A model-theoretic characterization of countable direct sums of finite cyclic groups

Abderezak OULD HOUCINE
Équipe de Logique Mathématique, UFR de Mathématiques, Université Denis-Diderot Paris 7, 2 place Jussieu 75251, Paris Cedex 05 France.
ould@logique.jussieu.fr

Abstract

We prove that an abelian group \( G \) is a countable direct sum of finite cyclic groups if and only if there exists a consistent existential theory \( \Gamma \) of abelian groups such that \( G \) is embeddable in every model of \( \Gamma \).

It is well known (cf. [4]) that a countable abelian group \( G \) is \( \aleph_0 \)-categorical if and only if \( G \) is of finite exponent, which implies that \( G \) is a countable direct sum of finite cyclic groups. What can be said about the model-theoretic properties of countable direct sums of finite cyclic groups in general? Here we give a model-theoretic characterization of countable direct sums of finite cyclic groups.

Let \( G \) be an abelian group. The order of an element \( g \) (of \( G \) or of any group) is denoted by \( o(g) \). For any prime \( p \) we denote by \( D_p(G) \), or just \( D(p) \) if there is no ambiguity, the \( p \) component of \( G \) in its primary decomposition. The cardinal of a set \( X \) is denoted by \(|X|\). By a 2-coloring of a set \( X \) we simply mean a map of \( X \) into \( \{0, 1\} \). By an existential theory of abelian groups we mean a set of existential sentences consistent with abelian group theory in the usual language of group theory with multiplication, inverse, and neutral element. We denote by \( Th_{\exists}(G) \) the set of existential sentences true in \( G \). We use only the rudiments of the theory of abelian groups (cf. the beginning of [3]) and model theory (cf. the beginning of [1]). Our result is:

**Theorem 1.** Let \( G \) be an abelian group. Then the following properties are equivalent:

1. \( G \) is a countable direct sum of finite cyclic groups.
2. \( G \) is embeddable in every abelian group which satisfies \( Th_{\exists}(G) \).
3. There exists an existential theory \( \Gamma \) of abelian groups such that \( G \) is embeddable in every abelian group which satisfies \( \Gamma \).

We need the following lemma, which is a straightforward application of the pigeon hole principle:
Lemma 1. Let \( m, n \) be integers where \( m \geq 1, n \geq 1 \). Then there exists an integer \( r(m, n) \) such that: for every set \( X \) such that \( |X| = r(m, n) \) and for every finite sequence of 2-colorings \( \zeta_1, \cdots, \zeta_m \) of \( X \) there exists a set \( Y \subseteq X \) such that \( |Y| = n \) and \( \zeta_i|Y \) is constant for every \( i = 1, \cdots, m \), i.e. a homogenous set having \( n \) elements. \( \square \)

Proof of Theorem 1.

(1)\( \Rightarrow \) (2). Let \( G \) be an abelian group which is a countable direct sum of finite cyclic groups. Let \( H \) be an abelian group which is a model of \( Th_{\exists}(G) \). It is enough to prove that \( D_p(G) = D(p) \) can be embedded in \( H \) for all \( p \).

We write \( D(p) = \bigoplus_{i \in \alpha} A_i \) where \( A_i \) is a non trivial \( p \)-primary cyclic group. If \( \alpha \) is finite then \( D(p) \) is finite and its isomorphism type is determined by an existential sentence \( \psi \), which is true in \( D(p) \), and which ‘translates’ its table. Hence if \( \alpha \) is finite one can embed \( D(p) \) in \( H \). So we can assume that \( \alpha \) is infinite and take \( \alpha = \mathbb{N} \).

Let \( I \) and \( J \) be the followings sets:

\[
I = \{ i \in \mathbb{N} : \text{the set } \{ j \mid A_i \hookrightarrow A_j \} \text{ is finite} \}
\]

\[
J = \{ i \in \mathbb{N} : \text{the set } \{ j \mid A_i \hookrightarrow A_j \} \text{ is infinite} \}
\]

Then \( I \cup J = \mathbb{N} \) and \( I \cap J = \emptyset \). Therefore \( D(p) = \bigoplus_{i \in I} A_i \bigoplus_{j \in J} A_j \).

