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Three Cocycles on Diff(S^1) Generalizing the Schwarzian Derivative

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1 Introduction

1.1. Consider the group Diff($\mathbf{RP}^1$) of diffeomorphisms of the circle $\mathbf{RP}^1 \cong S^1$. The well-known expression

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2,$$

where $f = f(x) \in \mathrm{Diff}(S^1)$, x is the affine parameter on $\mathbf{RP}^1$, and $f' = df(x)/dx$, is called the *Schwarzian derivative*.

The main properties of the Schwarzian derivative are as follows:

(a) It satisfies the relation

$$S(f \circ g) = S(f) \circ g \cdot (g')^2 + S(g).$$

That means S is a *1-cocycle* on Diff($\mathbf{RP}^1$) with values in the space $\mathcal{F}_2$ of quadratic differentials (see [12]);

(b) The kernel of the cocycle S is the group of Möbius (linear-fractional or projective) transformations $PSL(2, \mathbf{R}) \subset \mathrm{Diff}(\mathbf{RP}^1)$: $S(f) \equiv 0$ if and only if $f \in PSL(2, \mathbf{R})$. Since the Schwarzian derivative is a 1-cocycle, this means that it is *projectively invariant*.

1.2. The following two remarks have already been known to classics (see, e.g., [18], [3], [2][1]):

(c) Given a Sturm-Liouville equation $2\psi'' + u(x)\psi = 0$, where $u(x) \in C^\infty(\mathbf{RP}^1)$, let ψ_1 and ψ_2 be two independent solutions; the potential $u(x)$ can be expressed as a function of the quotient $u = S(\psi_1/\psi_2)$.

[1] We are grateful to B. Kostant for this reference.

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(d) Let us formulate the same fact in other words. Consider the space of Sturm-Liouville operators

$$A_u = 2\frac{d^2}{dx^2} + u(x).$$

The natural action of the group $\mathrm{Diff}(\mathbf{RP}^1)$ on the space of Sturm-Liouville operators is

$$f^{-1}(A_u) = A_{u\circ f\cdot(f')^2} + S(f). \tag{1}$$

To define this action, one considers the arguments of the Sturm-Liouville operators as $-1/2$-densities, and their images as $3/2$-densities on $\mathbf{RP}^n$ (see Section 2 for details).

1.3. We calculate the first group of differentiable cohomology of $\mathrm{Diff}(\mathbf{RP}^1)$ with coefficients in the modules of linear differential operators on tensor-densities, vanishing on $\mathrm{PSL}(2,\mathbf{R})$.

The main result of this paper is an explicit construction of three families of nontrivial cocycles on $\mathrm{Diff}(\mathbf{RP}^1)$ generalizing the Schwarzian derivative. These cocycles are with values in the space of linear differential operators on $\mathbf{RP}^1$. They satisfy the main property of the Schwarzian derivative, $\mathrm{PSL}(2,\mathbf{R})$-invariance.

2 $\mathrm{Diff}(\mathbf{RP}^1)$-module structures on the space of differential operators

The space of linear differential operators on a manifold considered as a module over the group of diffeomorphisms is a classical subject. In the one-dimensional case, we refer to [18] and [3]. This subject is closely related to quantization (cf. [13]). The modules of linear differential operators on the space of tensor-densities on a smooth manifold was studied in a series of recent papers (see [5], [14], [15], [11], [9], [16]).

In this section, we recall the definition of the natural two-parameter family of modules over the group of diffeomorphisms on the space of linear differential operators.

2.1. Consider a one-parameter family of $\mathrm{Diff}(\mathbf{RP}^1)$-actions on $C^\infty(\mathbf{RP}^1)$:

$$f_\lambda^*(\phi) = \phi\circ f^{-1}\cdot (f^{-1\prime})^\lambda.$$

Denote by $\mathcal{F}_\lambda$ the defined $\mathrm{Diff}(\mathbf{RP}^1)$-module structure on $C^\infty(\mathbf{RP}^1)$. It is called the *module tensor-densities of degree* λ on $\mathbf{RP}^1$. We will use the standard notation, $\phi = \phi(x)(dx)^\lambda$.

For example, according to (1), the potential u of a Sturm-Liouville operator should be considered as an element of $\mathcal{F}_2$ (the so-called *quadratic differential*: $u = u(x)(dx)^2$; see, e.g., [12]).

2.2. Denote by $\mathcal{D}^k$ the space of kth-order differential operators

$$A = a_k(x)\frac{d^k}{dx^k} + \cdots + a_0(x) \tag{2}$$

where $a_i(x) \in C^\infty(\mathbf{RP}^1)$.

Definition. A two-parameter family of actions of Diff($\mathbf{RP}^1$) on the space of differential operators (2) is defined by

$$g_{\lambda,\mu}(A) = g_\mu^* \circ A \circ (g_\lambda^*)^{-1}. \tag{3}$$

In other words, we consider differential operators acting on tensor-densities: $A\colon \mathcal{F}_\lambda \to \mathcal{F}_\mu$.

