

# Space of Second-Order Linear Differential Operators as a Module over the Lie Algebra of Vector Fields

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Received November 28, 1994; accepted January 2, 1995

The space of linear differential operators on a smooth manifold  $M$  has a natural one-parameter family of  $\text{Diff}(M)$ - (and  $\text{Vect}(M)$ -) module structures, defined by their action on the space of tensor densities. It is shown that, in the case of second-order differential operators, the  $\text{Vect}(M)$ -module structures are equivalent, for any degree of tensor densities except for three critical values;  $\{0, \frac{1}{2}, 1\}$ . A second-order analogue of the Lie derivative appears as an intertwining operator between the spaces of second-order differential operators on tensor densities. © 1997 Academic Press

## 1. INTRODUCTION: MAIN PROBLEM

Let  $M$  be an oriented manifold of dimension  $n$ . Consider the space  $\mathcal{D}^k(M)$  of  $k$ th order linear differential operators on  $M$ . In local coordinates, such an operator is given by

$$A(\phi) = a_k^{i_1 \cdots i_k} \partial_{i_1} \cdots \partial_{i_k} \phi + \cdots + a_1^i \partial_i \phi + a_0 \phi, \quad (1)$$

where  $\partial_i = \partial/\partial x^i$  and  $a_k^{i_1 \cdots i_k}$ ,  $\phi \in C^\infty(M)$  with  $l = 0, 1, \dots, k$ . (From now on we suppose a summation over repeated indices.)

The group  $\text{Diff}(M)$  of all diffeomorphisms of  $M$  and the Lie algebra  $\text{Vect}(M)$  of all smooth vector fields naturally act on the space  $\mathcal{D}^k(M)$ . Let  $G \in \text{Diff}(M)$ , then the action is defined by

$$G(A) := G^*{}^{-1}AG^*.$$

A vector field  $\zeta \in \text{Vect}(M)$  acts on differential operators by the commutator with the operator of Lie derivative:

$$\text{ad } L_\zeta(A) := L_\zeta \circ A - A \circ L_\zeta. \tag{2}$$

It is interesting to take as arguments *tensor densities* of degree  $\lambda$  instead of functions. This defines a family of  $\text{Diff}(M)$  and  $\text{Vect}(M)$ -module structures on  $\mathcal{D}^k(M)$  depending on  $\lambda$ .

Studying these module structures is a very important problem since a number of different examples naturally appear in differential geometry and mathematical physics (see below). To our knowledge, the classification problem of such module structures for different values of  $\lambda$  has never been considered (at least in the multidimensional case). In this paper we solve this problem for the space  $\mathcal{D}^2(M)$  of second-order linear differential operators.

### 1.1. Tensor Densities: Definition

Consider the determinant bundle  $A^n TM \rightarrow M$ . The group  $\mathbf{R}^*$  acts on the fibers by multiplication.

**DEFINITION.** A homogeneous function of degree  $\lambda$  on the complement of the zero section,  $A^n TM \setminus M$ , of the determinant bundle,

$$F(\kappa w) = \kappa^\lambda F(w),$$

is called the tensor density of degree  $\lambda$  on  $M$ .

Let us denote by  $\mathcal{F}_\lambda(M)$  the space of tensor densities of degree  $-\lambda$ . It is evident that  $\mathcal{F}_0(M) = C^\infty(M)$ , the space  $\mathcal{F}_{-1}(M)$  coincides with the space of differential  $n$ -forms on  $M$ :  $\mathcal{F}_{-1}(M) = \Omega^n(M)$ .

In local coordinates, one uses the following notation for a tensor density of degree  $\lambda$ :

$$\phi = \phi(x_1, \dots, x_n)(dx_1 \wedge \dots \wedge dx_n)^\lambda.$$

The group  $\text{Diff}(M)$  acts on the determinant bundle by homogeneous diffeomorphisms. Therefore, it acts on the space  $\mathcal{F}_\lambda$ . One has

$$G^*\phi = \phi \circ G \cdot J_G^\lambda$$

where  $J_G = (DG/Dx)$  is the Jacobian.

The corresponding action of the Lie algebra  $\text{Vect}(M)$  is given by the Lie derivative:

$$L_\zeta \phi = \zeta^i \partial_i \phi - \lambda \phi \partial_i \zeta^i. \tag{3}$$

Note that this formula does not depend on the choice of coordinates.

It is evident that, for an oriented manifold,  $\mathcal{F}_\lambda \cong \mathcal{F}_\mu$  as linear spaces (but not as modules) for any  $\lambda, \mu$ .

*Remark.* If  $M$  is compact, then there exists a natural isomorphism of  $\text{Vect}(M)$ - (and  $\text{Diff}(M)$ -) modules

$$\mathcal{F}_\lambda \cong \mathcal{F}_{-1-\lambda}^*$$

(tautological for  $\lambda = -\frac{1}{2}$ ). Indeed, there exists a non degenerate invariant pairing  $\mathcal{F}_\lambda \otimes \mathcal{F}_{-1-\lambda} \rightarrow \mathbf{R}$  given by  $\langle \phi, \psi \rangle = \int_M \phi \psi$ .

### 1.2. Linear Differential Operators on Tensor Densities

**DEFINITION.** Each vector field  $\xi$  defines an operator  $L_\xi$  on the space of tensor densities  $\mathcal{F}_\lambda$ . Define the space  $\mathcal{D}_\lambda^k$  of  $k$ th-order linear differential operators on  $\mathcal{F}_\lambda$  as the space of all  $k$ th order polynomials in different operators  $L_\xi$  and operators of multiplication by functions.

Each differential operator  $A \in \mathcal{D}_\lambda^k$  is given by (1) in any system of local coordinates.

**EXAMPLES.** (1) A classical example is the theory of the Sturm–Liouville equation:  $\phi''(x) + u(x)\phi(x) = 0$ . The argument  $\phi$ , in this case, is a  $-\frac{1}{2}$ -density:  $\phi = \phi(x)(dx)^{-1/2}$  (see [2, 6]).

(2) Another example is given by geometric quantization [9, 15], the algebra of linear differential operators acting on a Hilbert space of  $\frac{1}{2}$ -densities [1, 10] (see also [14, 7, 16]).

The space of differential operators on  $\mathcal{F}_\lambda$  does not depend on  $\lambda$  as a linear space. We shall use the notation  $\mathcal{D}_\lambda^k$  for  $\text{Diff}(M)$ - and  $\text{Vect}(M)$ -module structures on this space.

