Coadjoint representation of Virasoro-type Lie algebras and differential operators on tensor-densities

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To my teacher Alexander Alexandrovich Kirillov

Abstract

We discuss the geometrical nature of the coadjoint representation of the Virasoro algebra and some of its generalizations. The isomorphism of the coadjoint representation of the Virasoro group to the Diff(S^1)-action on the space of Sturm-Liouville operators was discovered by A.A. Kirillov and G. Segal. This deep and fruitful result relates this topic to the classical problems of projective differential geometry (linear differential operators, projective structures on S^1 etc.) The purpose of this talk is to give a detailed explanation of the A.A. Kirillov method [14] for the geometric realization of the coadjoint representation in terms of linear differential operators. Kirillov's method is based on Lie superalgebras generalizing the Virasoro algebra. One obtains the Sturm-Liouville operators directly from the coadjoint representation of these Lie superalgebras. We will show that this method is universal. We will consider a few examples of infinite-dimensional Lie algebras and show that the Kirillov method can be applied to them. This talk is purely expository: all the results are known.

Mathematical Subject Classification (2000) Primary: 17B68.
Secondary: 17B65, 81R10.

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Introduction

The coadjoint representation of infinite-dimensional Lie groups and Lie algebras is one of the most interesting subjects of Kirillov's orbit method. Geometrical problems related to this subject link together such fundamental domains as: symplectic and Kähler geometry, harmonic analysis, integrable systems and many others.

The main purpose of this talk is to describe a "geometrical picture" due to Kirillov, for the coadjoint representation of the Virasoro group and the Virasoro algebra. We will also consider some of their generalizations.

1. The Virasoro group is the unique (modulo equivalence) nontrivial central extension of the group of diffeomorphisms of the circle. The corresponding Lie algebra, called the Virasoro algebra, is defined as the unique (modulo equivalence) nontrivial central extension of the Lie algebra of vector fields on S^1. The coadjoint representation of the Virasoro group and the Virasoro algebra was studied in pioneering works by A.A. Kirillov [13] and G. Segal [27]. Their result is as follows.

The dual space to the Virasoro algebra can be realized as the space of Sturm-Liouville operators:

$$L = c\frac{d^2}{dx^2} + u(x) \tag{1}$$

where $u(x+2\pi) = u(x)$ is a periodic function, $c \in \mathbf{R}$ (or $\mathbf{C}$) is a constant. The coadjoint representation of the Virasoro group coincides with the natural action of the group of diffeomorphisms of S^1 on the space of operators (1).

This realization gives a geometric interpretation of the coadjoint representation of the Virasoro group (and the Virasoro algebra). It relates the coadjoint representation of the Virasoro group to classical works on differential operators and projective differential geometry [29],[3]. The main object in this theory which links together its different parts, is the classical Schwarzian derivative. It appears (in the Virasoro context) as a 1-cocycle on the group of diffeomorphisms of S^1 with values in the coadjoint representation.

2. The realization of the coadjoint representation of the Virasoro group was first discovered as a simple coincidence. Soon after, A.A. Kirillov has suggested a systematic method using Lie superalgebras (see [14]). He considered two Lie superalgebras (called now Ramond and Neveu-Schwarz superalgebras) containing the Virasoro algebra as the even part. Sturm-Liouville operator appears in the coadjoint action of Ramond and Neveu-Schwarz superalgebras.

To my knowledge, Kirillov's method is the only known way to obtain the Sturm-Liouville operators (in an automatic way) directly from the coadjoint representation. This makes this method particularly useful for generalizations. However, for a long time, Kirillov's method has not been tested in any other case then for the Virasoro algebra.

3. We consider two different generalizations of the above geometrical picture.

A. There exist series of infinite-dimensional groups, Lie algebras and Lie superalgebras generalizing the Virasoro group and the Virasoro algebra (see [21], [25]). Geometrical realization of the coadjoint representation leads to interesting generalization of the Sturm-Liouville operator and projective structures.

B. The space of higher order linear differential operators has an interesting structure of infinite-dimensional Poisson manifold with respect to the Adler-Gelfand-Dickey Poisson bracket. This Poisson structure is related to so-called W-algebras and is very popular in Mathematical Physics (see e.g. [28]). We will discuss the relations of the Adler-Gelfand-Dickey bracket to the $\mathrm{Diff}(S^1)$-module structure on the space of linear differential operators on S^1 (studied by classics, see [29], [3]). Following [24], we show that the Adler-Gelfand-Dickey Poisson structure can be defined in terms of the Moyal-Weyl star-product.

4. An important point, common for all known examples is the following "tensor sense". Arguments of differential operators are considered as tensor-densities on S^1. The well-known classical example is the Sturm-Liouville operator, (1) acting from the space of $-1/2$-densities to the space of $3/2$-densities. This defines a natural $\mathrm{Diff}(S^1)$-action on the space of differential operators and intrinsically contains all the information about the related algebraic structures.

Remark here that the structure of the module over the group of diffeomorphisms on the space of linear differential operators on a manifold was studied in a series of recent papers [5], [20], [6].

Acknowledgments. I am grateful to A.A. Kirillov for his constant help and to Ch. Duval, L. Guieu, P. Marcel and C. Roger for collaboration and numerous stimulating discussions.

1 Coadjoint representation of Virasoro group and Sturm-Liouville operators; Schwarzian derivative as a 1-cocycle

This introductory section is based on the articles of A.A. Kirillov [13], [14] and G. Segal [27]. We will give the definition of the Virasoro group and the Virasoro algebra and prove the following result.

Theorem 1. *The coadjoint representation of the Virasoro group is naturally isomorphic, i.e., isomorphic as a module over the group of diffeomorphisms, to the space of Sturm-Liouville operators.*

The classical Schwarzian derivative appears as a 1-cocycle on the group of diffeomorphisms with values in the coadjoint representation.

1.1 Virasoro group and Virasoro algebra

Consider the Lie algebra $\mathrm{Vect}(S^1)$ of smooth vector fields on the circle:

$$X = X(x)\frac{d}{dx},$$

where $X(x+2\pi) = X(x)$. The commutator in $\mathrm{Vect}(S^1)$ is given by the formula:

$$[X(x)\frac{d}{dx}, Y(x)\frac{d}{dx}] = (X(x)Y'(x) - X'(x)Y(x))\frac{d}{dx},$$

where $X' = dX/dx$.

Definition 1.1. The *Virasoro algebra* is the unique (up to isomorphism) non-trivial central extension of $\mathrm{Vect}(S^1)$. It is given by the *Gelfand-Fuchs cocycle*:

$$\omega(X(x)\frac{d}{dx}, Y(x)\frac{d}{dx}) = \frac{1}{2}\int_0^{2\pi} \begin{vmatrix} X'(x) & Y'(x) \\ X''(x) & Y''(x) \end{vmatrix} dx.$$

The Virasoro algebra is, therefore a Lie algebra on the space $\mathrm{Vect}(S^1) \oplus \mathbf{R}$ defined by the commutator

$$[(X,\alpha),(Y,\beta)] = ([X,Y]_{\mathrm{Vect(S^1)}}, \omega(X,Y)),$$

where α and $\beta \in \mathbf{R}$ are elements of the center.

One easily checks (using integration by part) the 2-cocycle condition:

$$\omega(X,[Y,Z]) + \omega(Y,[Z,X]) + \omega(Z,[X,Y]) = 0,$$

equivalent to the Jacobi identity for the Virasoro algebra.

Remark. The defined Lie algebra has been discovered by I.M. Gelfand and D.B. Fuchs [8] and was later rediscovered in the physics literature.

