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Sturm Theory, Ghys Theorem on Zeroes of the Schwarzian Derivative and Flattening of Legendrian Curves

V. Ovsienko and S. Tabachnikov†

Etienne Ghys has recently discovered a beautiful theorem: *given a diffeomorphism of the projective line, there exist at least four distinct points in which the diffeomorphism is unusually well approximated by projective transformations* [8]. The points in question are the ones in which the 3-jet of the diffeomorphism is that of a projective transformation; in a generic point the order of approximation is only 2. In other words, the Schwarzian derivative of every diffeomorphism of $\mathbf{RP}^1$ has at least four distinct zeroes.

The theorem of Ghys is analogous to the classical four vertex theorem: a closed convex plane curve has at least four curvature extrema [11]. The proof presented by E. Ghys was purely geometrical: it was inspired by the Kneser theorem on oscullating circles of a plane curve.

We will prove an amazing strengthening of the classical Sturm comparison theorem and deduce Ghys' theorem from it.

Sturm theory is related to the geometry of curves (see [2], [3], [4], [5], [9], [13], [14]).

This relation is based on the following general idea. A geometrical problem is reformulated as the problem on the least number of zeroes of a certain function; this function turns out to be orthogonal to a number of functions enjoying a disconjugacy property. This implies an estimate below on the number of zeroes.

An example of such an estimate is the Hurwitz theorem: *the number of zeroes of a function on the circle is minorated by that of its first harmonic* [10].

We will show that the Ghys theorem is equivalent to the following result.

Consider linear symplectic $(\mathbf{R}^4, \omega)$, choose two orthogonal (with respect to the symplectic structure) symplectic subspaces $\mathbf{R}^2_1$ and $\mathbf{R}^2_2$. Denote by $\mathbf{RP}^1_1$ and $\mathbf{RP}^1_2$ their projectivizations in the contact $\mathbf{RP}^3$.

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Theorem 1. *Let $C \subset \mathbf{RP}^3$ be a closed Legendrian curve such that its projection on $\mathbf{RP}^1_1$ from $\mathbf{R}P^1_2$ and its projection on $\mathbf{RP}^1_2$ from $\mathbf{RP}^1_1$ are diffeomorphisms. Then C has at least 4 flattening points.*

1. Symplectization of the projective line diffeomorphism and the Sturm–Liouville equation

The projective line $\mathbf{RP}^1$ is the space of lines through the origin in the plane $\mathbf{R}^2$. A linear transformation of the plane induces a transformation of $\mathbf{RP}^1$ called a projective transformation. Projective transformations form a group PSL_2.

Given a diffeomorphism $f : \mathbf{RP}^1 \to \mathbf{RP}^1$ and a point x of $\mathbf{RP}^1$, there exists a unique projective transformation $g \in PSL_2$ that approximates f in x up to the second order. That is, $j^2_x(f) = j^2_x(g)$. We are interested in the points in which $j^3_x(f) = j^3_x(g)$. Call such points *projective points* of the diffeomorphism f. Without loss of generality assume that f preserves the orientation.

Given an orientation preserving diffeomorphism $f : \mathbf{RP}^1 \to \mathbf{RP}^1$, there exists a unique area preserving homogeneous (of degree 1) diffeomorphism F of the punctured plane $\mathbf{R}^2 \setminus \{0\}$ that projects to f. Let α be the angular parameter on $\mathbf{RP}^1$ so that α and $\alpha + \pi$ correspond to the same point. Denote by $f(\alpha)$ the lift of the diffeomorphism f to a diffeomorphism of the line such that $f(\alpha + \pi) = f(\alpha) + \pi$. The diffeomorphism F is given in polar coordinates by

$$F : (\alpha, r) \mapsto (f(\alpha),\ r\dot{f}^{-1/2}(\alpha))$$

where the dot denotes $d/d\alpha$. Projective points of the diffeomorphism f give rise to lines along which the symplectomorphism F is unusually well (up to the second jet) approximated by a linear area preserving transformation of $\mathbf{R}^2$.

Let $\gamma(\alpha)$ be the image of the unit circle under F. It is a centrally symmetric curve: $\gamma(\alpha+\pi) = -\gamma(\alpha)$, and it bounds area π. If $F \in SL_2$ then the corresponding curve is a central ellipse of area π. Thus for every point $x \in \gamma$, there exists a unique central ellipse of area π tangent to γ in x. The projective points of the diffeomorphism f correspond to the points of γ in which the tangent ellipse has the second order contact with γ.