### 1.3. The Lie Derivative as a Vect(M)-Equivariant Operator

All the modules  $\mathcal{D}_\lambda^1$  of first-order differential operators are isomorphic to each other: there exists a  $\text{Vect}(M)$ -equivariant linear mapping

$$\mathcal{L}_{\mu\lambda}: \mathcal{D}_\mu^1 \rightarrow \mathcal{D}_\lambda^1.$$

Let  $A = a_1^i \partial_i + a_0 \in \mathcal{D}_\mu^1$ ; define

$$\mathcal{L}_{\mu\lambda}(A) = a_1^i \partial_i + a_0 + (\mu - \lambda) \partial_i a_1^i. \tag{4}$$

In fact, the expression  $A(\phi)$  can be written in an invariant way:  $A(\phi) = L_{a_1} \phi + (a_0 + \mu \partial_i a_1^i) \phi$  (the quantity  $a_0 + \mu \partial_i a_1^i$  transforms as a function

by coordinate transformations). The operator  $\mathcal{L}_{\mu\lambda}(A)$  is defined by a simple application of the same operator to tensor-densities of degree  $\lambda$ . Let  $\psi \in \mathcal{F}_\lambda$ , put  $\mathcal{L}_{\mu\lambda}(A)\psi := L_{a_1}\psi + (a_0 + \mu\partial_i a_1^i)\psi$ . One obtains the explicit formula (4) from (3).

The mapping  $\mathcal{L}_{\mu\lambda}$  generalizes the classical Lie derivative. In fact,

$$L_\xi = \mathcal{L}_{0\lambda}(\xi).$$

More generally, if  $L_\xi^{(\lambda)}$  is the Lie derivative on  $\mathcal{F}_\lambda$ , then  $\mathcal{L}_{\mu\lambda}(L_\xi^{(\mu)}) = L_\xi^{(\lambda)}$ .

We are now looking for a higher-order analogue of the Lie derivative, in other words, for an equivariant linear mapping

$$\mathcal{L}_{\mu\lambda}^k : \mathcal{D}_\mu^k \rightarrow \mathcal{D}_\lambda^k.$$

*Remark.* Let us recall that a classic theorem (see [8, 13]) states that the only  $\text{Vect}(M)$ -equivariant differential operator in *one* argument (*unar* operator) on the space of “geometric quantities” (tensors, tensor densities, etc.) is the standard differential:  $d: \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ . Now, the space of differential operators turns out to be a richer module, so that the Lie derivative (4) is admitted as a *unar* equivariant differential operator. The purpose of this paper is to find another equivariant operator extending the ordinary Lie derivative.

## 2. MAIN THEOREMS

The main results of this paper correspond to the space  $\mathcal{D}^2$  of second-order linear differential operators on an oriented manifold  $M$ :

$$A(\phi) = a_2^{ij}\partial_i\partial_j\phi + a_1^i\partial_i\phi + a_0\phi.$$

The formula (2) defines a family of  $\text{Vect}(M)$ -module structures  $\mathcal{D}_\lambda^2$  on this space.

### 2.1. Critical Values of the Degree

The following theorem gives the classification of the  $\text{Vect}(M)$ -modules  $\mathcal{D}_\lambda^2$  on the space of second-order linear differential operators.

**THEOREM 1.** (i) *If  $\dim M \geq 2$ , then all  $\text{Vect}(M)$ -modules  $\mathcal{D}_\lambda^2$  with  $\lambda \neq 0, -\frac{1}{2}, -1$  are isomorphic to each other, but are not isomorphic to*

$$\mathcal{D}_0^2 \cong \mathcal{D}_{-1}^2 \not\cong \mathcal{D}_{-1/2}^2.$$

(ii) If  $\dim M = 1$ , then all  $\text{Vect}(M)$ -modules  $\mathcal{D}_\lambda^2$  with  $\lambda \neq 0, -1$  are isomorphic to each other, but are not isomorphic to

$$\mathcal{D}_0^2 \cong \mathcal{D}_{-1}^2.$$

Therefore, there is one stable  $\text{Vect}(M)$ -module structure on the space of second-order linear differential operators and two exceptional modules corresponding to functions and to  $\frac{1}{2}$ -densities, if  $\dim M \geq 2$ . If  $\dim M = 1$ , there are only two different  $\text{Vect}(M)$ -module structures.

DEFINITION. Let us call critical values the following values of the degree:  $\{0, \frac{1}{2}, 1\}$  if  $\dim M \geq 2$  and  $\{0, 1\}$  if  $\dim M = 1$ .

### 2.2. Second-Order Lie Derivative

We propose here an explicit formula for the equivariant linear mapping

$$\mathcal{L}_{\mu\lambda}^2 : \mathcal{D}_\mu^2 \rightarrow \mathcal{D}_\lambda^2$$

which can be considered as an analogue of the Lie derivative. Let us introduce the following notation:  $\tilde{A} = \mathcal{L}_{\mu\lambda}^2(A)$ .

THEOREM 2. (i)  $\dim M \geq 2$ . Let  $\lambda, \mu \neq 0, -\frac{1}{2}, -1$ . There exists a unique (up to a constant) equivariant linear mapping  $\mathcal{L}_{\mu\lambda}^2 : \mathcal{D}_\mu^2 \rightarrow \mathcal{D}_\lambda^2$  given by the formula:

$$\begin{cases} \tilde{a}_2^{ij} = a_2^{ij} \\ \tilde{a}_1^i = \frac{2\lambda + 1}{2\mu + 1} a_1^i + 2 \frac{\mu - \lambda}{2\mu + 1} \partial_i a_2^{ij} \\ \tilde{a}_0 = \frac{\lambda(\lambda + 1)}{\mu(\mu + 1)} a_0 + \frac{\lambda(\mu - \lambda)}{(2\mu + 1)(\mu + 1)} (\partial_i a_1^i - \partial_i \partial_j a_2^{ij}). \end{cases} \tag{5}$$

(ii)  $\dim M = 1$ . Let  $\lambda, \mu \neq 0, -1$ . There exists a one-parameter family of equivariant linear mappings from  $\mathcal{D}_\mu^2$  to  $\mathcal{D}_\lambda^2$ .

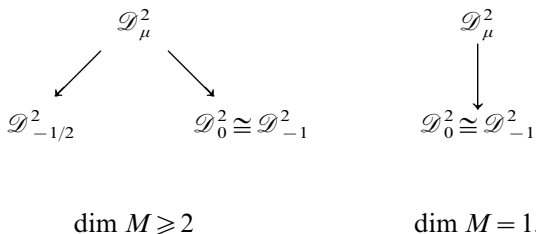
As a result of equivariance, the formula (5) does not depend on the choice of coordinates.

