Limit Groups of Equationally Noetherian Groups

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Abstract

We collect in this paper some remarks and observations about limit groups of equationally noetherian groups. We show in particular, that some known properties of limit groups of a free group or, more generally, of a torsion-free hyperbolic group can be seen as consequences of the fact that such groups are equationally noetherian. Especially, such properties are still true for linear groups and finitely generated abelian-by-nilpotent groups.

1 Introduction

Limit groups of free groups have been introduced by Sela [20] for his study of equations over free groups. They can be seen, geometrically and algebraically, as *limits* of free groups. This class coincides with the class of fully residually-free groups; a class of groups introduced by Baumslag [1] and studied by Kharlampovich and Myasnikov [12, 13, 14], Remeslennikov [14], Sela [20] and by many others [5, 9, 11, 19].

Some properties of limit groups, as the stationarity of any sequence of epimorphisms, can be deduced from the linearity of free groups. In fact they are consequences of the noetherian nature of the Zariski topology of the field of complex numbers; that is any system of equations in a free group is equivalent to a finite subsystem [8]. This last property is very interesting and a group satisfying it is called *equationally noetherian*; a notion introduced by Baumslag, Myasnikov and Remeslennikov [2]. The purpose of this paper is to see that some known properties of limit groups of free groups or of a torsion-free hyperbolic group can be seen as consequences of the fact such groups are equationally noetherian. These properties are still true for limit groups of any equationally noetherian group and in particular of a linear group and a finitely generated abelian-by-nilpotent group.

In [2] Baumslag, Myasnikov and Remeslennikov have developed algebraic geometry over groups. We recall here some definitions and results that we require. Let G be a fixed group and $\bar{x} = (x_1, \ldots, x_n)$. We denote by $G[\bar{x}]$ the group $G * F(\bar{x})$ where $F(\bar{x})$ is the free group with basis $\{x_1, \ldots, x_n\}$.

For an element $s(\bar{x}) \in G[\bar{x}]$ and a tuple $\bar{g} = (g_1, \ldots, g_n) \in G^n$ we denote by $s(\bar{g})$ the element of G obtained by replacing each x_i by g_i $(1 \le i \le n)$. Let S

be a subset of $G[\bar{x}]$. Then the set

$$V(S) = \{\bar{g} \in G^n \mid s(\bar{g}) = 1 \text{ for all } s \in S\}$$

is called the *algebraic set* over G defined by S. For instance the centralizer of any element in G is an algebraic set. Subsets of $G[\bar{x}]$ will be also seen as *system of equations* with parameters from G. A group G is called *equationally noetherian* if for every $n \geq 1$ and every subset S of $G[\bar{x}]$ there exists a finite subset $S_0 \subseteq S$ such that $V(S) = V(S_0)$. A subset of G^n is *closed* if it is the intersection of an arbitrary number of finite unions of algebraic sets. This defines a topology on G^n, called the *Zariski topology*. Then a group G is equationally noetherian if and only if for each $n \geq 1$ the Zariski topology on G^n is noetherian [2, Theorem D1].

Guba has shown that free groups are equationally noetherian using the fact that a free group is linear [8]. Using the same argument Baumslag, Myasnikov and Remeslennikov have proved that linear groups over a commutative, noetherian, unitary ring, e. g. a field are equationally noetherian [2, Theorem B1]. They have also shown that abelian groups are equationally noetherian. Bryant proved that finitely generated abelian-by-nilpotent groups are equationally noetherian [4]. Since the wreath product of a nontrivial finitely generated abelian group G by a finitely generated nilpotent group H is linear if and only if H is virtually abelian [22], this provides examples of equationally noetherian groups that are not linear [2, p. 38]. Sela has shown that torsion-free hyperbolic groups are equationally noetherian [21, Theorem 1.22]. This answers a question in [2] and gives another proof of the fact that a free group is equationally noetherian. All of this shows that the class of equationally noetherian groups is very large.

We recall the definition of an H-*limit* group. Let H and G be groups. Let $(f_{i\in\mathbb{N}}|f_i : G \to H)$ be a sequence of morphisms from G to H. We say that $f_{i\in\mathbb{N}}$ is *convergent* or *stable* if for every $g \in G$ one of the following sets is finite

$$\{i \in \mathbb{N}|f_i(g) = 1\}, \{i \in \mathbb{N}|f_i(g) \neq 1\}.$$

We let $\underrightarrow{\ker}(f_i) = \{g \in G| \text{ the set } \{i \in \mathbb{N}|f_i(g) \neq 1\} \text{ is finite }\}$. A group K is said to be H-*limit* if there exist a group G and a converging sequence $(f_{i\in\mathbb{N}}|f_i : G \to H)$ such that $K = G/\underrightarrow{\ker}(f_i)$. In general limit groups are supposed finitely generated, but since some results do not depend on the finite generation property, we do not suppose that and we work in the general case. Thus, in our definition, the free group $F_{\aleph_0}$ of rank $\aleph_0$ is a limit group of free groups, though it is not finitely generated.

Limit groups are heavily connected to *residual* properties. Let G be a group and $\mathcal{K}$ a class of groups. We say that G is *residually-$\mathcal{K}$* (or that $\mathcal{K}$ *separates* G) if for every $g \in G \setminus \{1\}$ there exist $K \in \mathcal{K}$ and a morphism $f : G \to K$ such that $f(g) \neq 1$. If $\mathcal{K}$ consists of the singleton K, then we say simply that G is *residually-K*. We say that G is *fully residually-$\mathcal{K}$* (or that $\mathcal{K}$ *discriminates* G) if for every finite subset $X \subseteq G \setminus \{1\}$ there exist $K \in \mathcal{K}$ and a morphism

$f : G \to K$ such that $1 \notin f(X)$. As before, if $\mathcal{K}$ consists of the singleton K, then we say simply that G is *fully residually-K*.

We finish with some notions from Model Theory needed in the sequel. For more details the reader is referred to [6, 10, 15].

The *universal theory* of a group H, denoted $\mathrm{Th}_\forall(H)$, is the set of all *universal* sentences true in H; i.e. sentences of the form

$$\forall \bar{x}(\bigvee_{1 \leq i \leq m} (\bigwedge_{w \in P_i} w(\bar{x}) = 1 \wedge \bigwedge_{v \in N_i} v(\bar{x}) \neq 1)),$$

(true in H), where P_i, N_i are finite sets of words on the variables $\{x_1, \ldots, x_n\}$ and their inverses. An universal sentence *with parameters from H* is defined analogously by allowing in the above words parameters from H.

The *universal Horn theory* of a group H, is the set of all sentences of the form

$$\forall \bar{x}((\bigwedge_{w \in P} w(\bar{x}) = 1) \Rightarrow v(\bar{x}) = 1),$$

true in H, where P is a finite set of words on the variables $\{x_1, \ldots, x_n\}$ and their inverses.