Denote by $\mathcal{D}^k_{\lambda,\mu}$ the space of operators (2) endowed with the defined Diff($\mathbf{RP}^1$)-module structure.

Remark. The complete classification of modules $\mathcal{D}^k_{\lambda,\mu}$ was obtained in [9].

In this paper, we consider the action of the group Diff($\mathbf{RP}^1$) on this space and study the cohomology groups arising in this context.

3 The main result

Studying the modules of differential operators leads to the cohomology group H^1(Diff($\mathbf{RP}^1$); Hom($\mathcal{F}_\lambda, \mathcal{F}_\mu$)) (see [15]). This cohomology can be considered as a measure of "nontriviality" for the modules of linear differential operators.

3.1. Let us formulate the precise problem considered in this paper.

We will study the *differentiable* (or local) cohomology (see [7] for the general definition). This means we consider only differentiable cochains on Diff($\mathbf{RP}^1$) with coefficients in the space of differential operators: $\mathcal{D}_{\lambda,\mu} \subset$ Hom($\mathcal{F}_\lambda, \mathcal{F}_\mu$).

We will impose one more condition: PSL($2, \mathbf{R}$)-*invariance*. In other words, we consider only the cohomology classes *vanishing* on PSL($2, \mathbf{R}$) (note that for the case of first cohomology, these two notions coincide).

Theorem 1. The differentiable cohomology of Diff($\mathbf{RP}^1$) with coefficients in the module of linear differential operators, vanishing on PSL($2, \mathbf{R}$),

$$H^1_{\mathrm{diff}}(\mathrm{Diff}(\mathbf{RP}^1), \mathrm{PSL}(2,\mathbf{R});\ \mathcal{D}_{\lambda,\mu}),$$

is one-dimensional in the following cases:

(a) $\mu - \lambda = 2, \lambda \neq -1/2$,

(b) $\mu - \lambda = 3, \lambda \neq -1$,

(c) $\mu - \lambda = 4, \lambda \neq -3/2$,

(d) $(\lambda, \mu) = (-4, 1), (0, 5)$.

Otherwise, this cohomology group is trivial. □

We will prove this theorem in Section 6.

Therefore, there exist three families of nontrivial cohomology classes depending on λ as a parameter (and with fixed $\mu = \lambda+2, \lambda+3$ or $\lambda+4$). We will call these cohomology classes *stable*.

Corollary. For the generic values of λ, one has

$$H^1_{\text{diff}}(\text{Diff}(\mathbf{RP}^1), \text{PSL}(2,\mathbf{R}); \mathcal{D}_{\lambda,\mu}) = \begin{cases} \mathbf{R}, & \mu - \lambda = 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$ □

Let us give here the list of 1-cocycles generating the stable nontrivial cohomology classes.

3.2. The main result of this paper is the following theorem.

Theorem 2. (i) For every λ, there exist unique (up to a multiple) 1-cocycles

$$\mathcal{S}_\lambda\colon \text{Diff}(\mathbf{RP}^1) \to \mathcal{D}_{\lambda,\lambda+2},$$
$$\mathcal{T}_\lambda\colon \text{Diff}(\mathbf{RP}^1) \to \mathcal{D}_{\lambda,\lambda+3},$$
$$\mathcal{U}_\lambda\colon \text{Diff}(\mathbf{RP}^1) \to \mathcal{D}_{\lambda,\lambda+4},$$

vanishing on $\text{PSL}(2,\mathbf{R})$. They are given by the formulae

$$\begin{aligned} \mathcal{S}_\lambda(f) &= S(f) \\ \mathcal{T}_\lambda(f) &= S(f)\frac{d}{dx} - \frac{\lambda}{2}S(f)' \\ \mathcal{U}_\lambda(f) &= S(f)\frac{d^2}{dx^2} - \frac{2\lambda+1}{2}S(f)'\frac{d}{dx} + \frac{\lambda(2\lambda+1)}{10}S(f)'' - \frac{\lambda(\lambda+3)}{5}S(f)^2, \end{aligned} \tag{4}$$

where $S(f)$ is the Schwarzian derivative.

(ii) The cocycles $\mathcal{S}_\lambda$, $\mathcal{T}_\lambda$, and $\mathcal{U}_\lambda$ are nontrivial for every λ except $\lambda = -1/2, \lambda = -1$, and $\lambda = -3/2$, respectively. □

Explicit formulae for the cocycles with values in the exceptional modules $\mathcal{D}_{-4,1}$ and $\mathcal{D}_{0,5}$ (cf. Theorem 1) will be given in the end of this paper.

3.3 Proof of existence

Let us show that the maps $\mathcal{S}_\lambda$, $\mathcal{T}_\lambda$, and $\mathcal{U}_\lambda$ are 1-cocycles.

(1) The first map $\mathcal{S}_\lambda(f)$ is just a zero-order differential operator of multiplication by the Schwarzian derivative: $\mathcal{S}_\lambda(f)(\phi) = S(f)\cdot\phi$. The condition of the 1-cocycle for $\mathcal{S}_\lambda$ follows from those for S.