Remark. If  $\lambda = \mu$ , then  $\tilde{A} = A$ ; if  $\lambda + \mu = -1$ , then the mapping (5) is the conjugation of differential operators:  $\tilde{A} = A^*$ . Thus, (5) realizes an interpolation between a differential operator and its conjugate.

### 2.3. Hierarchy of Modules

For the critical values of the degree  $-\lambda$ , the  $\text{Vect}(M)$ -module  $\mathcal{D}_\lambda^2$  is not isomorphic to  $\text{Diff}(M)_\mu^2$  (with general  $\mu$ ). The mapping  $\mathcal{L}_{\mu,\lambda}^2$  in this case is

an *invariant projection* from  $\mathcal{D}_\mu^2$  to  $\mathcal{D}_\lambda^2$ . The following diagram represents the hierarchy of  $\text{Vect}(M)$ -module structures on the space of second-order differential operators:



### 2.4. Automorphisms of Exceptional Modules

Recall that a linear mapping  $\mathcal{F}: \mathcal{D}_\lambda^2 \rightarrow \mathcal{D}_\lambda^2$  is called an *automorphism* of the  $\text{Vect}(M)$ -module  $\mathcal{D}_\lambda^2$  if

$$[\mathcal{F}, \text{ad } L_\xi] = 0$$

for any  $\xi \in \text{Vect}(M)$ .

An important property of the  $\text{Vect}(M)$ -module structures on  $\mathcal{D}^2$ , corresponding to critical values of the degree, is the existence of nontrivial automorphisms.

The uniqueness of the mapping (5) implies the following fact:

**COROLLARY OF THEOREM 2.** *The  $\text{Vect}(M)$ -module  $\mathcal{D}_\lambda^2$  with  $\lambda \neq 0, -\frac{1}{2}, -1$  has no nontrivial automorphisms.*

The following statement gives the classification of automorphisms for the modules  $\mathcal{D}_0^2 \cong \mathcal{D}_{-1}^2$  and  $\mathcal{D}_{-1/2}^2$ .

**PROPOSITION 1.** (i) *Each automorphism of the module  $\mathcal{D}_0^2$  is proportional to the automorphism*

$$\mathcal{F} \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \\ c \cdot a_0 \end{pmatrix}, \tag{6}$$

where  $c = \text{const.}$

(ii) *Each automorphism of the module  $\mathcal{D}_{-1/2}^2$  is proportional to the automorphism*

$$\mathcal{F} \begin{pmatrix} a_2^i \\ a_1^i \\ a_0 \end{pmatrix} = \begin{pmatrix} a_2^i \\ a_1^i - 2c(a_1^i - \partial_j a_2^i) \\ a_0 - c(\partial_i a_1^i - \partial_i \partial_j a_2^i) \end{pmatrix}, \tag{7}$$

where  $c = \text{const.}$ , in terms of the components  $a_2, a_1, a_0$  of the operator  $A = a_2^{ij} \partial_i \partial_j + a_1^i \partial_i + a_0$ .

The next part of the paper is devoted to proofs. We give the explicit formulae for the  $\text{Vect}(M)$ -actions on the space of second-order linear differential operators. This family of actions, depending on  $\lambda$ , can be considered as a *one-parameter deformation* of a  $\text{Vect}(M)$ -module structure. This approach leads to the cohomology of the Lie algebra  $\text{Vect}(M)$  with some nontrivial operator coefficients.

### 3. ACTION OF THE LIE ALGEBRA $\text{Vect}(M)$ ON THE SPACE OF OPERATORS

The space of second-order linear differential operators  $\mathcal{D}^2$  is isomorphic, as a vector space, to a direct sum of some spaces of tensor fields,

$$\mathcal{D}_\lambda^2 \cong S^2(M) \oplus \text{Vect}(M) \oplus C^\infty(M),$$

where  $S^2(M)$  is the space of second-order symmetric contravariant tensor fields.

The  $\text{Vect}(M)$ -action (2) on the space of differential operators  $\mathcal{D}_\lambda^2$  is therefore a “modification” of the standard  $\text{Vect}(M)$ -action on this direct sum.

#### 3.1. Explicit Formulae

LEMMA 1. *The action  $\text{ad } L_\xi = \text{ad } L_\xi^{(\lambda)}$  of  $\text{Vect}(M)$  on  $\mathcal{D}_\lambda^2$  (see (2)) is given by*

$$\begin{cases} \text{ad } L_\xi(A)_2^{ij} = (L_\xi a_2)^{ij} \\ \text{ad } L_\xi(A)_1^l = (L_\xi a_1)^l - a_2^{ij} \partial_i \partial_j \xi^l + 2\lambda a_2^{li} \partial_i \partial_j \xi^j \\ \text{ad } L_\xi(A)_0 = L_\xi a_0 + \lambda(a_1^r \partial_r + a_2^{ij} \partial_i \partial_j) \partial_k \xi^k, \end{cases} \quad (8)$$

where  $(L_\xi a_2)^{ij} = \xi^r \partial_r a_2^{ij} - a_2^{rj} \partial_r \xi^i - a_2^{ri} \partial_r \xi^j$  and  $(L_\xi a_1)^l = \xi^r \partial_r a_1^l - a_1^r \partial_r \xi^l$  and  $L_\xi a_0 = \xi^r \partial_r a_0$  are the Lie derivatives of tensor fields along the vector field  $\xi$ .

*Proof.* By definition, the result of the action  $\text{ad } L_\xi$  is given by  $\text{ad } L_\xi(A)(\phi) = [L_\xi, A](\phi)$ . From (2) one has:

$$\begin{aligned} [L_\xi, A](\phi) &= \xi^r \partial_r (a_2^{ij} \partial_i \partial_j \phi + a_1^i \partial_i \phi + a_0 \phi) \\ &\quad - \lambda \partial_k (\xi^k) (a_2^{ij} \partial_i \partial_j \phi + a_1^i \partial_i \phi + a_0 \phi) \\ &\quad - (a_2^{ij} \partial_i \partial_j + a_1^i \partial_i + a_0) (\xi^r \partial_r \phi - \lambda \partial_k (\xi^k) \phi). \end{aligned}$$

One gets immediately the formula (8).

*Remark.* The homogeneous part of the operator  $A$  transforms as a symmetric contravariant 2-tensor:  $a_2 \in S^2(M)$ .

### 3.2. Invariant Normal Form

There exists a canonical form of the  $\text{Vect}(M)$ -action on the space  $\mathcal{D}_\lambda^2$ . To obtain it, we introduce here some “transformation of variables” in the space  $\mathcal{D}_\lambda^2$ .

