Deforming the Lie algebra of vector fields on S^1 inside the Lie algebra of pseudodifferential symbols on S^1

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Abstract

We classify nontrivial deformations of the standard embedding of the Lie algebra $\mathrm{Vect}(S^1)$ of smooth vector fields on the circle, into the Lie algebra $\Psi\mathrm{D}(S^1)$ of pseudodifferential symbols on S^1. This approach leads to deformations of the central charge induced on $\mathrm{Vect}(S^1)$ by the canonical central extension of $\Psi\mathrm{D}(S^1)$. As a result we obtain a quantized version of the second Bernoulli polynomial.

1   Introduction

The classical deformation theory usually deals with formal deformations of associative (and Lie) algebras. Another part of this theory which studies deformations of Lie algebra homomorphisms, is less known. It should be stressed, however, that, sometimes, this second view point is more interesting and leads to richer results. The Lie algebra of vector fields on the circle $\mathrm{Vect}(S^1)$ gives an example when such situation occurs. It is well-known that $\mathrm{Vect}(S^1)$ itself is rigid, but it has many interesting embeddings to other remarkable Lie algebras that can be nontrivially deformed.

The aim of this article is to study the embeddings of $\mathrm{Vect}(S^1)$ into the Lie algebra of pseudodifferential symbols $\Psi\mathrm{D}(S^1)$. We will classify the deformations of the standard embedding which polynomially depend on the parameters of deformation. It turns out that there exists a three-parameter family of nontrivial deformations. We compute a universal explicit formula describing this family.

This work can be considered as the second part of [12] where deformations of $\mathrm{Vect}(S^1)$ inside the Poisson Lie algebra of Laurent series on T^*S^1 has been classified. The latter Lie algebra can be considered as the semi-classical limit of $\Psi\mathrm{D}(S^1)$, so the present article is, in some sense, the quantum version of [12].

It turns out that the space of infinitesimal deformations of $\mathrm{Vect}(S^1)$ inside $\Psi\mathrm{D}(S^1)$ is four-dimensional. The integrability condition, i.e., the condition of existence of a polynomial deformation corresponding to a given infinitesimal one, distinguishes an interesting algebraic surface in the space of parameters.

The Lie algebra of pseudodifferential symbols has a well-known nontrivial central extension $\widehat{\Psi\mathrm{D}(S^1)}$. The restriction of this central extension to the subalgebra $\mathrm{Vect}(S^1) \hookrightarrow \Psi\mathrm{D}(S^1)$ defines the famous central extension of $\mathrm{Vect}(S^1)$ called the Virasoro algebra. As an application of our results, we obtain a "deformed" expression for the Virasoro central charge induced by the deformations of the standard embedding we have constructed. It is given by a quantized version of the second Bernoulli polynomial.

Similar results hold for the semi-direct product $\mathrm{Vect}(S^1) \ltimes C^\infty(S^1)$ by the space of smooth functions on S^1.

Let us also mention that a few examples of non-standard embeddings of $\mathrm{Vect}(S^1)$ to the Lie algebra of differential operators have been considered in [3].

2   Definitions and notations

Let us first recall the definition of the Poisson algebra of Laurent series on S^1 (cf. [12]). Consider the space $\mathbb{C}[\xi, \xi^{-1}]]$ of (formal) Laurent series of finite order in ξ:

$$a(\xi) = \sum_{k \in \mathbb{Z}} a_k \xi^k,$$

with $a_k = 0$ for sufficiently large n. We put

$$\mathcal{A}(S^1) := C^\infty(S^1) \otimes \mathbb{C}[\xi, \xi^{-1}]] \tag{2.1}$$

with its natural multiplication and the Poisson bracket defined as follows. Any element of $\mathcal{A}(S^1)$ can be written in the following form:

$$F(x, \xi) = \sum_{k \in \mathbb{Z}} f_k(x) \xi^k \tag{2.2}$$

where $f_k(x) \in C^\infty(S^1)$, then for $F, G \in \mathcal{A}(S^1)$

$$\{F, G\} = \frac{\partial F}{\partial \xi}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial \xi}.$$

The associative algebra of pseudodifferential symbols $\Psi\mathrm{D}(S^1)$ has the same underlying vector space, but the multiplication is now defined by the following formula

$$(F \circ G)(x, \xi) = \sum_{k=0}^{\infty} \frac{1}{k!} : \frac{\partial^k F}{\partial \xi^k}(x, \xi)\frac{\partial^k G}{\partial x^k}(x, \xi) : \tag{2.3}$$

Here the double comma indicates the ordered product:

$$: f(x)\xi^k g(x)\xi^\ell : = \ f(x)g(x)\xi^{k+\ell}.$$

Remark. This is not, strictly speaking, the ordered product of Wick, but rather a natural generalization of it. Note that, here, variables x and ξ are no longer commutative.

As usual, one can consider this associative algebra $\Psi\mathrm{D}(S^1)$ as a Lie algebra with the commutator

$$[F, G] = F \circ G - G \circ F. \tag{2.4}$$

Definition 2.1. The contraction of the algebra $\Psi\mathrm{D}(S^1)$ will be constructed using the linear isomorphism $\Phi_h : \Psi\mathrm{D}(S^1) \to \Psi\mathrm{D}(S^1)$ defined by

$$\Phi_h(f(x)\xi^k) = f(x)h^k\xi^k, \tag{2.5}$$

where h is some real number belonging to $]0, 1]$. One can modify the multiplication of $\Psi\mathrm{D}(S^1)$ according to the formula

$$F \circ_h G = \Phi_h^{-1}(\Phi_h(F) \circ \Phi_h(G)). \tag{2.6}$$

Let us denote by $\Psi\mathrm{D}_h(S^1)$ the algebra of pseudodifferential symbols equipped with the multiplication $\circ_h$. It is clear that all the associative algebras $\Psi\mathrm{D}_h(S^1)$ are isomorphic to each other, but, for the commutator $[F, G]_h := F \circ_h G - G \circ_h F$, one has:

$$[F, G]_h = h\{F, G\} + hO(h)$$

and, therefore, $\lim_{h\to 0} \frac{1}{h}[F, G]_h = \{F, G\}$. This is what we will mean by contraction: the Lie algebra $\Psi\mathrm{D}(S^1)$ contracts to the Poisson algebra $\mathcal{A}(S^1)$.

This notion of contraction was introduced by Wigner and Inönu and further developed by Levy-Nahas [10].

Furthermore, the Lie algebra $\Psi\mathrm{D}(S^1)$ admits an analogue of the Killing form, known as the Adler trace (cf. [1]).

