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Projectively Equivariant Quantization and Symbol Calculus: Noncommutative Hypergeometric Functions

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Abstract. We extend projectively equivariant quantization and symbol calculus to symbols of pseudo-differential operators. An explicit expression in terms of hypergeometric functions with noncommutative arguments is given. Some examples are worked out, one of them yielding a quantum length element on S^3.

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1. Introduction

Let M be a smooth manifold and $\mathcal{S}(M)$ the space of smooth functions on T^*M, polynomial on the fibers; the latter is usually called the space of symbols of differential operators. Let us furthermore assume that M is endowed with an action of a Lie group G. The aim of equivariant quantization [7, 8, 12] (see also [2–4]) is to associate to each symbol a differential operator on M in such a way that this quantization map intertwines the G-action.

The existence and uniqueness of equivariant quantization in the case where M has a flat projective (resp. conformal) structure, i.e., when $G = \mathrm{SL}(n+1, \mathbb{R})$ with $n = \dim(M)$ (resp. $G = \mathrm{SO}(p+1, q+1)$ with $p + q = \dim(M)$) has recently been proved in the above references.

More precisely, let $\mathcal{F}_\lambda(M)$ stand for the space of (complex-valued) tensor densities of degree λ on M and $\mathcal{D}_{\lambda,\mu}(M)$ for the space of linear differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. These spaces are naturally modules over the group of all diffeomorphisms of M. The space of symbols corresponding to $\mathcal{D}_{\lambda,\mu}(M)$ is therefore $\mathcal{S}_\delta(M) = \mathcal{S}(M) \otimes \mathcal{F}_\delta(M)$ where $\delta = \mu - \lambda$. There is a filtration

$$\mathcal{D}^0_{\lambda,\mu} \subset \mathcal{D}^1_{\lambda,\mu} \subset \cdots \subset \mathcal{D}^k_{\lambda,\mu} \subset \cdots$$

and the associated module $\mathcal{S}_\delta(M) = \mathrm{gr}(\mathcal{D}_{\lambda,\mu})$ is graded by the degree of polynomials:

$$\mathcal{S}_\delta = \mathcal{S}_{0,\delta} \oplus \mathcal{S}_{1,\delta} \oplus \cdots \oplus \mathcal{S}_{k,\delta} \oplus \cdots$$

The problem of equivariant quantization is the quest for a *quantization map*:

$$\mathcal{Q}_{\lambda,\mu}\colon \mathcal{S}_\delta(M) \to \mathcal{D}_{\lambda,\mu}(M) \tag{1.1}$$

that commutes with the G-action. In other words, it amounts to an identification of these two spaces which is canonical with respect to the geometric structure on M.

The inverse of the quantization map

$$\sigma_{\lambda,\mu} = \left(\mathcal{Q}_{\lambda,\mu}\right)^{-1} \tag{1.2}$$

is called the symbol map.

In this Letter, we will restrict considerations to the projectively equivariant case. Without loss of generality, we will assume $M = S^n$ endowed with its standard $\mathrm{SL}(n+1, \mathbb{R})$-action. The explicit formulæ for the maps (1.1) and (1.2) can be found in [4] for $n = 1$ and in [12] for $\lambda = \mu$ in any dimension. Our purpose is to rewrite the expressions for $\mathcal{Q}_{\lambda,\mu}$ and $\sigma_{\lambda,\mu}$ in a more general way which, in particular, extends the quantization to a bigger class of symbols of pseudo-differential operators.

2. Projectively Equivariant Quantization Map

In terms of affine coordinates on S^n, the vector fields spanning the canonical action of the Lie algebra $\mathrm{sl}(n+1, \mathbb{R})$ are as follows

$$\frac{\partial}{\partial x^i}, \quad x^i \frac{\partial}{\partial x^j}, \quad x^i x^j \frac{\partial}{\partial x^j},$$

with $i, j = 1, \ldots, n$ (the Einstein summation convention is understood).