The intuitive idea is as follows. First, remark that the transformation law of the quantity  $a_0 + \lambda \partial_i a_1^i$  contains only terms with  $a_2$ . Second, the action (8) is a deformation of the standard action of  $\text{Vect}(M)$  on the space  $S^2(M) \oplus \text{Vect}(M) \oplus C^\infty(M)$ . We are therefore trying to find some canonical form of the cocycles on  $\text{Vect}(M)$  generating this deformation.

#### A. $\dim M \geq 2$

LEMMA 2. *The quantities*

$$\begin{cases} \bar{a}_2^{ij} = a_2^{ij} \\ \bar{a}_1^i = a_1^i + 2\lambda \partial_i a_2^i \\ \bar{a}_0 = a_0 + \lambda \partial_i a_1^i + \lambda^2 \partial_i \partial_j a_2^{ij} \end{cases} \quad (9)$$

are transformed by the  $\text{Vect}(M)$ -action in the following way:

$$\begin{cases} \text{ad } L_\xi(\bar{a}_2)^{ij} = \xi^k \partial_k \bar{a}_2^{ij} - \bar{a}_2^{ik} \partial_k \xi^j - \bar{a}_2^{kj} \partial_k \xi^i \\ \text{ad } L_\xi(\bar{a}_1)^i = \xi^k \partial_k \bar{a}_1^i - \bar{a}_1^k \partial_k \xi^i - (2\lambda + 1) \bar{a}_2^{kj} \partial_k \partial_j \xi^i \\ \text{ad } L_\alpha(\bar{a}_0) = \xi^k \partial_k \bar{a}_0 - \lambda(\lambda + 1) \partial_i (\bar{a}_2^{jk}) \partial_j \partial_k \xi^i. \end{cases} \quad (10)$$

*Proof.* From the formula (8), one has  $\partial_k L_\xi a_2^{ki} = \xi^r \partial_r \partial_k a_2^{ki} - \partial_k a_2^{kr} \partial_r \xi^i - a_2^{kr} \partial_k \partial_r \xi^i - a_2^{ir} \partial_r \partial_k \xi^k$ . The transformation law for  $\bar{a}_1$  in the formula (9) follows immediately from this expression.

The transformation law for  $\bar{a}_0$  can be easily verified in the same way.

*Important Remark.* The mapping

$$\gamma: \xi^r \partial_r \mapsto \partial_i \partial_j \xi^r \cdot dx^i \otimes dx^j \otimes \partial_r$$

is a nontrivial 1-cocycle on  $\text{Vect}(M)$  with values in the space of 2-covariant, 1-contravariant tensors. This cocycle appears in the Lie derivative of connections (see Section 6).

*Remark.* The formula (10) can be written in more invariant way. It is sufficient to note that the terms depending on  $\bar{a}_2$  in the expressions  $\text{ad } L_\xi(\bar{a}_1)$  and  $\text{ad } L_\xi(\bar{a}_0)$  are  $-(2\lambda + 1)\langle \bar{a}_2, \gamma(\xi) \rangle$  and  $-\lambda(\lambda + 1)[\partial_i \langle \bar{a}_2, \gamma(\xi) \rangle^i - \bar{a}_2(\partial_i \xi^i)]$ , respectively, where  $\bar{a}_2(f) := \bar{a}_2^{ij} \partial_j \partial_k f$ .



B.  $\dim M = 1$ .

In the one-dimensional case, let us simply give the normal form.

LEMMA 3. *The quantities*

$$\begin{cases} \bar{a}_2 = a_2 \\ \bar{a}_1 = a_1 + \frac{1}{2}(2\lambda - 1) a'_2 \\ \bar{a}_0 = a_0 + \lambda a'_1 + \frac{1}{3}\lambda(2\lambda - 1) a''_2 \end{cases}$$

transform under the action of  $\text{Vect}(M)$  according to:

$$\begin{cases} \text{ad } L_\xi(\bar{a}_2) = \xi \bar{a}'_2 - 2\xi' \bar{a}_2 \\ \text{ad } L_\xi(\bar{a}_1) = \xi \bar{a}'_1 - \xi' \bar{a}_1 \\ \text{ad } L_\xi(\bar{a}_0) = \xi \bar{a}'_0 + \frac{2}{3}\lambda(\lambda + 1) \bar{a}_2 \xi''' \end{cases}$$

4. PROOF OF THEOREM 1

First of all, it is evident that all the modules  $\mathcal{D}_\lambda^2$  with  $\lambda \neq 0, -\frac{1}{2}, -1$  are isomorphic. In fact, the mapping  $A \mapsto \tilde{A}$  given in normal coordinates (9) by  $\tilde{a}_2 = \bar{a}_2, \tilde{a}_1 = ((2\mu + 1)/(2\lambda + 1)) \bar{a}_1,$  and  $\tilde{a}_0 = ((\mu(\mu + 1))/(\lambda(\lambda + 1))) \bar{a}_0$  defines an isomorphism between  $\mathcal{D}_\lambda^2$  and  $\mathcal{D}_\mu^2$ .

In the same way, one has  $\mathcal{D}_0^2 \cong \mathcal{D}_{-1}^2$ . Here the mapping is the conjugation.

4.1. *Relation with Cohomology*

To prove that the modules  $\mathcal{D}_{-1/2}^2$  and  $\mathcal{D}_0^2 \cong \mathcal{D}_{-1}^2$  are not isomorphic to any other module, we use the approach of the general theory of deformations (see e.g. [5]).

Let us denote the coefficients  $-(2\lambda + 1)$  and  $-\lambda(\lambda + 1)$  by  $\alpha_1$  and  $\alpha_2$  respectively. Take  $\alpha_1$  and  $\alpha_2$  as *independent parameters*. One gets a two-parameter family of actions of  $\text{Vect}(M)$  more general than the action (10) on the space of differential operators:

LEMMA 4. *For each value of  $\alpha_1, \alpha_2,$  the following expression defines a  $\text{Vect}(M)$ -action:*

$$\begin{cases} T_\xi^{\alpha_1\alpha_2}(a_2)^{ij} = (L_\xi a_2)^{ij} \\ T_\xi^{\alpha_1\alpha_2}(a_1)^l = (L_\xi a_1)^l + \alpha_1 a_2^{ij} \partial_i \partial_j \xi^l \\ T_\xi^{\alpha_1\alpha_2}(a_0) = L_\xi a_0 + \alpha_2 \partial_i (a_2^{jm}) \partial_j \partial_m \xi^i. \end{cases} \tag{11}$$

*Proof.* This fact is evident since the formula (10) defines an action of  $\text{Vect}(M)$  and the coefficients denoted by  $\alpha_1$  and  $\alpha_2$  are independent.

The action (11) is a *two-parameter deformation* of the standard  $\text{Vect}(M)$ -module structure on the space  $S^2(M) \oplus \text{Vect}(M) \oplus C^\infty(M)$ .