(2) In general, if $k \geq 2$, the modules $\mathcal{D}^k_{\lambda,\mu}$ are not isomorphic to the modules of tensor-densities. However, for $k = 1$, this is still the case. The result is as follows:

$$\text{if} \quad \mu - \lambda \neq 1, \quad \text{then} \quad \mathcal{D}^1_{\lambda,\mu} \cong \mathcal{F}_{\mu-\lambda-1} \oplus \mathcal{F}_{\mu-\lambda}.$$

Given an operator $A = a_1(x)d/dx + a_0(x) \in \mathcal{D}^1_{\lambda,\mu}$, the principal symbol a_1 transforms under the action (3) as a $\mu-\lambda-1$-density. Verify that the quantity

$$\bar{a}_0 := a_0 - \frac{\lambda}{\mu-\lambda-1}a_1'$$

transforms as a $\mu-\lambda$-density. The isomorphism is as follows:

$$\sigma(A) = \left(a_1(x)(dx)^{\mu-\lambda-1}, \bar{a}_0(x)(dx)^{\mu-\lambda}\right). \tag{7'}$$

Existence of the cocycle $\mathcal{T}_\lambda$ is a corollary of the isomorphism σ. Namely, the map $\overline{\mathcal{T}}_\lambda = \sigma \circ \mathcal{T}_\lambda$: Diff($\mathbf{RP}^1$) $\to \mathcal{F}_2 \oplus \mathcal{F}_3$ is given by the formula

$$\overline{\mathcal{T}}_\lambda(f) = \left(S(f)(dx)^2, 0\right). \tag{4'}$$

This is obviously a 1-cocycle.

(3) Let us show that the map $\mathcal{U}_\lambda$ is a 1-cocycle for every λ.

The modules of second order differential operators $\mathcal{D}^2_{\lambda,\lambda+3}$ are not isomorphic to a direct sum of tensor-densities for every λ except $\lambda = 0, -3$. However, if $\mu - \lambda \neq 1, 3/2, 2$, then there exists a linear map

$$\sigma\colon\ \mathcal{D}^2_{\lambda,\mu} \to \mathcal{F}_{\mu-\lambda-2} \oplus \mathcal{F}_{\mu-\lambda-1} \oplus \mathcal{F}_{\mu-\lambda},$$

which is $PSL(2,\mathbf{R})$-equivariant (cf. [4], [15], [9], and Section 7). This defines a symbol of a differential operator

$$A = a_2\frac{d^2}{dx^2} + a_1\frac{d}{dx} + a_0 \in \mathcal{D}^2_{\lambda,\mu}$$

in a canonical way:

$$\sigma(A) = \left(a_2(dx)^{\mu-\lambda-2},\ \bar{a}_1(dx)^{\mu-\lambda-1},\ \bar{a}_0(dx)^{\mu-\lambda}\right), \tag{7''}$$

where

$$\bar{a}_1 = a_1 + \frac{2\lambda+1}{\mu-\lambda-2} a_2' \quad \text{and} \quad \bar{a}_0 = a_0 + \frac{\lambda}{\mu-\lambda-1} a_1 + \frac{\lambda(2\lambda+1)}{(\mu-\lambda)(2(\mu-\lambda)-3)} a_2''.$$

These expressions are a partial case of the formula (7) below.

One can easily calculate the $\mathrm{Diff}(\mathbf{RP}^1)$-action (3) on the right-hand side:

$$\sigma \circ f_{\lambda,\mu} \circ \sigma^{-1} \colon (a_2, \bar{a}_1, \bar{a}_0) \mapsto \left(f^*(a_2), f^*(a_1), f^*(a_0) + \beta S(f^{-1}) f^*(a_2)\right), \tag{3'}$$

where

$$\beta = \frac{2\lambda(\mu-1)}{2(\mu-\lambda)-3}.$$

In the particular case $\mu - \lambda = 4$, one obtains

$$\beta = \frac{2\lambda(\lambda+3)}{5}.$$

Verify that the map $\overline{\mathcal{U}}_\lambda = \sigma \circ \mathcal{U}_\lambda$ is of the form

$$\overline{\mathcal{U}}_\lambda(f) = \left(S(f)(dx)^2,\ 0,\ -\frac{\lambda(\lambda+3)}{5} S(f)^2 (dx)^4\right). \tag{4''}$$

Now it is very easy to check that this map is a 1-cocycle with respect to the action (3′).

It is proven that the maps $\mathcal{S}_\lambda$, $\mathcal{T}_\lambda$, and $\mathcal{U}_\lambda$ are indeed 1-cocycles.

Part (ii) of Theorem 2 and the uniqueness of the cocycles (4) will be proven in the next section.

Remarks. (1) Recall that there exist two more analogues of the Schwarzian derivative (see [7])

$$f \mapsto f'(x) \quad \text{and} \quad f \mapsto \frac{f''(x)}{f'(x)} dx$$

with values in $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. These 1-cocycles, however, do not vanish on $\mathrm{PSL}(2,\mathbf{R})$. The 1-cocycles with values in $\mathcal{D}_{\lambda,\lambda}$ and $\mathcal{D}_{\lambda,\lambda+1}$ defined through multiplication by these 1-cocycles turn out to be trivial for every $\lambda \neq 0$.