Consider the group $\mathrm{Diff}^+(S^1)$ of diffeomorphisms of the circle preserving its orientation: $x \mapsto f(x)$, where $x(\mathrm{mod}\ 2\pi)$ is a parameter on S^1, $f(x+2\pi) = f(x) + 2\pi$.

Definition 1.2. The *Virasoro group* is the unique (up to isomorphism) non-trivial central extension of $\mathrm{Diff}^+(S^1)$. It is given by so-called *Thurston-Bott cocycle* (see [2]):

$$B(f,g) = \int_0^{2\pi} \log((f\circ g)')d\log(g')$$

By definition, the Virasoro group is given by the following product on $\mathrm{Diff}^+(S^1) \times \mathbf{R}$:

$$(f,\alpha)(g,\beta) = (f\circ g, \beta + B(f,g)).$$

Associativity of this product is equivalent to the condition: $B(f, g\circ h) + B(g,h) = B(f\circ g, h)$ which means that B is a 2-cocycle.

Notation. Let us denote the Virasoro algebra by vir.

1.2 Regularized dual space

The dual space to the Virasoro algebra:

$$vir^* \cong \mathrm{Vect}(S^1)^* \oplus \mathbf{R}$$

consists of pairs: (u,c) where u is a distribution on S^1 and $c \in \mathbf{R}$. Following A.A. Kirillov (cf. [13]), we will consider only the *regular part* of the dual space, vir^*_{reg} corresponding to distributions given by smooth functions. In other words, $vir^*_{\mathrm{reg}} \cong C^\infty(S^1) \oplus \mathbf{R}$.

Geometrically, objects dual to vector fields on S^1 have the sense of quadratic differentials: $u = u(x)(dx)^2$ (cf. [13], formulae (2) and (4) below). Note, that as a vector space the space of quadratic differentials $\mathcal{F}_2 \cong C^\infty(S^1)$. One obtains the following realization:

$$vir^*_{\mathrm{reg}} = \mathcal{F}_2(S^1) \oplus \mathbf{R}$$

with the pairing:

$$\langle (u(x)(dx)^2, c), (X(x)\frac{d}{dx}, \alpha)\rangle = \int_0^{2\pi} u(x)X(x)dx + c\alpha.$$

Let us calculate the coadjoint action of the Virasoro algebra and of the Virasoro group on space vir^*_{reg}.

1.3 Coadjoint representation of the Virasoro algebra

Let us recall the definition. The coadjoint representation of a Lie algebra g is the action on its dual space g^*, defined by:

$$\langle ad^*_X(\mu), Y\rangle = -\langle \mu, [X,Y]\rangle,$$

for every $X \in g$ and $\mu \in g^*$.

The coadjoint action of the Virasoro algebra preserves the regular part of the dual space.

Lemma 1.3. *The coadjoint action of the Virasoro algebra on the regular part of its dual space is given by the formula*

$$ad^*_{(X(x)\frac{d}{dx},\alpha)}(u(x)(dx)^2, c) = (L_X(u) - c\cdot X'''(x)(dx)^2,\ 0) \tag{2}$$

where $L_X(u)$ is the Lie derivative of a quadratic differential u:

$$L_X(u) = (X(x)u'(x) + 2X'(x)u(x))(dx)^2.$$

Proof. By definition

$$\langle ad^*_{(X(x)\frac{d}{dx},\alpha)}(u(x),c),\ (Y(x)\frac{d}{dx},\beta)\rangle = -\langle (u(x),c),\ \left[(X(x)\frac{d}{dx},\alpha),(Y(x)\frac{d}{dx},\beta)\right]\rangle$$

$$= -\int_0^{2\pi} u(XY' - X'Y)dx - \frac{c}{2}\int_0^{2\pi}(X'Y'' - X''Y')dx.$$

Integrating by parts, one obtains the expression:

$$\int_0^{2\pi}(Xu' + 2X'u - cX''')Y\,dx.$$

The lemma follows.

Note, that the coadjoint action of the Virasoro algebra is in fact, a $\mathrm{Vect}(S^1)$-action (the center acts trivially).

Remarks. (a) The case $c = 0$ corresponds to the coadjoint action of $\mathrm{Vect}(S^1)$ (without central extension). This is just the natural $\mathrm{Vect}(S^1)$-action by the Lie derivative on the space $\mathcal{F}_2$ of quadratic differentials.

(b) The linear map:

$$s : X(x)\frac{d}{dx} \ \mapsto\ X'''(x)(dx)^2 \tag{3}$$

is a *1-cocycle* on $\mathrm{Vect}(S^1)$ with values in $\mathcal{F}_2$. It satisfies the relation: $L_X(s(Y)) - L_Y(s(X)) = s([X,Y])$.

1.4 The coadjoint action of Virasoro group and Schwarzian derivative

The coadjoint action of the Virasoro group on the regular part of vir^* is the "group version" of the $\mathrm{Vect}(S^1)$-action (2). As in the case of the Virasoro algebra, the center acts trivially and therefore, the coadjoint representation of the Virasoro group is just a $\mathrm{Diff}^+(S^1)$-representation.

It is clear that this action is of the form:

$$Ad^*_{f^{-1}}(u,c) = (u \circ f - c \cdot S(f),\ c) \tag{4}$$

where

$$u \circ f = u(f(x))(df)^2$$

is the natural $\mathrm{Diff}^+(S^1)$-action on $\mathcal{F}_2$ and S is some 1-cocycle on $\mathrm{Diff}^+(S^1)$ with values in $\mathcal{F}_2$. Indeed, this action corresponds to the coadjoint action of the Virasoro algebra (2).

The explicit formula for the 1-cocycle S was calculated in [13] and [27].

Proposition 1.4 [13]. *The coadjoint action of the Virasoro group on the regular dual space vir^*_{reg} is defined by the 1-cocycle*

$$S(f) = \big(f'''/f' - \frac{3}{2}(f''/f')^2\big)(dx)^2 \tag{5}$$

Notation. The cocycle (5) is called the *Schwarzian derivative.*

An elegant proof of the formulae (4), (5) directly from the definition of the coadjoint representation can be found in [13].

One can also deduce these formulae from (2). To do this, it is sufficient to check the following two properties:

(a) The formula (4) indeed defines an action of $\mathrm{Diff}^+(S^1)$:

$$Ad^*_f \circ Ad^*_g = Ad^*_{f\circ g}.$$

This follows from the well-known property of the Schwarzian derivative:

$$S(f \circ g) = S(f) \circ g + S(g).$$

Which means that the mapping $f \mapsto S(f)$ is a *1-cocycle* on $\mathrm{Diff}^+(S^1)$ with values in vir^*.

(b) The action (2) is the infinitesimal version of (4).

1.5 Space of Sturm-Liouville equations as a $\mathrm{Diff}^+(S^1)$-module

It turns out that the formulae (2) and (4) has already been known to classics for a long time before the discovering of the Virasoro algebra.

Consider the (affine) space of Sturm-Liouville operators (1). There exists a natural $\mathrm{Diff}^+(S^1)$-action on this space (cf. [3],[29]). It turns out that this action coincides with the coadjoint action (4).

Definition 1.5. Consider a one-parameter family of actions of $\mathrm{Diff}^+(S^1)$ on the space of functions on S^1:

$$g_\lambda^* a = a \circ g^{-1} \left((g^{-1})' \right)^\lambda \tag{6}$$

Notation. Denote $\mathcal{F}_\lambda$ the $\mathrm{Diff}^+(S^1)$-module structure (6) on space $C^\infty(S^1)$.

Remark. Geometrically speaking, a has the sense of tensor-density of degree λ on S^1:

$$a = a(x)(dx)^\lambda$$

and the action (6) becomes simply $g^* a = a \circ g^{-1}$.