LEMMA 5. *The two mappings*

$$C_1: \text{Vect}(M) \rightarrow \text{Hom}(S^2(M), \text{Vect}(M))$$

$$C_2: \text{Vect}(M) \rightarrow \text{Hom}(S^2(M), C^\infty(M))$$

given by  $C_1(\xi)(A) = a_2^{ij} \partial_i \partial_j(\xi^l) \partial_l$  and  $C_2(\xi)(A) = \partial_i(a_2^{jm}) \partial_j \partial_m \xi^i$  are 1-cocycles.

*Proof.* One should check that for any  $\xi, \eta \in \text{Vect}(M)$ ,

$$[L_\xi, C(\eta)] - [L_\eta, C(\xi)] = C([\xi, \eta]),$$

whenever  $C = C_1, C_2$ . This relation readily follows from the fact that the formula (11) defines an action of  $\text{Vect}(M)$ .

Standard arguments show that the structures of  $\text{Vect}(M)$ -module given by (11) with  $\alpha_1 \neq 0$  and with  $\alpha_1 = 0$  are isomorphic if and only if the cocycle  $C_1$  is a coboundary. In the same way, the  $\text{Vect}(M)$ -modules (11) with  $\alpha_2 \neq 0$  and with  $\alpha_2 = 0$  iff the cocycle  $C_2$  is a coboundary.

Moreover, let us prove that it is sufficient to study  $C_1$  and  $C_2$  as *differentiable* cocycles. This means that we consider the groups

$$H^1_A(\text{Vect}(M); \text{Hom}(S^2(M), \text{Vect}(M)))$$

and

$$H^1_A(\text{Vect}(M); \text{Hom}(S^2(M), C^\infty(M)))$$

of *differentiable (or diagonal) cohomology*. In other words, it is sufficient to consider the cohomology classes of  $C_1$  and  $C_2$  only module coboundaries given by differential operators (see [5] for details).

LEMMA 6. *If  $C_1$  and  $C_2$  represent nontrivial classes of differentiable cohomology, then there exist three nonisomorphic structures of  $\text{Vect}(M)$ -module given by (11):*

- (1)  $\alpha_1, \alpha_2 \neq 0,$
- (2)  $\alpha_1 = 0, \alpha_2 \neq 0,$
- (3)  $\alpha_1 \neq 0, \alpha_2 = 0.$

*Proof.* Suppose that the  $\text{Vect}(M)$ -module structures (1) and (2) are isomorphic. Then the cocycle  $C_1$  is a coboundary: there exists an operator  $B \in \text{Hom}(S^2(M), \text{Vect}(M))$  such that  $C_1(\xi) = L_\xi \circ B - B \circ L_\xi$ . Moreover, the isomorphism between the  $\text{Vect}(M)$ -modules (1) and (2) is given by a differential operator:  $A \mapsto J(A)$ . In fact, any such isomorphism is local:  $\text{supp } J(A) = \text{supp } A$ . Thus,  $B$  is also a differential operator and  $C_1$  is a coboundary as a differentiable cocycle.

In the same way one proves that if  $C_2$  is nontrivial as a differentiable cocycle, then the  $\text{Vect}(M)$ -modules (1) and (2) are not isomorphic to the module (3). The lemma is proven.

4.2. Nontrivial Cohomology Class in  $H^1_\Delta(\text{Vect}(M); \text{Hom}(S^2(M), \text{Vect}(M)))$

PROPOSITION 2. (i) *If  $\dim M \geq 2$ , then the cocycle  $C_1$  represents a nontrivial cohomology class of the differentiable cohomology group*

$$H^1_\Delta(\text{Vect}(M); \text{Hom}(S^2(M), \text{Vect}(M))).$$

(ii) *If  $\dim M = 1$ , then  $C_1$  is a coboundary.*

*Proof.* To prove the second statement, note that in the one-dimensional case  $S^2 \cong \mathcal{F}_2$  and consider the following operator:

$$B(a(x)(dx)^{-2}) = \frac{1}{2}a'(dx)^{-1}.$$

It is easy to check that  $C_1(\xi) = L_\xi \circ B - B \circ L_\xi = (\delta B)(\xi)$ .

Let now  $\dim M \geq 2$ . Suppose that there exists a differential operator  $B: S^2(M) \rightarrow \text{Vect}(M)$  such that  $\delta B(\xi) = C_1(\xi)$ . Let  $a^{ij}\partial_i \otimes \partial_j \in S^2(M)$ ; we have in general

$$B(a) = b^{ki_1 \dots i_m}_{ij} \partial_{i_1} \dots \partial_{i_m} a^{ij} \partial_k$$

where  $b^{ki_1 \dots i_m}_{ij} = b^{ki_1 \dots i_m}_{ji}$  and

$$\begin{aligned} (\delta B)(\xi) a &:= L_\xi \circ Ba - B \circ L_\xi a \\ &= [\xi^r \partial_r (b^{ki_1 \dots i_m}_{ij} \partial_{i_1} \dots \partial_{i_m} a^{ij}) - b^{ri_1 \dots i_m}_{ij} \partial_{i_1} \dots \partial_{i_m} a^{ij} \partial_r \xi^k \\ &\quad - b^{ki_1 \dots i_m}_{ij} \partial_{i_1} \dots \partial_{i_m} (\xi^r \partial_r a^{ij} - a^{ir} \partial_r \xi^j - a^{jr} \partial_r \xi^i)] \partial_k. \end{aligned}$$

The condition  $\delta B = C_1$  implies immediately  $m = 1$ , i.e.,  $B$  is a first-order differential operator. Indeed, the highest order (in the derivatives of  $\xi$ ) term in  $(\delta B(\xi))(a)$  is

$$2a^{ir} b^{ki_1 \dots i_m}_{ij} \partial_{i_1} \dots \partial_{i_m} \partial_r \xi^j \partial_k.$$

From  $\delta B = C_1$  one gets that if  $m > 1$ , then this term vanishes for any  $a$  and  $\xi$ , so  $b^{ki_1 \dots i_m}_{ij} \equiv 0$ .

For a first-order operator  $B(a) = b_{ij}^{kr} \partial_r a^{ij} \partial_k$  with  $b_{ij}^{kr} = b_{ji}^{kr}$ , the condition  $\delta B = C_1$  implies  $2b_{ij}^{kr} a^{it} \partial_r \partial_t \xi^j = a^{rt} \partial_r \partial_t \xi^k$  for any  $a$  and  $\xi$  (again, we consider the highest order term in  $\xi$ ). One readily finds that this equation has no solution if the dimension is  $n \geq 2$ . Indeed, take  $\xi$  such that  $\xi^j = 0$  with  $j \neq j_0$ . Then  $b_{ij}^{kr} = 0$  if  $k \neq j_0$ . Comparing this property for different values of  $j_0$ , one finds a contradiction:  $b_{ij}^{kr} \equiv 0$ .