When ϕ is a sentence, we write $G \models \phi$ to mean that ϕ is true in G and we say that G is a *model* of ϕ. $G \models \mathrm{Th}_\forall(H)$ means that G satisfies every sentence $\phi \in \mathrm{Th}_\forall(H)$ and we say that G is a *model* of $\mathrm{Th}_\forall(H)$ or that G is a model of the universal theory of H.

An *ultrafilter* on a set I is a finitely additive probability measure $\mu : \mathcal{P}(I) \to \{0, 1\}$. An ultrafilter μ is called *nonprincipal* if $\mu(X) = 0$ for every finite subset $X \subseteq I$.

Given an ultrafilter μ on I and a sequence of groups $(G_i)_{i \in I}$ we define an equivalence relation $\sim_\mu$ on $\prod_{i \in I} G_i$ by

$$\hat{a} = (a_i \in G_i)_{i \in I} \sim_\mu \hat{b} = (b_i \in G_i)_{i \in I} \ \text{ if and only if } \mu(\{i \in I | a_i = b_i\}) = 1.$$

The set of equivalence classes $(\prod_{i \in I} G_i)/ \sim_\mu$ is endowed with a structure of group by defining

$$\hat{a}.\hat{b} = \hat{c} \ \text{ if and only if } \mu(\{i \in I | a_i.b_i = c_i\}) = 1.$$

The group $(\prod_{i \in I} G_i)/ \sim_\mu$ is called the *ultraproduct* of the family $(G_i)_{i \in I}$. When $G_i = G$ for all $i \in I$, $(\prod_{i \in I} G_i)/ \sim_\mu$ is called an *ultrapower* and it is denoted by $^*G$. If μ is nonprincipal, then $^*G$ is called a *nonprincipal ultrapower*.

Define $\pi : G \to {}^*G$ by $\pi(g) = (g_i = g | i \in I)$. Then π is an embedding. Moreover, a theorem of Łos [6, Theorem 4.1.9] claims that G is an *elementary subgroup* of $^*G$; that is, any sentence with parameters from G which is true in G is also true in $^*G$. As consequence, we get the property, needed in the sequel, that if $G \models \phi$, where ϕ is an universal sentence with parameters from G, then $^*G \models \phi$. In particular $^*G$ and G have the same universal theory.

Another theorem in Model theory, due to Keisler ([6, Theorem 6.1.1] or [15, Exercice 4.5.37]), claims that when I is countable, a *nonprincipal* ultrapower

$^*G$ of G is $\aleph_1$-*saturated*. In what follows, we need only a special case of that saturation property (compactness): if

$$^*G \models \forall \bar{x}(\bigvee_{i \in \mathbb{N}} \phi_i(\bar{x})),$$

where each ϕ_i is of the form $(\bigwedge_{1 \leq j \leq p} w_j(\bar{x}) = 1 \wedge \bigwedge_{1 \leq j \leq q} v_j(\bar{x}) \neq 1)$, then there exists $n \in \mathbb{N}$ such that

$$^*G \models \forall \bar{x}(\bigvee_{0 \leq i \leq n} \phi_i(\bar{x})).$$

(The words $w_j(\bar{x}), v_j(\bar{x})$ are allowed to have parameters from $^*G$.)

Conventions.

• In what follows, for a finitely generated group G we write $G = \langle \bar{x} | P(\bar{x}) \rangle$ to mean that G is generated by $\bar{x} = (x_1, \ldots, x_n)$ and $P(\bar{x})$ is a presentation *closed by deduction*; that is, if $G \models w(\bar{x}) = 1$ then $w \in P$. If $S(\bar{x})$ is a set of words we write $S(\bar{x}) = 1$ as abbreviation of the formula $\bigwedge_{w \in S} w(\bar{x}) = 1$.

• Let G be an equationally noetherian group and let $S \subseteq G[\bar{x}]$. A subset $S_0 \subseteq S$ is called a G-*witness* of S if S_0 is finite and $V(S) = V(S_0)$. Notice that we have

$$G \models \forall \bar{x}(S_0(\bar{x}) = 1 \Leftrightarrow S(\bar{x}) = 1).$$

• In all of the rest of this paper, all the ultrafilter considered will be on $\mathbb{N}$.

2 First Properties

We begin this section by the following theorem which regroups several relations between the different notions given in the introduction.

Theorem 2.1 *Let H be a group.*
(1) A countable fully residually-H group is an H-limit group.
(2) An H-limit group is embeddable in all nonprincipal ultrapowers of H; in particular it is a model of $Th_\forall(H)$.
(3) Let G be a finitely generated group. The following properties are equivalents:
    (i) G is a model of $Th_\forall(H)$,
    (ii) G is embeddable in some nonprincipal ultrapower of H,
    (iii) G is embeddable in all nonprincipal ultrapowers of H.
(4) If H is equationally noetherian then for every finitely generated group G the following properties are equivalent:
    (i) G is a model of $Th_\forall(H)$,
    (ii) G is fully residually-H,
    (iii) G is an H-limit group.
(5) If H is equationally noetherian then for every finitely generated group G the following properties are equivalent:

(i) G is a model of the universal Horn theory of H,

(ii) G is residually-H.

The following Lemma is classical in Model Theory, but for completeness we provide a proof.

Lemma 2.2 *Let $G = \langle \bar{x} | P(\bar{x}) \rangle$ be a finitely generated group and H a group. Let $N(\bar{x})$ be the set of all words $w(\bar{x})$ such that $G \models w(\bar{x}) \neq 1$. Suppose that for every finite subset $X \subseteq P(\bar{x}) \cup N(\bar{x})$, there exists a tuple $\bar{a}$ in H such that $H \models \bigwedge_{w \in X \cap P} w(\bar{a}) = 1 \wedge \bigwedge_{v \in X \cap N} v(\bar{a}) \neq 1$. Then G is embeddable in all nonprincipal ultrapowers of H. In particular G is a model of the universal theory of H.*

Proof

Let μ be a nonprincipal ultrafilter on $\mathbb{N}$ and let $^*H$ be the ultrapower of H relatively to μ.

Write $P(\bar{x}) \cup N(\bar{x}) = \bigcup_{i \in \mathbb{N}} S_i$, where each S_i is finite and $S_i \subseteq S_{i+1}$. For every $i \in \mathbb{N}$, pick $\bar{a}_i \in H$ such that $H \models \bigwedge_{w \in S_i \cap P} w(\bar{a}_i) = 1 \wedge \bigwedge_{v \in S_i \cap N} v(\bar{a}_i) \neq 1$.