Let us look for a $\mathrm{Diff}^+(S^1)$-action on the space of Sturm-Liouville operators in the form: $g_{\lambda\mu}^*(L) = g_\lambda^* \circ L \circ (g_\mu^*)^{-1}$ for some λ, μ. It is easy to check that this formula preserves the space of Sturm-Liouville operators (this means, the differential operator $g_{\lambda\mu}^*(L)$ is again an operator of the form (1)) if and only if $\lambda = 3/2, \mu = -1/2$.

Definition 1.6. The action of group $\mathrm{Diff}^+(S^1)$ on the space of differential operators (1) is defined by:

$$g^*(L) := g_{\frac{3}{2}}^* \circ L \circ (g_{-\frac{1}{2}}^*)^{-1} \tag{7}$$

(see [29],[3]).

In other words, Sturm-Liouville operators are considered as acting on tensor-densities:

$$L : \mathcal{F}_{-1/2} \to \mathcal{F}_{3/2}.$$

The following statement has already been known to classics.

Proposition 1.7. *The result of the action (7) is again a Sturm-Liouville operator:* $g^*(L) = c \cdot d^2/dx^2 + u^g$ *with the potential*

$$u^g = u \circ g^{-1} \left((g^{-1})' \right)^2 + \frac{c}{2} \cdot S(g^{-1}).$$

Proof. Straightforward.

1.6 The isomorphism

The last formula coincides with the coadjoint action of the Virasoro group (4) (up to the multiple $-1/2$ in the last term). This remarkable coincidence shows that the space of Sturm-Liouville operators is isomorphic as a $\mathrm{Diff}^+(S^1)$-module to the coadjoint representation.

The isomorphism is given by the formula:

$$(u, c) \quad \longmapsto \quad -2c \cdot d^2/dx^2 + u(x) \tag{8}$$

Theorem 1 is proven.

1.7 $\mathrm{Vect}(S^1)$-action on the space of Sturm-Liouville operators

The infinitesimal version of the $\mathrm{Diff}^+(S^1)$-action on the space of Sturm-Liouville operators is given by the commutator with the Lie derivative:

$$\mathrm{ad}L_X(L) := L_X^{3/2} \circ L - L \circ L_X^{-1/2} \tag{9}$$

where L_X^λ is the operator of $\mathrm{Vect}(S^1)$-action on $\mathcal{F}_\lambda$. In other words, L_X^λ is the operator of Lie derivative on the space of tensor-densities of degree λ:

$$L_X^\lambda = X\frac{d}{dx} + \lambda \cdot X' \tag{10}$$

Proposition 1.8. *The result of the action (9) is a scalar operator of multiplication by:*

$$\mathrm{ad}L_X(L) = Xu' + 2X'u - c \cdot X'''$$

Proof. This formula can be proven by simple direct calculations.

The proposition follows also from the isomorphism (8). Indeed, the coadjoint action of X associates to the pair $(u(x)(dx)^2, c)$ the expression $((Xu' + 2X'u - c \cdot X''')(dx)^2, 0)$ corresponding to the scalar operator.

Remarks. (a) The operator L (and therefore $\mathrm{ad}L_X(L)$) maps from $\mathcal{F}_{-1/2}$ to $\mathcal{F}_{3/2}$. This means that the scalar operator $\mathrm{ad}L_X(L)$ is rather an operator of multiplication by a tensor-density of degree 2 (a quadratic differential) then by a function: $\mathrm{ad}L_X(L) \in \mathcal{F}_2$.

(b) The formula (9) corresponds to the $\mathrm{Diff}^+(S^1)$-action (7). However, the "point of view" of Lie algebras is much more universal: it works in the case (of Lie algebras more general then the Virasoro algebra) when there is no corresponding Lie group and there is no analogue of the formula (7).

2 Projectively invariant version of the Gelfand-Fuchs cocycle and of the Schwarzian derivative

In this section we follow G. Segal (see [27]).

As it was mentioned, the Virasoro algebra is the unique nontrivial central extension of $\mathrm{Vect}(S^1)$. However, the 2-cocycle on $\mathrm{Vect}(S^1)$ defined this extension can be chosen in different ways (up to a coboundary). There exists a unique way to define the Gelfand-Fuchs cocycle and the cocycles (3),(5) such that they are projectively invariant.

Consider the subalgebra of $\mathrm{Vect}(S^1)$ generated by the vector fields:

$$\frac{d}{dx}, \quad \sin x \frac{d}{dx}, \quad \cos x \frac{d}{dx}. \tag{11}$$

It is isomorphic to $sl_2(\mathbf{R})$ and the corresponding Lie group is $PSL(2, \mathbf{R})$ acting on S^1 ($\cong \mathbf{RP}^1$) by *projective transformation.*

2.1 Modified Gelfand-Fuchs cocycle

Consider the following "modified" Gelfand-Fuchs cocycle on $\mathrm{Vect}(S^1)$:

$$\bar{\omega}(X(x)\frac{d}{dx}, Y(x)\frac{d}{dx}) = \int_0^{2\pi} (X''' + X')Y \, dx \tag{12}$$

It is clear that this cocycle is cohomologous to the Gelfand-Fuchs cocycle and therefore, the corresponding central extension is isomorphic to the Virasoro algebra. Indeed, the additional term in (12) is a coboundary: the functional

$$\int_0^{2\pi} X'Y dx = \frac{1}{2}\int_0^{2\pi} (X'Y - XY')dx$$

depends only on the commutator of X and Y.

The cocycle (12) is sl_2*-equivariant.* This means,

$$\bar{\omega}([Z, X], Y) + \bar{\omega}(X, [Z, Y]) = 0$$

for every $X, Y \in \mathrm{Vect}(S^1)$ and $Z \in sl_2$.

Proposition 2.1. *The cocycle (12) is the unique (up to a constant)* sl_2*-equivariant 2-cocycle on* $\mathrm{Vect}(S^1)$.

Proof. Let $\widetilde{\omega}$ be a sl_2-equivariant 2-cocycle on $\mathrm{Vect}(S^1)$. The equivariance condition is equivalent to:

$$\widetilde{\omega}(X,Z) \equiv 0, \quad Z \in sl_2$$

Indeed, since $\widetilde{\omega}$ is a cocycle, one has: $\widetilde{\omega}([Z,X],Y)+\widetilde{\omega}([X,Y],Z)+\widetilde{\omega}([Y,Z],X) = 0$ The sl_2-equivariance condition gives now: $\widetilde{\omega}([X,Y],Z) = 0$ for every $X,Y \in \mathrm{Vect}(S^1)$ and $Z \in sl_2$. But the commutant in $\mathrm{Vect}(S^1)$ coincides with $\mathrm{Vect}(S^1)$.

The Gelfand-Fuchs theorem (see [8]) states that $H^2(\mathrm{Vect}(S^1)) = \mathbf{R}$, and therefore, every nontrivial cocycle is proportional to the Gelfand-Fuchs cocycle up to a coboundary. One has:

$$\widetilde{\omega} = \kappa\omega + b,$$

where b is a coboundary: $b(X,Y) = \langle u, [X,Y]\rangle$ for some $u \in \mathrm{Vect}(S^1)^*$.

The sl_2-equivariance condition means that $b(X,Z) = 0$ for $Z \in sl_2$ and an arbitrary $X \in \mathrm{Vect}(S^1)$. This implies $u = 0$.