(2) The module of second order differential operators $\mathcal{D}^2_{\lambda,\mu}$ is a direct sum of modules of tensor-densities: $\mathcal{D}^2_{\lambda,\mu} \cong \mathcal{F}_{\mu-\lambda-2} \oplus \mathcal{F}_{\mu-\lambda-1} \oplus \mathcal{F}_{\mu-\lambda}$ if and only if $\lambda = 0$ or $\mu = 1$ (cf. Section 7). If $\lambda - \mu = 3$, these conditions correspond to the case when the 1-cocycle $\mathcal{U}_\lambda$ is a *linear* map (namely, $\lambda = 0$ or $\lambda = -3$).

(3) The unique module of third-order operators, isomorphic to a direct sum of modules of tensor-densities, is $\mathcal{D}^3_{0,1} \cong \mathcal{F}_{-2} \oplus \mathcal{F}_{-1} \oplus \mathcal{F}_0 \oplus \mathcal{F}_1$.

4 PSL(2, **R**)-invariant differential operators and trivialization of the cocycles $\mathcal{S}_\lambda, \mathcal{T}_\lambda$, and $\mathcal{U}_\lambda$

4.1 Proof of Theorem 2, Part (ii)

Let $\mathcal{C}$: Diff($\mathbf{RP}^1$) $\to \mathcal{D}_{\lambda,\mu}$ be a 1-cocycle vanishing on PSL(2, **R**). Suppose that $\mathcal{C}$ is a coboundary; $\mathcal{C} = \delta(B)$ for some $B \in \mathcal{D}_{\lambda,\mu}$. This means, for every $f \in$ Diff($\mathbf{RP}^1$), that

$$\mathcal{C}(f) = f_{\lambda,\mu}(B) - B.$$

In particular, for $f \in$ PSL(2, **R**), it follows that the *operator* B *is* PSL(2, **R**)-*equivariant*.

PSL(2, **R**)-equivariant linear differential operators on tensor-densities were classified in [1]. Let us recall here the classical result.

Bol's theorem. (i) For every k there exists a unique (up to a constant) PSL(2, **R**)-invariant linear differential operator of order k: $\partial^k\colon \mathcal{F}_{(1-k)/2} \to \mathcal{F}_{(1+k)/2}$, such that

$$\partial^k \left(\phi(x)(dx)^{(1-k)/2}\right) = \phi^{(k)}(x)(dx)^{(1+k)/2}.$$

(ii) If $(\lambda, \mu) \neq ((1-k)/2, (1+k)/2)$, then there is no nonzero PSL(2, **R**)-invariant linear differential operator from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$. □

The operator ∂^k is called the *Bol operator*.

Let us show how Theorem 2, Part (ii) follows from the Bol theorem.

(1) Verify that the cocycles $\mathcal{S}_{-1/2}, \mathcal{T}_{-1}$, and $\mathcal{U}_{-3/2}$ are differentials of the Bol operators:

$$\mathcal{S}_{-1/2} = \delta\left(\partial^2\right), \quad \mathcal{T}_{-1} = \delta\left(\partial^3\right), \quad \mathcal{U}_{-3/2} = \delta\left(\partial^4\right).$$

(2) On the other hand, Bol's theorem implies that the cocycles $\mathcal{S}_\lambda$, $\mathcal{T}_\lambda$, and $\mathcal{U}_\lambda$ with $\lambda \neq -1/2, -1$ and $-3/2$, respectively, are nontrivial. Indeed, these cocycles vanish on PSL(2, **R**). Therefore, if these cocycles are trivial, they have to be differentials of some PSL(2, **R**)-invariant operators. This contradicts the Bol classification.

Theorem 2, Part (ii) is proven. ■

4.2 Proof of uniqueness

Let us show that uniqueness of the cocycles $\mathcal{S}_\lambda$, $\mathcal{T}_\lambda$, and $\mathcal{U}_\lambda$ follows from Theorem 1 and the Bol theorem.

Let C_1 and C_2 be two differentiable 1-cocycles on Diff($\mathbf{RP}^1$) vanishing on PSL(2, **R**), with values in the same module $\mathcal{D}_{\lambda,\mu}$. We will show that C_1 and C_2 are proportional to each other (it does not matter whether C_1 and C_2 are trivial or not).

Since the considered cohomology group is at most one-dimensional, there exists a linear combination $C = \alpha_1 C_1 + \alpha_2 C_2$ which is a coboundary. Since the cocycle C vanishes on $PSL(2, \mathbf{R})$, this means $C = \delta(B)$, where $B \in \mathcal{D}_{\lambda,\mu}$ is a $PSL(2, \mathbf{R})$-invariant operator. In the case $\lambda \neq -1/2, -1, -3/2, \ldots$, Bol's theorem implies $C \equiv 0$. If $\lambda = -1/2, -1, -3/2, \ldots$, then the cohomology is trivial. In this case, Bol's theorem implies that both of the cocycles C_1 and C_2 are proportional to the differential of one of the Bol operators.