We claim that the map $f : G \to {}^*H$ defined by $f(\bar{x}) = (\bar{a}_i | i \in \mathbb{N})$ extends to an embedding.

If $w(\bar{x}) \in P$, then there exists $n \in \mathbb{N}$ such that $w \in S_n$ and thus $H \models w(\bar{a}_i) = 1$ for every $i \geq n$. Therefore $\mu(\{i \in \mathbb{N} | H \models w(\bar{a}_i) = 1\}) = 1$ and thus $^*H \models w((\bar{a}_i | i \in \mathbb{N})) = 1$.

Similarly, if $v(\bar{x}) \in N$, then there exists $n \in \mathbb{N}$ such that $v \in S_n$ and thus $H \models v(\bar{a}_i) \neq 1$ for every $i \geq n$. As before, $\mu(\{i \in \mathbb{N} | H \models v(\bar{a}_i) \neq 1\}) = 1$ and thus $^*H \models v((\bar{a}_i | i \in \mathbb{N})) \neq 1$. This ends the proof of our claim.

Now, since universal sentences are conserved by taking subgroups, G is a model of the universal theory of $^*H$. As H and $^*H$ have the same universal theory, G is a model of the universal theory of H. $\square$

Proof of Theorem 2.1.

(1). Let G be a countable fully residually-H group and write $G \setminus \{1\} = \bigcup_{i \in \mathbb{N}} S_i$ where each S_i is finite and $S_i \subseteq S_{i+1}$. Then there exists a convergent sequence $(f_{i \in \mathbb{N}} | f_i : G \to H)$ such that $1 \notin f_i(S_i)$. Clearly $\underrightarrow{\ker}(f_i) = 1$ and thus G is an H-limit group.

(2). Let K be an H-limit group and G be a group for which there exists a convergent sequence $(f_{i \in \mathbb{N}} | f_i : G \to H)$ such that $K = G / \underrightarrow{\ker}(f_i)$. Let μ be a nonprincipal ultrafilter on $\mathbb{N}$, and let $^*G$ (resp.$^*H$) be the ultrapower of G (resp.H) relatively to μ. Let $\pi : G \to {}^*G$ be the natural embedding defined by $g \to \hat{g} = (g_i = g | i \in \mathbb{N})$. The map $f : {}^*G \to {}^*H$ defined by $f((g_i | i \in \mathbb{N})) = (f_i(g_i) | i \in \mathbb{N})$ is a morphism and satisfies

$$f((g_i | i \in \mathbb{N})) = 1 \Leftrightarrow \mu(\{i \in \mathbb{N} | f_i(g_i) = 1\}) = 1.$$

Therefore, the morphism $\varphi = f \circ \pi : G \to {}^*H$ satisfies

$$\varphi(g) = 1 \Leftrightarrow \mu(\{i \in \mathbb{N} | f_i(g) = 1\}) = 1.$$

5

Since $(f_{i\in\mathbb{N}}|f_i : G \to H)$ is a convergent sequence and μ is nonprincipal we get

$$\varphi(g) = 1 \Leftrightarrow \{i \in \mathbb{N}|f_i(g) \neq 1\} \text{ is finite.}$$

Hence $\ker(\varphi) = \underrightarrow{\ker}(f_i)$. Thus $K = G/\ker(\varphi)$ and K is embeddable in $^*H$. In particular $K \models \text{Th}_\forall(H)$.

$(3)(iii) \Rightarrow (ii)$ is obvious, $(3)(ii) \Rightarrow (i)$ is also obvious as universal sentences are preserved by subgroups and $\text{Th}_\forall(H)=\text{Th}_\forall(^*H)$. The implication $(3)(i) \Rightarrow (ii)$ is a consequence of $(3)(i) \Rightarrow (iii)$ and of the existence of nonprincipal ultrafilters on $\mathbb{N}$; so it remains to show $(3)(i) \Rightarrow (iii)$.

Write $G = \langle \bar{x}|P(\bar{x})\rangle$ and let $N(\bar{x})$ be the set of words $v(\bar{x})$ such that $G \models v(\bar{x}) \neq 1$. Since G is a model of the universal theory of H, for every finite subset $X \subseteq P(\bar{x}) \cup N(\bar{x})$, there exists a tuple $\bar{a}$ in H such that $H \models \bigwedge_{w\in X\cap P} w(\bar{a}) = 1 \wedge \bigwedge_{v\in X\cap N} v(\bar{a}) \neq 1$. Therefore, by Lemma 2.2, G is embeddable in all nonprincipal ultrapowers of H.

$(4)(ii) \Rightarrow (iii)$ is a consequence of (1) and $(4)(iii) \Rightarrow (i)$ is a consequence of (2), it remains to show $(4)(i) \Rightarrow (ii)$. Write $G = \langle \bar{x}|P(\bar{x})\rangle$ and let $P_0(\bar{x})$ be an H-witness of $P(\bar{x})$. Since $G \models \text{Th}_\forall(H)$, for any words $v_1(\bar{x}), \ldots, v_n(\bar{x})$ such that $G \models \bigwedge_{1\leq i\leq n} v_i(\bar{x}) \neq 1$, H is a model of

$$\exists\bar{x}(P_0(\bar{x}) = 1 \wedge \bigwedge_{1\leq i\leq n} v_i(\bar{x}) \neq 1).$$

Therefore there exists $\bar{a} \in H$ satisfying $P(\bar{a}) = 1 \wedge \bigwedge_{1\leq i\leq n} v_i(\bar{a}) \neq 1$. Since $H \models \forall\bar{x}(P_0(\bar{x}) = 1 \Leftrightarrow P(\bar{x}) = 1)$ there exists a morphism $\bar{f} : G \to H$ such that $f(v_i(\bar{x})) = v_i(\bar{a}) \neq 1$, $1 \leq i \leq n$. Thus G is fully residually-H.

$(5)(i) \Rightarrow (ii)$. Write $G = \langle \bar{x}|P(\bar{x})\rangle$ and let $P_0(\bar{x})$ be an H-witness of $P(\bar{x})$. Since G is a model of the universal Horn theory of H, for any word $v(\bar{x})$ such that $G \models v(\bar{x}) \neq 1$, H is a model of $\exists\bar{x}(P_0(\bar{x}) = 1 \wedge v(\bar{x}) \neq 1)$. As before, there exists $\bar{a} \in H$ satisfying $P(\bar{a}) = 1 \wedge v(\bar{a}) \neq 1$. Thus there exists a morphism $f : G \to H$ such that $f(v(\bar{x})) \neq 1$. Thus G is residually-H.