We will denote by $\mathrm{aff}(n, \mathbb{R})$ the affine subalgebra spanned by the first-order vector fields. We will find it convenient to identify locally, in each affine chart, the spaces $\mathcal{S}_\delta$ and $\mathcal{D}_{\lambda,\mu}$ via the 'normal ordering' isomorphism

$$\mathcal{I}\colon P(x)^{i_1\ldots i_k}\, \xi_{i_1} \cdots \xi_{i_k} \longmapsto (-i\hbar)^k\, P(x)^{i_1\ldots i_k} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_k}} \tag{2.1}$$

which is already equivariant with respect to $\mathrm{aff}(n, \mathbb{R})$. An equivalent means of identification is provided by the Fourier transform

$$(\mathcal{I}(P)\phi)(x) = \frac{1}{(2\pi\hbar)^{n/2}} \int_{\mathbb{R}^{2n}} \mathrm{e}^{(i/\hbar)\langle \xi, x-y\rangle} P(y, \xi)\phi(y)\, \mathrm{d}y\mathrm{d}\xi, \tag{2.2}$$

where

$$P(y, \xi) = \sum_{k=0}^{\infty} P(y)^{i_1\ldots i_k}\, \xi_{i_1} \cdots \xi_{i_k}$$

and where ϕ is a compactly supported function (representing a λ-density in the coordinate patch). This mapping extends to the space of pseudo-differential symbols (defined in the chosen affine coordinate system).

The purpose of projectively equivariant quantization is to modify the map $\mathcal{I}$ in (2.1) in order to obtain an identification of $\mathcal{S}_\delta$ and $\mathcal{D}_{\lambda,\mu}$ that does not depend upon a chosen affine coordinate system, and is, therefore, globally defined on S^n.

Recall [13] that the (locally defined) operators on $\mathcal{S}_\delta$, namely

$$\mathcal{E} = \xi_i \frac{\partial}{\partial \xi_i}, \qquad \mathrm{D} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i}, \tag{2.3}$$

(where the ξ_i are the coordinates dual to the x^i) commute with the $\mathrm{aff}(n, \mathbb{R})$-action on T^*S^n. The Euler operator, $\mathcal{E}$, is the degree operator on $\mathcal{S}_\delta = \bigoplus_{k=0}^{\infty} \mathcal{S}_{k,\delta}$ while the divergence operator D lowers this degree by one.

Let us now recall (in a slightly more general context) the results obtained in [4, 12]. The $\mathrm{SL}(n+1, \mathbb{R})$-equivariant quantization map (1.1) is given on every homogeneous component by

$$\mathcal{Q}_{\lambda,\mu}\big|_{\mathcal{S}_{k,\delta}} = \sum_{m=0}^{k} C_m^k \,(i\hbar \mathrm{D})^m|_{\mathcal{S}_{k,\delta}}, \tag{2.4}$$

where the constant coefficients C_m^k are determined by the following relation

$$C_{m+1}^k = \frac{k - m - 1 + (n+1)\lambda}{(m+1)(2k - m - 2 + (n+1)(1-\delta))}\, C_m^k \tag{2.5}$$

and the normalization condition: $C_0^k = 1$.

As to the projectively equivariant symbol map (1.2), it retains the form

$$\sigma_{\lambda,\mu}\big|_{\mathcal{S}_{k,\delta}} = \sum_{m=0}^{k} \widetilde{C}_m^k \left(\frac{\mathrm{D}}{i\hbar}\right)^m \Big|_{\mathcal{S}_{k,\delta}}, \tag{2.6}$$

where the coefficients $\widetilde{C}_m^k$ are such that

$$\widetilde{C}_{m+1}^{k+1} = -\frac{k + (n+1)\lambda}{(m+1)(2k - m + (n+1)(1-\delta))}\, \widetilde{C}_m^k \tag{2.7}$$

and, again, $\widetilde{C}_0^k = 1$ for all k.

Remark 2.1. Expressions (2.4) and (2.6) make sense if $\delta \neq 1 + \ell/(n+1)$ with $\ell = 0, 1, 2, \ldots$ For these values of δ, the quantization and symbol maps do not exist for generic λ and μ; see [10] for a detailed classification.

In contradistinction with the operators $\mathcal{E}$ and D defined in (2.3), the quantization map $\mathcal{Q}_{\lambda,\mu}$ and the symbol map $\sigma_{\lambda,\mu}$ are globally defined on T^*S^n, i.e., they are independent of the choice of an affine coordinate system.

