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Generalizations of Virasoro group and Virasoro algebra through extensions by modules of tensor-densities on S^1

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ABSTRACT

We classify non-trivial (non-central) extensions of the group $\mathrm{Diff}^+(S^1)$ of all diffeomorphisms of the circle preserving its orientation and of the Lie algebra $\mathrm{Vect}(S^1)$ of vector fields on S^1, by the modules of tensor-densities on S^1. The result is: 4 non-trivial extensions of $\mathrm{Diff}^+(S^1)$ and 7 non-trivial extensions of $\mathrm{Vect}(S^1)$. Analogous results hold for the Virasoro group and the Virasoro algebra. We also classify central extensions of the constructed Lie algebras.

1. INTRODUCTION

The Lie group $\mathrm{Diff}^+(S^1)$ of all orientation preserving diffeomorphisms of the circle, has a unique (up to isomorphism) non-trivial central extension, so-called *Bott-Virasoro group*. It is defined by the Thurston-Bott cocycle [B]:

$$B(\Phi, \Psi) = \int_{S^1} log((\Phi \circ \Psi)')dlog(\Psi')$$

where $\Phi, \Psi \in \mathrm{Diff}^+(S^1)$, the function $\Phi' = \frac{d\Phi(x)}{dx}$ being well defined on S^1.

The corresponding Lie algebra is called the *Virasoro algebra*. It is given by the unique (up to isomorphism) non-trivial central extension of the Lie algebra $\mathrm{Vect}(S^1)$ of all vector fields on the circle. This central extension is defined by the Gelfand-Fuchs cocycle [GF 1]:

$$\omega(f, g) = \int_{S^1} \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} dx$$

In this paper we study (non-central) extensions of the group $\mathrm{Diff}^+(S^1)$ and the Lie algebra $\mathrm{Vect}(S^1)$ by the modules of tensor densities on S^1.
Let $\mathcal{F}_\lambda$ be the space of all tensor-densities on S^1 of degree λ:

$$a = a(x)(dx)^\lambda$$

This space has natural structures of $\mathrm{Diff}(S^1)$ and $\mathrm{Vect}(S^1)$-module. The $\mathrm{Diff}(S^1)$-action on $\mathcal{F}_\lambda$ is given by

$$\Phi^* a = a(\Phi)(\Phi')^\lambda$$

The Lie algebra $\mathrm{Vect}(S^1)$ acts on this space by the Lie derivative: let $f = f(x)d/dx$ be a vector field, then

$$L_f a = (fa' + \lambda f' a)(dx)^\lambda$$

We consider the problem of classification of all non-trivial extensions

$$0 \to \mathcal{F}_\lambda \to G_\lambda \to \mathrm{Diff}^+(S^1) \to 0 \tag{1}$$

of the group $\mathrm{Diff}^+(S^1)$ by $\mathrm{Diff}^+(S^1)$-modules $\mathcal{F}_\lambda$.

In other words, we are looking for group structures on $\mathrm{Diff}^+(S^1) \times \mathcal{F}_\lambda$ given by associative product of the following form:

$$(\Phi, a)(\Psi, b) = (\Phi \circ \Psi, b + \Psi^* a + B_\lambda(\Phi, \Psi))$$

The expression $B_\lambda(\Phi, \Psi) \in \mathcal{F}_\lambda$ satisfies the condition:

$$B_\lambda(\Phi, \Psi \circ \Xi) + B_\lambda(\Psi, \Xi) = B_\lambda(\Phi \circ \Psi, \Xi) + \Xi^* B_\lambda(\Phi, \Psi)$$

which means that $B_\lambda(\Phi, \Psi)$ is a 2-cocycle on $\mathrm{Diff}^+(S^1)$ with values in $\mathcal{F}_\lambda$. If $B_\lambda = 0$ then the group G_λ is called the semi-direct product: $G_\lambda = \mathrm{Diff}^+(S^1) \rhd \mathcal{F}_\lambda$.

The extension (1) is *non-trivial* if the Lie group G_λ is not isomorphic to $\mathrm{Diff}^+(S^1) \rhd \mathcal{F}_\lambda$. The cocycle B_λ in this case, represents a non-trivial cohomology class of the group $H^2(\mathrm{Diff}^+(S^1); \mathcal{F}_\lambda)$. The classification problem for the extensions (1) is equivalent to the problem of computing this cohomology group.