$(5)(ii) \Rightarrow (i)$. Clearly if G is residually-H, then G is embeddable in some product $\prod_{i\in I} H$ and thus G is a model of the universal Horn theory of H. $\square$

Remarks 2.3

(1). The implications $(4)(ii) \Rightarrow (i)$ and $(5)(ii) \Rightarrow (i)$ do not depend on the equationally noetherian nature of H neither on the finite generation property of G; that is $(4)(ii) \Rightarrow (i)$ and $(5)(ii) \Rightarrow (i)$ are still true for any group H and any group G. The implication $(5)(ii) \Rightarrow (i)$ is very clear. Let us show that $(4)(ii) \Rightarrow (i)$ is true for any group H and any group G. Suppose that G is fully residually-H. Suppose

$$(*) \qquad H \models \forall\bar{x}(\bigvee_{1\leq i\leq m}(\bigwedge_{w\in P_i} w(\bar{x}) = 1 \wedge \bigwedge_{v\in N_i} v(\bar{x}) \neq 1)),$$

where P_i, N_i are finite sets of words, and suppose towards a contradiction

$$G \models (\bigwedge_{1 \leq i \leq m} (\bigvee_{w \in P_i} w(\bar{a}) \neq 1 \vee \bigvee_{v \in N_i} v(\bar{a}) = 1)), \text{ for some tuple } \bar{a} \text{ in } G.$$

Let

$$X = \{w(\bar{a}) | w(\bar{x}) \in P_1 \cup \cdots \cup P_m, \ G \models w(\bar{a}) \neq 1\}.$$

Since G is fully residually-H, there exists a morphism $f : G \to H$ such that $1 \notin f(X)$. Then

$$H \models (\bigwedge_{1 \leq i \leq m} (\bigvee_{w \in P_i} w(f(\bar{a})) \neq 1 \vee \bigvee_{v \in N_i} v(f(\bar{a})) = 1)),$$

contradicting $(*)$.

(2). Note that the equivalences $(4)(i) - (iii)$ are not true if G is not finitely generated: $\mathbb{Q}$ is a model of the universal theory of $\mathbb{Z}$ but $\mathbb{Q}$ is not residually-$\mathbb{Z}$. Furtheremore, $\mathbb{Q}$ is embeddable in all nonprincipal ultrapowers of $\mathbb{Z}$.

Corollary 2.4 *Let H be an equationally noetherian group. Then a group G is a model of $Th_\forall(H)$ if and only if G is locally fully residually-H.* $\square$

Definition 2.5 *A group G is said H-pseudo-limit if G is a model of the universal Horn theory of H.*

In order to prove the next theorem we need the following definitions from [2]. Let $Y \subseteq G^n$. We define

$$I(Y) = \{w(\bar{x}) \in G[\bar{x}] | w(\bar{y}) = 1 \text{ for all } \bar{y} \in Y\}.$$

Notice that when $S \subseteq G[\bar{x}]$, then

$$w(\bar{x}) \in I(V(S)) \Leftrightarrow G \models \forall \bar{x}(S(\bar{x}) = 1 \Rightarrow w(\bar{x}) = 1).$$

For $S \subseteq G[\bar{x}]$ the *coordinate group* of S, denoted G_S, is $G[\bar{x}]/I(V(S))$. The proof of the following Lemma can certainly be extracted from [16] but for completeness we provide a proof.

Lemma 2.6 *For any group H and any $S \subseteq H[\bar{x}]$, H_S is residually-H. In particuliar H_S is an H-pseudo-limit group.*

Proof

Let $v(\bar{x})$ be a word such that $H_S \models v(\bar{x}) \neq 1$. Clearly we have $v(\bar{x}) \notin I(V(S))$ and thus

$$H \models \exists \bar{x}(S(\bar{x}) = 1 \wedge v(\bar{x}) \neq 1).$$

Thus there exists a morphism $f : H_S \to H$ which fixes every element of H and $f(v(\bar{x})) \neq 1$. Therefore H_S is residually-H as desired.

By Remarks 2.3 (1), H_S is an H-pseudo-limit group. $\square$

Theorem 2.7 *If H is equationally noetherian then any sequence of epimorphisms of finitely generated H-pseudo-limit groups*

$$G_1 \to_{\phi_{1,2}} G_2 \to_{\phi_{2,3}} \cdots$$

terminates after finitely many steps.

Conversely, if H is finitely generated and any sequence of epimorphisms of finitely generated resiudally-H groups terminates after finitely many steps, then H is equationally noetherian.

Proof

Suppose that H is equationally noetherian and write $\bar{x} = (x_1, \ldots, x_n)$, $G_i = \langle \bar{x} | P_i(\bar{x}) \rangle$, $\phi_{i,i+1}(x_k) = x_k$. Since H is equationally noetherian there exists a finite subset $S(\bar{x}) \subseteq P(\bar{x}) = \bigcup_i P_i(\bar{x})$ such that

$$H \models \forall \bar{x}(S(\bar{x}) = 1 \Rightarrow w(\bar{x}) = 1)) \text{ for every } w \in P.$$

Let $m \in \mathbb{N}$ such that $S(\bar{x}) \subseteq P_m(\bar{x})$. We claim that for every $k \geq m$, the epimorphism $\phi_{k,k+1} : G_k \to G_{k+1}$ is an isomorphism. Let $v(\bar{x})$ be a word on $\bar{x}$ such that $G_{k+1} \models v(\bar{x}) = 1$. Then there exists a finite subset $A \subseteq P_{k+1}$ such that $\forall \bar{x}(A(\bar{x}) = 1 \Rightarrow v(\bar{x}) = 1)$ is true in any group. Since $k + 1 \geq m$ we have

$$H \models \forall \bar{x}(S(\bar{x}) = 1 \Rightarrow A(\bar{x}) = 1)),$$

thus we get

$$H \models \forall \bar{x}(S(\bar{x}) = 1 \Rightarrow v(\bar{x}) = 1)).$$

Since G_k is a model of the universal Horn theory of H, we get $G_k \models v(\bar{x}) = 1$.

