A model-theoretic characterization of countable direct sums of finite cyclic groups

Abderezak OULD HOUCINE
Équipe de Logique Mathématique, UFR de Mathématiques,
Université Denis-Diderot Paris 7,
2 place Jussieu 75251, Paris Cedex 05 France.
ould@logique.jussieu.fr

Abstract

We prove that an abelian group G is a countable direct sum of finite cyclic groups if and only if there exists a consistent existential theory Γ of abelian groups such that G is embeddable in every model of Γ.

It is well known (cf. [4]) that a countable abelian group G is $\aleph_0$-categorical if and only if G is of finite exponent, which implies that G is a countable direct sum of finite cyclic groups. What can be said about the model-theoretics properties of countable direct sums of finite cyclic groups in general? Here we give a model-theoretic characterization of countable direct sums of finite cyclic groups.

Let G be an abelian group. The order of an element g (of G or of any group) is denoted by $o(g)$. For any prime p we denote by $D_p(G)$, or just $D(p)$ if there is no ambiguity, the p component of G in its primary decomposition. The cardinal of a set X is denoted by $|X|$. By a 2-coloring of a set X we simply mean a map of X into $\{0,1\}$. By an existential theory of abelian groups we mean a set of existential sentences consistent with abelian group theory in the usual language of group theory with multiplication, inverse, and neutral element. We denote by $Th_\exists(G)$ the set of existential sentences true in G. We use only the rudiments of the theory of abelian groups (cf. the beginning of [3]) and model theory (cf. the beginning of [1]). Our result is:

Theorem 1. *Let G be an abelian group. Then the following properties are equivalent:*

(**1**) *G is a countable direct sum of finite cyclic groups.*

(**2**) *G is embeddable in every abelian group which satisfies $Th_\exists(G)$.*

(**3**) *There exists an existential theory Γ of abelian groups such that G is embeddable in every abelian group which satisfies Γ.*

We need the following lemma, which is a straightforward application of the pigeon hole principle:

Lemma 1. *Let m, n be integers where $m \geq 1, n \geq 1$. Then there exists an integer $r(m, n)$ such that: for every set X such that $|X| = r(m, n)$ and for every finite sequence of 2-colorings $\zeta_1, \cdots, \zeta_m$ of X there exists a set $Y \subseteq X$ such that $|Y| = n$ and $\zeta_i|Y$ is constant for every $i = 1, \cdots, m$, i.e. a homogenous set having n elements.* □

Proof of Theorem 1.

(1)⇒(2). Let G be an abelian group which is a countable direct sum of finite cyclic groups. Let H be an abelian group which is a model of $Th_{\exists}(G)$. It is enough to prove that $D_p(G) = D(p)$ can be embedded in H for all p.

We write $D(p) = \oplus_{i \in \alpha} A_i$ where A_i is a non trivial p-primary cyclic group. If α is finite then $D(p)$ is finite and its isomorphism type is determined by an existential sentence ψ, which is true in $D(p)$, and which 'translates' its table. Hence if α is finite one can embed $D(p)$ in H. So we can assume that α is infinite and take $\alpha = \mathbb{N}$.

Let I and J be the followings sets:

$$I = \{i \in \mathbb{N} : \text{ the set } \{j \mid A_i \hookrightarrow A_j\} \text{ is finite }\}$$

$$J = \{i \in \mathbb{N} : \text{ the set } \{j \mid A_i \hookrightarrow A_j\} \text{ is infinite }\}$$

Then $I \cup J = \mathbb{N}$ and $I \cap J = \emptyset$. Therefore $D(p) = (\oplus_{i \in I} A_i) \oplus (\oplus_{j \in J} A_j)$.

We begin with the following facts:

Fact 1. *I is finite.*

Proof. Suppose towards a contradiction that I is infinite.

Let $l = \min \{n_k : A_k = \mathbb{Z}_{p^{n_k}},\ k \in I\}$. Let i_0 be an element of I such that $A_{i_0} = \mathbb{Z}_{p^l}$. We have $A_{i_0} \hookrightarrow A_k$ for every $k \in I$. Since I is infinite then i_0 is in J, which contradicts $I \cap J = \emptyset$. □

Fact 2. *Let $n, k \in \mathbb{N}$, where $n \geq 1, k \geq 2$. Then there exists an existential sentence $\psi_{n,k} = \exists x_1 \cdots \exists x_n \varphi_{n,k}(x_1, \cdots, x_n)$ such that for every abelian group K and every tuple $a_1, \cdots, a_n \in K$ we have:*

$$K \models \varphi_{n,k}(a_1, \cdots, a_n) \text{ if and only if } (\langle a_1, \cdots, a_n \rangle = \oplus_{i=1}^{i=n} \langle a_i \rangle) \wedge \bigwedge_{i=1}^{i=n} (o(a_i) = k)$$