We calculate the group $H^2_c(\mathrm{Diff}^+(S^1); \mathcal{F}_\lambda)$ of differentiable cohomology in Van-Est's sense. It means, that we classify all the extensions given by differentiable cocycles. We find four non-isomorphic infinite-dimensional Lie groups. We give explicit formulae for non-trivial cocycles on $\mathrm{Diff}^+(S^1)$.
We also consider non-trivial extensions of the Lie algebra $\mathrm{Vect}(S^1)$:

$$0 \to \mathcal{F}_\lambda \to g_\lambda \to \mathrm{Vect}(S^1) \to 0 \tag{2}$$

One obtains the classification of these extensions as a corollary of some general theorems in the cohomology theory of infinite-dimensional Lie algebras.

On classifying the extensions (2), one finds a series of seven Lie algebras. They are defined on the space $\mathrm{Vect}(S^1) \oplus \mathcal{F}_\lambda$. The commutator is given by:

$$[(f, a)(g, b)] = ([f, g], L_f b - L_g a + c(f, g))$$

where c is a 2-cocycle on $\mathrm{Vect}(S^1)$ with values in $\mathcal{F}_\lambda$.

We also classify non-trivial central extensions of all the Lie algebras given by the extensions (2). Some of these Lie algebras have already been considered in the mathematical literature, some of them are probably new.

All the Lie groups and the Lie algebras defined by the extensions (1) and (2) seem to be interesting generalizations of the Virasoro group and algebra. Their representations, coadjoint orbits etc. appear to be interesting subjects, and deserve further study.

2. MAIN THEOREMS

Let us formulate the main results of this paper.

2.1. Extensions of the group $\mathrm{Diff}^+(S^1)$

The following theorem gives a classification of non-trivial extensions (1) up to an isomorphism.

Theorem 1. *One has for cohomology of the group* $\mathrm{Diff}^+(S^1)$*:*

$$H_c^2(\mathrm{Diff}^+(S^1); \mathcal{F}_\lambda) = \begin{cases} \mathbf{R}, \ \lambda = 0, 1, 2, 5, 7 \\ 0, \ \lambda \neq 0, 1, 2, 5, 7 \end{cases}$$

In other words, there exists a uniue (modulo isomorphism) non-trivial extension of $\mathrm{Diff}^+(S^1)$ by the module $\mathcal{F}_\lambda$ for each value: $\lambda = 0, 1, 2, 5, 7$. If $\lambda \neq 0, 1, 2, 5, 7$, there is no non-trivial extension.

Let us describe here the 2-cocycles

$$B_\lambda : \mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1) \to \mathcal{F}_\lambda$$

which generate all possible non-trivial cohomology classes.
First of all, recall that the following mappings:

$$l : \Phi \mapsto log(\Phi'(x))$$

$$dl : \Phi \mapsto d\,log(\Phi'(x)) = \frac{\Phi''}{\Phi'} dx$$

$$S : \Phi \mapsto \left[\frac{\Phi'''}{\Phi'} - \frac{3}{2} \left(\frac{\Phi''}{\Phi'} \right)^2 \right] (dx)^2 \tag{3}$$

define 1-cocycles on $\mathrm{Diff}^+(S^1)$ with values in $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, correspondingly. They represent unique (modulo coboundaries) non-trivial classes of the cohomology groups: $H_c^1(\mathrm{Diff}^+(S^1); \mathcal{F}_\lambda)$, $\lambda = 0, 1, 2$. The cocycle S is called the *Schwarzian derivative*, dl is called the logarithmic derivative.

To construct non-trivial cocycles on $\mathrm{Diff}^+(S^1)$, we shall use the following version of the general *cap-product* (see e.i. [Br]) on group cocycles. Consider a Lie group G. Let: U, V, W be G-modules; $u : G \to U$, $v : G \to V$ be 1-cocycles on G $\langle .,. \rangle : U \otimes V \to W$ be G-invariant bilinear mapping. Then

$$u \cap v(g, h) = \langle h(u(g)). v(h) \rangle$$

is a 2-cocycle on G with values in W.

We are going to use the following invariant bilinear mapping on the spaces $\mathcal{F}_\lambda$.

(a) The product of tensor-densities: $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_{\lambda+\mu}$

$$a(x)(dx)^\lambda \otimes b(x)(dx)^\mu \mapsto a(x)b(x)(dx)^{\lambda+\mu}.$$

(b) The Poisson bracket: $\mathcal{F}_\lambda \otimes \mathcal{F}_\mu \rightarrow \mathcal{F}_{\lambda+\mu+1}$

$$\{a(x)(dx)^\lambda, b(x)(dx)^\mu\} = (\lambda a(x)b'(x) - \mu a'(x)b(x))(dx)^{\lambda+\mu+1}$$

(cf. [F]).