Lemma 1.1. *The parameterized curve $\gamma(\alpha)$ satisfies the equation*

$$\ddot{\gamma}(\alpha) = -k(\alpha)\gamma \tag{1}$$

where $k(\alpha)$ is a π-periodic smooth function.

Proof. Let $\gamma_0(\alpha)$ be the parameterized unit circle. Then $[\gamma_0, \dot{\gamma}_0] = 1$ where $[\ ,\ ]$ is the oriented area of the parallelogram generated by two vectors. Applying the symplectomorphism F one obtains $[\gamma, \dot{\gamma}] = 1$. Differentiate to obtain $[\gamma, \ddot{\gamma}] = 0$. Hence, the vector $\ddot{\gamma}$ is collinear to γ. The result follows.

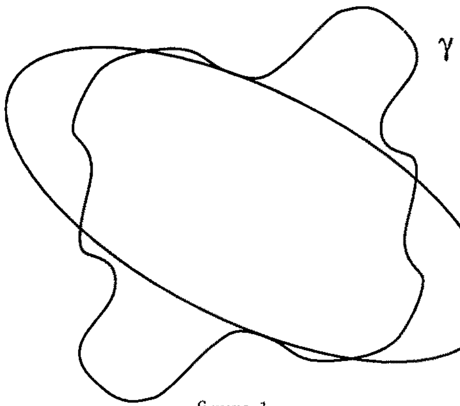


figure 1

Lemma 1.2. *The projective points of the diffeomorphism f correspond to the points in which $k(\alpha) = 1$.*

Proof. If $F \in SL_2$ the curve γ is a central ellipse of area π. In this case $k(\alpha) \equiv 1$. Projective points are points of second order contact with central ellipses of area π. Thus $k(\alpha) = 1$ in these points.

In view of the above lemmas it suffices to prove that $k(\alpha) - 1$ has at least 4 distinct zeroes on the interval $[0, \pi]$.

2. Strengthened Sturm comparison theorem

The Sturm comparison theorem is stated as follows. *Given two Sturm–Liouville equations $\ddot{\phi}(\alpha) = -k(\alpha)\phi(\alpha)$ with potentials $k_1(\alpha) > k_2(\alpha)$, let ϕ_1 and ϕ_2 be solutions of the respective equations. Then between any two zeros of ϕ_2 there exists a zero of ϕ_1* [12]. Equivalently, if ϕ_1 and ϕ_2 have two coinciding consecutive zeroes (see figure 2), then there is a zero of the function $k_1 - k_2$ on this interval.

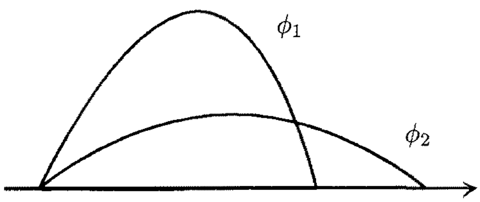


figure 2(a): $k_1 > k_2$

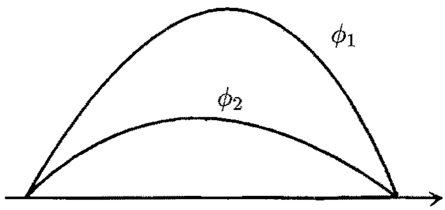


figure 2(b)

This already implies the existence of two projective points of a diffeomorphism of $\mathbf{RP}^1$. Indeed, every solution of the equation $\ddot{\phi}(\alpha) = -k(\alpha)\phi(\alpha)$ is a linear coordinate (e.g. the projection on the vertical axis) of the curve $\gamma(\alpha)$ satisfying equation (1). Therefore, the solutions of $\ddot{\phi}(\alpha) = -k(\alpha)\phi(\alpha)$ are antiperiodic on $[0, \pi]$ as well as those of the equation $\ddot{\phi}(\alpha) = -\phi(\alpha)$ (see figure 3).