We begin with the following facts:

**Fact 1.** \( I \) is finite.

**Proof.** Suppose towards a contradiction that \( I \) is infinite.

Let \( l = \min \{ n_k : A_k = \mathbb{Z}_{p^{n_k}}, \ k \in I \} \). Let \( i_0 \) be an element of \( I \) such that \( A_{i_0} = \mathbb{Z}_{p^l} \). We have \( A_{i_0} \hookrightarrow A_k \) for every \( k \in I \). Since \( I \) is infinite then \( i_0 \) is in \( J \), which contradicts \( I \cap J = \emptyset \). \( \square \)

**Fact 2.** Let \( n, k \in \mathbb{N} \), where \( n \geq 1, k \geq 2 \). Then there exists an existential sentence \( \psi_{n, k} = \exists x_1 \cdots \exists x_n \varphi_{n, k}(x_1, \cdots, x_n) \) such that for every abelian group \( K \) and every tuple \( a_1, \cdots, a_n \in K \) we have:

\[
K \models \varphi_{n, k}(a_1, \cdots, a_n) \text{ if and only if } \langle (a_1, \cdots, a_n) = \bigoplus_{i=1}^n \langle a_i \rangle \rangle \land \bigwedge_{i=1}^n (o(a_i) = k)
\]

**Proof.** It is sufficient to take \( \varphi_{n, k}(x_1, \cdots, x_n) \) to be the following formula:

\[
\left( \bigwedge_{\pi \in S(k)} (\sum_{i=1}^n \alpha_i x_i \neq 0) \right) \land \bigwedge_{i=1}^n (k x_i = 0 \land \bigwedge_{1 \leq i \leq k-1} (x_i \neq 0),
\]

where \( S(k) = \{ (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n : 0 \leq \alpha_i < k, \sum_{i=1}^n \alpha_i \neq 0 \} \). \( \square \)

**Fact 3.** Let \( s \in J \) and \( k = |A_s| \). Then for every \( n \in \mathbb{N}^* \), \( H \models \psi_{n, k} \) where \( \psi_{n, k} \) is the sentence given in Fact 2.
Proof. Since $s \in J$, the set $\{ j : A_s \mapsto A_j \}$ is infinite. Therefore the group $(A_s)^{(B_0)}$ is isomorphic to a subgroup of $D(p)$, hence to a subgroup of $G$. Then for every $n \in \mathbb{N}^*$, there exists a tuple $g_1, \ldots, g_n$ in $G$ such that:

$$\langle (g_1, \ldots, g_n) = \bigoplus_{i=1}^{n} \langle g_i \rangle \rangle \wedge o(g_i) = k.$$  

Hence by Fact 2, for every $n \in \mathbb{N}^*$, $G \models \psi_{n,k}$ and since $H \models Th_{\exists}(G)$ we have for every $n \in \mathbb{N}^*$, $H \models \psi_{n,k}$. □

Fact 4. Let $s \in J$ and $k = |A_s|$. Let $B$ be a non trivial finite subgroup of $H$. Then there exists $h \in H$ such that $o(h) = k$ and $\langle B, h \rangle = B \oplus \langle h \rangle$.

Proof. Let $U = \{ (b, \alpha) \in B \times \mathbb{Z} : 0 < \alpha < k \}$, and $m = |U|$. By Fact 3, $H \models \psi_{r(m,2),k}$ where $r(m,2)$ is the integer given in Lemma 1. Hence, by Fact 2, $H$ contains a tuple $h_1, \ldots, h_{r(m,2)}$ such that:

$$\langle h_1, \ldots, h_{r(m,2)} \rangle = \bigoplus_{i=1}^{r(m,2)} \langle h_i \rangle \wedge \bigwedge_{i=1}^{r(m,2)} (o(h_i) = k).$$  

Let $X = \{ h_1, \ldots, h_{r(m,2)} \}$. For every $(b, \alpha) \in U$ let $\zeta_{b,\alpha}$ be the 2-coloring, defined on $X$, by:

$$\zeta_{b,\alpha}(h_i) = \begin{cases} 1, & \text{if } b + \alpha h_i \neq 0; \\ 0, & \text{if } b + \alpha h_i = 0. \end{cases}$$  

Since $|X| = r(m,2)$ by Lemma 1, there exists in $X$ a homogeneous subset $Y = \{ h_{i_1}, h_{i_2} \}$, i.e. for every $(b, \alpha) \in U$, $\zeta_{b,\alpha}(h_{i_1}) = \zeta_{b,\alpha}(h_{i_2}) = 1$ or $\zeta_{b,\alpha}(h_{i_1}) = \zeta_{b,\alpha}(h_{i_2}) = 0$.

Suppose towards a contradiction that there exists $(b, \alpha) \in U$ such that $\zeta_{b,\alpha}(h_{i_1}) = \zeta_{b,\alpha}(h_{i_2}) = 0$. Then $\alpha h_{i_1} = \alpha h_{i_2}$ which is a contradiction as $0 < \alpha < k$, $o(h_{i_1}) = o(h_{i_2}) = k$ and $\langle h_{i_1} \rangle \cap \langle h_{i_2} \rangle = 0$.