Definition 2.2. Let $F(x, \xi)$ as given by (2.2) be a pseudodifferential symbol, its residue $\mathrm{Res}\, F$ is just the coefficient $f_{-1}(x) \in C^\infty(S^1)$ and the Adler trace is then defined by

$$\mathrm{Tr}(F) := \int_{S^1} \mathrm{Res}\, F\, dx = \int_{S^1} f_{-1}(x)dx. \tag{2.7}$$

One can readily check that $(F, G) \mapsto \mathrm{Tr}(F \circ G)$ defines a bilinear, invariant, and nondegenerate form.

3 Statement of the problem

The main purpose of this paper is to study deformations the canonical embedding $\pi : \mathrm{Vect}(S^1) \to \Psi\mathrm{D}(S^1)$ defined by

$$\pi(f(x)\partial = f(x)\xi, \tag{3.1}$$

where $\partial = d/dx$, into a (one-)parameter family of Lie algebra homomorphisms $\widetilde{\pi}_t : \mathrm{Vect}(S^1) \to \Psi\mathrm{D}(S^1)[[t]]$, where as usual, the symbol $[[t]]$ indicates formal power series in t.

3.1 Formal deformations

Let us put

$$\widetilde{\pi}_t = \pi + \sum_{k=1}^{\infty} t^k \pi_k, \tag{3.2}$$

where π_k are linear maps $\pi_k : \mathrm{Vect}(S^1) \to \Psi\mathrm{D}(S^1)$ and the condition of homomorphism reads

$$\widetilde{\pi}_t([X,Y]) = [\widetilde{\pi}_t(X), \widetilde{\pi}_t(Y)], \tag{3.3}$$

where the bracket in the right hand side is the natural (formal) extension to $\Psi\mathrm{D}(S^1)[[t]]$ of the Lie bracket of $\Psi\mathrm{D}(S^1)$.

Definition 3.1. Two formal deformations $\widetilde{\pi}_t$ and $\widetilde{\pi}_t'$ are called equivalent when there exists a formal inner automorphism $\mathrm{I}_t : \Psi\mathrm{D}(S^1)[[t]] \to \Psi\mathrm{D}(S^1)[[t]]$ of the form

$$\mathrm{I}_t = \exp(t\,\mathrm{ad}F_1 + t^2\mathrm{ad}F_2 + \cdots) \tag{3.4}$$

with $F_i \in \Psi\mathrm{D}(S^1)$, such that

$$\widetilde{\pi}_t' = \mathrm{I}_t \circ \widetilde{\pi}_t \tag{3.5}$$

3.2 Polynomial deformations

In many cases we will have to consider polynomial deformations instead of formal ones: the formal series can sometimes be cut off, and the formal parameter be replaced by complex coefficients. More precisely, a polynomial deformation has the following form:

$$\widetilde{\pi}(c) = \pi + \sum_{k \in \mathbb{Z}} \widetilde{\pi}_k(c)\xi^k, \tag{3.6}$$

where each linear map $\widetilde{\pi}_k(c) : \mathrm{Vect}(S^1) \to C^\infty(S^1)$ is polynomial in $c \in \mathbb{C}^n$ and satisfy the following two conditions: $\widetilde{\pi}_k(0) = 0$ and $\widetilde{\pi}_k \equiv 0$ for sufficiently large k. It defines a Lie algebra homomorphism $\widetilde{\pi}(c) : \mathrm{Vect}(S^1) \to \Psi\mathrm{D}(S^1)$.

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Definition 3.2. To define the notion of equivalence in the case of polynomial deformations, one simply replaces the formal automorphism I_t in (3.5) by an automorphism $I(c) : \Psi D(S^1) \to \Psi D(S^1)$ depending on $c \in \mathbb{C}^n$ in the following way:

$$I(c) = \exp\left(\sum_{i=1}^{n} c_i \, \mathrm{ad} F_i + \sum_{i,j=1}^{n} c_i c_j \, \mathrm{ad} F_{ij} + \cdots\right), \qquad (3.7)$$

with $F_i, F_{ij}, \ldots, F_{i_1,\ldots,i_k}$ belonging to $\Psi D(S^1)$.

3.3   Contraction of deformations

Given a deformation of one of the types, (3.2) or (3.6), one can modify it so that it becomes a deformation with values in the deformed algebra $\Psi D_h(S^1)$. Namely, if one sets $\pi_t^h = \Phi_h^{-1} \circ \widetilde{\pi}_t$, then one has

$$
\begin{aligned}
\pi_t^h([X,Y]) &= \Phi_h^{-1}([\widetilde{\pi}_t(X), \widetilde{\pi}_t(Y)]) \\
&= \Phi_h^{-1}([\Phi_h \pi_t^h(X), \Phi_h \pi_t^h(Y)]) \\
&= [\pi_t^h(X), \pi_t^h(Y)]_h,
\end{aligned}
$$

and so $\pi_t^h : \mathrm{Vect}(S^1) \to \Psi D_h(S^1)$ is a (formal) deformation of the standard embedding. If one sets

$$\pi_t = \lim_{h \to 0} \pi_t^h,$$

one obtains $\pi_t : \mathrm{Vect}(S^1) \to \mathcal{A}(S^1)$ which is a (formal) deformation of the standard embedding of $\mathrm{Vect}(S^1)$ into $\mathcal{A}(S^1)$ as we considered and classified in [12]. We will say that the deformation $\widetilde{\pi}_t$ contracts to the deformation π_t.

We will classify the deformations of the standard embedding $\mathrm{Vect}(S^1)$ into $\Psi D(S^1)$ and show that they all contract to some deformations of the standard embedding $\mathrm{Vect}(S^1)$ into $\mathcal{A}(S^1)$. This enables us to interpret these deformations as "quantization" of deformations inside $\mathcal{A}(S^1)$. Conversely, the deformations inside $\mathcal{A}(S^1)$ are "semi-classical limits" of deformations inside $\Psi D(S^1)$.

4   Nijenhuis - Richardson theory of deformation

The theory of formal and polynomial deformations of Lie algebra homomorphisms was developed for the first time by Nijenhuis and Richardson [11]; we will make extensive use of their methods and results, and we now recall the main points of the cohomological apparatus for clarity and comfort of the reader.