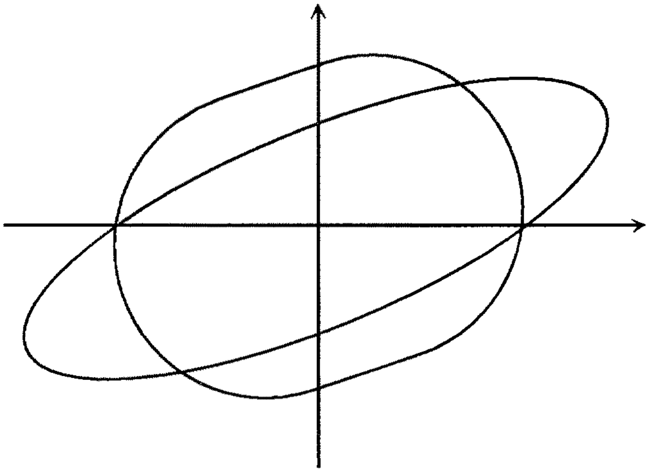


figure 3

We prove the following strengthened version of the Sturm theorem which implies the theorem of Ghys.

Call a Sturm–Liouville equation with a π-periodic potential *disconjugate* if, for every solution $\phi(\alpha)$, one has $\phi(\alpha+\pi) = -\phi(\alpha)$, and every solution has exactly one zero on $[0, \pi)$. Notice that the Sturm–Liouville equation corresponding to a diffeomorphism of $\mathbf{RP}^1$ is disconjugate (since the curve γ in figure 1 is star-shaped).

Theorem 2.1. *Given two disconjugate Sturm–Liouville equations $\ddot{\phi}\ (\alpha) = -k_1(\alpha)\phi(\alpha)$ and $\ddot{\phi}(\alpha) = -k_2(\alpha)\phi(\alpha)$, there exist at least* 4 *distinct zeroes of the function $k_1 - k_2$ on every interval $[\alpha, \alpha+\pi)$.*

Proof. Let ϕ_1, ϕ_2 be solutions of the two Sturm–Liouville equation, respectively. Then,

$$\int_\alpha^{\alpha+\pi} (k_1 - k_2)\phi_1\phi_2 \, d\alpha = 0.$$

Indeed,

$$\begin{aligned} 0 &= \int_\alpha^{\alpha+\pi} \Big(\phi_1(\ddot{\phi}_2 + k_2\phi_2) - \phi_2(\ddot{\phi}_1 + k_1\phi_1)\Big) \, d\alpha \\ &= \Big(\phi_1\dot{\phi}_2 - \phi_2\dot{\phi}_1\Big)\Big|_\alpha^{\alpha+\pi} + \int_\alpha^{\alpha+\pi} (k_2 - k_1)\phi_1\phi_2 \, d\alpha. \end{aligned}$$

The first term at the right hand side vanishes because $\phi_1\phi_2$ is π-periodic.

The theorem is a corollary of the following fact.

Lemma 2.2. *If a π-periodic function ψ is orthogonal to the product of any two solutions ϕ_1 and ϕ_2 of two disconjugate Sturm–Liouville equation, respectively, then ψ has at least 4 distinct zeroes on any interval $[\alpha, \alpha+\pi)$.*

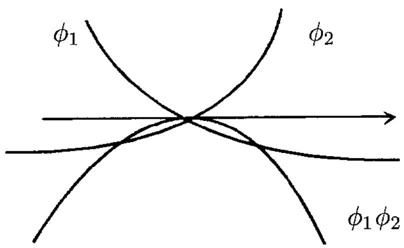


figure 4(a)

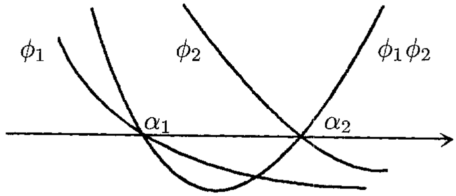


figure 4(b)

Proof. First prove that ψ has 2 zeroes. If not, choose ϕ_1 and ϕ_2 to have the same zero on $[\alpha, \alpha+\pi)$ (figure 4). Since both equations are disconjugate, neither ϕ_1 nor ϕ_2 have other zeroes on the interval. Then the product $\psi\phi_1\phi_2$ is either positive almost everywhere or negative almost everywhere.