2.2 Modified Schwarzian derivative

It is easy to check that the modified action of $\mathrm{Vect}(S^1)$ on vir_{reg} is as follows:

$$\overline{ad}^*_{X\frac{d}{dx}}(u(dx)^2, c) = (L_X(u) - c\cdot(X''' + X')(dx)^2,\ 0) \tag{13}$$

The modified $\mathrm{Diff}(S^1)$-action on vir_{reg} is:

$$\overline{Ad}^*_{f^{-1}}(u,c) = (u\circ f - c\cdot \bar{S}(f),\ c) \tag{14}$$

where $\bar{S}(f)$ is the following (modified) Schwarzian derivative:

$$\bar{S}(f) = \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 + \frac{1}{2}(f'^2 - 1)\right)(dx)^2 \tag{15}$$

The 1-cocycles S and $\bar{S}$ on $\mathrm{Diff}^+(S^1)$ are cohomologous.

Remarks. (a) The following amazing fact often leads to confusion. Take the affine parameter $t = \mathrm{tg}(x/2)$. Then, the modified Schwarzian derivative $\bar{S}$ is given by the expression: $\bar{S}(f(t)) = \dddot{f}/\dot{f} - (3/2)(\ddot{f}/\dot{f})^2$, where $\dot{f} = df/dt$. This expression coincides with the formula (5) for S (but $\bar{S} \neq S$).

(b) The PSL_2-equivariant Schwarzian (15) has been considered in [27] (see also [17]).

2.3 Energy shift

The projectively invariant coadjoint action corresponds to another realization of the dual space to the Virasoro algebra as the space of Sturm-Liouville operators. The map

$$(u, c) \longmapsto -2c\frac{d^2}{dx^2} + u(x) + \frac{c}{2} \tag{16}$$

is an isomorphism of $\mathrm{Diff}^+(S^1)$-module (16) and the module of Sturm-Liouville operators.

Indeed, the fact that the quantity $U = (u(x) + c/2)(dx)^2$ transforms according to the formulae (13) and (14), means that the quadratic differential $u(x)(dx)^2$ transforms under the $\mathrm{Diff}^+(S^1)$-action via the formulae (2) and (4).

2.4 Projective structures

Let us recall well-known definitions.

An atlas (U_i, t_i) on S^1 is called a *projective atlas* if the coordinate transformations $t_j \circ t_i^{-1}$ are linear-fractional functions. Two atlas' are called equivalent if their union is again a projective atlas. A class of equivalent projective atlas' is called a *projective structure* on S^1.

Every projective structure on S^1 defines a (local) action of the Lie algebra $sl_2(\mathbf{R})$ generated by the vector fields

$$\frac{d}{dt}, \quad t\frac{d}{dt}, \quad (t)^2\frac{d}{dt},$$

where $t = t_i$ is a local coordinate of the projective structure. This action is invariant under the linear-fractional transformations of t_i.

Remark. This sl_2-action coincides with the action (11) for the angular parameter $x = \mathrm{arctg}(t)$.

There exists a natural isomorphism between the space of Sturm-Liouville operators and the space of projective structures on S^1. Given a Sturm-Liouville operator (1), consider the corresponding differential equation: $c \cdot \phi'' + u(x)\phi = 0$. Local coordinates of projective structure associated to this operator are defined as functions of two independent solutions:

$$t = \frac{\phi_1}{\phi_2},$$

on an interval with $\phi_2 \neq 0$. An important remark is that for the local coordinate t, the potential of the Sturm-Liouville operators is identically zero:

$$c\frac{d^2}{dx^2} + u(x) \;=\; c\frac{d^2}{dt^2}.$$

A beautiful definition of the corresponding sl_2-action was proposed by A.A. Kirillov (see [14]). It is given by products of solutions: the generators are as follows:

$$\phi_1^1, \quad \phi_1\phi_2, \quad \phi_2^1$$

Note, that the solutions are $-1/2$-tensor-densities, therefore their product is a vector field. Use $W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = \text{const}$ to verify that the these vector fields indeed generate a sl_2-subalgebra.

3 Kirillov's method of Lie superalgebras

The "mysterious" coincidence between the coadjoint representation of the Virasoro group and the natural $\mathrm{Diff}^+(S^1)$-action on the space of Sturm-Liouville operators (Theorem 1) was explained by A.A. Kirillov in [14], where furthermore an algebraic explanation of the fact that Sturm-Liouville operators act on tensor-densities is given. It is an amazing fact, that the natural interpretation of these geometrical results uses Lie superalgebras.

3.1 Lie superalgebras

A Lie superalgebra is a $\mathbf{Z}_2$-graded algebra

$$g = g_0 \oplus g_1$$

with the multiplication called the commutator satisfying two conditions:
(1) superized skew symmetry: $[X, Y] + (-1)^{\widetilde{X}\widetilde{Y}}[Y, X] = 0$,
(2) superized Jacobi identity:

$$(-1)^{\widetilde{X}\widetilde{Z}}[X, [Y, Z]] + (-1)^{\widetilde{X}\widetilde{Y}}[Y, [Z, X]] + (-1)^{\widetilde{Y}\widetilde{Z}}[Z, [X, Y]] = 0,$$

where $\widetilde{X}$ is the degree ($\widetilde{X} = 0$ for $X \in g_0$ and $\widetilde{X} = 1$ for $X \in g_1$).

The simplest properties of Lie superalgebras are as follows:
(a) $g_0 \subset g$ is a Lie subalgebra,
(b) g_0 acts on g_1 via the commutator: if $X \in g_0$ and $\xi \in g_1$ is $[X, \xi] \in g_1$
(c) the product of $\xi, \eta \in g_1$ called an *anticommutator* is defined by a symmetric bilinear map

$$[\ ,\ ]_+ : g_1 \otimes g_1 \to g_0.$$

3.2 Ramond and Neveu-Schwarz superalgebras

Consider the space $\mathcal{F}_{-1/2}$ of $-1/2$-tensor densities on S^1: $\phi = \phi(x)(dx)^{-1/2}$. As a vector space $\mathcal{F}_{-1/2}$ is isomorphic to $C^\infty(S^1)$ and the $\mathrm{Vect}(S^1)$-action on $\mathcal{F}_{-1/2}$ is given by the Lie derivative:

$$L_X^{-1/2}(\phi) = (X\phi' - (1/2)X'\phi)(dx)^{-1/2}.$$

There exists a natural Lie superalgebra structure on the space $\mathrm{Vect}(S^1) \oplus \mathcal{F}_{-1/2}$, The anticommutator

$$[\ ,\ ]_+ : \mathcal{F}_{-1/2} \otimes \mathcal{F}_{-1/2} \longrightarrow \mathrm{Vect}(S^1)$$

is just the product of tensor-densities:

$$[\xi(x)(dx)^{-1/2}, \eta(x)(dx)^{-1/2}]_+ := \xi(x)\eta(x)\frac{d}{dx}.$$

Thus, the commutator in the defined Lie superalgebra is given by the following formula:

$$\Big[(X,\xi),(Y,\eta)\Big] = ([X,Y]_{\mathrm{Vect}(S^1)} + \xi \cdot \eta,\ L_X(\eta) - L_Y(\xi))$$

Definition 3.1. There exists a unique (modulo isomorphism) nontrivial central extension of the defined Lie superalgebra. It can be given by the following 2-cocycle:

$$\Omega\Big((X\,d/dx, \xi(dx)^{-1/2}), (Y\,d/dx, \eta(dx)^{-1/2})\Big) = \int_0^{2\pi} (X''Y' + 2\xi'\eta')dx \qquad (17)$$

The Lie superalgebra defined by this central extension is called the *Ramond* algebra.

Remark. The even part of the Ramond Lie superalgebra coincides with the Virasoro algebra.