Now suppose that H is finitely generated and that any sequence of epimorphisms of finitely generated residually-H groups terminates after finitely many steps. Let $S(\bar{x})$ be a system of equations with parameters from H. Since H is finitely generated $S(\bar{x})$ is countable. Write $S(\bar{x}) = \bigcup_{i \in \mathbb{N}} S_i(\bar{x})$ where S_i is finite and $S_i \subseteq S_{i+1}$. Let $P_i(\bar{x})$ be the set of words $w(\bar{x})$ with parameters from H such that

$$H \models \forall \bar{x}(S_i(\bar{x}) = 1 \Rightarrow w(\bar{x}) = 1),$$

i.e. $P_i(\bar{x}) = I(V(S_i))$. Let $H_i = \langle H * F(\bar{x}) | P_i(\bar{x}) \rangle = H_{S_i}$. By Lemma 2.6, H_i is residually-H. Now the sequence

$$H_1 \to H_2 \to H_3 \cdots$$

is a sequence of epimorphisms of finitely generated residually-H groups, and thus stabilizes. Therefore there exists n such that for every $i, j \geq n$, $P_i(\bar{x}) = P_j(\bar{x})$. Hence for every $i \geq n$ we have

$$H \models \forall \bar{x}(S_n(\bar{x}) = 1 \Rightarrow S_i(\bar{x}) = 1).$$

Thus $S(\bar{x})$ is equivalent to a finite subsystem as desired. $\qquad \square$

By Theorem 2.1 (2) an H-limit group is a model of the universal theory of H and thus it is H-pseudo-limit. Thus we get:

Corollary 2.8 *Let H be an equationally noetherian group. Then any sequence of epimorphisms of finitely generated H-limit groups*

$$G_1 \rightarrow G_2 \rightarrow \cdots$$

terminates after finitely many steps. □

Corollary 2.9 *Let H be an equationally noetherian group. Then every finitely generated H-pseudo-limit group is Hopfian. In particular every H-limit group is Hopfian.*

Proof

Let G be a finitely generated H-pseudo-limit group and $\phi : G \rightarrow G$ be a surjective morphism. Then the sequence $G \rightarrow_\phi G \rightarrow_\phi G \rightarrow_\phi G \cdots$ terminates after finitely many steps. Thus ϕ is an isomorphism. □

Corollary 2.10 *Let H be an equationally noetherian group. Then any finitely generated H-pseudo-limit group is equationally noetherian. In particular any finitely generated H-limit group is equationally noetherian.*

Proof

Let G be finitely generated H-pseudo-limit group. Then any sequence of epimorphisms of finitely generated residually-G groups is also a sequence of epimorphisms of finitely generated H-pseudo-limit groups; thus terminates after finitely many steps, by Theorem 2.7. Again by Theorem 2.7, G is equationally noetherian. □

Theorem 2.11 *Let H be an equationally noetherian group. Then there exists at most countably many nonisomorphic finitely generated H-pseudo-limit groups. In particular there exists at most countably many nonisomorphic finitely generated H-limit groups.*

Proof

Suppose towards a contradiction that the opposite is true. Then there exists $n \in \mathbb{N}$ such that there exists at least λ nonisomorphic n-generated H-pseudo-limit groups for some $\lambda > \aleph_0$, (n-generated means generated by n elements). (In fact we can assume that $\lambda = 2^{\aleph_0}$, see [18]). Let $(G_i = \langle \bar{x} | P_i(\bar{x}) \rangle | i \in \lambda > \aleph_0)$ be the list of nonisomorphic n-generated H-pseudo-limit groups. For every $i \in \lambda$ there exists a finite subset $S_i \subseteq P_i$ such that $H \models \forall \bar{x}(S_i(\bar{x}) = 1 \Leftrightarrow P_i(\bar{x}) = 1)$.

Since for every $i \in \lambda$ the set S_i is finite, the set $\{S_i | i \in \lambda\}$ is countable. Therefore the map $f : \{P_i | i \in \lambda\} \rightarrow \{S_i | i \in \lambda\}$ defined by $P_i \mapsto S_i$ is not injective and thus there exist $i, j \in \lambda, i \neq j$ such that $S_i = S_j$.

Since G_i, G_j are models of the universal Horn theory of H we get $P_i = P_j$, a contradiction with the fact that G_i and G_j are not isomorphic. □

3 Factor sets

Let H be a group equipped with the Zariski topology. Recall that a closed set is called *irreducible* if it is not the union of two proper, nonempty, closed subsets. In our context, if $S \subseteq H[\bar{x}]$ is a system of equations such that $V(S)$ is irreducible, then whenever $S_1, \ldots, S_n \subseteq H[\bar{x}]$ are systems of equations and if

$$H \models \forall \bar{x}(S(\bar{x}) = 1 \Rightarrow S_1(\bar{x}) = 1 \vee \cdots \vee S_n(\bar{x}) = 1),$$

then there exists i such that $H \models \forall \bar{x}(S(\bar{x}) = 1 \Rightarrow S_i(\bar{x}) = 1)$.

Now suppose that H is equationally noetherian. As noticed in the introduction the Zariski topology on H^n is noetherian. This implies that every closed subset of H^n is a finite union of algebraic sets. Recall also that every closed set in a noetherian topological space is a finite union of irreducible closed subsets. Thus, in our context, for any system $S(\bar{x})$ of equations there exist systems of equations $S_1, \ldots, S_n$ such that $(V(S_i)|1 \leq i \leq n)$ are irreducible and

$$H \models \forall \bar{x}(S(\bar{x}) = 1 \Leftrightarrow S_1(\bar{x}) = 1 \vee \cdots \vee S_n(\bar{x}) = 1).$$

Some properties contained in the next Lemma, can certainly be extracted from [2].

Lemma 3.1 *Let H be a group and $S \subseteq H[\bar{x}]$ such that $V(S)$ is irreducible. Then H_S is fully residually-H. In particular, H_S is a model of the universal theory of H, and if H is countable then H_S is an H-limit group.*

Proof

Let $v_1(\bar{x}), \ldots, v_p(\bar{x})$ be words such that $H_S \models \bigwedge_{1 \leq j \leq p} v_j(\bar{x}) \neq 1$. Suppose that
$$H \models \forall \bar{x}(S(\bar{x}) = 1 \Rightarrow \bigvee_{1 \leq j \leq p} v_j(\bar{x}) = 1).$$

Then, since $V(S)$ is irreducible, there exists j such that

$$H \models \forall \bar{x}(S(\bar{x}) = 1 \Rightarrow v_j(\bar{x}) = 1),$$

contradicting the fact that $H_S \models v_j(\bar{x}) \neq 1$. Therefore

$$H \models \exists \bar{x}(S(\bar{x}) = 1 \wedge \bigwedge_{1 \leq j \leq p} v_j(\bar{x}) \neq 1).$$

Consequently, there exists a morphism $f : H_S \to H$, which fixes every element of H, such that $f(v_j) \neq 1$ $(1 \leq j \leq p)$. Thus H_S is fully residually-H as desired.

It follows, by Remarks 2.3 (1), that H_S is a model of the universal theory of H. Furtheremore, if H is countable then H_S is also countable and by Theorem 2.1 (1), H_S is an H-limit group. $\qquad\square$

Factors sets are the first step in the construction of Makanin-Razborov diagrams. Let H be a group, K an H-limit group and G a finitely generated

group. A *factor set* of G relatively to K is a finite collection of proper quotients $\{f_i : G \to L_i\}$ of H-limit groups such that any morphism $f : G \to K$ factors through some f_i after precomposition with some automorphism of G.