Theorem 2. *The cohomology groups $H^2_c(\mathrm{Diff}^+(S^1); \mathcal{F}_\lambda)$, where $\lambda = 0, 1, 2, 5, 7$ are generated by the following non-trivial 2-cocycles:*

$$B_0(\Phi, \Psi) = const(\Phi, \Psi) = B(\Phi, \Psi) \tag{4}$$

$$B_1(\Phi, \Psi) = \Psi^*(l\Phi) \cdot dl\Phi \tag{5}$$

$$B_2(\Phi, \Psi) = \Psi^*(l\Phi) \cdot S\Phi \tag{6}$$

$$B_5(\Phi, \Psi) = \begin{vmatrix} \Psi^* S\Phi & S\Psi \\ (\Psi^* S\Phi)' & (S\Psi)' \end{vmatrix} \tag{7}$$

$$\begin{aligned} B_7(\Phi, \Psi) = 2\begin{vmatrix} \Psi^* S\Phi & S\Psi \\ (\Psi^* S\Phi)''' & (S\Psi)''' \end{vmatrix} - 9\begin{vmatrix} (\Psi^* S\Phi)' & (S\Psi)' \\ (\Psi^* S\Phi)'' & (S\Psi)'' \end{vmatrix} \\ - \tfrac{32}{3}(S\Psi + S(\Phi \circ \Psi))B_5(\Phi, \Psi) \end{aligned} \tag{8}$$

Remark. The central extension of $\mathrm{Diff}^+(S^1)$ by $\mathcal{F}_0 \simeq C^\infty(S^1)$ is in fact, a semi-direct product of the Bott-Virasoro group by the module of functions: it is given by the Thurston-Bott cocycle.

2.2. Extensions of the Lie algebra Vect(S^1)

The following theorem classifies non-trivial extensions (2).

Proposition 1. *The cohomology group*

$$H^2(\mathrm{Vect}(S^1); \mathcal{F}_\lambda) = \begin{cases} \mathbf{R}^2 & , \lambda = 0, 1, 2 \\ \mathbf{R} & , \lambda = 5, 7 \\ 0 & . \lambda \neq 0, 1, 2, 5, 7 \end{cases}$$

In other words, there exist 2 non-isomorphic non-trivial extensions of $\mathrm{Vect}(S^1)$ by the module $\mathcal{F}_\lambda$ for $\lambda = 1, 2$; and an unique non-trivial extension for each $\lambda = 0, 5, 7$.

Let us describe 2-cocycles on $\mathrm{Vect}(S^1)$ with values in $\mathcal{F}_\lambda$ representing the non-trivial cohomology classes.

Theorem 3. *The cohomology groups $H^2(\mathrm{Vect}(S^1); \mathcal{F}_\lambda)$, where $\lambda = 0, 1, 2, 5, 7$ are generated by the following 8 non-trivial 2-cocycles:*

$$\bar{c}_0(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \tag{9}$$

$$c_0(f,g) = const(f,g) = \omega(f,g) \tag{10}$$

$$c_1(f,g) = \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} dx \tag{11}$$

$$\bar{c}_1(f,g) = \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix} dx \tag{12}$$

$$c_2(f,g) = \begin{vmatrix} f' & g' \\ f''' & g''' \end{vmatrix} (dx)^2 \tag{13}$$

$$\bar{c}_2(f,g) = \begin{vmatrix} f & g \\ f''' & g''' \end{vmatrix} (dx)^2 \tag{14}$$

$$c_5(f,g) = \begin{vmatrix} f''' & g''' \\ f^{(IV)} & g^{(IV)} \end{vmatrix} (dx)^5 \tag{15}$$

$$c_7(f,g) = \left(2\begin{vmatrix} f''' & g''' \\ f^{(VI)} & g^{(VI)} \end{vmatrix} - 9\begin{vmatrix} f^{(IV)} & g^{(IV)} \\ f^{(V)} & g^{(V)} \end{vmatrix}\right)(dx)^7 \tag{16}$$

Let us denote g_i the Lie algebras given by the non-trivial cocycles c_i and $\bar{g}_i$ the Lie algebras given by the non-trivial cocycles $\bar{c}_i$.

Remark 1. The algebra cocycles c_0, c_1, c_2, c_5, c_7 correspond to the group cocycles B_0, B_1, B_2, B_5, B_7. The algebra cocycles $\bar{c}_0$, $\bar{c}_1$, $\bar{c}_2$ can not be 'integrated' to the group $\mathrm{Diff}^+(S^1)$.