4.1   Infinitesimal deformations and the first cohomology

Consider the relation (3.3) and take the terms of lower degree. One has

$$\pi_1([X,Y]) = [\pi_1(X), \pi(Y)] + [\pi(X), \pi_1(Y)]. \tag{4.1}$$

In the same way, the relation (3.5) for equivalence implies

$$\pi_1'(X) = \pi_1(X) + [F_1, \pi(X)]. \tag{4.2}$$

Now comes a cohomological interpretation: if $\pi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then $\mathfrak{h}$ is naturally a $\mathfrak{g}$-module through π, and $\pi_1 : \mathfrak{g} \to \mathfrak{h}$ satisfying (4.1) is a 1-cocycle on $\mathfrak{g}$ with values in $\mathfrak{g}$-module $\mathfrak{h}$, so $\pi_1 \in \mathrm{Z}^1(\mathfrak{g}; \mathfrak{h})$. Similarly, the relation (4.2) means that $\pi_1' - \pi_1$ is a coboundary, so for each formal deformation one associates a well-defined cohomology class in $\mathrm{H}^1(\mathfrak{g}; \mathfrak{h})$.

Definition 4.1. A map $\pi + t\pi_1 : \mathfrak{g} \to \mathfrak{h}$ where $\pi_1 \in \mathrm{Z}^1(\mathfrak{g}; \mathfrak{h})$ is then a Lie algebra homomorphism up to quadratic terms in t, it is called an infinitesimal deformation.

The same formalism works in the case of polynomial deformations, one simply has to consider $\frac{d\pi(c)}{dc}|_{c=0}$ and derive the formula expressing $\pi(c)$ as a homomorphism.

4.2   Obstructions

The problem is now to find higher order prolongations of these infinitesimal deformations, there exists an obstruction theory.

One can rewrite the relation (3.3) in the following way:

$$[\widetilde{\pi}_t(X), \pi(Y)] + [\pi(X), \widetilde{\pi}_t(Y)] - \widetilde{\pi}_t([X,Y]) + \sum_{i,j>0} [\pi_i(X), \pi_j(Y)] t^{i+j} = 0.$$

The first three terms read $(d\widetilde{\pi}_t)(X,Y)$ where d stands for the cohomological boundary for cochains on $\mathfrak{g}$ with values in $\mathfrak{h}$. In order to rewrite this expression in a more compact form and to understand its cohomological nature, we will need the following

Definition 4.2. For arbitrary linear maps $\varphi, \varphi' : \mathfrak{g} \to \mathfrak{h}$, one can associate a bilinear map $[\![\varphi, \varphi']\!] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{h}$ in the following way[1]

$$[\![\varphi, \varphi']\!](X,Y) = [\varphi(X), \varphi'(Y)] + [\varphi'(X), \varphi(Y)]. \tag{4.3}$$

The relation (3.3) becomes now equivalent to:

$$d\widetilde{\pi}_t + \frac{1}{2}[\![\widetilde{\pi}_t, \widetilde{\pi}_t]\!] = 0 \tag{4.4}$$

familiar in the deformation theory (see e.g. [9]) as the deformation relation.

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[1]Warning: this bracket is not a Lie algebra bracket but a graded Lie algebra one.

One can now develop (4.4) according to degree in t, and one obtains an infinite series of relations

$$d\pi_k + \sum_{i+j=k} [\![\pi_i, \pi_j]\!] = 0 \tag{4.5}$$

for each $k \geq 1$. These are cohomological relations for the coboundary of π_k, viewed as a 1-cochain on $\mathfrak{g}$ with values in $\mathfrak{h}$.

In particular, the first nontrivial relation gives $d\pi_2 + [\![\pi_1, \pi_1]\!] = 0$ and one gets the first obstruction to integration of an infinitesimal deformation. Indeed, it is quite easy to check that for any two 1-cocycles γ_1 and $\gamma_2 \in \mathrm{Z}^1(\mathfrak{g}; \mathfrak{h})$ the bilinear map $[\![\gamma_1, \gamma_2]\!]$ is a 2-cocycle. The first nontrivial relation (4.5) is precisely the condition for this cocycle to be a coboundary. Moreover, if one of the cocycles γ_1 or γ_2 is a coboundary, the 2-cocycle $[\![\gamma_1, \gamma_2]\!]$ is a 2-coboundary. We, therefore, naturally obtain the following

Definition 4.3. The operation (4.3) defines a bilinear map $\mathrm{H}^1(\mathfrak{g}; \mathfrak{h}) \otimes \mathrm{H}^1(\mathfrak{g}; \mathfrak{h}) \to \mathrm{H}^2(\mathfrak{g}; \mathfrak{h})$ called the cup-product, by analogy with the case of differential forms. This cup-product can be extended to the whole cohomology group $\mathrm{H}^*(\mathfrak{g}; \mathfrak{h})$ (see e.g. [5]).

All the obstructions lie in $\mathrm{H}^2(\mathfrak{g}; \mathfrak{h})$ and they are in the image of $\mathrm{H}^1(\mathfrak{g}; \mathfrak{h})$ through the cup-product.

So, in our case, we have to compute $\mathrm{H}^1(\mathrm{Vect}(S^1); \Psi\mathrm{D}(S^1))$ and the product classes in $\mathrm{H}^2(\mathrm{Vect}(S^1); \Psi\mathrm{D}(S^1))$; our first task will, therefore, be to describe $\Psi\mathrm{D}(S^1)$ as a $\mathrm{Vect}(S^1)$-module.

5 Structure of $\Psi\mathrm{D}(S^1)$ as a $\mathrm{Vect}(S^1)$-module

5.1 Filtration on $\Psi\mathrm{D}(S^1)$

The order of pseudodifferential operators defines a natural filtration on $\Psi\mathrm{D}(S^1)$. Recall, that the order is defined by $\mathrm{ord}(F) = \{\sup k \in \mathbb{Z} \,|\, f_k(x) \not\equiv 0\}$ for every $F(x, \xi) = \sum_{k \in \mathbb{Z}} f_k(x)\xi^k$. One sets

$$\mathrm{F}_n\left(\Psi\mathrm{D}(S^1)\right) = \{F \in \Psi\mathrm{D}(S^1) \,|\, \mathrm{ord}(F) \leq -n\},$$

where $n \in \mathbb{Z}$. Thus, one has a decreasing filtration,

$$\cdots \subset \mathrm{F}_{n+1} \subset \mathrm{F}_n \subset \cdots, \tag{5.1}$$

compatible with multiplication, if $F \in \mathrm{F}_n$ and $G \in \mathrm{F}_m$, then $F \circ G \in \mathrm{F}_{n+m}$, and $\{F, G\} \in \mathrm{F}_{n+m-1}$ (check the symbolic terms!).