The following Theorem gives a weak version of factor sets.

Theorem 3.2 *Let H be an equationally noetherian group. Then for any finitely generated group G there exists a finite collection of epimorphisms $\{f_i : G \to L_i\}$, where each L_i is an H-limit group, such that for any H-limit group L any morphism $f : G \to L$, factors through some f_i.*

Proof

Put $G = \langle \bar{x}|P(\bar{x})\rangle$ and let $S(\bar{x})$ be an H-witness of P. Let $S_1, \ldots, S_n$ be systems of equations (maybe with parameters from H) such that each S_i is finite, $(V(S_i)|1 \leq i \leq n)$ are irreducible and

$$H \models \forall \bar{x}(S(\bar{x}) = 1 \Leftrightarrow S_1(\bar{x}) = 1 \vee \cdots \vee S_n(\bar{x}) = 1).$$

Let P_i be the set of words $w(\bar{x})$, *without parameters from H*, such that

$$H \models \forall \bar{x}(S_i(\bar{x}) = 1 \Rightarrow w(\bar{x}) = 1).$$

Let $L_i = \langle \bar{x}|P_i(\bar{x})\rangle$. We claim that the sequence $L_1, \ldots, L_n$ satisfies the desired properties.

First we claim that L_i is an H-limit group. Let us show that L_i is the subgroup of H_{S_i} generated by $\bar{x}$. Clearly the map $\phi : L_i \to H_{S_i}$ which sends x_i to x_i is a morphism. We claim that it is injective. Indeed if $v(\bar{x})$ is a word, without parameters from H, such that $H_{S_i} \models v(\bar{x}) = 1$ then $v(\bar{x}) \in I(V(S_i))$ and thus $v(\bar{x}) \in P_i$. Therefore $L_i \models v(\bar{x}) = 1$ and thus ϕ is injective.

By Lemma 3.1, H_{S_i} is a model of $\mathrm{Th}_\forall(H)$ and thus L_i is also a model of $\mathrm{Th}_\forall(H)$. As L_i is finitely generated, by Theorem 2.1 $(4)(i) \Leftrightarrow (iii)$, L_i is an H-limit group.

Since L_i is an H-limit group and $L_i \models S(\bar{x}) = 1$, there is an epimorphism $f_i : G \to L_i$ such that $f_i(x_k) = x_k$.

Now let L be an H-limit group and $h : G \to L$ be an epimorphism. Since $L \models \mathrm{Th}_\forall(H)$, by Theorem 2.1 (3) L is embeddable in some nonprincipal ultrapower ${}^*H$ of H. Without loss of generality we may assume that $L \subseteq {}^*H$. Write $L = \langle \bar{y}|Q(\bar{y})\rangle$ such that $h(x_i) = y_i$. Then $L \models P(\bar{y}) = 1$, and thus $L \models S(\bar{y}) = 1$. Since every universal sentence with parameters from H is true in ${}^*H$ (see the introduction) we get

$$ {}^*H \models \forall \bar{x}(S(\bar{x}) = 1 \Leftrightarrow S_1(\bar{x}) = 1 \vee \cdots \vee S_n(\bar{x}) = 1).$$

Therefore there exists j such that ${}^*H \models S_j(\bar{y}) = 1$. Again, since every universal sentence with parameters from H is true in ${}^*H$, we get ${}^*H \models P_j(\bar{y}) = 1$ and thus $L \models P_j(\bar{y}) = 1$. Hence there exists a morphism $\varphi : L_j \to L$ which sends x_i to y_i. Now clearly $h = \varphi \circ f_i$. $\qquad\square$

Note that the proof of the above Theorem is slightly different of those presented in [7] for free groups, as here we do not use the fact that some ultrapower of H is equationally noetherian.

Corollary 3.3 *Let H be an equationally noetherian group. Then a finitely generated group G is residually-H if and only if there exist H-limit groups $L_1, \ldots, L_n$ such that G is embeddable in $L_1 \times \cdots \times L_n$.*

Proof

Clearly if G is embeddable in $L_1 \times \cdots \times L_n$ for some H-limit groups $L_1, \ldots, L_n$ then G is a model of the universal Horn theory of H and thus it is residually-H by Theorem 2.1(5).

Suppose that G is residually-H. By Theorem 3.2, there exists a finite collection of epimorphisms $\{f_i : G \to L_i\}$, where each L_i is an H-limit group, such that for any H-limit group L any morphism $h : G \to L$, factors through some f_i. We claim that G is embeddable in $K = L_1 \times \cdots \times L_m$.

Write $G = \langle x_1, \ldots, x_n | P(\bar{x}) \rangle$ and $L_i = \langle y_{i,1}, \ldots, y_{i,n} | P_i(\bar{y}_i) \rangle$ such that $f_i(x_j) = y_{i,j}$. Then the map $f : G \to L_1 \times \cdots \times L_m$ defined by

$$f(x_j) = \hat{y}_j = (y_{1,j}, \ldots, y_{m,j})$$

is a morphism. It remains to show that f is an embedding. Let $w(\hat{y}_1, \ldots, \hat{y}_n)$ be a word on the variables $\{\hat{y}_1, \ldots \hat{y}_n\}$ such that $w(\hat{y}_1, \ldots, \hat{y}_n) = 1$ in K. Suppose towards a contradiction that $G \models w(x_1, \ldots, x_n) \neq 1$. Since G is residually-H there exists a morphism $\phi : G \to H$ such that $\phi(w(x_1, \ldots, x_n)) \neq 1$. Now there exist f_i and a morphism $h : L_i \to H$ such that $\phi = h \circ f_i$ and thus $h(w(y_{i,1}, \ldots, y_{i,n})) \neq 1$. Therefore $L_i \models w(y_{i,1}, \ldots, y_{i,n}) \neq 1$ and thus $K \models w(\hat{y}_1, \ldots, \hat{y}_n) \neq 1$, contradiction. Hence f is an embedding as claimed. $\square$

A natural question arise from Theorem 3.2: when can we say that the groups L_i are a proper quotients of G ? Clearly if G is not H-limit, then the groups L_i are proper quotients. When G is an H-limit group, to answer the question, we need the following notion.

Definition 3.4 *A finitely generated H-limit group G is said H-determined if there exists a finite subset $X \subseteq G \setminus \{1\}$ such that for any morphism $f : G \to L$, where L is an H-limit group, if $1 \notin f(X)$ then f is an embedding. We denote by $\mathcal{D}_H$ the class of H-determined groups.*

Notice that every H-determined group is a subgroup of H. In particular if H is free then any H-determined group is also free. For instance $\mathbb{Z}, F_2$ are F_2-determined but $\mathbb{Z}^2$ is not. Thus, in some sense, H-determined groups play the same role in general case, as free groups in the special case of limit groups of free groups. Notions related to determined groups are investigated in [3] and [17].