Proof. It is sufficient to take $\varphi_{n,k}(x_1, \cdots, x_n)$ to be the following formula:

$$\Big(\bigwedge_{\overline{\alpha} \in S(k)} \Big(\sum_{i=1}^{i=n} \alpha_i x_i \neq 0\Big)\Big) \wedge \bigwedge_{i=1}^{i=n} \Big(kx_i = 0 \wedge \bigwedge_{1 \leq l \leq k-1} lx_i \neq 0\Big),$$

where $S(k) = \{(\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n : 0 \leq \alpha_i < k,\ \sum_{i=1}^{i=n} \alpha_i \neq 0\}$. □

Fact 3. *Let $s \in J$ and $k = |A_s|$. Then for every $n \in \mathbb{N}^*$, $H \models \psi_{n,k}$ where $\psi_{n,k}$ is the sentence given in Fact 2.*

Proof. Since $s \in J$, the set $\{j : A_s \hookrightarrow A_j\}$ is infinite. Therefore the group $(A_s)^{(\aleph_0)}$ is isomorphic to a subgroup of $D(p)$, hence to a subgroup of G. Then for every $n \in \mathbb{N}^*$, there exists a tuple $g_1, \cdots, g_n$ in G such that:

$$(\langle g_1, \cdots, g_n\rangle = \oplus_{i=1}^{i=n}\langle g_i\rangle) \wedge \bigwedge_{i=1}^{i=n} o(g_i) = k.$$

Hence by Fact 2, for every $n \in \mathbb{N}^*$, $G \models \psi_{n,k}$ and since $H \models Th_\exists(G)$ we have for every $n \in \mathbb{N}^*$, $H \models \psi_{n,k}$. □

Fact 4. *Let $s \in J$ and $k = |A_s|$. Let B be a non trivial finite subgroup of H. Then there exists $h \in H$ such that $o(h) = k$ and $\langle B, h\rangle = B \oplus \langle h\rangle$.*

Proof. Let

$$U = \{(b, \alpha) \in B \times \mathbb{Z} : 0 < \alpha < k\},$$

and $m = |U|$. By Fact 3, $H \models \psi_{r(m,2),k}$ where $r(m, 2)$ is the integer given in Lemma 1. Hence, by Fact 2, H contains a tuple $h_1, \cdots, h_{r(m,2)}$ such that:

$$\langle h_1, \cdots, h_{r(m,2)}\rangle = \oplus_{i=1}^{i=r(m,2)}\langle h_i\rangle \wedge \bigwedge_{i=1}^{i=r(m,2)} (o(h_i) = k).$$

Let $X = \{h_1, \cdots, h_{r(m,2)}\}$. For every $(b, \alpha) \in U$ let $\zeta_{b,\alpha}$ be the 2-coloring, defined on X, by:

$$\zeta_{b,\alpha}(h_i) = \begin{cases} 1, & \text{if } b + \alpha h_i \neq 0; \\ 0, & \text{if } b + \alpha h_i = 0. \end{cases}$$

Since $|X| = r(m, 2)$ by Lemma 1, there exists in X a homogeneous subset $Y = \{h_{i_1}, h_{i_2}\}$, i.e. for every $(b, \alpha) \in U$, $\zeta_{b,\alpha}(h_{i_1}) = \zeta_{b,\alpha}(h_{i_2}) = 1$ or $\zeta_{b,\alpha}(h_{i_1}) = \zeta_{b,\alpha}(h_{i_2}) = 0$.

Suppose towards a contradiction that there exists $(b, \alpha) \in U$ such that $\zeta_{b,\alpha}(h_{i_1}) = \zeta_{b,\alpha}(h_{i_2}) = 0$. Then $\alpha h_{i_1} = \alpha h_{i_2}$ which is a contradiction as $0 < \alpha < k$, $o(h_{i_1}) = o(h_{i_2}) = k$ and $\langle h_{i_1}\rangle \cap \langle h_{i_2}\rangle = 0$.

Therefore for every $(b, \alpha) \in U$ we have $b + \alpha h_{i_1} \neq 0$. Thus by taking $h = h_{i_1}$ we have $o(h) = k$ and $\langle B, h\rangle = B \oplus \langle h\rangle$ as required. □

By Fact 1 I is finite thus $\oplus_{i\in I}A_i$ is finite and, as it was remarked before, $\oplus_{i\in I}A_i$ is embeddable in H. Without loss of generality we can assume that $\oplus_{i\in I}A_i$ is a subgroup of H. Let $J = \{j_t \mid t \in \mathbb{N}\}$. To prove that $D(p)$ is embeddable in H, it is sufficient to prove that H contains a sequence $(B_t : t \in \mathbb{N})$ of subgroups which satisfies:

1. $B_t \cong A_{j_t}$.
2. For every $n \in \mathbb{N}$, $\langle B_0, \cdots, B_n, \bigcup_{i\in I} A_i\rangle = (\oplus_{t=0}^{t=n} B_t) \oplus (\oplus_{i\in I}A_i)$.