3. Noncommutative Hypergeometric Function

Our main purpose is to obtain an expression for $\mathcal{Q}_{\lambda,\mu}$ and $\sigma_{\lambda,\mu}$ valid for a larger class of symbols, namely for symbols of *pseudo-differential* operators. We will rewrite the formulæ (2.4), (2.5) and (2.6), (2.7) in terms of the $\mathrm{aff}(n, \mathbb{R})$-invariant operators $\mathcal{E}$ and D in a form independent of the degree, k, of polynomials.

It turns out that our quantization map (1.1) involves a certain hypergeometric function; let us now recall this classical notion. A hypergeometric function with $p+q$ parameters is defined (see, e.g., [9]) as the power series in z given by

$$F\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z\right) = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{z^m}{m!} \tag{3.1}$$

with $(a)_m = a(a+1)\ldots(a+m-1)$. This hypergeometric function is called confluent if $p = q = 1$.

THEOREM 3.1. *The projectively equivariant quantization map is of the form*

$$\mathcal{Q}_{\lambda,\mu} = F\left(\begin{matrix} A_1, A_2 \\ B_1, B_2 \end{matrix} \middle| Z\right), \tag{3.2}$$

where the parameters

$$\begin{aligned} A_1 &= \mathcal{E} + (n+1)\lambda, & A_2 &= 2\mathcal{E} + (n+1)(1-\delta) - 1, \\ B_1 &= \mathcal{E} + \tfrac{1}{2}(n+1)(1-\delta) - \tfrac{1}{2}, & B_2 &= \mathcal{E} + \tfrac{1}{2}(n+1)(1-\delta), \end{aligned} \tag{3.3}$$

are operator-valued, as well as the variable

$$Z = \frac{i\hbar \mathrm{D}}{4}. \tag{3.4}$$

Proof. Recall that for a hypergeometric function (3.1), one has

$$F\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| z\right) = \sum_{m=0}^{\infty} c_m z^m$$

with

$$\frac{c_{m+1}}{c_m} = \frac{1}{m+1}\left[\frac{(a_1+m)\cdots(a_p+m)}{(b_1+m)\cdots(b_q+m)}\right].$$

Let us replace $k-m$ by the degree operator $\mathcal{E}$ in the coefficients C_m^k; the expression (2.4) can be therefore rewritten as $\mathcal{Q}_{\lambda,\mu} = \sum_{m=0}^{k} C_m(\mathcal{E})\,(i\hbar \mathrm{D})^m$. From the recursion relation (2.5), one readily obtains

$$\frac{C_{m+1}(\mathcal{E})}{C_m(\mathcal{E})} = \frac{1}{4(m+1)}\left[\frac{(\mathcal{E}+(n+1)\lambda+m)(2\mathcal{E}+(n+1)(1-\delta)-1+m)}{\left(\mathcal{E}+\frac{1}{2}(n+1)(1-\delta)-\frac{1}{2}+m\right)\left(\mathcal{E}+\frac{1}{2}(n+1)(1-\delta)+m\right)}\right],$$

completing the proof. □

COROLLARY 3.2. *The quantization map is given by the series*

$$\mathcal{Q}_{\lambda,\mu} = \sum_{m=0}^{k} C_m(\mathcal{E})\,(i\hbar \mathrm{D})^m, \tag{3.5}$$

where

$$C_m(\mathcal{E}) = \frac{1}{m!}\frac{(\mathcal{E}+(n+1)\lambda)_m}{(2\mathcal{E}+(n+1)(1-\delta)+m-1)_m}. \tag{3.6}$$

Remark 3.3. Let us stress that the operator-valued parameters (3.3) and the variable (3.4) entering the expression (3.2) do not commute. We have therefore chosen an ordering that assigns the divergence operator D to the right.

In the particular and most interesting case of half-densities (cf. [7, 8]), the expression (3.2) takes a simpler form.

COROLLARY 3.4. *If* $\lambda = \mu = \frac{1}{2}$, *the quantization map* (3.2) *reduces to the confluent hypergeometric function*

$$\mathcal{Q}_{\frac{1}{2},\frac{1}{2}} = F\left(\begin{matrix} 2\mathrm{E} \\ \mathrm{E} \end{matrix}\,\middle|\, \frac{i\hbar \mathrm{D}}{4}\right) \tag{3.7}$$

with the notation: $\mathrm{E} = \mathcal{E} + \frac{1}{2}\, n$.