A *primitive-quantifier-free formula* is a formula $\vartheta(\bar{x})$ of the form

$$\left(\bigwedge_{w \in P} w(\bar{x}) = 1 \wedge \bigwedge_{v \in N} v(\bar{x}) \neq 1 \right),$$

where P, N are finite sets of words on the variables $\bar{x} = \{x_1, \ldots, x_n\}$ and their inverses. We begin with the following proposition.

Proposition 3.5 [17, Proposition 2.3.1] *Let H be a group such that $Th_\forall(H)$ has at most countably many nonisomorphic finitely generated models. Then for any primitive-quantifier-free formula $\vartheta(\bar{x})$ such that $H \models \exists \bar{x}(\vartheta(\bar{x}))$, there exists a primitive-quantifier-free formula $\xi(\bar{x})$ such that $H \models \exists \bar{x}(\vartheta(\bar{x}) \wedge \xi(\bar{x}))$ and for any word $w(\bar{x})$ on the variables $\bar{x} = \{x_1, \ldots, x_n\}$ and their inverses one has*

$$H \models \forall \bar{x}(\vartheta(\bar{x}) \wedge \xi(\bar{x}) \Rightarrow w(\bar{x}) = 1) \text{ or } H \models \forall \bar{x}(\vartheta(\bar{x}) \wedge \xi(\bar{x}) \Rightarrow w(\bar{x}) \neq 1).$$

Proof

Let $\vartheta(\bar{x})$ be a primitive-quantifier-free formula such that $H \models \exists \bar{x}(\vartheta(\bar{x}))$ and suppose towards a contradiction that $\vartheta(\bar{x})$ does not satisfies the conclusions of the proposition. We are going to construct a tree. By hypothesis there exists a word $\alpha_1(\bar{x})$ such that $H \models \exists \bar{x}(\vartheta(\bar{x}) \wedge \alpha_1(\bar{x}) = 1)$ and $H \models \exists \bar{x}(\vartheta(\bar{x}) \wedge \alpha_1(\bar{x}) \neq 1)$ (to simplify notation we omit $\bar{x}$). We can do the same thing with $\vartheta \wedge \alpha_1 = 1$ and $\vartheta \wedge \alpha_1 \neq 1$.

Thus we have:
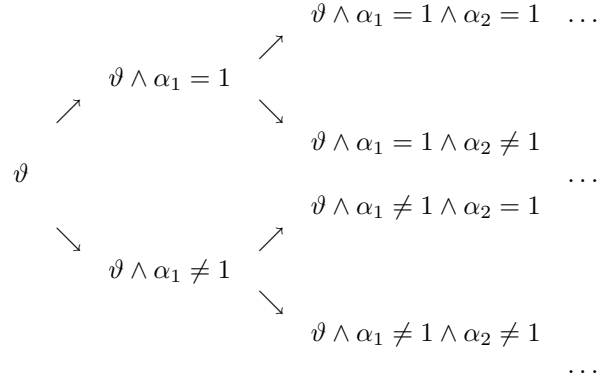
Now every branch B in the above tree defines a finitely generated group G_B, by taking G_B the group generated by $\bar{x}$ and with presentation the set of all words equal to 1 which appear in the branch B. Then G_B satisfies any inequality which occurs in B. Since any finite segment in any branch B is consistent in H, by Lemma 2.2, G_B is a model of the universal theory of H. Since there exists $2^{\aleph_0}$ branch, we get $2^{\aleph_0}$ nonisomorphic finitely generated models of $Th_\forall(H)$. Contradiction with the property satisfied by H. $\qquad \square$

Theorem 3.6 *Let H be an equationally noetherian group and G a nontrivial finitely generated H-limit group. Then for any finite subset $X \subseteq G \setminus \{1\}$ there exists an epimorphism $f : G \to L$ where L is an H-determined group such that $1 \notin f(X)$.*

Proof

Write $G = \langle \bar{x} | P(\bar{x}) \rangle$ and let $S(\bar{x})$ be an H-witness of P. Let $X \subseteq G \setminus \{1\}$ be finite and let $v_1(\bar{x}), \ldots, v_n(\bar{x})$ be words representing the elements of X.

Let $\vartheta(\bar{x}) = (S(\bar{x}) = 1 \wedge \bigwedge_{1 \le i \le n} v_i(\bar{x}) \ne 1)$. By Theorem 2.11, there exists countably many nonisomorphic H-limit groups and thus by Proposition 3.5 there exists a primitive-quantifier-free formula $\xi(\bar{x})$ such that $H \models \exists \bar{x}(\vartheta(\bar{x}) \wedge \xi(\bar{x}))$ and for any word $w(\bar{x})$ on the variables $\bar{x} = \{x_1, \ldots, x_n\}$ and their inverses, one has

(1) $H \models \forall \bar{x}(\vartheta(\bar{x}) \wedge \xi(\bar{x}) \Rightarrow w(\bar{x}) = 1)$ or $H \models \forall \bar{x}(\vartheta(\bar{x}) \wedge \xi(\bar{x}) \Rightarrow w(\bar{x}) \ne 1)$.

Let $\xi(\bar{x}) = (\bigwedge_{w \in Q} w(\bar{x}) = 1 \wedge \bigwedge_{u \in N} u(\bar{x}) \ne 1))$, where Q, N are finite sets of words. Since $H \models \exists \bar{x}(\vartheta(\bar{x}) \wedge \xi(\bar{x}))$ there exists $\bar{a} \in H^n$ such that

$$H \models (S(\bar{a}) = 1 \wedge \bigwedge_{w \in Q} w(\bar{a}) = 1 \wedge \bigwedge_{1 \le i \le n} v_i(\bar{a}) \ne 1 \wedge \bigwedge_{u \in N} u(\bar{a}) \ne 1)).$$

Let $L = \langle \bar{a} \rangle$. Then L is an H-limit group as it is a subgroup of H and since $L \models S(\bar{a}) = 1$ there is an epimorphism $f : G \to L$ which sends x_i to a_i.

We claim that L is an H-determined group. Let C be the set of elements of L which are represented by the words $v_1(\bar{a}), \ldots, v_n(\bar{a})$ and $u(\bar{a}), u \in N$. Then for any morphism $h : L \to K$ where K is an H-limit group if $1 \notin h(C)$ then

$$K \models \vartheta(h(\bar{a})) \wedge \xi(h(\bar{a})),$$

and thus, since K is H-limit, by (1) we get

$$L \models w(\bar{a}) = 1 \Leftrightarrow K \models w(h(\bar{a})) = 1.$$

Therefore h is an embedding. Thus L is H-determined. $\qquad \square$

Theorem 3.7 *Let H be an equationally noetherian group. Then a finitely generated group G is H-limit if and only if G is fully residually-$\mathcal{D}_H$. Furtheremore, a finitely generated H-limit group G has the same universal theory as H if and only if every H-determined group is embeddable in G.*

Proof

Let G be a finitely generated H-limit group. Then, by Theorem 3.6, G is fully residually-$\mathcal{D}_H$.