Theorem 2 is proven. ■

5 Exceptional modules of differential operators

The exceptional values $\lambda = -1/2, -1, -3/2$ correspond to the particular modules of differential operators studied by classics. In this section, we interpret these modules in terms of the constructed cocycles.

Consider the space of differential operators of the form

$$A = \frac{d^k}{dx^k} + a_{k-2}\frac{d^{k-2}}{dx^{k-2}} + a_{k-3}\frac{d^{k-3}}{dx^{k-3}} + \cdots + a_0, \tag{5}$$

where A is acting on the space of $(1-k)/2$-densities with values in $(1+k)/2$-densities:

$$A \in \mathcal{D}_{(1-k)/2,(1+k)/2}.$$

The structure of the $\mathrm{Diff}(\mathbf{RP}^1)$-module on this space has already been studied in [18] and [3].

Note that the modules (5) are closely related to so-called Adler-Gelfand-Dickey (or classical W) Poisson structure.

Example (a). The module of Sturm-Liouville operators considered in the introduction is a submodule $\mathcal{D}^2_{-1/2,3/2}$. Indeed, verify that the formula of the $\mathrm{Diff}(\mathbf{RP}^1)$-action (1) coincides with the action $f_{-1/2,3/2}$. In the case $u \equiv 0$, one has

$$f_{-1/2,3/2}\left(\partial^2\right) - \partial^2 = \mathcal{S}_{-1/2}(f).$$

This is just the coboundary relation $\mathcal{S}_{-1/2} = \delta(\partial^2)$.

Example (b). The operators (5) for $k = 3$ can be written in the form

$$A_{u,v} = \frac{d^3}{dx^3} + 4u(x)\frac{d}{dx} + 2u(x)' + v(x),$$

where $A \in \mathcal{D}_{\lambda,\lambda+3}$. It is easy to check that the formula of the $\mathrm{Diff}(\mathbf{RP}^1)$-action reads as

$$g^{-1}_{\lambda,\lambda+3}(A_{u,v}) = A_{u\circ g\cdot(g')^2, v\circ g\cdot(g')^3} + \mathcal{T}_{-1}(g).$$

In the case $u = v \equiv 0$, one obtains the relation $\mathcal{T}_{-1} = \delta(\partial^3)$.

Example (c). Every operator (5) for $k = 4$ can be written in the form

$$A_{u,v,w} = \frac{d^3}{dx^3} + 5u(x)\frac{d^2}{dx^2} + 5u(x)'\frac{d}{dx} + \frac{3}{2}u(x)'' + \frac{9}{4}u(x)^2 + v(x)\frac{d}{dx} + w(x).$$

The Diff($\mathbf{RP}^1$)-action reads as follows:

$$g^{-1}_{\lambda,\lambda+4}(A_{u,v,w}) = A_{u\circ g\cdot(g')^2, v\circ g\cdot(g')^3, w\circ g\cdot(g')^4} + \mathcal{U}_{-3/2}(g).$$

In the case $u = v \equiv 0$, one obtains the relation $\mathcal{U}_{-3/2} = \delta(\partial^4)$.

Note that the coefficients of the terms containing $u(x)$ coincide with the coefficients of the cocycle $\mathcal{U}_\lambda$ for $\lambda = -3/2$.

Remark. In the same way, one can interpret the modules of operators (5) for an arbitrary value of k in terms of 1-cocycles defined as differentials of the Bol operators.

6 Bilinear differential $PSL(2, \mathbf{R})$-invariant operators and cohomology of vector fields Lie algebra

In this section, we prove Theorem 1.

Consider the differentiable cohomology of the Lie algebra of vector fields vanishing on the Möbius subalgebra $sl(2, \mathbf{R}) \subset \text{Vect}(\mathbf{RP}^1)$:

$$H^1_{\text{diff}}(\text{Vect}(\mathbf{RP}^1), sl(2, \mathbf{R}); \mathcal{D}_{\lambda,\mu}). \tag{6}$$

Every class of differentiable cohomology $H^1_{\text{diff}}(\text{Diff}(\mathbf{RP}^1), PSL(2, \mathbf{R}); \mathcal{D}_{\lambda,\mu})$ corresponds to some nontrivial class of Vect($\mathbf{RP}^1$)-cohomology (6) (cf. [7]).

To prove Theorem 1, let us first show that a similar result holds for the cohomology group (6).

The following proposition has been proven in [15].

Proposition 6.1. The differentiable cohomology (6) is one-dimensional in the following cases:

(a) $\mu - \lambda = 2, \lambda \neq -1/2$,

(b) $\mu - \lambda = 3, \lambda \neq -1$,

(c) $\mu - \lambda = 4, \lambda \neq -3/2$,

(d) $(\lambda, \mu) = (-4, 1), (0, 5)$.