Secondly, assume that ψ changes sign only twice at points α_1, α_2. Choose ϕ_1 to have zero at α_1 and ϕ_2 – at α_2. Again, ϕ_1 and ϕ_2 have no extra zeroes. As before, the integral of $\psi\phi_1\phi_2$ cannot vanish. The lemma is proved.

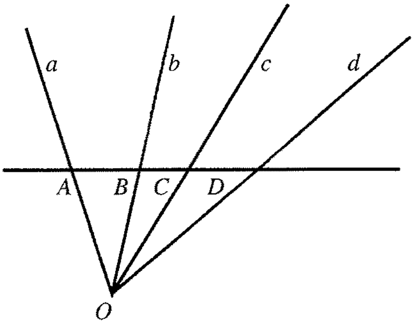
Remark. The above lemma is a particular case of a hierarchy of theorems estimating below the number of zeroes of functions orthogonal to products of solutions of disconjugate linear differential equations; see [9] where such theorems are used to prove the existence of 6 affine vertices of a closed convex plane curve.

3. Zeroes of the Schwarzian derivative

The Schwarzian derivative measures the failure of a diffeomorphism of $\mathbf{RP}^1$ to be projective. More specifically, let f be a diffeomorphism, $\alpha \in \mathbf{RP}^1$ a point. Consider four "infinitely close" points α, $\alpha+\epsilon$, $\alpha+2\epsilon$, $\alpha+3\epsilon$. Apply f to obtain the points $f(\alpha)$, $f(\alpha+\epsilon)$, $f(\alpha+2\epsilon)$, $f(\alpha+3\epsilon)$. Define the Schwarzian derivative $S(f)(\alpha)$ by

$$\Big[f(\alpha), f(\alpha+\epsilon), f(\alpha+2\epsilon), f(\alpha+3\epsilon)\Big] = \Big[\alpha, \alpha+\epsilon, \alpha+2\epsilon, \alpha+3\epsilon\Big] + \epsilon^2 S(f)(\alpha) + O(\epsilon^3)$$

where [, , ,] denotes the cross-ratio of four lines through the origin.



$$[a,b,c,d] = [A,B,C,D] = \frac{(A-C)(B-D)}{(B-C)(A-D)}$$

figure 5

By definition *projective points of a diffeomorphism are zeroes of its Schwarzian derivative.*

The following formula is the result of a direct computation.

Lemma 3.1. *In the angular parameter* α

$$S(f) = \frac{\dddot{f}}{\dot{f}} - \frac{3}{2}\left(\frac{\ddot{f}}{\dot{f}}\right)^2 + 2(\dot{f}^2 - 1). \tag{2}$$

Notice that, in the affine parameter $x = \operatorname{tg}\alpha$, the Schwarzian derivative is given by the standard formula

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

where $' = d/dx$.

Remark. The Schwarzian derivative is invariantly defined as the unique 1-cocycle on the group of diffeomorphisms of $\mathbf{RP}^1$ whose kernel is PSL_2 with values in the space of quadratic differentials.

The relation between the Schwarzian derivative and the potential of the Sturm–Liouville equation (1) is as follows.

Lemma 3.2.

$$k = \frac{1}{2}S(f) + 1.$$

Proof. The curve $\gamma(\alpha)$ is the image of the circle under the symplectomorphism F. In Cartesian coordinates,

$$\gamma(\alpha) = \left(\dot{f}^{-1/2}(\alpha)\cos f(\alpha), \quad \dot{f}^{-1/2}(\alpha)\sin f(\alpha)\right).$$

Differentiate twice to obtain the result.

Remark. It is interesting to consider the infinitesimal analog of the condition $S(f)(\alpha) = 0$. Let $f(\alpha) = \alpha + \epsilon a(\alpha)$ where $a(\alpha)$ is a π-periodic function. Then, in the first order in ϵ one gets from (2) the equation

$$\dddot{a}(\alpha) + 4\dot{a}(\alpha) = 0.$$

The existence of at least 4 roots of this equation on $[0, \pi)$ follows from the Hurwitz theorem. Indeed, the space of π-periodic functions has the basis

$$1,\ \sin 2\alpha,\ \cos 2\alpha,\ \sin 4\alpha,\ \dots$$

and the function $\dddot{a} + 4\dot{a}$ does not contain the first harmonics.