Therefore for every $(b, \alpha) \in U$ we have $b + \alpha h_{i_1} \neq 0$. Thus by taking $h = h_{i_1}$ we have $o(h) = k$ and $\langle B, h \rangle = B \oplus \langle h \rangle$ as required. □

By Fact 1 $I$ is finite thus $\bigoplus_{i \in I} A_i$ is finite and, as it was remarked before, $\bigoplus_{i \in I} A_i$ is embeddable in $H$. Without loss of generality we can assume that $\bigoplus_{i \in I} A_i$ is a subgroup of $H$. Let $J = \{ J_t \mid t \in \mathbb{N} \}$. To prove that $D(p)$ is embeddable in $H$, it is sufficient to prove that $H$ contains a sequence $(B_t : t \in \mathbb{N})$ of subgroups which satisfies:

1. $B_t \cong A_{j_t}$.
2. For every $n \in \mathbb{N}$, $\langle B_0, \ldots, B_n, \bigcup_{i \in I} A_i \rangle = (\bigoplus_{t=0}^{n} B_t) \oplus (\bigoplus_{i \in I} \bigoplus_{t} A_i)$.

To do this we argue by induction on $t$ and we use Fact 4.
For \( t = 0 \).

Put \( B = \oplus_{i \in I} A_i \). Then by Fact 4 there exists \( h \in H \) such that \( o(h) = |A_{j_0}| \) and \( \langle B, h \rangle = B \oplus \langle h \rangle \). By putting \( B_0 = \langle h \rangle \) we have \( B_0 \cong A_{j_0} \) and \( \langle B_0, \bigcup_{i \in I} A_i \rangle = \langle B_0, \oplus_{i \in I} A_i \rangle = \langle B_0 \oplus (\oplus_{i \in I} A_i) \rangle \) which are the required properties. \( \square \)

For \( t + 1 \).

Let \( B_0, \cdots, B_t \) be the constructed sequence. Put \( B = \langle B_0, \cdots, B_t, \oplus_{i \in I} A_i \rangle \). Then by Fact 4 there exists \( h \in H \) such that \( o(h) = |A_{j_{t+1}}| \) and \( \langle B, h \rangle = B \oplus \langle h \rangle \). By putting \( B_{t+1} = \langle h \rangle \) we have \( B_{t+1} \cong A_{j_{t+1}} \) and \( \langle B_0, \cdots, B_t, B_{t+1}, \bigcup_{i \in I} A_i \rangle = \langle B_{t+1}, \oplus_{i=0}^{t} B_i \oplus (\oplus_{i \in I} A_i) \rangle = \langle B_{t+1} \oplus \oplus_{i=0}^{t} B_i \oplus (\oplus_{i \in I} A_i) \rangle \) which are the required properties. \( \square \)

This completes the proof of (1)\( \Rightarrow \) (2).

(2)\( \Rightarrow \) (3). Obvious.

(3)\( \Rightarrow \) (1). Let \( G \) be an abelian group and \( \Gamma \) an existential consistent theory of abelian groups such that \( G \) is embeddable in every abelian group which satisfies \( \Gamma \). Let \( \Gamma = \{ \psi_i \mid i \in \omega \} \). It is well known that every existential sentence true in an abelian group is true in a finite abelian group, i.e. in a finite direct sum of finite cyclic groups. Hence every \( \psi_i \) is true in such a finite direct sum \( K_i \) of finite cyclic groups and \( \Gamma \) has as model \( K = \oplus_{i \in \omega} K_i \) which is a direct sum of finite cyclic groups. Then \( G \) is isomorphic to a subgroup of \( K \) and is by a classical theorem (cf. Theorem 13 in [3]) a countable direct sum of finite cyclic groups. \( \square \)

**Corollary 1.** Let \( G \) be a countable direct sum of finite cyclic groups and let \( H \) be an abelian group. Then the following properties are equivalent:

(1) \( G \) is embeddable in \( H \).

(2) For every prime number \( p \) and for every \( n \), \( \dim p^n G/p \leq \dim p^n H/p \).

(Here for an abelian group \( K \), \( p^n K = \{ p^n x \mid x \in K \} \); \( K/p = \{ x \in K \mid px = 0 \} \); and "\( \dim \)" means dimension over \( \mathbb{Z}/p\mathbb{Z} \)).

**Proof.** It is an immediate consequence of Theorem 1 and [2, Theorem 4]. \( \square \)

**Remarks.**

(1) The model theory of abelian groups has been extensively studied. While the implication (3)\( \Rightarrow \) (1) can be seen as a consequence of [2] we have been unable to derive (1)\( \Rightarrow \) (3) or (1)\( \Rightarrow \) (2) from [2] or from other known results.

(2) Conditions (2) and (3) of the theorem make sense in a general model theoretic setting and are investigated in [5]. For any countable theory they give rise to two classes of countable structures. In general these classes are distinct: it is not difficult to see (and it is shown in [5]) that they are distinct for the theory of (all) groups, since the non trivial countable free groups satisfy (3) (with \( \Gamma \) finitely axiomatizable) and do not satisfy (2).
References