Otherwise, this cohomology group is trivial. □

6.1. The first remark is as follows.

Lemma 6.2. Given a differentiable 1-cocycle c on $\mathrm{Vect}(\mathbf{RP}^1)$ vanishing on $sl(2,\mathbf{R})$, with values in $\mathcal{D}_{\lambda,\mu}$, the bilinear differential operator $J\colon \mathrm{Vect}(\mathbf{RP}^1)\otimes\mathcal{F}_\lambda\to\mathcal{F}_\mu$, defined by

$$J(X,\phi)=c(X)(\phi),$$

is $sl(2,\mathbf{R})$-invariant. □

Proof. Since c is a 1-cocycle, it satisfies the relation

$$L_X\circ c(Y)-c(Y)\circ L_X-L_Y\circ c(X)+c(X)\circ L_Y=c([X,Y])$$

for every $X,Y\in\mathrm{Vect}(\mathbf{RP}^1)$; then for $X\in sl(2,\mathbf{R})$, one gets

$$L_X(J(Y,\phi))=J([X,Y],\phi)+J(Y,L_X(\phi)).$$

That means J is $sl(2,\mathbf{R})$-invariant.

6.2. All the $sl(2,\mathbf{R})$-invariant bilinear differential operators on tensor-densities, $\mathcal{F}_\lambda\otimes\mathcal{F}_\mu\to\mathcal{F}_\nu$, were classified in [11] (see also [8] for a clear and detailed exposition). The result is as follows.

Gordan's theorem. (i) For every λ,μ, and integer $m\geq 0$, there exists a $sl(2,\mathbf{R})$-invariant bilinear differential operator $J_m^{\lambda,\mu}\colon \mathcal{F}_\lambda\otimes\mathcal{F}_\mu\to\mathcal{F}_{\lambda+\mu+m}$ given by

$$J_m^{\lambda,\mu}(\phi,\psi)=\sum_{i+j=m}(-1)^i m!\binom{2\lambda+m-1}{i}\binom{2\mu+m-1}{j}\phi^{(i)}\psi^{(j)}.$$

(ii) If either λ or $\mu\not\in\{-1/2,-1,-3/2,\ldots\}$, then the operator $J_m^{\lambda,\mu}$ is the unique (up to a constant) $sl(2,\mathbf{R})$-invariant bilinear differential operator from $\mathcal{F}_\lambda\otimes\mathcal{F}_\mu$ to $\mathcal{F}_{\lambda+\mu+m}$. □

The operator $J_m^{\lambda,\mu}$ is called the *transvectant*.

Therefore, one obtains a series of bilinear $sl(2,\mathbf{R})$-invariant maps:

$$J_m^{-1,\lambda}\colon \mathrm{Vect}(\mathbf{RP}^1)\otimes\mathcal{F}_\lambda\to\mathcal{F}_{\lambda+m-1}.$$

If $\lambda\neq -1/2,-1,-3/2,\ldots$, the operator $J_m^{-1,\lambda}$ is a *unique* (up to a constant) operator invariant with respect to $sl(2,\mathbf{R})$.

An important property of the operators $J_m^{-1,\lambda}$ with $m\geq 2$ is that they vanish on the subalgebra $sl(2,\mathbf{R})$.

6.3 Proof of Proposition 6.1

(1) Let us consider the case $\lambda\neq -1/2,-1,-3/2,\ldots$.

Every differentiable 1-cocycle c on Vect($\mathbf{RP}^1$) vanishing on $sl(2,\mathbf{R})$, with values in $\mathcal{D}_{\lambda,\mu}$, is proportional to the map c_m such that

$$c_m(X)(\phi) := J_m^{-1,\lambda}(X,\phi).$$

Indeed, it follows from Lemma 6.2 that c is $sl(2,\mathbf{R})$-invariant, and it follows from the Gordan theorem that it is proportional to $J_m^{-1,\lambda}$. Now let us check if each of the maps c_m are 1-cocycles.

It is easy to calculate their explicit formulae:

$$\begin{aligned}
J_2^{-1,\lambda}\left(X\frac{d}{dx},\phi\right) &\equiv 0,\\
J_3^{-1,\lambda}\left(X\frac{d}{dx},\phi\right) &= X'''\phi,\\
J_4^{-1,\lambda}\left(X\frac{d}{dx},\phi\right) &= X'''\phi' - \frac{\lambda}{2}X^{IV}\phi,\\
J_5^{-1,\lambda}\left(X\frac{d}{dx},\phi\right) &= X'''\phi'' - \frac{2\lambda+1}{2}X^{IV}\phi' + \frac{\lambda(2\lambda+1)}{10}X^{V}\phi.
\end{aligned}$$

These maps are the 1-cocycles on Vect($\mathbf{RP}^1$) corresponding to the cocycles $\mathcal{S}_\lambda$, $\mathcal{T}_\lambda$, and $\mathcal{U}_\lambda$. They are nontrivial if and only if $\lambda \neq -1/2, -1, -3/2$, respectively.

One can verify the following lemma by direct calculation.