This filtration makes $\Psi\mathrm{D}(S^1)$ an associative filtered algebra, one can consider, as usual, the associated graded algebra. Each quotient space $\mathrm{F}_n/\mathrm{F}_{n+1}$ is canonically isomorphic to $C^\infty(S^1)$, any function $f \in C^\infty(S^1)$ induces a pseudodifferential symbol

$f\xi^{-n}$ which has a well-defined image in F_n/F_{n+1}. The associated graded algebra is then

$$\mathrm{Gr}\left(\Psi\mathrm{D}(S^1)\right) = \widetilde{\bigoplus_{n\in\mathbb{Z}}} F_n/F_{n+1}\,, \qquad \text{where} \qquad \widetilde{\bigoplus_{n\in\mathbb{Z}}} = \left(\bigoplus_{n<0}\right) \oplus \left(\prod_{n\geq 0}\right)$$

but the induced multiplication is nothing but the restriction on the symbolic part (i.e. the terms without derivatives in the formula (2.3)) so, it is commutative and one has

$$\mathrm{Gr}\left(\Psi\mathrm{D}(S^1)\right) = \mathcal{A}(S^1) \tag{5.2}$$

as an associative algebra.

One can use this construction to recover the Poisson bracket on $\mathcal{A}(S^1)$ from a purely algebraic version of symbol calculus. Let $\Phi : \Psi\mathrm{D}(S^1) \to \mathcal{A}(S^1)$ be the map which associates to each pseudodifferential symbol its symbolic term (i.e. its principal symbol) and let Ψ be an arbitrary section, that is, $\Psi : \mathcal{A}(S^1) \to \Psi\mathrm{D}(S^1)$ satisfying $\Phi \circ \Psi = \mathrm{Id}_{\mathcal{A}(S^1)}$, one obtains the Poisson bracket on $\mathcal{A}(S^1)$ through the formula

$$\{F,G\} = \Phi\left(\Psi(F) \circ \Psi(G) - \Psi(G) \circ \Psi(F)\right)$$

5.2 Algebra $\mathcal{A}(S^1)$ as a $\mathrm{Vect}(S^1)$-module

The defined filtration is also a filtration of $\Psi\mathrm{D}(S^1)$ as a $\mathrm{Vect}(S^1)$-module (i.e. compatible with the natural action of $\mathrm{Vect}(S^1)$ on $\Psi\mathrm{D}(S^1)$). Indeed, if $X \in \mathrm{Vect}(S^1)$ and $F \in F_n(\Psi\mathrm{D}(S^1))$, then

$$X \cdot F = [X,F] \in F_n(\Psi\mathrm{D}(S^1)).$$

One induces an action of $\mathrm{Vect}(S^1)$ on the associative algebra $\mathrm{Gr}(\Psi\mathrm{D}(S^1)) = \mathcal{A}(S^1)$ and a simple computation shows that we recover the canonical action of $\mathrm{Vect}(S^1)$ on the Poisson algebra $\mathcal{A}(S^1)$ considered in [12].

More explicitly, as a $\mathrm{Vect}(S^1)$-module, $F_n/F_{n+1} = \mathcal{F}_{-n}$, where $\mathcal{F}_{-n}$ is the space of tensor densities of degree n on S^1:

$$\mathcal{F}_{-n} = \left\{a(x)dx^n \,|\, a \in C^\infty(S^1)\right\}$$

and the action of $\mathrm{Vect}(S^1)$ reads

$$f(x)\partial \cdot a(x)dx^n = \left(f(x)a'(x) + nf'(x)a(x)\right)dx^n. \tag{5.3}$$

Note, that the expression in the right hand side is just the standard Lie derivative of a tensor density along a vector field. So, finally,

$$\mathcal{A}(S^1) = \widetilde{\bigoplus_{n\in\mathbb{Z}}} \mathcal{F}_{-n} \tag{5.4}$$

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as a $\text{Vect}(S^1)$-module (cf. [12], Lemma 4.1).

The cohomology of $\text{Vect}(S^1)$ with coefficients in $\mathcal{A}(S^1)$ then follows from determination of $\text{H}^*(\text{Vect}(S^1); \mathcal{F}_{-n})$ by D.B. Fuchs [4], we shall recall later the necessary results. The important fact here is that one can deduce the cohomology for the filtered module from the cohomology of the associated graded module, as we shall see now.

6 Cohomology of $\text{Vect}(S^1)$ with coefficients in the module of pseudodifferential operators

In this section we compute the first cohomology group $\text{H}^1(\text{Vect}(S^1); \Psi\text{D}(S^1))$. The Nijenhuis - Richardson theory implies (cf. Section 4.1) that this is equivalent to classification of infinitesimal deformations of the embedding (3.1).

6.1 The spectral sequence for a filtered module over a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra and M a filtered module with decreasing filtration $\{\text{M}_n\}_{n \in \mathbb{Z}}$ so that $\text{M}_{n+1} \subset \text{M}_n$, $\bigcup_{n \in \mathbb{Z}} \text{M}_n = \text{M}$ and $\mathfrak{g} \cdot \text{M}_n \subset \text{M}_n$. Let

$$\text{Gr}(\text{M}) = \widetilde{\bigoplus}_{n \in \mathbb{Z}} \text{M}_n/\text{M}_{n+1}$$

be the associated graded $\mathfrak{g}$-module. One can then naturally construct a filtration on the space of cochains by setting $F_n(\text{C}^*(\mathfrak{g}; \text{M})) = \text{C}^*(\mathfrak{g}; \text{M}_n)$, this filtration being obviously compatible with the Chevalley-Eilenberg differential. So, the corresponding spectral sequence satisfies:

$$\text{E}_0^{p,q} = F_p(\text{C}^{p+q}(\mathfrak{g}; \text{M}))/F_{p-1}(\text{C}^{p+q}(\mathfrak{g}; \text{M})) \tag{6.1}$$

and one has the following

Proposition 6.1. *Let* $\mathfrak{g}$ *be a Lie algebra and* M *a graded* $\mathfrak{g}$-*module as above, then one has a spectral sequence such that*

$$\text{E}_1^{p,q} = \text{H}^{p+q}(\mathfrak{g}; \text{Gr}^p(\text{M}))$$

converging to $\text{H}^*(\mathfrak{g}; \text{M})$. *It induces on* $\text{H}^n(\mathfrak{g}; \text{M})$ *a filtration coming from all the spaces* $\text{E}_\infty^{p,q}$ *such that* $p + q = n$.

Proof. This is a simple adaptation of the usual construction of the spectral sequence of a filtered complex. The reader can refer to any classical textbook on homological algebra, for example [5]. $\square$

We will use the constructed spectral sequence to compute $\text{H}^1(\text{Vect}(S^1); \Psi\text{D}(S^1))$ in Section 6.3 below.