4. Projective points of diffeomorphisms as flattening points of Legendrian curves in projective space

In this section we prove Theorem 1.

Consider the graph of the homogeneous symplectomorphism F. It is a conical Lagrangian submanifold in the symplectic space $\mathbf{R}^2 \times \mathbf{R}^2$, the symplectic structure being $\omega_1 \ominus \omega_2$ where ω_1 and ω_2 are the symplectic structures on the factors. Denote by $\mathcal{F}$ the projectivization of the graph. $\mathcal{F}$ is a Legendrian curve in contact $\mathbf{RP}^3$.

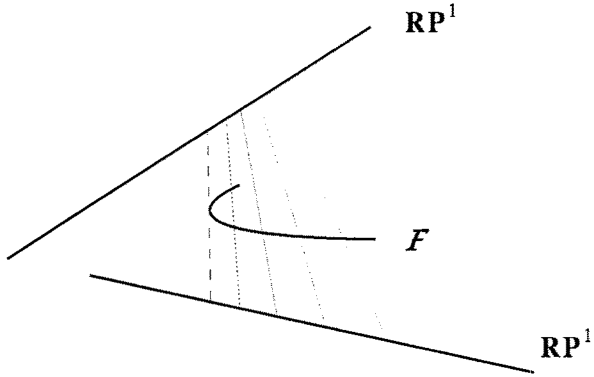


figure 6

Space $\mathbf{RP}^3$ contains two distinguished projective lines, the projectivizations of the source and target $\mathbf{R}^2$. Both projections of $\mathcal{F}$ from one of these lines to another are diffeomorphisms.

Projective points of the diffeomorphism f correspond to lines along which the graph of the symplectomorphism F has contact of second order with graphs of linear symplectic mappings. In projectivization projective points of the diffeomorphism f correspond to *inflection points* of the curve $\mathcal{F}$.

An inflection point of a curve in $\mathbf{RP}^3$ is a point in which the acceleration vector is proportional to the velocity vector. Said differently, the curve has the second order contact with its tangent line. A generic curve in $\mathbf{RP}^3$ does not have inflection points (since the inflection point condition has codimension 2).

Conversely, let $C \subset \mathbf{RP}^3$ be a closed Legendrian curve such that its projection on $\mathbf{RP}^1_1$ from $\mathbf{RP}^1_2$ and its projection on $\mathbf{RP}^1_2$ from $\mathbf{RP}^1_1$ are diffeomorphisms. Then the composition of one projection with the inverse of another is a diffeomorphism of $\mathbf{RP}^1$. The inflection points of C are the projective points of this diffeomorphism. Theorem 1 is a consequence of the Ghys theorem and Proposition 4.1 below.

To formulate this proposition recall that *flattening points* of a curve $C \subset \mathbf{RP}^3$ are the points in which the curve has contact of second order with its osculating plane (i.e., the vectors $\dot{C}, \ddot{C}, \dddot{C}$ are linearly dependent). Clearly, inflection points are flattening points. It turns out that for Legendrian curves the two notions coincide.

Proposition 4.1. *The flattening points of a Legendrian curve in the standard contact* $\mathbf{RP}^3$ *are its inflection points.*

Proof. Consider a parameterization $C(t)$. We need to prove that if the vectors $\dot{C}, \ddot{C}, \dddot{C}$ are linearly dependent then so are $\dot{C}, \ddot{C}$ (which makes sense in any affine chart and does not depend on its choice). Let θ be the standard contact form. The flattening condition reads: $\theta \wedge d\theta(\dot{C}, \ddot{C}, \dddot{C}) = 0$.

Let $\widetilde{C}$ be a lift of C to linear symplectic space $\mathbf{R}^4$, ω — the linear symplectic structure. Since C is Legendrian, $\omega(\widetilde{C}, \dot{\widetilde{C}}) = 0$. Differentiate to obtain $\omega(\widetilde{C}, \ddot{\widetilde{C}}) = 0$. This means that the acceleration vector $\ddot{C}$ belongs to the contact plane (cf. [1]).