Consider now the space of *anti-periodic* $-1/2$-densities on S^1:

$$\xi(x)(dx)^{-1/2}, \qquad \xi(x+2\pi) = -\xi(x).$$

This space is also a $\mathrm{Vect}(S^1)$-module. Let us denote it: $\mathcal{F}^{(-)}_{-1/2}$. Note that the product of two anti-periodic $-1/2$-densities is a ("periodic") vector field well-defined on S^1.

Definition 3.2. The same formulae as above define a Lie superalgebra structure on the space

$$\mathrm{Vect}(S^1) \oplus \mathbf{R} \oplus \mathcal{F}^{(-)}_{-1/2}.$$

This Lie superalgebra is called the *Neveu-Schwarz* algebra.

Remarks. (a) The Lie superalgebras on $\mathrm{Vect}(S^1) \oplus \mathcal{F}_{-1/2}$ and $\mathrm{Vect}(S^1) \oplus \mathcal{F}^{(-)}_{-1/2}$ can be defined as the Lie superalgebras of contact vector fields on $S^{1|1}$ and $\mathbf{RP}^{1|1}$ reciprocally (see [21]).

(b) The Ramond and Neveu-Schwarz superalgebras are particular cases of a series of so-called string superalgebras (see [21]).

3.3 Coadjoint representation

The (regularized) dual space to the Ramond algebra is naturally isomorphic to:

$$\mathcal{F}_2 \oplus \mathbf{R} \oplus \mathcal{F}_{3/2}.$$

Indeed, the module $\mathcal{F}_{3/2}$ is dual to $\mathcal{F}_{-1/2}$ with respect to the pairing

$$\langle \phi(x)(dx)^{3/2}, \xi(x)(dx)^{-1/2} \rangle = \int_0^{2\pi} \phi(x)\xi(x)dx.$$

Thus, the regular dual space to the Ramond algebra consists of the elements: $(u, c, \phi) = (u(x)(dx)^2, c, \phi(x)(dx)^{3/2})$.

In the same way, the regular dual space to the Neveu-Schwarz algebra is:

$$\mathcal{F}_2 \oplus \mathbf{R} \oplus \mathcal{F}_{3/2}^{(-)},$$

where $\mathcal{F}_{3/2}^{(-)}$ is the space of antiperiodic 3/2-densities.

Lemma 3.3. *The coadjoint representation of the Ramond and Neveu-Schwarz superalgebras are given by the formula:*

$$ad^*_{(X,\xi)} \begin{pmatrix} u(dx)^2 \\ c \\ \phi(dx)^{3/2} \end{pmatrix} = \begin{pmatrix} (Xu' + 2X'u - c \cdot X''' \ + \xi\phi'/2 + 3\xi'\phi/2)(dx)^2 \\ 0 \\ (X\phi' + 3X'/2 \ + u\xi - 2c \cdot \xi'')(dx)^{3/2} \end{pmatrix} \quad (18)$$

Proof. The formula (18) can be obtained directly from the definition of the coadjoint representation. The easy calculations are similar to those from the proof of Lemma 1.3.

The Sturm-Liouville operator appears as the coadjoint action of the odd part of the Ramond and Neveu-Schwarz superalgebras. Indeed,

$$ad^*_{(0,\xi)}(u, c, 0) = \Big(- 2c\frac{d^2}{dx^2} + u \Big)\xi.$$

3.4 Projective equivariance and Lie superalgebra $osp(1|2)$

Consider the Lie superalgebra generated by the vector fields (11) and two more odd generators:

$$\sin\left(\frac{x}{2}\right)(dx)^{-1/2} \quad \text{and} \quad \cos\left(\frac{x}{2}\right)(dx)^{-1/2}$$

This Lie superalgebra is a subalgebra of the Neveu-Schwarz superalgebra isomorphic to the $osp(1|2)$, that has a natural interpretation as the algebra of symmetries of the projective superspace $\mathbf{P}^{1|1}$.

As in the case of the Virasoro algebra (cf. Section 2) it is possible to write the cocycle giving the central extension in a canonic ($osp(1|2)$-invariant) form.

Lemma 3.4. *The 2-cocycle*

$$\bar{\Omega}\Big((Xd/dx, \xi(dx)^{-1/2}), (Yd/dx, \eta(dx)^{-1/2})\Big) = \int_0^{2\pi}((X'''+X')Y+2(\xi''+4\xi)\eta)dx$$

is the unique (up to a constant) nontrivial 2-cocycle on the Lie superalgebra $\mathrm{Vect}(S^1) \oplus \mathcal{F}^{(-)}_{-1/2}$ *equivariant with respect to the subalgebra* $osp(1|2)$.

Proof. Similar to those of the proof of Proposition 2.1.

4 Invariants of coadjoint representation of the Virasoro group

It follows from Theorem 1 that the invariants of the coadjoint representation of the Virasoro group are the invariants of the $\mathrm{Diff}^+(S^1)$-action on the space of Sturm-Liouville operator (1). This is quite old and classical problem was considered in [18],[19] and in [13],[27], [30], [10] in the context of the Virasoro algebra.

In this section we will describe the invariants of the $\mathrm{Diff}^+(S^1)$-action following [13] and [27].

4.1 Monodromy operator as a conjugation class of $\widetilde{SL}(2, \mathbf{R})$

Consider the Sturm-Liouville equation $2c\psi'' + u(x)\psi = 0$. Since the potential $u(x)$ is a periodic function, the translation

$$M\psi(x) = \psi(x + 2\pi)$$

defines is a linear operator on the space of solutions. This operator is called the *monodromy operator.*

We need the following two remarks.

(1) The monodromy operator defines a conjugation class of the group $SL(2,\mathbf{R})$. Indeed, the Wronsky determinant of any two solutions

$$W(\psi_1,\psi_2) = \psi_1\psi_2' - \psi_1'\psi_2$$

is a constant function. Thus, W defines a bilinear skew-symmetric form on the space of solutions and operator M preserves W. Now, an arbitrary choice of the basis ψ_1, ψ_2 such that $W(\psi_1,\psi_2) = 1$ associates to M a matrix from $SL(2,\mathbf{R})$. The conjugation class of this matrix does not depend on the choice of the basis.

(2) Moreover, the monodromy operator defines a conjugation class of the universal covering $\widetilde{SL}(2,\mathbf{R})$. Indeed, for every value $x = x_0$, identify the space of solutions with $\mathbf{R}^2$ choosing the initial conditions: $T_{x_0} : \psi \mapsto (\psi(x_0), \psi'(x_0))$. Define a family of linear operators on the space of solutions:

$$T(x) := T_x^{-1} \circ T_0$$

The family $T(x)$ joins the monodromy operator: $M = T(2\pi)$ with the identity: $T(0) = \mathrm{Id}$. It can be lift (up to a conjugation) to $\widetilde{SL}(2,\mathbf{R})$.

We will confound the monodromy operator with the corresponding conjugation class of $\widetilde{SL}(2,\mathbf{R})$.

4.2 Classification theorem

The following theorem is the classification of the invariants of the $\mathrm{Diff}^+(S^1)$-action on the space of Sturm-Liouville operators. According to Theorem 1, it classifies also the invariants of the coadjoint action of the Virasoro group. Various approaches to the classification see in [18],[19],[13] and [27].

Theorem 2. *The monodromy operator is the unique invariant of the $Diff^+(S^1)$-action.*

Proof. Let us give a simple proof (different from those of [18],[19],[13] and [27]) based on [23].

First, it is clear that the monodromy operator *is* an invariant, since the $\mathrm{Diff}^+(S^1)$-action is just a coordinate transformation and the monodromy operator is defined intrinsically.