6.2 $\mathrm{Vect}(S^1)$-cohomology with coefficients in $\mathcal{F}_{-k}$

In our case we can use the results of D.B. Fuchs to determine the cohomology with coefficients in the graded module, namely $\mathrm{H}^k(\mathrm{Vect}(S^1); \mathrm{Gr}^p(\Psi\mathrm{D}(S^1)))$ for $k = 1, 2$. One has:

$$\mathrm{H}^1(\mathrm{Vect}(S^1); \mathcal{F}_{-k}) = \begin{cases} \mathbb{R}^2, & k = 0 \\ \mathbb{R}, & k = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

represented by the cocycles

$$\bar{c}_0(f\partial) = f, \quad c_0(f\partial) = f', \quad c_1(f\partial) = f''dx, \quad c_2(f\partial) = f'''dx^2. \qquad (6.2)$$

Now, the isomorphisms (5.2) and (5.4) imply:

$$\mathrm{H}^1(\mathrm{Vect}(S^1); \mathrm{Gr}^p(\Psi\mathrm{D}(S^1))) = \mathbb{R}^4 \qquad (6.3)$$

For the second cohomology one has:

$$\mathrm{H}^2(\mathrm{Vect}(S^1); \mathcal{F}_{-k}) = \begin{cases} \mathbb{R}^2, & k = 0, 1, 2 \\ \mathbb{R}, & k = 5, 7 \\ 0, & \text{otherwise} \end{cases}$$

see [13] for explicit formulæ of generators. Therefore $\mathrm{H}^2(\mathrm{Vect}(S^1); \mathrm{Gr}^p(\Psi\mathrm{D}(S^1)))$ is eight-dimensional.

6.3   Computing $\mathrm{H}^1(\mathrm{Vect}(S^1); \Psi\mathrm{D}(S^1))$

One has to check now the behavior of the above cocycles through the successive differentials of the spectral sequence. Cocycles $\bar{c}_0, c_0, c_1$ and c_2 belong to $\mathrm{E}_1^{0,1}, \mathrm{E}_1^{0,1}, \mathrm{E}_1^{1,0}$ and $\mathrm{E}_1^{2,-1}$, respectively. The differential d_1 is

$$d_1 : \mathrm{E}_1^{p,q} \to \mathrm{E}_1^{p+1,q}.$$

One can check it in the following way: consider a cocycle with values in $\mathcal{A}(S^1)$, but compute its boundary as if it was with values in $\Psi\mathrm{D}(S^1)$ and keep the symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one, and its class will represent its image by d_1. The higher order differentials

$$d_r : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1} \qquad (6.4)$$

can be constructed by iteration of this procedure, the space $\mathrm{E}_r^{p+r,q-r+1}$ contains the subspace coming from $\mathrm{H}^{p+q+1}(\mathfrak{g}; \mathrm{Gr}^{p+1}(\mathrm{M}))$.

It is now easy to see that the cocycles $\bar{c}_0$ and c_0 will survive in the same form, we will denote them θ_0 and θ_1 when seen as cocycles with values in $\Psi\mathrm{D}(S^1)$.

For extension of the cocycles c_1 and c_2 to the algebra $\Psi\mathrm{D}(S^1)$, potential obstructions may come from $\mathrm{H}^2(\mathrm{Vect}(S^1); \mathcal{F}_\lambda)$ for $\lambda \leq 7$, so we must check the terms

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$E_r^{1,0} \to E_r^{1+r,1-r}$ for $r \le 6$ and $E_r^{2,-1} \to E_r^{2+r,-r}$ for $r \le 5$. A lengthy but straightforward calculations show that both classes survive in $H^1(\mathrm{Vect}(S^1); \Psi D(S^1))$; explicitly, every potential obstruction turns out to be trivial in cohomology, so one can always find the supplementary higher order terms to obtain two cocycles θ_2 and θ_3 coming from c_1 and c_2 respectively. So, we have obtained

Proposition 6.2. *The first cohomology group* $H^1(\mathrm{Vect}(S^1); \Psi D(S^1))$ *is four-dimensional and generated by the classes of cocycles* $\theta_0, \theta_1, \theta_2$ *and* θ_3 *defined above.*

Another way to obtain this result is to find explicit expressions for the nontrivial cocycles, since the cohomology group $H^1(\mathrm{Vect}(S^1); \Psi D(S^1))$ is obviously upper-bounded by $H^1(\mathrm{Vect}(S^1); \mathcal{A}(S^1))$.

6.4 Explicit formulæ for the $\mathrm{Vect}(S^1)$-cocycles

We have already seen, the cocycles t_0 and t_1 are given by the same formulæ as $\bar{c}_0$ and c_0, namely,

$$\theta_0(f\partial) = f, \quad \theta_1(f\partial) = f'. \tag{6.5}$$

One can find the explicit formulæ for θ_2 and θ_3

$$\theta_2(f(x)\partial) \;=\; \sum_{n=2}^{\infty} (-1)^{n-1} \frac{2(n-3)}{n} f^{(n)}(x)\xi^{-n+1}, \tag{6.6}$$

$$\theta_3(f(x)\partial) \;=\; \sum_{n=2}^{\infty} (-1)^{n} \frac{3(n-1)}{n+1} f^{(n+1)}(x)\xi^{-n}. \tag{6.7}$$

Remark 6.3. The symbolic terms of $\theta_1(f\partial)$ and $\theta_2(f\partial)$ are $f''\xi^{-1}$ and $f'''\xi^{-2}$ respectively. Recall, that $f\partial \mapsto f'''\xi^{-2}$ is the Souriau cocycle of the Virasoro algebra since $\mathcal{F}_2$ is the regular dual of $\mathrm{Vect}(S^1)$ (see[8] and also [6]). This class can be interpreted as infinitesimal version of the well-known Schwarzian derivative.

6.5 Relations with outer derivations of the Lie algebra $\Psi D(S^1)$

The inclusion map $\mathrm{Vect}(S^1) \hookrightarrow \Psi D(S^1)$ induces the following map in cohomology: $H^1(\Psi D(S^1); \Psi D(S^1)) \to H^1(\mathrm{Vect}(S^1); \Psi D(S^1))$. The space $H^1(\Psi D(S^1); \Psi D(S^1))$ of outer derivations of the Lie algebra $\Psi D(S^1)$ has been determined by O. Kravchenko and B. Khesin [7]: it is two-dimensional and generated by the linear operators on $\Psi D(S^1)$ denoted by $\mathrm{ad}(x)$ and $\mathrm{ad}(\log \xi)$. They are defined by

$$[\mathrm{ad}(x), f(x)\xi^p] \;:=\; -pf(x)\xi^{p-1}, \tag{6.8}$$

$$[\mathrm{ad}(\log \xi), f(x)\xi^p] \;:=\; \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} f^{(n)}(x)\xi^{-n+p}. \tag{6.9}$$

The restriction to $\mathrm{Vect}(S^1)$ gives

$$[\mathrm{ad}(x), f(x)\xi] \;=\; -f(x),$$

$$[\mathrm{ad}(\log\xi), f(x)\xi] \;=\; f'(x) - \frac{f''(x)}{2}\xi^{-1} + \frac{f'''(x)}{3}\xi^{-2} + \cdots$$

$$\;=\; \left(\theta_1 - \frac{\theta_2}{2} + \frac{\theta_3}{3}\right)(f(x)\partial).$$

So, the image of $\mathrm{H}^1(\Psi\mathrm{D}(S^1); \Psi\mathrm{D}(S^1))$ in $\mathrm{H}^1(\mathrm{Vect}(S^1); \Psi\mathrm{D}(S^1))$ is two-dimensional.