To do this we argue by induction on t and we use Fact 4.

For $t = 0$**.**

Put $B = \oplus_{i \in I} A_i$. Then by Fact 4 there exists $h \in H$ such that $o(h) = |A_{j_0}|$ and $\langle B, h \rangle = B \oplus \langle h \rangle$. By putting $B_0 = \langle h \rangle$ we have $B_0 \cong A_{j_0}$ and $\langle B_0, \bigcup_{i \in I} A_i \rangle = \langle B_0, \oplus_{i \in I} A_i \rangle = (B_0) \oplus (\oplus_{i \in I} A_i)$ which are the required properties. □

For $t + 1$.

Let $B_0, \cdots, B_t$ be the constructed sequence. Put $B = \langle B_0, \cdots, B_t, \oplus_{i \in I} A_i \rangle$. Then by Fact 4 there exists $h \in H$ such that $o(h) = |A_{j_{t+1}}|$ and $\langle B, h \rangle = B \oplus \langle h \rangle$. By putting $B_{t+1} = \langle h \rangle$ we have $B_{t+1} \cong A_{j_{t+1}}$ and $\langle B_0, \cdots, B_t, B_{t+1}, \bigcup_{i \in I} A_i \rangle = \langle B_{t+1}, \oplus_{t=0}^{t=n} B_t \oplus (\oplus_{i \in I} A_i) \rangle = (B_{t+1} \oplus \oplus_{t=0}^{t=n} B_t) \oplus (\oplus_{i \in I} A_i)$ which are the required properties. □

This completes the proof of (1)⇒(2).

(2)⇒(3). Obvious.

(3)⇒(1). Let G be an abelian group and Γ an existential consistent theory of abelian groups such that G is embeddable in every abelian group which satisfies Γ. Let $\Gamma = \{\psi_i | i \in \omega\}$. It is well known that every existential sentence true in an abelian group is true in a finite abelian group, i.e. in a finite direct sum of finite cyclic groups. Hence every ψ_i is true in such a finite direct sum K_i of finite cyclic groups and Γ has as model $K = \oplus_{i \in \omega} K_i$ which is a direct sum of finite cyclic groups. Then G is isomorphic to a subgroup of K and is by a classical theorem (cf. Theorem 13 in [3]) a countable direct sum of finite cyclic groups. □

Corollary 1. *Let G be a countable direct sum of finite cyclic groups and let H be an abelian group. Then the following properties are equivalent:*

(1) *G is embeddable in H.*

(2) *For every prime number p and for every n, $\dim p^n G[p] \leq \dim p^n H[p]$.*

(Here for an abelian group K, $p^n K = \{p^n x : x \in K\}$; $K[p] = \{x \in K : px = 0\}$; and "$\dim$" means dimension over $\mathbb{Z}/p\mathbb{Z}$).

Proof. It is an immediate consequence of Theorem 1 and [2, Theorem 4]. □

Remarks.

(1) The model theory of abelian groups has been extensively studied. While the implication (3)⇒(1) can be seen as a consequence of [2] we have been unable to derive (1)⇒(3) or (1)⇒(2) from [2] or from other known results.

(2) Conditions (2) and (3) of the theorem make sense in a general model theoretic setting and are investigated in [5]. For any countable theory they give rise to two classes of countable structures. In general these classes are distinct: it is not difficult to see (and it is shown in [5]) that they are distinct for the theory of (all) groups, since the non trivial countable free groups satisfy (3) (with Γ finitely axiomatizable) and do not satisfy (2).

References

[1] C.C.Chang, H.J.Keisler, *Model Theory,* North-Holland, Amsterdam, 1973.

[2] P.C.Eklof, *Some model theory of abelian groups*, J.Symbolic.Logic, vol.37 (1972), pp.335-342.

[3] I.Kaplansky, *Infinite Abelian Groups*, The University of Michigan Press, 1968.

[4] D. Marker, *Model Theory : An Introduction,* Graduate Texts in Mathematics, Springer-Verlag, New york, 2002.

[5] A.Ould Houcine, *Sur quelques problèmes de plongement dans les groupes*, PhD thesis, Université Paris 7, 2003.