Proposition 2 is proven.

*Remark.* Define  $\tilde{C}_1: \text{Vect}(M) \rightarrow \text{Hom}(S^2(M), \text{Vect}(M))$  by

$$(\tilde{C}_1(\xi))(a) = a^{kr} \partial_r \partial_i \xi^i \partial_k.$$

Then  $C_1$  is a 1-cocycle cohomologous to  $\tilde{C}_1$ : if  $B \in \text{Hom}(S^2(M), \text{Vect}(M))$  is given by  $B(a) = \partial_i a^{ik} \partial_k$ , then  $\delta B = C_1 + \tilde{C}_1$ .

### 4.3. Nontrivial Cohomology Class in $H^1_A(\text{Vect}(M); \text{Hom}(S^2(M), C^\infty(M)))$

**PROPOSITION 3.** *The cocycle  $C_2$  represents a nontrivial cohomology class of the differentiable cohomology group*

$$H^1_A(\text{Vect}(M); \text{Hom}(S^2(M), C^\infty(M))).$$

*Proof.* Suppose there exists a differential operator  $B: S^2(M) \rightarrow C^\infty(M)$  such that  $\delta B(\xi) = C_2(\xi)$ . It is given by  $B(a) = b_{ij}^{i_1 \dots i_m} \partial_{i_1} \dots \partial_{i_m} a^{ij}$ . Then,

$$(\delta B)(\xi) a = \xi^r \partial_r (b_{ij}^{i_1 \dots i_m} \partial_{i_1} \dots \partial_{i_m} a^{ij}) \\ - b_{ij}^{i_1 \dots i_m} \partial_{i_1} \dots \partial_{i_m} (\xi^r \partial_r a^{ij} - a^{ir} \partial_r \xi^j - a^{jr} \partial_r \xi^i)$$

where  $b_{ij}^{i_1 \dots i_m} = b_{ji}^{i_1 \dots i_m}$ . The highest order term (in the derivatives of  $\xi$ ) in this expression is  $2a^{ir} b_{ij}^{i_1 \dots i_m} \partial_{i_1} \dots \partial_{i_m} \partial_r \xi^j$ . The condition  $\delta B = C_2$  implies that this term equals zero (for any value of  $m$  and for any  $a$  and  $\xi$ ) which entails  $b_{ij}^{i_1 \dots i_m} \equiv 0$ . This contradiction proves Proposition 3.

Theorem 1 is proven.

*Remarks.* (1) The cocycle  $C_2$  is related to the coadjoint action of the Virasoro algebra. Its group analogue is given by the Schwarzian derivative (see [6]).

(2) Let us recall that, in the one-dimensional case, the cohomology group  $H^1(\text{Vect}(M), \mathcal{F}_\lambda)$  is nontrivial for  $\lambda = 0, -1, -2$ . In these three cases it has dimension one and is generated by the following 1-cocycles:

$$c_0(\xi(x) \partial_x) = \xi'(x), \\ c_1(\xi(x) \partial_x) = \xi''(x) dx, \\ c_2(\xi(x) \partial_x) = \xi'''(x)(dx)^2$$

respectively. Three corresponding cocycles with values in the operator space  $C_k: \text{Vect}(M) \rightarrow \text{Hom}(\mathcal{F}_\lambda, \mathcal{F}_{\lambda-k})$  are given by  $(C_k(\xi))(a) := c_k(\xi) \cdot a$ . It is interesting to note that the cocycles  $C_0$  and  $C_2$  are nontrivial for any value of  $\lambda$ ,  $\lambda \neq 1/2$ , but the cocycle  $C_1$  is nontrivial only for  $\lambda = 0$  (cf. [4]).

### 5. PROOF OF THEOREM 2

#### A. $\dim M \geq 2$

It is easy to show that the mapping (5) is equivariant. In fact, it becomes especially simple in the normal coordinate system (9). It multiplies each normal component by a constant:

$$\mathcal{L}^2_{\mu\lambda}(\bar{A})_2 = \bar{a}_2, \quad \mathcal{L}^2_{\mu\lambda}(\bar{A})_1 = \frac{2\mu + 1}{2\lambda + 1} \bar{a}_1, \quad \mathcal{L}^2_{\mu\lambda}(\bar{A})_0 = \frac{\mu(\mu + 1)}{\lambda(\lambda + 1)} \bar{a}_0.$$

Let us prove the uniqueness.

#### 5.1. Automorphisms of the Modules $\mathcal{D}^2_\lambda$

We show in this section a remarkable property of the “critical” modules  $\mathcal{D}^2_0$ ,  $\mathcal{D}^2_{-1}$ , and  $\mathcal{D}^2_{-1/2}$ , namely the existence of nontrivial automorphisms of these modules.

**PROPOSITION 4.** *Let  $\dim M \geq 2$ .*

- (i) *The modules  $\mathcal{D}^2_\lambda$  have no automorphisms other than multiplication by a constant for  $\lambda \neq 0, -\frac{1}{2}, -1$ .*
- (ii) *All automorphisms of the modules  $\mathcal{D}^2_0 \cong \mathcal{D}^2_{-1}$  and  $\mathcal{D}^2_{-1/2}$  are proportional to (6) and (7) respectively.*

*Proof.* Let  $\mathcal{I} \in \text{End}(\mathcal{D}^2_\lambda)$  be an automorphism. This means that  $\mathcal{I}$  commutes with the Lie derivative:  $\mathcal{I} \circ \text{ad } L_\xi - \text{ad } L_\xi \circ \mathcal{I} = 0$ . We first give the general formula for these automorphisms.