7 Integrability of infinitesimal deformations

Proposition 6.2 gives us all the infinitesimal deformations of the standard embedding of $\mathrm{Vect}(S^1)$ into $\Psi\mathrm{D}(S^1)$ given by (3.1). In this section we will calculate the condition under which an infinitesimal deformation corresponds to a polynomial one.

7.1 Nontrivial deformation generated by cocycles θ_0 and θ_1

Following the Richardson-Nijenhuis theory, one has to determine the cup-products $[\![\theta_i, \theta_j]\!]$ for $i, j = 0, 1, 2, 3$ of the cocycles (6.5-6.7) in $\mathrm{H}^2(\mathrm{Vect}(S^1); \Psi\mathrm{D}(S^1))$.

First of all, since zero-order operators commute in $\Psi\mathrm{D}(S^1))$, it is evident that the the cup-products $[\![\theta_0, \theta_0]\!]$, $[\![\theta_0, \theta_1]\!]$ and $[\![\theta_1, \theta_1]\!]$ vanish identically (and not only in cohomology). Therefore, the map

$$f(x)\partial \mapsto f(x)\xi + \nu f(x) + \lambda f'(x) \tag{7.1}$$

is, indeed, a nontrivial deformation of the standard embedding. This deformation is, in fact, both formal and polynomial, since it is of degree one.

Real difficulties begin when we deal with integrability of the infinitesimal deformations corresponding to the cocycles θ_1, θ_2 and θ_3 into polynomial or formal ones.

7.2 The integrability condition

Consider an infinitesimal deformation of the standard embedding of $\mathrm{Vect}(S^1)$ into $\Psi\mathrm{D}(S^1)$ defined by the cocycles $\theta_0, \theta_1, \theta_2, \theta_3$ and depending on three complex parameters c_0, c_1, c_2, c_3:

$$\widetilde{\pi}(c)(f\partial) = f\xi + c_0\theta_0(f\partial) + c_1\theta_1(f\partial) + c_2\theta_2(f\partial) + c_3\theta_3(f\partial). \tag{7.2}$$

Theorem 7.1. *The infinitesimal deformation (7.2) corresponds to a polynomial deformation, if and only if the following (quartic) relation is satisfied:*

$$6c_1^3c_3 - 3c_1^2c_2^2 - 18c_1c_2c_3 + 8c_2^3 + 9c_3^2 + 3c_1^3c_2 - 6c_1c_2^2$$

$$-9c_1^2c_3 + 18c_2c_3 + 2c_1c_3 - 5c_1^2c_2 + 8c_2^2 + 2c_1c_2 = 0 \tag{7.3}$$

Let us first prove that the condition (7.3) is necessary for integrability of infinitesimal deformations; we will show in Section 8 that this condition is, sufficient by exhibiting explicit deformations.

Proof. The infinitesimal deformation (7.2) is clearly of the form:

$$\widetilde{\pi}(c)(f\partial) = f\xi + c_0 f + c_1 f' + c_2 f''\xi^{-1} + c_3 f'''\xi^{-2} + \cdots, \tag{7.4}$$

where "$\cdots$" means the terms in ξ^{-3}, ξ^{-4}. To compute the obstructions for integration of the infinitesimal deformation (7.2), one has to add the first nontrivial terms and impose the homomorphism condition. So put

$$\overline{\pi}(c)(f\partial) = \pi(c)(f\partial) + P_4(c)f^{(IV)}\xi^{-3} + P_5(c)f^{(V)}\xi^{-4},$$

where $P_4(c)$ and $P_5(c)$ are some polynomials in $c = (c_0, c_1, c_2, c_3)$ and compute the difference $[\overline{\pi}(c)(f\partial), \overline{\pi}(c)(g\partial)] - \overline{\pi}(c)([f\partial, g\partial])$. Collecting the terms in ξ^{-3} and ξ^{-4} yields:

$$2P_4(c) = 2c_1 c_3 - c_2^2 + c_1 c_2 - 3c_3 - c_2, \tag{7.5}$$

$$5P_5(c) = -c_2 c_3 + 3c_1 P_4(c) + c_2^2 - 6P_4(c) - c_1 c_2 + c_2. \tag{7.6}$$

Note that these expressions do not contain c_0.

Let us go one step further, expand our deformation up to ξ^{-6}, that is, put $\overline{\overline{\pi}}(c)(f\partial) = \overline{\pi}(c)(f\partial) + P_6(c)f^{(VI)}\xi^{-5}$, the homomorphism condition leads to a non-trivial relation for the parameters:

$$5(-c_2 P_4 + 4c_1 P_5 + c_2 c_3 - 10P_5 - c_2^2 + c_1 c_2 - c_2)$$
$$= 9\left(-2c_3^2 + 3c_2 P_4 + 3c_2 c_3 - 6c_1 P_4 - 4c_1^2 c_3 + 10P_4 + 5c_3\right)$$

Substituting the expressions (7.5) and (7.6) for P_4 and P_4, one gets the formula (7.3). We have thus shown that this condition is necessary for integrability of infinitesimal deformations. $\qquad\square$

Remark 7.2. In a cohomological interpretation, the obstructions to integrability of an infinitesimal deformation (7.2) which does not satisfy the condition (7.3), corresponds to a nontrivial class of $\mathrm{H}^2(\mathrm{Vect}(S^1); \mathcal{F}_5)$.