Lemma 6.3. The map $J_6^{-1,\lambda}$ defines a 1-cocycle if and only if $\lambda = -4, 0, -2$. □

The first two cocycles define nonzero cohomology classes of (6). In the last case, this cocycle is trivial (equals the differential of the Bol operator).

Lemma 6.4. The map $J_m^{-1,\lambda}$, with $m \geq 7$, defines a 1-cocycle if and only if $\lambda = (1-m)/2$. □

The corresponding 1-cocycle is trivial: it is just the differential of the Bol operator.

Proposition 6.1 is proven for the case $\lambda \neq -1/2, -1, -3/2, \ldots$.

(2) Let $\lambda \in \{-1/2, -1, -3/2, \ldots\}$. In this case, the property of $sl(2,\mathbf{R})$-invariance does not define a unique operator from Vect($\mathbf{RP}^1$) $\otimes \mathcal{F}_\lambda$ to $\mathcal{F}_{\lambda+m-1}$. However, the two properties of $sl(2,\mathbf{R})$-invariance and vanishing on $sl(2,\mathbf{R})$ determine the transvectants $J_m^{-1,\lambda}$ uniquely.

Lemma 6.5. The operators $J_m^{-1,\lambda}$ with $m \geq 2$ are the unique (up to a constant) $sl(2,\mathbf{R})$-invariant bilinear differential operators vanishing on $sl(2,\mathbf{R})$. □

Proof. The proof is straightforward (see [15]). ■

Now Proposition 6.1 follows from Lemmas 6.3 and 6.4 and is trivial for $\lambda = -1/2, -1, -3/2$, respectively.

This gives an upper boundary for the dimension of the cohomology group. After that, Theorem 1 follows from the explicit construction of the cocycles generating nontrivial cohomology classes. ■

6.4. Remark. Another way to prove Theorem 1 is to use the results of Feigin-Fuchs [6] and Roger [17]. The cohomology group $H^1(W; \mathrm{Hom}(F_\lambda, F_\mu))$, where W is the Lie algebra of formal vector fields on $\mathbf{R}$, and F_λ is a module of formal tensor-densities on $\mathbf{R}$, were calculated in [6]. The analogous results were obtained in [17] in the differentiable case. One can obtain Theorem 1 from the result of [6] and [17] by selecting the cohomology classes trivial on $sl(2, \mathbf{R})$.

7 Relations to the modules of higher order differential operators

What follows is an illustration of what has been done. We will show how the cocycles $\mathcal{S}_\lambda$, $\mathcal{T}_\lambda$, and $\mathcal{U}_\lambda$ appear in the modules of higher order differential operators. In this section, we will not give the details of calculations.

7.1. Let us recall the following result from [4] (see also [15] and [9]) concerning the restriction of the $\mathrm{Diff}(\mathbf{RP}^1)$-module structure to the subgroup $\mathrm{PSL}(2, \mathbf{R})$.

Cohen-Manin-Zagier's theorem. For $\rho - \nu \neq 1, 3/2, 2, \ldots, k$, there exists an isomorphism of $\mathrm{PSL}(2, \mathbf{R})$-modules

$$\sigma\colon\ \mathcal{D}^k_{\nu,\rho} \to \mathcal{F}_{\rho-\nu-k} \oplus \mathcal{F}_{\rho-\nu-k+1} \oplus \cdots \oplus \mathcal{F}_{\rho-\nu}. \qquad \square$$

This isomorphism is called the $\mathrm{PSL}(2, \mathbf{R})$-*equivariant symbol map.*

The explicit formula of the $\mathrm{PSL}(2, \mathbf{R})$-equivariant symbol map is

$$\bar{a}_i(x) = \sum_{j=i}^{k} \alpha_i^j a_j^{(j-i)}(x) \quad \in \mathcal{F}_{\rho-\nu+i}, \tag{7}$$

where the constants α_i^j are written in terms of binomial coefficients

$$\alpha_i^j = \frac{\binom{j}{i}\binom{2\nu+i}{2\nu+j}}{\binom{2(\rho-\nu)-i-j-1}{2(\rho-\nu)-2i-1}}.$$

Note that the isomorphisms (7′) and (7″) are particular cases of the $\mathrm{PSL}(2, \mathbf{R})$-equivariant symbol map.

7.2. Now consider the action (3) of the group Diff($\mathbf{RP}^1$) on the modules $\mathcal{D}^k_{\lambda,\mu}$ with $\rho - \nu \neq 1, 3/2, 2, \ldots, k$, written in terms of the PSL(2, $\mathbf{R}$)-equivariant symbol: $\sigma \circ f_{\nu,\rho} \circ \sigma^{-1}$. We will be interested in the transformation low for the first five coefficients.