Differentiate once again:

$$\omega(\dot{\widetilde{C}}, \ddot{\widetilde{C}}) + \omega(\widetilde{C}, \dddot{\widetilde{C}}) = 0.$$

The flattening condition reads: $\omega \wedge \omega(\widetilde{C}, \dot{\widetilde{C}}, \ddot{\widetilde{C}}, \dddot{\widetilde{C}}) = 0$. In view of the two previous formulæ

$$0 = \omega \wedge \omega(\widetilde{C}, \dot{\widetilde{C}}, \ddot{\widetilde{C}}, \dddot{\widetilde{C}}) = \omega(\widetilde{C}, \dddot{\widetilde{C}})\omega(\dot{\widetilde{C}}, \ddot{\widetilde{C}}) = -\omega(\dot{\widetilde{C}}, \ddot{\widetilde{C}})^2.$$

If this is zero then $d\theta(\dot{C}, \ddot{C})^2 = 0$. Since $\dot{C}, \ddot{C}$ lie in the contact plane and $d\theta$ in nondegenerate therein, the vectors $\dot{C}, \ddot{C}$ are linearly dependent. The proposition is proved.

Remark. Parameterize $\mathbf{RP}^1_1$ by the angular parameter α. Then α also parameterizes C. Let $\widetilde{C}(\alpha)$ be a lift of $C(\alpha)$ to $\mathbf{R}^4$ such that its projection to $\mathbf{R}^2_1$ has coordinates $(\cos(\alpha), \sin(\alpha))$. Let $(\phi_1(\alpha), (\phi_2(\alpha))$ be the projection of $\widetilde{C}(\alpha)$ to $\mathbf{R}^2_2$. Then

$$\begin{vmatrix} \phi_1 & \phi_2 \\ \dot{\phi}_1 & \dot{\phi}_2 \end{vmatrix} = \text{const}$$

and ϕ_1 and ϕ_2 satisfy a Sturm–Liouville equation: $\ddot{\phi} + k\phi = 0$ where $k(\alpha)$ is a π-periodic function.

One gets

$$\omega \wedge \omega(\widetilde{C}, \dot{\widetilde{C}}, \ddot{\widetilde{C}}, \dddot{\widetilde{C}}) = \begin{vmatrix} \phi_1 + \ddot{\phi}_1 & \phi_2 + \ddot{\phi}_2 \\ \dot{\phi}_1 + \dddot{\phi}_1 & \dot{\phi}_2 + \dddot{\phi}_2 \end{vmatrix}. \tag{3}$$

This in turn is equal to $\mathrm{const}(k(\alpha) - 1)^2$ (cf. Theorem 2.1).

The Legendrian condition on the curve C in Theorem 1 was essential.

Example. Consider the curve $\widetilde{C}(\alpha) \subset \mathbf{R}^4$ whose projection to $\mathbf{R}_1^2$ is $(\cos\alpha, \sin\alpha)$, and to $\mathbf{R}_2^2$ is $(\cos\alpha + \epsilon\cos 3\alpha,\ \sin\alpha + \epsilon \sin 3\alpha)$. Then for sufficiently small ϵ both projections of C to $\mathbf{RP}_1^1$ and $\mathbf{RP}_2^1$ are diffeomorphisms. However, C has no flattening points: the determinant (3) equals $192\epsilon^2$.

5. Inflections of the characteristic curve of a projective line diffeomorphism

Consider the Hopf fibration $\pi : \mathbf{RP}^3 \to S^2$ whose fibers are the characteristic lines of the standard contact form in $\mathbf{RP}^3$. Consider also the fibration $\widetilde{\pi} : \mathbf{R}_1^2 \times \mathbf{R}_2^2 \setminus \{0\} \to S^2$ that covers π.

Given a Legendrian curve in $\mathbf{RP}^3$ its projection is an immersed smooth curve on S^2 that bounds the area which is a multiple of half that of the sphere. The Legendrian lines project to great circles in S^2 (a one-parameter family of Legendrian lines over each great circle). The points $\widetilde{\pi}(\mathbf{R}_1^2) = \pi(\mathbf{RP}_1^1)$ and $\widetilde{\pi}(\mathbf{R}_2^2) = \pi(\mathbf{RP}_2^1)$ are antipodal; we think of them as the poles of the sphere.

We have associated the Legendrian curve $\mathcal{F} \subset \mathbf{RP}^3$ with an orientation preserving diffeomorphism of the projective line.

Definition. The characteristic curve c_f of a diffeomorphism f is the curve $\pi(\mathcal{F})$.