To prove that there is no more (independent of M) invariants, we will use the homotopy method. One should show that:

(a) Every two Sturm-Liouville operators with the same monodromy are homotopic to each other in the class of operators with the fixed monodromy. In other words, the set of operators with fixed monodromy is connected.

(b) Given a smooth family of operators with fixed monodromy:

$$L_s = 2c\frac{d^2}{dx^2} + u_s(x), \quad s \in [0,1],$$

there exists a vector field $X \in \mathrm{Vect}(S^1)$ such that

$$Xu' + 2X'u + cX''' = \dot{u}, \quad \text{where} \quad \dot{u} = \frac{\partial}{\partial s} u_s(x)\Big|_{s=0} \tag{19}$$

Statements (a) and (b) imply that every two Sturm-Liouville operators L_1 and L_1 with the same monodromy are on the same $\mathrm{Diff}^+(S^1)$-orbit. Indeed, there exists a family L_s of operators with the fixed monodromy and a family $X_s \mathrm{Vect}(S^1)$ of solutions of the homotopy equation (19) for each s. Now, there exists a flow corresponding to this family (S^1 is compact!), this is a diffeomorphism which maps L_1 to L_2.

Let us first prove (a). To each Sturm-Liouville operator, associate a family of mapping $T(x)$ (see Section 4.1). If L_1, L_2 are two operators with the same monodromy, then, the corresponding families are homotopic (as two curves in $PSL_2(\mathbf{R})$). Fixe this homotopy: $T(x)^\tau$. Now, to define a family of Sturm-Liouville operators L_τ joining L_1 and L_2, one associates a Sturm-Liouville oparator to each family $T(x)^\tau \in PSL_2(\mathbf{R})$, for fixed τ (this is a standard procedure).

Let us now prove (b).

Lemma 4.1. *Given a basis ψ_1^s, ψ_2^s of solutions of the equation $L_s\psi = 0$ such that $W(\psi_1^s, \psi_2^s) \equiv 1$, the vector field*

$$X = \frac{1}{2} \begin{vmatrix} \dot{\psi}_1 & \dot{\psi}_2 \\ \psi_1 & \psi_2 \end{vmatrix} \tag{20}$$

is a solution of the equation (19).
Proof. Taking the derivative of the equality $L_s\psi^s = 0$, one gets:

$$\dot{u}\psi^{s=0} + L_{s=0}\dot{\psi} = 0.$$

To solve the equation (19), it is sufficient to find a vector fields X such that the Lie derivative $L_X(\psi) = \dot{\psi}$. Indeed, it follows from the definition (9) of the $\mathrm{Vect}(S^1)$-action on the space of Sturm-Liouville operators.

Let us look for a vector field X such that $L_X(\psi_1^{s=0}) = \dot{\psi}_1$ and $L_X(\psi_2^{s=0}) = \dot{\psi}_2$. This gives a system of linear equations:

$$\begin{cases} X\psi_1' - (1/2)X'\psi_1 = \dot{\psi}_1 \\ X\psi_2' - (1/2)X'\psi_2 = \dot{\psi}_2\,. \end{cases}$$

Taking X and X' as independent arguments, one obtains formally:

$$X = \frac{1}{2} \begin{vmatrix} \dot{\psi}_1 & \dot{\psi}_2 \\ \psi_1 & \psi_2 \end{vmatrix}, \quad X' = \begin{vmatrix} \dot{\psi}_1 & \dot{\psi}_2 \\ \psi_1 & \psi_2 \end{vmatrix}\,.$$

Now, let us verify that $X' = dX/dx$. Indeed,

$$\frac{dX}{dx} = \frac{1}{2}\begin{vmatrix} \dot{\psi}_1' & \dot{\psi}_2' \\ \psi_1 & \psi_2 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} \dot{\psi}_1 & \dot{\psi}_2 \\ \psi_1' & \psi_2' \end{vmatrix}.$$

The two terms in the right hand side coincide since:

$$\frac{d}{ds}\begin{vmatrix} \psi_1' & \psi_2' \\ \psi_1 & \psi_2 \end{vmatrix} = \begin{vmatrix} \dot{\psi}_1' & \dot{\psi}_2' \\ \psi_1 & \psi_2 \end{vmatrix} + \begin{vmatrix} \psi_1' & \psi_2' \\ \dot{\psi}_1 & \dot{\psi}_2 \end{vmatrix} = 0,$$

and therefore $X' = dX/dx$.

Lemma 4.1 is proven.

Let us show that Theorem 2 follows from the lemma. Indeed, since the monodromy operator does not depend on s, the basis ψ_1^s, ψ_2^s can be chosen in such a way that the corresponding monodromy matrix does not depend on s. Then, the solution (20) is periodic: $X(x + 2\pi) = \det(M)X(x) = X(x)$.

Theorem 2 is proven.

Remark. Recall that the solutions of a Sturm-Liouville equation have a sense of $-1/2$-tensor-densities. Therefore, the quadratic expression (20) is indeed a vector field.

The Kähler geometry of the coadjoint orbits of the Virasoro group has been studied in A.A. Kirillov's works [15],[16].

5 Extension of the Lie algebra of first order linear differential operators on S^1 and matrix analogue of the Sturm-Liouville operator

This section follows the recent work [22]. We will show that the Kirillov method is valid in a more general framework then the Virasoro algebra.

5.1 Lie algebra of first order differential operators on S^1 and its central extensions

Consider the Lie algebra of first order linear differential operators on S^1:

$$A = X(x)\frac{d}{dx} + a(x) \tag{21}$$

(This Lie algebra is in fact the semi-direct product of $\mathrm{Vect}(S^1)$ by the module of functions $\mathcal{F}_0$).

This Lie algebra has three nonisomorphic central extensions (cf. [25]). The first one is given by the Gelfand-Fuchs cocycle and two more extensions are given by the non-trivial 2-cocycles:

$$\begin{aligned} \omega'((X\tfrac{d}{dx},\ a),\ (Y\tfrac{d}{dx},\ b)) &= \int_{S^1} (X''(x)b(x) - Y''(x)a(x))dx \\ \omega''((X\tfrac{d}{dx},\ a),\ (Y\tfrac{d}{dx},\ b)) &= 2\int_{S^1} a(x)b'(x)dx \end{aligned} \tag{22}$$

Definition 5.1. Let us denote $\mathcal{G}$ the Lie algebra defined on the space $Vect(S^1) \oplus C^\infty(S^1) \oplus \mathbf{R}^3$ as the *universal central extension* of the Lie algebra of the operators (21). This means, $\mathcal{G}$ is the Lie algebra defined by the commutator:

$$\left[(X\frac{d}{dx}, a, \alpha),\ (Y\frac{d}{dx}, b, \beta)\right] = \left((XY' - X'Y)\frac{d}{dx},\ Xb' - Ya',\ \omega\right)$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3),\ \beta = (\beta_1, \beta_2, \beta_3) \in \mathbf{R}^3$ are in the center and

$$\omega = (\omega((X\tfrac{d}{dx},\ a),\ (Y\tfrac{d}{dx},\ b)),\ \ \omega'((X\tfrac{d}{dx},\ a),\ (Y\tfrac{d}{dx},\ b)),\ \ \omega''((X\tfrac{d}{dx},\ a),\ (Y\tfrac{d}{dx},\ b))).$$

5.2 Matrix Sturm-Liouville operators

The space of matrix linear differential operators on $C^\infty(S^1) \oplus C^\infty(S^1)$:

$$\mathcal{L} = \begin{pmatrix} -2c_1\dfrac{d^2}{dx^2} + u(x) & 2c_2\dfrac{d}{dx} + v(x) \\ -2c_2\dfrac{d}{dx} + v(x) & 4c_3 \end{pmatrix} \tag{23}$$

where $c_1, c_2, c_3 \in \mathbf{R}$ and $u = u(x), v = v(x)$ are 2π-periodic functions was defined in [22]. It was shown that this space gives a geometric realization for the dual space of the Lie algebra $\mathcal{G}$.