It is a remarkable fact that the expression for inverse symbol map (1.2) is much simpler. It is given by a confluent hypergeometric function for any λ and μ.

THEOREM 3.5. *The projectively equivariant symbol map* (1.2) *is given by*

$$\sigma_{\lambda,\mu} = F\left(\begin{matrix} \mathcal{E}+(n+1)\lambda \\ 2\mathcal{E}+(n+1)(1-\delta) \end{matrix}\,\middle|\, -\frac{D}{i\hbar}\right). \tag{3.8}$$

The proof is analogous to that of Theorem 3.1.

It would be interesting to obtain expressions of the projectively equivariant quantization and symbol maps as integral operators similar to (2.2).

4. Some Examples

We wish to present here a few applications of the projectively equivariant quantization to some special Hamiltonians on T^*S^n.

The first example deals with the geodesic flow. Denote by g the standard round metric on the unit n-sphere and by $H = \mathrm{g}^{ij}\xi_i\xi_j$ the corresponding quadratic Hamiltonian. In an affine coordinate system, it takes the following form

$$H = \big(1 + \|x\|^2\big)\big(\delta^{ij} + x^i x^j\big)\xi_i\xi_j, \tag{4.1}$$

where $\|x\|^2 = \delta_{ij}x^i x^j$ with $i, j = 1, \dots, n$. Moreover, we will consider a family of such Hamiltonians belonging to $\mathcal{S}_\delta$, namely $H_\delta = H\sqrt{g^\delta}$ where $g = \det(\mathrm{g}_{ij})$.

In order to provide explicit formulæ, we need to recall the expression of the covariant derivative of λ-densities, namely $\nabla_i = \partial_i - \lambda\Gamma^j_{ij}$.

PROPOSITION 4.1. *The projectively equivariant quantization map* (1.1) *associates to* H_δ *the following differential operator*

$$\mathcal{Q}_{\lambda,\mu}(H_\delta) = -\hbar^2\big(\Delta + C_{\lambda,\mu}\,R\big), \tag{4.2}$$

where $\Delta = \mathrm{g}^{ij}\nabla_i\nabla_j$ *is the Laplace operator; the constant coefficient is*

$$C_{\lambda,\mu} = \frac{(n+1)^2\lambda(\mu-1)}{(n-1)((1-\delta)(n+1)+1)} \tag{4.3}$$

and $R = n(n-1)$ *is the scalar curvature of* S^n.

Proof. The quantum operator (4.2) is obtained, using (3.2)–(3.4), by a direct computation. However, the formula (4.2) turns out to be a particular case of (5.4) in [1] since the Levi-Civita connection is projectively flat. □

Another example is provided by the αth power H^α of the Hamiltonian H, where $\alpha \in \mathbb{R}$. We will only consider the case $\lambda = \mu$ in the sequel.

PROPOSITION 4.2. *For*

$$\alpha = \frac{1-n}{4} \tag{4.4}$$

one has $\mathcal{Q}_{\lambda,\lambda}(H^\alpha) = H^\alpha$.

Proof. Straightforward computation leads to

$$D(H^\alpha) = 2\alpha(4\alpha + n - 1)H^{\alpha-1}(1 + \|x\|^2)\langle\xi, x\rangle$$

and (2.4) therefore yields the result. □

We have just shown that the Fourier transform (2.2) of $H^{(1-n)/4}$ is well-defined on S^n and actually corresponds to the projectively equivariant quantization of this pseudo-differential symbol.

If we want to deal with operators acting on a Hilbert space, we have to restrict now considerations to the case $\lambda = \mu = \frac{1}{2}$.

For the 3-sphere only, the above quantum Hamiltonian on $T^*S^n \setminus S^n$ is as follows

$$\mathcal{Q}_{\frac{1}{2},\frac{1}{2}}(H^{-\frac{1}{2}}) = \frac{1}{\hbar}\frac{1}{\sqrt{-\Delta}} \tag{4.5}$$

and can be understood as a quantized 'length element' in the sense of [5].

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