7.3 Introducing the parameter h

One can now modify the relation in order to get a deformation in $\Psi\mathrm{D}_h(S^1)$, the scalar h then appears with different powers according to the "weight" of the successive terms in the formula (7.3). One gets finally:

$$\begin{aligned}
& 6c_1^3 c_3 - 3c_1^2 c_2^2 - 18c_1 c_2 c_3 + 8c_2^3 + 9c_3^2 \\
+\ & h(3c_1^3 c_2 - 6c_1 c_2^2 - 9c_1^2 c_3 + 18c_2 c_3) \\
+\ & h^2(2c_1 c_3 - 5c_1^2 c_2 + 8c_2^2) \\
+\ & h^3(2c_1 c_2) = 0
\end{aligned} \tag{7.7}$$

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This relation is a necessary integrability condition for infinitesimal deformations (7.2) in $\Psi D_h(S^1)$.

Remark 7.3. Let us remark that by setting: weight$(c_i) = i$ (for $i = 1, 2, 3$) and weight$(h) = 1$, this polynomial is homogeneous of weight 6. Moreover, put $h = 0$, one gets the polynomial we have obtained in the semi-classical (Poisson) case (cf. [12], Theorem 5.1).

We will now give two natural descriptions of the surface defined by the equation (7.7) in order to unveil its algebraic nature.

7.4   Cubic curve in the space of parameters

Let us change the parameters c_1, c_2 and c_3 and rewrite the expression (7.7) in a canonical form. One checks by an elementary straightforward calculations the

Proposition 7.4. *The relation (7.7) is equivalent to the following cubic*

$$Y^2 = X^3 + \frac{h^2}{4} X^2, \tag{7.8}$$

where the new parameters X and Y are given by

$$
\begin{aligned}
X &= c_1^2 - 2c_2 - hc_1, & (7.9) \\
Y &= c_1^3 - 3(c_1 c_2 - c_3) - \tfrac{3}{2}h(c_1^2 - 2c_2) + \tfrac{1}{2}h^2 c_1, & (7.10)
\end{aligned}
$$

One thus obtains a cubic curve on the plane (X, Y). This curve contracts for $h \to 0$ to the semi-cubic parabola found in [12].

7.5   Rational parameterization

There exists another, parametric, way to define the surface (7.7).

Proposition 7.5. *(i) For all λ and $\mu \in \mathbb{C}$, the constants*

$$
\begin{aligned}
c_1 &= \lambda + \mu & (7.11) \\
c_2 &= \lambda\mu + \frac{\lambda(\lambda - h)}{2} & (7.12) \\
c_3 &= \frac{\lambda\mu(\lambda - h)}{2} + \frac{\lambda(\lambda - h)(\lambda - 2h)}{6} & (7.13)
\end{aligned}
$$

satisfy the relation (7.7).
(ii) Any triple $c_1, c_2, c_3 \in \mathbb{C}$ satisfying the relation (7.7) is of the form (7.11-7.13) for some $\lambda, \mu \in \mathbb{C}$.

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Proof. (i) The first statement can be easily checked directly.

(ii) To prove the converse statement, fix c_1 and c_2 as in (7.11) and (7.12) respectively and solve (7.7) as a quadratic equation with c_3 indeterminate. Then verify, that one of the solutions of this equation coincides with (7.13) and the other one is as follows:

$$c_3 = \frac{\lambda\mu(\lambda - h)}{2} + \frac{\lambda(\lambda - h)(\lambda - 2h)}{6} - \frac{\mu(\mu - h)(2\mu - h)}{3}.$$

But, this second solution also coincides with (7.13) after the involution

$$(\lambda, \mu) \mapsto (\lambda + 2\mu - h, -\mu + h) \qquad (7.14)$$

which preserves the expressions (7.11) and (7.12). $\qquad\qquad\square$

Remark 7.6. We have shown that λ and μ parameterize the surface (7.7); note finally, that X and Y defined by (7.9) and (7.10) are of the form:

$$
\begin{aligned}
X &= \mu^2 - h\mu, \\
Y &= \mu^3 - \tfrac{3}{2}h\mu^2 + \tfrac{1}{2}h^2\mu,
\end{aligned}
$$

and so μ is a parameter on the curve (7.8).

Now, the Richardson-Nijenhuis theory prescribes us to compute the second cohomology group $\mathrm{H}^2(\mathrm{Vect}(S^1); \Psi\mathrm{D}(S^1))$ in order to obtain the complete information concerning the cohomological obstructions. This, however, seems to be a quite difficult problem. If one tries to compute $\mathrm{H}^2(\mathrm{Vect}(S^1); \Psi\mathrm{D}(S^1))$ using the spectral sequence, just as we did it in Section 6.3 for the first cohomology group, one has to check $\mathrm{H}^3(\mathrm{Vect}(S^1); \mathcal{F}_n)$ for $n \leq 15$ (see [4], p.176) in order to decide whether the nontrivial classes from $\mathrm{H}^2(\mathrm{Vect}(S^1); \mathcal{A}(S^1))$ will survive or not. We shall not do that; an explicit construction of polynomial deformations will allow us to avoid the standard framework.

8 The universal formula for nontrivial deformations

In this section we will give explicit formulæ of genuine polynomial deformations of the standard embedding (3.1) for all infinitesimal deformations satisfying the condition (7.7). According to the Richardson-Nijenhuis theory, any polynomial deformation is equivalent to a deformation from this class.

8.1 Construction of deformations

Outer derivations (6.8) and (6.9) can be integrated in one-parameter families of outer automorphisms denoted by Φ_ν and Ψ_μ respectively, and defined by

$$\Phi_\nu(F) = e^{x\nu} \circ F \circ e^{-x\nu} \tag{8.1}$$

$$\Psi_\mu(F) = \xi^\mu \circ F \circ \xi^{-\mu}. \tag{8.2}$$

Note that these formulæ should be understood as Laurent series in ξ (depending on the parameters ν and μ respectively); one computes the product in (8.1) and (8.2) formally, using the composition formula (2.3), as if $e^{x\nu}$ and ξ^μ were elements of $\Psi D(S^1)$. One obtains, in particular:

$$\Phi_\nu(f\xi) = f\xi + \nu f \qquad \text{and} \qquad \Psi_\mu(f\xi) = \sum_{n=0}^{\infty} \binom{\mu}{n} f^{(n)} \xi^{-n+1}.$$

Let us apply the automorphism (8.2) to the elementary deformation (7.1):

$$\begin{aligned}
\widetilde{\pi}_{\lambda,\mu}(f\partial) &= \Psi_\mu(f\xi + \lambda f') \\
&= f\xi + (\lambda + \mu)f' \\
&\quad + \left(\lambda\mu + \frac{\lambda(\lambda - h)}{2} \right) f'' \xi^{-1} \\
&\quad + \left(\frac{\lambda\mu(\lambda - h)}{2} + \frac{\lambda(\lambda - h)(\lambda - 2h)}{6} \right) f''' \xi^{-2} \\
&\quad + \cdots
\end{aligned} \tag{8.3}$$