The $Vect(S^1)$-action on the space of operators (23) is defined, as in the case of Sturm-Liouville operators (1), by commutator with the Lie derivative. We consider $\mathcal{L}$ as an operator on $Vect(S^1)$-modules:

$$\mathcal{L} : \mathcal{F}_{-\frac{1}{2}} \oplus \mathcal{F}_{\frac{1}{2}} \to \mathcal{F}_{\frac{3}{2}} \oplus \mathcal{F}_{\frac{1}{2}}.$$

Remark. The choice of degrees of tensor-densities in this formula is the *unique* choice such that the operators (23) are *selfadjoint*.

There exists a natural action of the Lie algebra of first order differential operators (21) the space of operators (23).

5.3 Action of Lie algebra of differential operators

There exists a nice family of modules over the Lie algebra of operators (21). Consider the space

$$\mathcal{F}_\lambda \oplus \mathcal{F}_{\lambda+1}$$

It is defined by the formula:

$$T^{(\lambda)}_{\left(X(x)\frac{d}{dx}+a(x)\right)} \begin{pmatrix} \phi(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} L^{(\lambda)}_{X\frac{d}{dx}}\ \phi(x) \\ L^{(\lambda+1)}_{X\frac{d}{dx}}\ \psi(x) - \lambda a'(x)\phi(x) \end{pmatrix} \tag{24}$$

The action on the space of operators (23) is defined in analogous way as the action of $\mathrm{Vect}(S^1)$ on the space of Sturm-Liouville operators given by the formula (9). Put:

$$\left[T_{\left(X\frac{d}{dx}+a\right)}, \mathcal{L}\right] := T^{(1/2)}_{\left(X\frac{d}{dx}+a\right)} \circ \mathcal{L} - \mathcal{L} \circ T^{(-1/2)}_{\left(X\frac{d}{dx}+a\right)} \tag{25}$$

Theorem 5.2 (see [22]). *The action (25) coincides with the coadjoint action of the Lie algebra of first order linear differential operators.*

Proof. The explicit formula for the action (25) is:

$$\left[T_{\left(X\frac{d}{dx}+a\right)}, \mathcal{L}\right] = \begin{pmatrix} \begin{array}{l} Xu' + 2X'u - c_1X''' \\ +va' + c_2a'' \end{array} & \begin{array}{l} Xv' + X'v - c_2X'' \\ +2c_3a' \end{array} \\ & \\ \begin{array}{l} Xv' + X'v - c_2X'' \\ +2c_3a' \end{array} & 0 \end{pmatrix}$$

One easily verifies that this is precisely the coadjoint action of the Lie algebra of differential operators (23) (see [22] for the details).

5.4 Generalized Neveu-Schwarz superalgebra

The space of matrix analogues of the Sturm-Liouville operators (23) was found in [22] using a Lie superalgebra generalizing the Neveu-Schwarz algebra.

Definition 5.3. Consider the $\mathbf{Z}_2$-graded vector space : $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$. where $\mathcal{S}_0 = \mathcal{G}$ the extension of the Lie algebra of operators (23). and $\mathcal{S}_1$ the $\mathcal{G}$-module:

$$\mathcal{S}_1 = \mathcal{F}_{-\frac{1}{2}} \oplus \mathcal{F}_{\frac{1}{2}}.$$

The even part $\mathcal{S}_0$ acts on $\mathcal{S}_1$ according to (24). Let us define the *anticommutator* $[\ ,\ ]_+ : \mathcal{S}_1 \otimes \mathcal{S}_1 \to \mathcal{S}_0$:

$$\Big[(\phi,\ \alpha),\ (\psi,\ \beta)\Big]_+ = (\phi\psi\frac{d}{dx},\ \phi\beta + \alpha\psi,\ \sigma_+)$$

where $\mathbf{\Omega}_+ = (\Omega, \Omega', \Omega'')$, where Ω is the Ramond – Neveu-Schwarz cocycle (17) and $\Omega', \Omega'')$ are the continuations of the cocycles (22):

$$\Omega'((\phi,\alpha),(\psi,\beta)) = -2\int_{S^1}(\phi'(x)\beta(x) + \alpha(x)\psi'(x))dx$$

$$\Omega''((\phi,\alpha),(\psi,\beta)) = 4\int_{S^1}\alpha(x)\beta(x)dx$$

Theorem 5.4 (see [22]). *$\mathcal{S}$ is a Lie superalgebra.*

The differential operators (23) can be defined as a part of the coadjoint action of the superalgebra $\mathcal{G}$. Namely, one obtains:

$$ad^*_{\begin{pmatrix}0\\0\\\phi(dx)^{-\frac{1}{2}}\\\alpha(dx)^{\frac{1}{2}}\end{pmatrix}}\begin{pmatrix}u\\v\\\mathbf{c}\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\-2c_1\phi'' + u\phi + v\alpha + 2c_2\alpha'\\-2c_2\phi' + v\phi + 4c_3\alpha\end{pmatrix}$$

Remark. The Lie algebra $\mathcal{G}$ considered in this section, is just an example from the series of seven Lie algebras generalizing the Virasoro algebra (see [25]). It turns out that Kirillov's method works for also for the other Lie algebra from this series (work in preparation of P. Marcel). It would be very interesting to apply this method to other Virasoro type Lie algebras (see [26]).

6 Geometrical definition of the Gelfand-Dickey bracket and the relation to the Moyal-Weil star-product

In this section we follow [24]. We consider another generalization of the Virasoro algebra: the so-called *second Adler-Gelfand-Dickey* Poisson structure, which is also known as the classical W-algebras in the physics literature. The Adler-Gelfand-Dickey bracket is (an infinite-dimensional) Poisson bracket on the space of n-th order differential operators on S^1. (We will consider here only the first nontrivial case corresponding to the space of third order linear differential operators)

We will show that the Gelfand-Dickey bracket is related to the well-known *Moyal-Weyl star-product*.

The main idea is to consider arguments of differential operators as tensor-densities and use the PSL_2-equivariance of all the operations.

6.1 Moyal-Weyl star-product

Consider the standard symplectic plane $(\mathbf{R}^2, dp \wedge dq)$, where p, g are linear coordinates. The space of functions on $\mathbf{R}^2$ is a Lie algebra with respect to the Poisson bracket:

$$\{F, G\} = F_p G_q - F_q G_p,$$

where $F_p = \partial F / \partial p$.

The following operation:

$$F \star_\hbar G = FG + \frac{\hbar}{2}\{F, G\} + \cdots + \frac{\hbar^m}{2^m m!}\{F, G\}_m + \cdots$$

where

$$\{F, G\}_m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{\partial^m F}{\partial p^{m-i} \partial q^i} \frac{\partial^m G}{\partial p^i \partial q^{m-i}}$$

is called the Moyal-Weyl star-product on $\mathbf{R}^2$. Here $\hbar$ is a formal parameter and the operation $\star_\hbar$ is with values in formal series in $\hbar$. (In the case polynomials, one can consider $\hbar$ as a number). The operation $\star_\hbar$ is associative.

The Moyal-Weyl star-product is a very popular object in deformation quantization.

