(M)[[\hbar]] \to \Gamma_{D\text{-}flat}(M, \mathcal{W}_M). \tag{4}$$

Corollary 6. *The noncommutative structure on the Weyl bundle determines a $*$–product on functions by the correspondence (4), namely for two functions $f, g \in \mathbb{C}^\infty(M)[[\hbar]]$;*
$$f * g = Q^{-1}(Q(f) \circ Q(g)). \tag{5}$$

Definition 7. The algebra $\mathbb{C}^\infty(M)[[\hbar]]$ with a noncommutative associative product $*$ is called a quantization of M and denoted $\mathbb{A}^\hbar(M)$.

Remark 8. The connection D can be rewritten as a sum of two terms – one being a derivation along the manifold, the differential d, and the other being an endomorphism of a fibre. Let us call it Γ. As a consequence of the flatness of the resolution (2) all endomorphisms are inner, one can write Γ as an adjoint action with respect to the Moyal product.
$$D = d + \Gamma = d + \frac{i}{\hbar}[\gamma, \cdot]_\circ, \tag{6}$$
where $\gamma \in \Gamma(M, \mathcal{T}^*M \otimes \mathcal{W})$. Then the equation $D^2 = 0$ becomes:
$$d\Gamma + \frac{1}{2}[\Gamma, \Gamma]_\circ = 0.$$
The same equation gives for γ :
$$\Omega + d\gamma + \frac{i}{\hbar}\frac{[\gamma, \gamma]_\circ}{2} = 0, \tag{7}$$
where Ω is a central 2–form, the characteristic class of deformation is then $[\Omega] \in H^2(M)[[\hbar]]$.

2. Hamiltonian action on F–manifolds.

Let G act on a symplectic manifold (M, ω) by symplectomorphisms, that is, there is a group homomorphism
$$G \to Symp(M, \omega) : \quad g \mapsto \psi_g.$$
The infinitesimal action determines the Lie algebra homomorphism
$$\mathfrak{g} \to Vect(M, \omega) : X \mapsto \hat{X}, \quad \text{defined by} \quad \hat{X} = \frac{d}{dt}\Big|_{t=0} \psi_{exp(tX)} \tag{8}$$
A symplectic form determines a correspondence between functions and certain vector fields, called Hamiltonian vector fields:
$$\mathbb{C}^\infty(M) \to Vect(M, \omega) : H \mapsto X^H, \quad \text{defined by} \quad \imath_{X^H}\omega = dH. \tag{9}$$

Definition 9. The action of G on an F–manifold $(M, \Omega = \omega + \hbar\alpha, \nabla)$ is called *F–Hamiltonian* if:

1. There is a lift $\mathfrak{g} \to C^\infty(M)[[\hbar]] : \quad X \mapsto \check{H}_X = H_X + \sum_{i>0} \hbar^i H_X^i$:
$$\iota_{\hat{X}}\Omega = d\check{H}_X,$$
where $\hat{X}$ is a field corresponding to X under the map $\mathfrak{g} \to Vect(M)$ (8). In particular, for H_X by definition of the Poisson bracket:
$$\{H_X, f\} = \hat{X}f, \quad \text{for any} \quad f \in C^\infty(M). \tag{10}$$

2. The restriction of this map

$$\mathfrak{g} \to C^{\infty}(M) \cong C^{\infty}(M)[[\hbar]]/\{\hbar C^{\infty}(M)[[\hbar]]\}$$

is a Lie algebra homomorphism:

$$H_{[X,Y]} = \{H_X, H_Y\}.$$

$[X, Y]$ is the Lie bracket in $\mathfrak{g}$ and $\{H_X, H_Y\}$ is the Poisson bracket in $C^{\infty}(M)$.

3. The connection ∇ is equivariant under the action of the group, namely

$$L_{\hat{X}}\nabla = 0$$

This definition can be reformulated as follows:

Definition 9'. F–Hamiltonian action on (M, Ω, ∇) is a Hamiltonian action on (M, ω) [1], such that

- For each ω_i in the decomposition

$$\Omega = \omega + \sum_{i>0} \hbar^i \omega_i,$$

there exists $H^i_X \in C^{\infty}(M) : \iota_{\hat{X}}\omega_i = dH^i_X$. in other words each H^i_X is a Hamiltonian with respect to corresponding ω_i (ω_i are closed 2–forms on M).