**LEMMA 7.** *Any automorphism of  $\mathcal{D}^2_\lambda$  has the following form:*

$$\mathcal{I}(A) = c_1 a_2^{ij} \partial_i \partial_j + (c_2 a_1^k + c_3 \partial_j a_2^{kj}) \partial_k + c_4 \partial_i \partial_j a_2^{ij} + c_5 \partial_i a_1^i + c_6 a_0 \quad (12)$$

*Proof.* (a) Consider the highest order term  $\mathcal{I}_2: \mathcal{D}^2_\lambda \rightarrow S^2(M)$ ,

$$\mathcal{I}_2(A) = \mathcal{I}^2_2(a_2) + \mathcal{I}^1_2(a_1) + \mathcal{I}^0_2(a_0).$$

Then the differential operators  $\mathcal{I}^2_2: S^2(M) \rightarrow S^2(M)$ ,  $\mathcal{I}^0_2: C^\infty(M) \rightarrow S^2(M)$ , and  $\mathcal{I}^1_2: \text{Vect}(M) \rightarrow S^2(M)$  are *unar equivariant differential operators*, since  $[\text{ad } L_\xi, \mathcal{I}](A) = 0$  for any  $\xi$  and  $A$  (indeed, consider  $a_2 \equiv 0$  to check the

invariance of  $\mathcal{F}_2^1$  and  $a_2 \equiv 0$ ,  $a_1 \equiv 0$  to check the invariance of  $\mathcal{F}_2^0$ ). A well known theorem (see [13] and the remark in Section 1.3) states that there is no such nonconstant equivariant operator. Thus,  $\mathcal{F}_2^0 = \mathcal{F}_2^1 \equiv 0$  and  $\mathcal{F}_2$  is an operator of multiplication by a constant. Put  $\mathcal{F}_2(A) = c_1 \cdot a_2$ .

(b) Consider the first-order term  $\mathcal{F}_1(A) = \mathcal{F}_1^2(a_2) + \mathcal{F}_1^1(a_1) + \mathcal{F}_1^0(a_0)$ . Again, one obtains that  $\mathcal{F}_1^0: C^\infty(M) \rightarrow \text{Vect}(M)$  and  $\mathcal{F}_1^1: \text{Vect}(M) \rightarrow \text{Vect}(M)$  are equivariant differential operators. Thus, one has  $\mathcal{F}_1^0 = 0$  and  $\mathcal{F}_1^1(a_1) = \text{const} \cdot a_1$ . Put  $\mathcal{F}_1^1(a_1) = c_2 \cdot a_1$ . The operator  $\mathcal{F}_1^2: S^2(M) \rightarrow \text{Vect}(M)$  verifies the following relation:

$$[\mathcal{F}_1^2, \text{ad } L_\xi](a_2) = [-(c_1 - c_2) a_2^{ij} \partial_i \partial_j \xi^l + 2\lambda(c_1 - c_2) a_2^{ij} \partial_i \partial_k \xi^k] \partial_l. \quad (13)$$

Since the right-hand side of this equation contains only second-order derivatives of  $\xi^j$ , one necessarily obtains that  $\mathcal{F}_1^2$  is a first-order differential operator, namely  $\mathcal{F}_1^2(a_2) = a_{ij}^{sl} \partial_s a_2^{ij} \partial_l$ . It follows easily from (13) that  $\alpha_{ij}^{sl} = \text{const} \cdot \delta_j^s \delta_i^l$ . Finally,  $\mathcal{F}_1^2(a_2)^l = c_3 \partial_j a_2^{lj}$ .

(c) In the same way, for the last term  $\mathcal{F}_0(A) = \mathcal{F}_0^2(a_2) + \mathcal{F}_0^1(a_1) + \mathcal{F}_0^0(a_0)$ , one gets  $\mathcal{F}_0^0(a_0) = \text{const} \cdot a_0$ ,  $\mathcal{F}_0^1(a_1) = \text{const} \cdot \partial_k a_1^k$ , and  $\mathcal{F}_0^2(a_2) = \text{const} \cdot \partial_i \partial_j a_2^{ij}$ . Lemma 7 is proven.

To finish the proof of Proposition 4, substitute the expression (12) for  $\mathcal{F}$  into the equation  $[\text{ad } L_\xi, \mathcal{F}]$ . The result reads:

$$\left\{ \begin{array}{l} [\text{ad } L_\xi, \mathcal{F}](A)_2^{ij} = 0 \\ [\text{ad } L_\xi, \mathcal{F}](A)_1^l = (c_2 + c_3 - c_1) a_2^{ij} \partial_i \partial_j \xi^l \\ \quad + (c_3 + 2\lambda(c_1 - c_2)) a_2^{ij} \partial_i \partial_k \xi^k \\ [\text{ad } L_\xi, \mathcal{F}](A)_0 = (c_5 - \lambda(c_2 - c_6)) a_1^r \partial_r \partial_k \xi^k \\ \quad + (2c_4 + (1 + 2\lambda) c_5 + \lambda(c_6 + c_1)) a_2^{ij} \partial_i \partial_j \partial_k \xi^k \\ \quad + (2c_4 + 2\lambda c_5 - \lambda c_3) \partial_i (a_2^{ir}) \partial_r \partial_k \xi^k \\ \quad + (c_4 + c_5) \partial_r (a_2^{ij}) \partial_i \partial_j \xi^r. \end{array} \right.$$

This expression must vanish for any  $\xi$  and  $A$ , yielding the following conditions for the constants  $c_1, \dots, c_6$ :

$$\left\{ \begin{array}{l} c_2 + c_3 - c_1 = 0 \\ c_3 - 2\lambda(c_2 - c_1) = 0 \\ c_5 + \lambda(c_2 - c_6) = 0 \\ 2c_4 + (1 - 2\lambda) c_5 - \lambda(c_6 - c_1) = 0 \\ 2c_4 - 2\lambda c_5 + \lambda c_3 = 0 \\ c_4 + c_5 = 0. \end{array} \right.$$

If  $\lambda \neq 0, -\frac{1}{2}, -1$ , then this system has the following solution:  $c_1 = c_2 = c_6, c_3 = c_4 = c_5 = 0$ . Thus, in this case the modules  $\mathcal{D}_\lambda^2$  have no nontrivial automorphisms.

If  $\lambda = 0$ , then the solution is given by  $c_1 = c_2$  and  $c_3 = c_4 = c_5 = 0$ , and  $c_6$  is a free parameter. One obtains the formula (6).

If  $\lambda = -\frac{1}{2}$ , then the solution is given by  $c_1 = c_6, c_3 = 2c_4, c_2 + c_3 = c_6$ , and  $c_4 + c_5 = 0$ . This corresponds to the formula (7) for the automorphisms of  $\mathcal{D}_{-1/2}$ .

Proposition 4 and Theorem 2 are proven.

### B. $\dim M = 1$

In the one-dimensional case, for each  $\lambda \neq 0, -1$ , there exists a two-parameter family of automorphisms of  $\mathcal{D}_\lambda^2$ . For each value  $\lambda = 0, -1$  there exists a three-parameter family. These facts follow from the normal form (10) of the  $\text{Vect}(M)$ -action.

## 6. DISCUSSION

### 6.1. A Few Ideas around Quantization

Maybe the most interesting corollary of Theorem 1 is the existence of two exceptional modules of second-order differential operators:  $\mathcal{D}_0^2$  and  $\mathcal{D}_{-1/2}^2$ . Recall that there is no nontrivial equivariant linear mapping  $\mathcal{D}_0^2 \rightarrow \mathcal{D}_{-1/2}^2$ .