6.2 Moyal-Weyl star-product on tensor-densities, the transvectants

Isomorphism 6.1. *There exists a natural isomorphism between the space $\mathcal{F}_\lambda$ (of tensor-densities of degree λ on S^1) and the space of functions on $\mathbf{R}^2 \setminus \{0\}$ homogeneous of degree -2λ. For the affine parameter on S^1: $t = tg(x)$ this isomorphism is given by the formula:*

$$\phi(t)(dt)^\lambda \ \longmapsto \ p^{-2\lambda}\phi(\frac{q}{p}) \tag{26}$$

Indeed, a function corresponding to a vector field X is: $p^2 X(q/p)$. Verify, that the Lie derivative corresponds to the Poisson bracket.

The isomorphism (26) lifts the Moyal-Weyl star-product to the space of tensor-densities.

Lemma 6.2. *The terms of this star-product are as follows:*

$$\{\phi, \psi\}_m = \frac{m!}{2^m} \sum_{i+j=m} (-1)^i m! \binom{2\lambda + m - 1}{i} \binom{2\mu + m - 1}{j} \phi^{(i)} \psi^{(j)} \tag{27}$$

where $\phi \in \mathcal{F}_\lambda, \psi \in \mathcal{F}_\mu$, $\phi^{(i)} = d^i\phi/dx^i$ *and*

$$\binom{k}{i} = k(k-1)\cdots(k-i+1).$$

Proof. Straightforward.

It turns out that the operations (27) coincides (up to the constant $m!/2^m$) with so-called Gordan's *transvectants*. This operations can be defined as bilinear maps

$$\mathcal{F}_\lambda \otimes \mathcal{F}_\mu \ \to \ \mathcal{F}_{\lambda+\mu+m}$$

equivariant with respect to the action of the Lie algebra $sl_2(\mathbf{R}$ defined by (11) (projectively invariant).

Remark. The isomorphism (26) is, in fact, given by the standard projective structure on S^1. Indeed, t is the corresponding projective parameter. Given an arbitrary projective structure on S^1, one defines an isomorphism (analogue of (26)) between tensor-densities on S^1 and homogeneous functions on $\mathbf{R}^2$.

6.3 Space of third order linear differential operators as a $\mathrm{Diff}^+(S^1)$-module

Consider the space of third order linear differential operators

$$A = \frac{d^3}{dx^3} + u(x)\frac{d}{dx} + v(x) \tag{28}$$

This space plays the same role that the space of Sturm-Liouville operators (1) in the case of Virasoro algebra. However, the Adler-Gelfand-Dickey bracket is not a Lie-Poisson structure. We refer [1] and [7] for the original definition and [4] for another one related to the Kac-Moody algebras.

The subject of this section was known to the classics (see [29],[3]) ... and was forgotten by the contemporary experts.

Definition 6.3. The $\mathrm{Diff}^+(S^1)$-action on the space of operators (28) is defined by the formula:

$$g^*(A) := g_2^* \circ A \circ (g_{-1}^*)^{-1}$$

This means that the operator A is considered as acting from the space of vector fields on S^1 with values in the space of quadratic differentials:

$$A : \mathcal{F}_{-1} \to \mathcal{F}_2.$$

The corresponding action of $X(x)d/dx \in \mathrm{Vect}(S^1)$ is:

$$\mathrm{ad}L_X(A) := L_X^{(2)} \circ A - A \circ L_X^{(-1)}$$

Let us give the explicit formulae for $\mathrm{Diff}^+(S^1)$- and $\mathrm{Vect}(S^1)$-action. It is convenient to decompose the operator (28) as a sum of its skew-symmetric and symmetric parts:

$$A = \frac{d^3}{dx^3} + u(x)\frac{d}{dx} + \frac{u(x)}{2} + w(x),$$

where $w(x) = v(x) - u(x)/2$.

Proposition 6.4 (see [29],[3]). *A diffeomorphism f transform an operator A into an operator of the form (28) with coefficients:*

$$\begin{aligned} u^f &= u \circ f (f')^2 + 2S(f) \\ w^f &= w \circ f (f')^3. \end{aligned}$$

This means, u transforms as a potential of the Sturm-Liouville operator: $4\frac{d^2}{dx^2} + u(x)$ and w has the sense of *cubic differential*: $w = w(x)(dx)^3$.

Corollary. *The projection from the space of third order operators (28) to the space of Sturm-Liouville operators:*

$$\frac{d^3}{dx^3} + u(x)\frac{d}{dx} + v(x) \longmapsto 4\frac{d^2}{dx^2} + u(x)$$

is $\mathrm{Diff}(S^1)$*-equivariant (does not depend on the choice of the parameter x).*

The corresponding action of a vector field $X(x)d/dx \in \mathrm{Vect}(S^1)$ associates to A a first order operator: $\mathrm{ad}L_X(A) = u^X \frac{d}{dx} + \frac{u^X}{2} + w^X$, where

$$\begin{aligned} u^X &= Xu' + 2X'u + 2X''' \\ w^X &= Xw' + 3X'w. \end{aligned}$$

Remark. The geometric interpretation of u and w is related to the projective projective differential geometry of plane curves (associated to differential operators (28)). Namely, u is interpreted as the projective curvature and w leads to the notion of projective length element: $ds = (w)^{1/3}$ (see [29],[3] and also [11]).

6.4 Second order Lie derivative

The notion of *second order Lie derivative* considered below was introduced in [24] (see [5] for a general definition in the multi-dimensional case). The question is as follows: given a second order contravariant tensor field $X \in \mathcal{F}_2$:

$$Z = Z(x)(dx)^{-2},$$

is it possible to define an "action" of Z on geometric quantities (like tensor-densities etc.) analogous to the Lie derivative along a vector field?

The answer is negative. There is no $\mathrm{Diff}(S^1)$-equivariant bilinear differential operators

$$\mathcal{F}_2 \otimes \mathcal{F}_\lambda \to \mathcal{F}_\lambda,$$

for general values of λ (cf. [12]) and so, one can not define such an action intrinsically.

To define the second order Lie derivative, we fix a projective structure on S^1.

Definition 6.5. The second order Lie derivative over contravariant tensor field of degree 2: $Z = Z(x)(dx)^{-2}$ is a linear map

$$L_Z^2 : \mathcal{F}_\lambda \to \mathcal{F}_\lambda$$

given by:

$$L_Z^2(\phi) := \{Z, \phi\}_2$$

Remark. Note, that the operations $\{\ ,\ \}_m$, $m \geq 2$ are defined if one fix a projective structure (cf. Section 6.2).

6.5 Adler-Gelfand-Dickey Poisson structure

A Poisson structure on a manifold is given by a linear map on each cotangent space with values in the tangent space (satisfying the Jacobi condition). Thus, to define a Poisson structure on a vector space, it is sufficient to associate a vector field to every linear functional.

Every linear functional on the space of operators (28) is a linear combination of:

$$\langle l_X^1, A \rangle = \int X(x)u(x)dx, \quad \langle l_Z^2, A \rangle = \int Z(x)w(x)dx$$

where $X = X(x)d/dx, Z = Z(x)(dx)^{-2}$.

Definition 6.6. The Adler-Gelfand-Dickey Poisson structure on the space of operators (28) associates to a linear functionals vector fields given by the commutator with the Lie derivative:

$$\begin{aligned} \dot{A}_X &:= [L_X, A] \\ \dot{A}_Z &:= [L_Z^2, A] \end{aligned}$$

(see [24] for the details).

The Adler-Gelfand-Dickey Poisson structure is a very interesting and popular object in Mathematical Physics. This way to define it seems to be natural in the spirit of Section 1.

Addendum. Recently Kirillov's method has been applied for a new class of infinite-dimensional Lie algebras, see [31, 32].

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