- The connection ∇ is G–equivariant.

Hamiltonian action determines a map $\mu : M \to \mathfrak{g}^*$, for each point $x \in M$ defined by

$$< \mu(x)|X > = H_X(x)$$

for any $X \in \mathfrak{g}$, where $< \cdot|\cdot >$ is the pairing: $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$. This map is usually called a moment map, however for our purposes of quantization we will call a map $\mathfrak{g} \to C^{\infty}(M)$ also a moment map and it is this map of algebras which we are going to quantize.

3. What is a quantum moment map?

Let (M, Ω, ∇) be an F–manifold. Consider $\mathbb{A}^{\hbar}(M)$ with $*$–product constructed as above (5). $\mathbb{A}^{\hbar}(M)$ is a noncommutative algebra of formal series in $\hbar$ where coefficients are C^{∞}–functions on M. The $*$–product on $\mathbb{A}^{\hbar}(M)$ defines the Lie algebra structure:

$$[f, g]_* = \frac{i}{\hbar}(f * g - g * f), \quad \text{for} \quad f, g \in \mathbb{A}^{\hbar}(M).$$

When G is a group which with an F–Hamiltonian action on (M, Ω, ∇) (Definition 9), there is the induced action of G on $\mathbb{A}^{\hbar}(M)$.

We want to quantize the moment map: $\mathfrak{g} \to C^{\infty}(M)$, namely, get a Lie algebra map from the algebra $\mathfrak{g}$ to the quantized algebra $\mathbb{A}^{\hbar}(M)$. However it is possible only up-to a two–cocycle in $C^{\infty}(M)[[\hbar]]$, so we get a projective representation. Otherwise, we could consider a central extension of $\mathbb{A}^{\hbar}(M)$ or slightly change a definition of the moment map. We eliminate the central elements by considering the map into adjoint representation of $\mathbb{A}^{\hbar}(M)$, the inner automorphisms $Inn\ \mathbb{A}^{\hbar}(M)$. They obviously inherit the Lie algebra structure from $\mathbb{A}^{\hbar}(M)$, so

Definition 10. A quantum moment map is a map of Lie algebras

$$\mu^{\vee} : \mathfrak{g} \to Inn\ \mathbb{A}^{\hbar}(M)$$

with the correspondence principle:

$$lim_{\hbar\to 0}\, \mu(X)^{\vee}(f) = \{H_X, f\}$$

for $X \in \mathfrak{g}$ and $f \in \mathbb{A}^{\hbar}(M)$.

Remark 11. This definition could be reformulated as an isomorphism of associative algebras:

$$U(\mathfrak{g}) \to \mathbb{A}^\hbar(M).$$

such that on vector fields it gives:

$$\mu(X)^\vee(f) = [\mu(X), f]_*.$$

4. Construction of quantum moment maps.

Proposition 12. *Given a Hamiltonian action of G on (M, ω) with the Hamiltonian function H_X. Let*

$$(M, \Omega = \omega + \hbar\alpha, \nabla)$$

be an F–manifold and the G–action on M could be lifted to be F–Hamiltonian, that is such that ∇ is a G–invariant connection and there is a series $\check{H}_X = H_X + \Sigma_{i=1}^{\infty}\hbar^i H_X^i$, with $H_X^i \in C^\infty(M)$ as follows

$$\imath_{\hat{X}}\Omega = d\check{H}_X. \tag{11}$$

Let $\mathbb{A}^\hbar(M)$ be the quantization of M. Then $\check{H}_X$, considered as an element of $\mathbb{A}^\hbar(M)$, gives the quantum moment map

$$\mu(X) = \check{H}_X, \tag{12}$$

That is $\mu^\vee : \mathfrak{g} \to Inn\ \mathbb{A}^\hbar(M)$, $\mu(X)^\vee(f) = [\mu(X), f]_*$ is a homomorphism of Lie algebras:

$$\mu^\vee([X,Y]) = [\mu^\vee(X), \mu^\vee(Y)]_*. \tag{13}$$

Moreover, there are no terms in $\hbar$:

$$[\mu(X), f]_* = [\check{H}_X, f]_* = \{H_X, f\}. \tag{14}$$

Proof. [1] We are going to prove (12–14) by lifting the Hamiltonian $\check{H}_X$ to a section of the Weyl algebra bundle, using the $*$–product (4) and (5). Namely, let $\Gamma_{\text{D-flat}}(M, \mathcal{W})$ be the space of flat sections Fedosov connection D constructed from ∇ corresponding to the quantization of M with the characteristic class Ω. The structure of a Lie algebra in Weyl algebra bundle $\mathcal{W}$ is defined by the fibrewise commutator

$$[\mathbf{a}, \mathbf{b}]_\circ = \mathbf{a} \circ \mathbf{b} - \mathbf{b} \circ \mathbf{a}, \quad \text{for } \mathbf{a}, \mathbf{b} \in \Gamma(M, \mathcal{W})$$

By construction, the Fedosov connection D is G–equivariant, since the initial connection ∇ is G–equivariant. Flat sections of an equivariant connection D are also equivariant. It also means that the Lie derivative of D is zero with respect to any vector field $\hat{X}$:

$$[L_{\hat{X}}, D]_\circ = 0.$$

To use this fact to prove (12) we need to express $L_{\hat{X}}$ through D. Since $L_{\hat{X}}$ and $D_{\hat{X}} = \imath_{\hat{X}} D + D\imath_{\hat{X}}$ are first order derivations commuting with D we can find an analogue of the Cartan homotopy formula for the Lie derivative on forms with values in the Weyl algebra bundle. The difference of $L_{\hat{X}}$ and $D_{\hat{X}}$ could only be an inner automorphism of the Weyl algebra bundle, we denote it $[Q(X), \cdot]_\circ$:

$$L_{\hat{X}}\mathbf{a} = (\imath_X D + D\imath_X)\mathbf{a} + [Q(X), \mathbf{a}]_\circ. \tag{15}$$

$Q(X)$ has to commute with D as well, that is to be a flat section. It is easy to see that the equality (15) is true in Darboux coordinates chosen so that the field $\hat{X}$ is just a pure derivation in the direction of only one of the coordinates. Then $D = D^0 = d + \delta$ and $Q(X) = Q^0(X) = \textit{central section}(X, \hbar) - \imath_X\delta$. Let $D = D^0 + [\Delta\gamma, \cdot]_\circ$ be another flat connection, $\Delta\gamma$ being an equivariant one form in $\mathcal{W}_F$. Then

[1] I am grateful to Boris Tsygan for pointing out a gap in the proof in an earlier version of the paper.

since the $L_{\hat{X}}$ does not change and commutes with the new D as well, the right hand side of (15) must not change either, so one has to subtract $\imath_X \Delta\gamma$ from $Q^0(X)$.
We want to show that we could chose a *central section*$(X, \hbar)$ in $Q(X)$ to be equal to $\mu(X) = \check{H}_X$ so that $Q(X) = \check{H}_X - \imath_X \gamma$ is a flat section corresponding to $\check{H}_X$ and hence is a quantization of the moment map.
Recall that locally we can write the connection D as $Df = df + \frac{i}{\hbar}[\gamma, f]_\circ$ (6). Using the equation $D^2 = 0$ (7) and $L_{\hat{X}} D = 0$ applying (15) to D in place of $\mathbf{a}$

$$0 = \{\imath_{\hat{X}}(d + \frac{i}{\hbar}[\gamma, \cdot]_\circ) + (d + \frac{i}{\hbar}[\gamma, \cdot]_\circ)\imath_{\hat{X}}\}(D) - [\imath_X\gamma, D]_\circ = (\imath_{\hat{X}} d + d\imath_{\hat{X}})(D)$$