Lemma 5.1. *The projective points of f correspond to the inflection points of its characteristic curve.*

Proof. The projective points are the inflection points of $\mathcal{F}$, i.e. the second order contacts with its Legendrian tangent line. In projection π the curve c_f has the second order contact with a great circle, that is, an inflection point.

Thus the Ghys theorem states that the characteristic curve of an orientation preserving projective line diffeomorphism has at least 4 distinct inflection points.

Proposition 5.2. *The characteristic curve c_f is an embedded curve transverse to every meridian of the sphere (a great circle through the poles) and it bisects the area of the sphere (figure 7).*

Proof. First we show that c_f is transverse to every meridian. Meridians correspond to 2-dimensional subspaces in $\mathbf{R}_1^2 \times \mathbf{R}_2^2$ that nontrivially intersect both factors $\mathbf{R}_1^2$

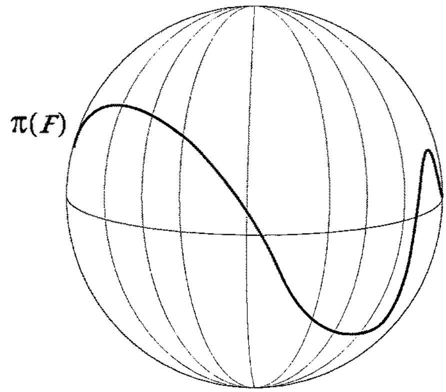


figure 7

and $\mathbf{R}_2^2$; such a space is automatically Lagrangian since it contains two linearly independent symplectically orthogonal vectors. The curve c_f lifts to $\mathbf{R}_1^2 \times \mathbf{R}_2^2$ as $\widetilde{C}(\alpha)$ from the previous section. Let $\widetilde{C}_1$ and $\widetilde{C}_2$ be its projections to $\mathbf{R}_1^2$ and $\mathbf{R}_2^2$.

If c_f is tangent to a meridian then the plane generated by $\widetilde{C}$ and $\dot{\widetilde{C}}$ nontrivially intersect $\mathbf{R}_1^2$ and $\mathbf{R}_2^2$ for some value of α. That is, a nontrivial linear combination of $\widetilde{C}_1$ and $\dot{\widetilde{C}}_1$ (or $\widetilde{C}_2$ and $\dot{\widetilde{C}}_2$) vanishes. But this is impossible because both curves $\widetilde{C}_1$ and $\widetilde{C}_2$ are star shaped.

Next we show that c_f makes exactly one turn around the sphere. Indeed f is isotopic to identity, therefore c_f is homotopic to c_{id} in the class of immersed curves transverse to the meridians. The curve c_{id} is the equator, and the result follows.

Inflection points of spherical curves is the subject of the *Tennis Ball Theorem* (see [3], [6], [7]): an embedded curve that bisects the area of the sphere has at least 4 distinct inflections. Therefore the tennis ball theorem implies the theorem on 4 zeroes of the Schwarzian derivative.

Remark. One may consider a Legendrian fibration $p : \mathbf{RP}^3 \to S^2$ which identifies $\mathbf{RP}^3$ with the space of oriented contact elements of the sphere. The curve $p(\mathcal{F})$ is a front; generically it has cusps. The flattening points of $\mathcal{F}$ correspond to the vertices of the front $p(\mathcal{F})$, i.e. its third order contacts with circles on the sphere. The derivative curve c_f is the *derivative* of the front $p(\mathcal{F})$ — see [7] for more details. Thus the Ghys theorem indeed concerns the vertices of a front on the sphere.

Acknowledgement. This paper is the result of a reflection over the lecture by E. Ghys [8]. We are grateful to him and to V. Arnold, D. Fuchs and M. Kazarian for stimulating discussions.

Added in proof. In preprint [15] the following generalization of Ghys' theorem is proved: if a real projective line diffeomorphism has n fixed points then its Schwarzian derivative has at least n zeroes.

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V. Ovsienko
Centre de Physique Théorique
Centre National de la
Recherche Scienifique
Luminy, Case 907
F–13288 Marseille Cedex 9
France

S. Tabachnikov
University of Arkansas (UARK)
Department of Mathematics
Fayetteville, AR 72701
USA