Now if G is fully residually-$\mathcal{D}_H$, then clearly it is fully residually-H as any H-determined group is a subgroup of H.

Let G be a finitely generated group having the same universal theory as H. Let $L = \langle \bar{x} | P(\bar{x}) \rangle$ be an H-determined group and let $X \subseteq G \setminus \{1\}$, given by words $v_1(\bar{x}), \cdots, v_n(\bar{x})$, such that for any morphism $f : G \to L$, where L is an H-limit group, if $1 \notin f(X)$ then f is an embedding. Let $S(\bar{x})$ be H-witness of P. Then

(1) $$H \models \exists \bar{x}(S(\bar{x}) = 1 \wedge \bigwedge_{1 \le i \le n} v_i(\bar{x}) \ne 1).$$

Since $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(H)$, G satisfies the sentence which appears in (1). Therefore, as L is H-determined and G is H-limit, L is embeddable in G.

Now suppose that every H-determined group is embeddable in G and let us show that H is a model of the universal theory of G. Since G is H-limit, by Corollary 2.10, G is equationally noetherian. Therefore, by Corollary 2.4, it remains to show that every finitely generated subgroup of H is G-limit. By the above result every finitely generated subgroup of H is fully residually-$\mathcal{D}_H$. Therefore, every finitely generated subgroup of H is fully residually-G and thus it is G-limit. $\qquad\square$

Corollary 3.8 *If H is equationally noetherian then $Th_\forall(H) = \bigcap\limits_{K \in \mathcal{D}_H} Th_\forall(K)$.*

$\qquad\square$

Corollary 3.9 *If H is equationally noetherian and if every H-determined group is residually-finite then every finitely generated H-limit group is residually-finite.*

$\qquad\square$

This justifies the following problem.

Problem. What are the H-determined groups for H an hyperbolic group ?

Definition 3.10 *A* strong factor set *for a group G is a finite collection of proper quotients $\{f_i : G \to G_i\}$ of H-limit groups such that for any H-limit group L and any morphism $f : G \to L$ either f is an embedding or f factors through some f_i.*

Theorem 3.11 *Let H an equationally noetherian group. A nontrivial finitely generated H-limit group G is H-determined if and only if G has a strong factor set.*

Proof

Let G be a nontrivial finitely generated H-limit group which is H-determined and let us prove that G has a strong factor set. If every proper quotient of G which is H-limit is trivial then we have the desired conclusion. So suppose that G has a nontrivial proper quotient which is H-limit. Write $G = \langle \bar{x} | P(\bar{x}) \rangle$. By Theorem 2.11, G has at most countably many nonisomorphic H-limit quotients. Let $(G_i = \langle \bar{x} | P_i(\bar{x}) \rangle \mid i \in \mathbb{N})$ be the list of all nontrivial proper quotients of G which are H-limit. Since H is equationally noetherian there exist finite subsets $S(\bar{x}) \subseteq P(\bar{x})$, $S_i(\bar{x}) \subseteq P_i(\bar{x})$ such that

$$H \models \forall \bar{x}(S(\bar{x}) = 1 \Leftrightarrow P(\bar{x}) = 1).$$

$$H \models \forall \bar{x}(S_i(\bar{x}) = 1 \Leftrightarrow P_i(\bar{x}) = 1).$$

Since G is H-determined there exists a finite number $v_1(\bar{x}), \ldots, v_n(\bar{x})$ of words such that if K is an H-limit group containing $\bar{a}$ which satisfies $S(\bar{a}) = 1 \wedge \bigwedge_{1 \le i \le n} v_i(\bar{a}) \neq 1$, then the morphism $f : G \to K$ defined by $f(x_i) = a_i$ is an embedding.

Let $^*H$ be a nonprincipal ultrapower of H. Then every H-limit group is embeddable in $^*H$. Now if $\bar{b} \in {}^*H$ is a tuple such that $^*H \models S(\bar{b}) = 1$, then

$$^*H \models \bigwedge_{1 \leq i \leq n} v_i(\bar{b}) \neq 1 \vee \bigvee_{j \in \mathbb{N}} S_j(\bar{b}) = 1.$$

Since $^*H$ is $\aleph_1$-saturated, there is $m \in \mathbb{N}$ such that for every $\bar{b} \in {}^*H$ satisfying $^*H \models S(\bar{b}) = 1$ we get

$$^*H \models \bigwedge_{1 \leq i \leq n} v_i(\bar{b}) \neq 1 \vee \bigvee_{1 \leq j \leq m} S_j(\bar{b}) = 1.$$

Therefore

$$(1) \qquad H \models \forall \bar{x}(S(\bar{x}) = 1 \Rightarrow \bigwedge_{1 \leq i \leq n} v_i(\bar{x}) \neq 1 \vee \bigvee_{1 \leq j \leq m} S_j(\bar{x}) = 1).$$

We claim that $\{G_1, \ldots, G_m\}$ is a strong factor set where f_i is defined by the obvious manner. Let $f : G \to L$ be a morphism, where L is an H-limit group. Then there exists $\bar{a} \in L$ such that $L \models S(\bar{a}) = 1$ and $f(x_i) = a_i$. Since L is an H-limit group it satisfies the sentence appearing in (1) and thus $\langle \bar{a} \rangle$ is either isomorphic to G or $\langle \bar{a} \rangle \models \bigvee_{1 \leq j \leq m} S_j(\bar{a}) = 1$. Thus if f is not an embedding, there exists j and a morphism $h : G_j \to \langle \bar{a} \rangle$ and we see that $f = f_i \circ h$. Thus $\{G_1, \ldots, G_m\}$ is a strong factor set as claimed.

Suppose now that G has a strong factor set $\{f_i : G \to G_i \mid 1 \leq i \leq m\}$ and let us show that G is H-determined.

Write $(G_i = \langle \bar{x} | P_i(\bar{x}) \rangle \mid 1 \leq i \leq m)$ with $f(x_i) = x_i$. Since every G_i is a proper quotient, there exists words $v_1(\bar{x}), \ldots, v_m(\bar{x})$ such that

$$G_i \models v_i(\bar{x}) = 1, G \models v_i(\bar{x}) \neq 1.$$

Therefore if $f : G \to L$ where L is an H-limit group such that $\bigwedge_{1 \leq i \leq m} f(v_i) \neq 1$ then clearly f is an embedding. $\qquad \square$

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