we get

$$(\imath_{\hat{X}} d + d\imath_{\hat{X}})\gamma = 0. \tag{16}$$

Using (11), and also (7) and (16) we find that indeed $Q(X)$ is a flat section of D :

$$\begin{aligned} D(\check{H}_X - \iota_X \gamma) \quad &= d\check{H}_X - d(\iota_X\gamma) + \frac{i}{\hbar}[\gamma, \check{H}_X]_\circ - \frac{i}{\hbar}[\gamma, \iota_X\gamma]_\circ \\ &= \imath_X \Omega + \imath_X d\gamma + 0 + \imath_X(\frac{i}{\hbar}\frac{[\gamma,\gamma]_\circ}{2}) = \imath_X(\Omega + d\gamma + \frac{i}{\hbar}\frac{[\gamma,\gamma]_\circ}{2}) = 0. \end{aligned}$$

We also get that $\mu(X) = H_X + \sum_{i=1}^{\infty} \hbar^i H^i{}_X$, so the commutator is the Poisson bracket:

$$\begin{aligned} [\mu(X), f]_* \quad &= \frac{i}{\hbar} Q^{-1}\{[Q(X), Q(f)]_\circ\} = \frac{i}{\hbar} Q^{-1}\{[(\check{H}_X - \imath_{\hat{X}}\gamma), Q(f)]_\circ\} \\ &= \frac{i}{\hbar} Q^{-1}\{[-\imath_{\hat{X}}\gamma, Q(f)]_\circ\} = Q^{-1}\{-\imath_{\hat{X}}(D - d)Q(f)\} \\ &= Q^{-1}\{\imath_{\hat{X}} dQ(f)\} = \hat{X}(f) = \{H_X, f\}. \end{aligned}$$

the fact that $\imath_{\hat{X}} d$ commutes with the map Q follows from the Fedosov's construction of flat sections. Using that the action is Hamiltonian we get on the quantum level:

$$[\mu^\vee(X), \mu^\vee(Y)]_* = ad_{[\check{H}_X, \check{H}_Y]_*} = ad_{\{H_X, H_Y\}} = ad_{\check{H}_{[X,Y]}} = \mu^\vee([X,Y]). \tag{17}$$

□

5. Discussion.

We lifted the Hamiltonian $H_X \in C^\infty(M)$ to $\check{H}_X \in \mathbb{A}^\hbar(M)$ and found that it gives the quantization of the moment map. We did it assuming that the quantization $\mathbb{A}^\hbar(M)$ is G–equivariant. For that it is enough to start from an F–manifold with an equivariant connection. Equivariant connection exists, in particular, when the group is compact: averaging any connection over the group action gives an equivariant connection.
For the case $\Omega = \omega$, in the absence of additional terms $\hbar^i H^i$ we do not need to consider the adjoint representation. The notion of the quantum moment map repeats the classical one, it is a map from $\mathfrak{g}$ to $\mathbb{A}^\hbar(M)$:

Corollary 13. *Consider a quantization $\mathbb{A}^\hbar(M)$ obtained from an F–manifold (M, ω, ∇). Let G act Hamiltonially on M with the Hamiltonian function H_X. Then the quantum moment map is a homomorphism of Lie algebras $\mu : \mathfrak{g} \to \mathbb{A}^\hbar(M)$:*

$$\mu(X) = H_X \quad \textit{and} \quad [\mu(X), f]_* = \{H_X, f\}.$$

Remark 14. We got that $\mu^\vee : \mathfrak{g} \to Inn\ \mathbb{A}^\hbar(M)$ is a map of Lie algebras. This gives the positive answer to a question posed in [10] that every classical moment map can be uniquely lifted to a quantum moment map to $\mathbb{A}^\hbar(M)$.

The result of Corollary 13 appeared also in Fedosov's preprint [4] in about the same time as Glanon meeting. In fact, one can consider the proof of Proposition 12 as a generalization of the proof of Fedosov [3, Propositions 5.8.1,2],[4] to the case of a deformed symplectic form.

This note is another attempt (after the paper [7]) to explain strange behavior of the moment map under quantization, namely the fact that equations of motion do not change.

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