Lemma 7.1. The action $\sigma \circ f_{\nu,\rho} \circ \sigma^{-1}$ of the group Diff($\mathbf{RP}^1$) on the quotient-module $\mathcal{D}^k_{\nu,\rho}/\mathcal{D}^{k-5}_{\nu,\rho}$ is given by

$$\begin{aligned}
f(\bar{a}_k) &= f^*(\bar{a}_k) \\
f(\bar{a}_{k-1}) &= f^*(\bar{a}_{k-1}) \\
f(\bar{a}_{k-2}) &= f^*(\bar{a}_{k-2}) + \beta^k_{k-2} S(f^{-1})\bar{a}_k \\
f(\bar{a}_{k-3}) &= f^*(\bar{a}_{k-3}) + \beta^{k-1}_{k-3} S(f^{-1})\bar{a}_{k-1} + \beta^k_{k-3} \mathcal{T}_{\rho-\nu-k}(f^{-1})(\bar{a}_k) \\
f(\bar{a}_{k-4}) &= f^*(\bar{a}_{k-4}) + \beta^{k-2}_{k-4} S(f^{-1})\bar{a}_2 + \beta^{k-1}_{k-4} \mathcal{T}_{\rho-\nu-k+1}(f^{-1})(\bar{a}_{k-1}) \\
&\quad + \beta^k_{k-4} \mathcal{U}_{\rho-\nu-k}(f^{-1})(\bar{a}_k)
\end{aligned} \tag{3''}$$

where $f^*(\bar{a}_i) = \bar{a}_i \circ f^{-1} \cdot (f^{-1\prime})^{\nu-\rho-i}$, and β^j_i are some constants depending on ν and ρ. □

7.3. Recall that the cohomology group $H^1(G; \mathrm{Hom}(A, B))$, where G is a group, and A and B are G-modules, classifies nontrivial extensions of G-modules

$$0 \to A \to \mathcal{E} \to B \to 0.$$

Given a 1-cocycle $\mathcal{C}$: $G \to \mathrm{Hom}(A, B)$, one has a G-module structure on $\mathcal{E} = A \oplus B$ defined by

$$\rho_g(\phi, \psi) := (g(\phi), g(\psi) + \gamma \mathcal{C}(g)(g(\phi))),$$

where γ is an arbitrary constant. Moreover, the condition $\rho_f \circ \rho_g = \rho_{f \circ g}$ is equivalent to the fact that $\mathcal{C}$ is a 1-cocycle.

Let us realize the extensions $\mathcal{E}_{\lambda,\lambda+2} = \mathcal{F}_\lambda \oplus \mathcal{F}_{\lambda+2}$ and $\mathcal{E}_{\lambda,\lambda+3} = \mathcal{F}_\lambda \oplus \mathcal{F}_{\lambda+3}$ defined by the cocycles $\mathcal{S}_\lambda$ and $\mathcal{T}_\lambda$, respectively, as some modules of differential operators.

7.4. Examples. (1) If $\lambda \neq 0, -1/2, -1$, then the submodule of $\mathcal{D}^2_{\nu,\nu+\lambda+2}$, consisting of differential operators

$$A = a_2(x)\frac{d^2}{dx^2} - \frac{2\nu+1}{\lambda} a_2'(x)\frac{d}{dx} + a_0(x),$$

is isomorphic to the module $\mathcal{E}_{\lambda,\lambda+2}$.

(2) If $\lambda \neq 0, -1/2, -1, -3/2, -2$, then the submodule of $\mathcal{D}^3_{\nu,\nu+\lambda+3}$, where ν is determined by the condition

$$3\nu^2 + 3\nu(\lambda + 2) + \lambda + 2 = 0,$$

and the differential operators are of the form

$$A = a_3(x)\frac{d^3}{dx^3} - 3\frac{\nu+1}{\lambda}a_3'(x)\frac{d^2}{dx^2} + 3\frac{(\nu+1)(2\nu+1)}{\lambda(2\lambda+1)}a_3''(x)\frac{d}{dx} + a_0(x),$$

is isomorphic to the module $\mathcal{E}_{\lambda,\lambda+3}$.

8 Appendix

It follows from Theorem 1 that there exist two 1-cocycles on $\mathrm{Diff}(\mathbf{RP}^1)$ vanishing on $PSL(2,\mathbf{R})$, with values on $\mathcal{D}_{\lambda,\lambda+5}$ for the particular values of $\lambda = 0$ and -4.

Let us give the explicit formulae for nontrivial cocycles on $\mathrm{Diff}(\mathbf{RP}^1)$ with values in $\mathcal{D}^3_{0,5}$ and $\mathcal{D}^3_{-4,1}$:

$$V_0(f) = S(f)\frac{d^3}{dx^3} - \frac{3}{2}S(f)'\frac{d^2}{dx^2} + \left(\frac{3}{10}S(f)'' - \frac{4}{5}S(f)^2\right)\frac{d}{dx}$$

and

$$V_{-4}(f) = S(f)\frac{d^3}{dx^3} + \frac{9}{2}S(f)'\frac{d^2}{dx^2} + \left(\frac{63}{10}S(f)'' - \frac{4}{5}S(f)^2\right)\frac{d}{dx} + \frac{14}{5}S(f)''' - \frac{8}{5}S(f)'S(f),$$

respectively.

The proof is straightforward. ■

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