Since Ψ_μ is an automorphism, it is, indeed, a polynomial deformation of the embedding (3.1) for any λ and $\mu \in \mathbb{C}$. In the same way, one also obtains a more general class of polynomial deformations:

$$\widetilde{\pi}_{\lambda,\mu,\nu}(f\partial) = \Psi_\mu \left(f\xi + \nu f + \lambda f' \right) = (\Psi_\mu \circ \Phi_\nu)(f\xi + \lambda f'). \tag{8.4}$$

Lemma 8.1. *Every infinitesimal deformation (7.2) satisfying the condition (7.7) can be realized as the infinitesimal part of the polynomial deformation $\widetilde{\pi}_{\lambda,\mu,\nu}$ for some λ, μ and $\nu \in \mathbb{C}$.*

Proof. The first coefficients in (8.3) coincide with (7.11-7.13). The formula (7.4) shows then that the parameters c_1, c_2, c_3 of the infinitesimal part of (8.3) (and (8.4)) are precisely given by these expressions. Now, Proposition 7.5 implies that any infinitesimal deformation (7.2) satisfying the condition (7.7) is the infinitesimal part of one of the deformations (8.4). $\square$

Remark 8.2. (a) The constructed deformations are not equivalent to each other for different values of the parameters λ, μ and ν, since the corresponding infinitesimal deformations are given by non-cohomological cocycles. This is due to the fact that Φ_ν and Ψ_μ are outer automorphisms.

(b) After the contraction procedure (for $h \to 0$) applied to the deformations (8.3,8.4) one gets precisely the deformations, found in the semi-classical case in [12] (see Theorem 5.1).

8.2 Proof of Theorem 7.1

We have constructed a polynomial deformation corresponding to any infinitesimal deformation satisfying the condition (7.3) (or (7.7)). This implies that the condition (7.3,7.7) is not only necessary, but also sufficient for integrability of infinitesimal deformations and completes the proof of Theorem 7.1.

9 Variation of the Virasoro central charge

The space $\mathrm{H}^2(\Psi\mathrm{D}(S^1); \mathbb{C})$, classifying central extensions of the Lie algebra $\Psi\mathrm{D}(S^1)$ has been determined in [7] (see also [14]). It is two-dimensional, every outer derivation δ_i (for $i = 1, 2$) given by the formulæ (6.8) and (6.9) defines a 2-cocycle with scalar values through the formula

$$c_i(F, G) = \int_{S^1} \mathrm{Res}(\delta_i(F) \circ G). \tag{9.1}$$

It is well known that the space $\mathrm{H}^2(\mathrm{Vect}(S^1); \mathbb{C})$ is one-dimensional and generated by the famous Gelfand-Fuchs cocycle, that we will define by the formula

$$c(f\partial, g\partial) = \frac{1}{12} \int_{S^1} f'''g \, dx. \tag{9.2}$$

Remark 9.1. The normalization term $1/12$ is traditional: it is very natural because of the values of central charges one encounters among the Virasoro algebra representations.

We will consider the cocycle $c_1 \in \mathrm{Z}^2(\Psi\mathrm{D}(S^1); \mathbb{C})$, associated with $\delta_1 = \log \xi$. A lengthy, but easy computation then proves the following

Proposition 9.2. *The restriction of the cocycle c_1 to $\mathrm{Vect}(S^1) \hookrightarrow \Psi\mathrm{D}(S^1)$ with respect to the embedding (8.4) is $\widetilde{\pi}^*_{\lambda,\mu,\nu}(c_1) = (-12\lambda^2 + 12\lambda - 2)\, c$, and in the case of h-deformed algebra $\Psi\mathrm{D}_h(S^1)$:*

$$\widetilde{\pi}^*_{\lambda,\mu,\nu}(c_1) = (-12\lambda^2 + 12h\lambda + 4 - 6h)\, c, \tag{9.3}$$

Remark 9.3. One then recovers for $h = 1$ the famous (second) Bernoulli polynomial which appears in central charges of some representations of the Virasoro algebra, as well as in the context of moduli spaces (see [2]). The formula (9.3) is, therefore, its "quantized" version.

10 Deformations of the embedding of $\mathrm{Vect}(S^1) \ltimes \mathcal{N}$

Denote $\mathcal{N}$ the space of C^∞-functions on S^1. The semi-direct product $\mathrm{Vect}(S^1) \ltimes \mathcal{N}$ is then the Lie algebra of differential operators of order smaller or equal to 1. One can look for deformations of the embedding of this Lie algebra into $\Psi\mathrm{D}(S^1)$.

Tedious, but without serious difficulties, computations allow to prove that every deformation of the canonical embedding can be written in the form

$$\widehat{\pi}_{\lambda,\mu,\nu}\Big(f(x)\partial + a(x)\Big) = f(x)\xi + \nu f(x) + \lambda f'(x) + (\mu+1)a(x). \tag{10.1}$$

One can now compute the induced map on second cohomology.

As above, the space $\mathrm{H}^2(\mathrm{Vect}(S^1) \ltimes \mathcal{N}; \mathbb{C})$ is well known (see [2]) and admits three generators: the first one is the Gelfand-Fuchs cocycle still denoted c, and the following two cocycles:

$$\widetilde{c} = \int_{S^1} (f''b - g''a)dx, \tag{10.2}$$

$$\widetilde{\widetilde{c}} = \int_{S^1} (a'b - ab')dx. \tag{10.3}$$

One has the following

Proposition 10.1. *The restriction of the cocycle c_1 to $\mathrm{Vect}(S^1) \ltimes \mathcal{N} \hookrightarrow \Psi\mathrm{D}(S^1)$ with respect to the embedding (10.1) is as follows*

$$\widehat{\pi}^*_{\lambda,\mu,\nu}(c_1) = (-12\lambda^2 + 12\lambda - 2)\,c + (\nu+1)(\lambda - \frac{1}{2})\,\widetilde{c} + \frac{(\nu+1)^2}{2}\,\widetilde{\widetilde{c}}$$

and in the case of h-deformed algebra $\Psi\mathrm{D}_h(S^1)$:

$$\widehat{\pi}^{h\,*}_{\lambda,\mu,\nu}(c_1) = (-12\lambda^2 + 12h\lambda + 4 - 6h)\,c + (\nu+1)(\lambda - \frac{h}{2})\,\widetilde{c} + \frac{(\nu+1)^2}{2}\,\widetilde{\widetilde{c}}$$

Remark 10.2. The coefficient $\lambda - \frac{1}{2}$ is also a Bernoulli polynomial, the first one. In terms of W-algebras, we computed central charges of W_1 and W_2.

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