However, these modules are of a great interest, e.g., in geometric quantization. So, to obtain such a mapping one needs an additional structure on  $M$ .

Given a linear connection  $\Gamma_{ij}^k$  on  $M$ , it is possible to define a  $\text{Vect}(M)$ -equivariant linear mapping:

$$\mathcal{L}_\Gamma: \mathcal{D}_0^2 \rightarrow \mathcal{D}_{-1/2}^2. \tag{14}$$

Let us give here the complete list of such mappings which are polynomial in  $\Gamma_{ij}^k$  and its partial derivatives.

**THEOREM.** *Let  $A = a_2^{ij} \partial_i \partial_j + a_1^i \partial_i + a_0 \in \mathcal{D}_0^2$ . All  $\text{Vect}(M)$ -equivariant linear mappings (14) polynomially depending on  $\Gamma_{ij}^k$  and its partial derivatives are given by:*

$$\begin{cases} \tilde{a}_2^{ij} = a_2^{ij} \\ \tilde{a}_1^i = \partial_k a_2^{ik} + c_1 \nabla_k a_2^{ik} + c_2 (a_1^i + a_2^{jk} \Gamma_{jk}^i) \\ \tilde{a}_0 = -\frac{1}{4} (2\partial_i (a_2^{ij} \Gamma_{jk}^k) + a_2^{ij} \Gamma_{ik}^k \Gamma_{jl}^l) \\ \quad + c_3 \nabla_i \nabla_j a_2^{ij} + c_4 \nabla_i (a_1^i + a_2^{jk} \Gamma_{jk}^i) + c_5 a_2^{ij} R_{ij} + c_6 a_0 \end{cases} \tag{15}$$

where the parameters  $c_1, \dots, c_6$  are arbitrary constants and  $R_{ij}$  is the Ricci tensor.

EXAMPLE. When the connection is given by a Riemannian metric  $g_{ij}$  on  $M$ , consider the Laplace operator  $\Delta = g^{ij}(\partial_i\partial_j - \Gamma_{ij}^k\partial_k) \in \mathcal{D}_0^2$ . Let  $\psi \in \mathcal{F}_{-1/2}$  be a  $\frac{1}{2}$ -density. One can write locally  $\psi = f \cdot (\det g)^{1/4}$ . The result of the mapping (15) is:

$$\tilde{\Delta}(f \cdot (\det g)^{1/4}) = (\Delta f + c_5 R f)(\det g)^{1/4}$$

where  $R$  is the scalar curvature.

The proof of the theorem and the properties of the mapping (15) will be discussed elsewhere.

Remark. This result is consistent with the various quantizations of the geodesic flow on Riemannian manifolds:  $-(\Delta + cR)$  where the constant  $c$  is actually determined by the chosen quantization procedure. For example, the infinitesimal Blattner–Kostant–Sternberg pairing of real polarizations leads to  $c = -\frac{1}{6}$  [14, 16]. In the case of  $n$ -dimensional spheres, the pairing of complex polarizations gives  $c = -(n - 1)/4n$  [12] while Weyl quantization and quantum reduction would lead to  $c = (n + 1)/4n$  (see [3] for a survey).

Remark. So far, we have only considered linear connections; it would be interesting to find a similar construction in terms of projective connections. For example, in the one-dimensional case there exists a natural mapping from  $\mathcal{D}_0^2$  to  $\mathcal{D}_\lambda^2$  using a projective connection. Fixing a Sturm–Liouville operator  $\partial^2 + u(x)$  is equivalent to fixing a projective connection. The mapping  $\mathcal{L}_u: \mathcal{D}_0^2 \rightarrow \mathcal{D}_\lambda^2$  defined by

$$\begin{aligned} \mathcal{L}_u(a_2\partial^2 + a_1\partial + a_0) &= a_2\partial^2 + (a_1 - \lambda a'_2)\partial + ca_0 + \lambda a'_1 \\ &\quad + \frac{1}{3}\lambda(\lambda + 1)[a''_2 - 4ua_2] \end{aligned}$$

is equivariant for any  $c = \text{const}$ . Indeed, the potential  $u$  transforms via the Lie derivative as follows:  $L_\xi u = \xi u' + 2\xi' u + \frac{1}{2}\xi'''$  (see [2]). This construction with  $\lambda = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$  is related to the Gelfand–Dickey bracket (see [11]).

### 6.2. Higher Order Operators

The study of the modules of differential operators leads to first cohomology groups

$$H^1(\text{Vect}(M); \text{Hom}(S^k(M), S^m(M)))$$



where  $S^k(M)$  is the space of  $k$ -contravariant symmetric tensors on  $M$ . Calculation of these cohomology groups seems to be a very interesting open problem. In the one-dimensional case this problem is solved by Feigin and Fuks in [4] for the Lie algebra of formal vector fields.

We have almost no information about  $\text{Vect}(M)$ -module structures on the space of higher-order linear-differential operators. Let us formulate the main problem:

*Is it true that two spaces of  $n$ th order differential operators on tensor densities are naturally isomorphic for any values of degree, except for a finite set of critical values?*

A positive answer would mean that there exist higher-order analogues of the Lie derivative. A negative answer would mean that second-order differential operators play a special role.

The only information that we have corresponds to the case of third-order differential operators on a one-dimensional manifold.

**PROPOSITION.** *If  $\dim M = 1$ , then the  $\text{Vect}(M)$ -module structures  $\mathcal{D}_\lambda^3$  on the space of differential operators*

$$a_3\partial^3 + a_2\partial^2 + a_1\partial + a_0$$

*are isomorphic to each other if  $\lambda \neq 0, -1, -\frac{1}{2}, -\frac{1}{2} \pm \frac{\sqrt{21}}{6}$ .*

Note that the value  $\lambda = -\frac{1}{2}$ , absent in the case of second-order differential operators on a one-dimensional manifold, is present here.

Let us finish with two simple remarks:

(1) In a particular case  $\lambda + \mu = 1$ , the modules  $\mathcal{D}_\lambda^n$  and  $\mathcal{D}_\mu^n$  are isomorphic for any  $n$ . The isomorphism is given by the conjugation.

(2) The module  $\mathcal{D}_{-1/2}^n$  is not isomorphic to any module  $\mathcal{D}_\lambda^n$  with  $\lambda \neq -\frac{1}{2}$ .

## ACKNOWLEDGMENTS

It is a pleasure to acknowledge enlightening discussions with A. A. Kirillov, Y. Kosmann-Schwarzbach, E. Mourre, and C. Roger.

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