2. The Lie algebra g_0 is a semi-direct product of the Virasoro algebra by the module of functions.

3. The Lie algebra g_5 was considered in [OR].

2.3. Central extensions

Each of the constructed Lie groups and Lie algebras (except G_0, g_0), has at least one non-trivial central extension given by prolongations of the Thurston-Bott and Gelfand-Fuchs cocycles. Thus, one has 4 Lie groups which are non-trivial extensions of the Virasoro group and 7 Lie algebras which are non-trivial extensions of the Virasoro algebra, as described through the following diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{F}_\lambda & = & \mathcal{F}_\lambda & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{R} & \to & \hat{g}_\lambda & \to & g_\lambda & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathbf{R} & \to & Vir & \to & Vect(S^1) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

An interesting fact is that the Lie algebras $\bar{g}_1$ and g_1 possess new central extensions.

Let us give here the complete list of non-trivial central extensions of the constructed Lie algebras.

Proposition 2. *Non-trivial central extensions of the Lie algebras g_i and $\bar{g}_i$ are given by the following list.*

(1) Each one of the Lie algebras g_1, g_2, g_5, g_7 and $\bar{g}_0$, $\bar{g}_1$, $\bar{g}_2$, has a non-trivial central extension given by:

$$c((f, a), (g, b)) = \omega(f, g)$$

There exist two more non-trivial central extensions:

(2) A central extension of the Lie algebra $\bar{g}_1$ given by the cocycle:

$$c((f, a), (g, b)) = \int_{S^1} (fb - ga)\, dx \tag{17}$$

(3) A central extension of the Lie algebra g_1 given by the cocycle:

$$c((f, a), (g, b)) = \int_{S^1} (f'b - g'a)\, dx \tag{18}$$

Let us consider also central extensions of a semi-direct product: $\mathrm{Vect}(S^1) \triangleright \mathcal{F}_\lambda$.

Proposition 3. *The cohomology group*

$$H^2(\mathrm{Vect}(S^1) \triangleright \mathcal{F}_\lambda; \mathbf{R}) = \begin{cases} \mathbf{R}^3, \ \lambda = 0, 1 \\ \mathbf{R}, \ \lambda \neq 0, 1 \end{cases}$$

The cocycles which define central extensions of $\mathrm{Vect}(S^1) \triangleright \mathcal{F}_0$ nonequivalent to the Virasoro extension, are:

$$c((f, a), (g, b)) = \int_{S^1} (f''b - g''a)\, dx \tag{19}$$

and

$$c((f, a), (g, b)) = \int_{S^1} (adb - bda) \tag{20}$$

One remarks, that the last cocycle defines the structure of infinite-dimensional Heisenberg algebra on the space $\mathcal{F}_0 \simeq C^\infty(S^1)$. This Lie algebra was considered in [ACKP].

In the case of $\mathrm{Vect}(S^1) \triangleright \mathcal{F}_1$ non-trivial cocycles are also given by the formulae (17) and (18).

3. LIE ALGEBRAS g_5 AND g_7 AND MOYAL BRACKET

Consider the standard Poisson bracket on $\mathbf{R}^2$:

$$\{F, G\} = F_q G_p - F_p G_q$$

The Lie algebra of functions on $\mathbf{R}^2$ has a non-trivial formal deformation which is called the Moyal bracket (see e.g. [FLS]).

Consider the following bilinear operations invariant under the action of the group $SL(2, \mathbf{R})$ of all linear symplectic transformations of $\mathbf{R}^2$:

$$\{F, G\}_m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{\partial^m F}{\partial^{m-i} q \partial^i p} \frac{\partial^m G}{\partial^i q \partial^{m-i} p} \tag{21}$$

For example, $\{F, G\}_0 = FG$, $\{F, G\}_1 = \{F, G\}$,

$$\{F, G\}_2 = F_{qq}G_{pp} - 2F_{qp}G_{qp} + F_{pp}G_{qq}$$

$$\{F, G\}_3 = F_{qqq}G_{ppp} - 3F_{qqp}G_{qpp} + 3F_{qpp}G_{qqp} - F_{ppp}G_{qqq}$$

etc.
The Moyal bracket is defined as a formal series:

$$\{F, G\}_t = \{F, G\}_1 + \frac{t}{6}\{F, G\}_3 + \frac{t^2}{5!}\{F, G\}_5 + \dots \tag{22}$$

It verifies the Jacobi identity and defines a Lie algebra structure on the space $C^\infty(\mathbf{R}^2)[[t]]$ of formal series in t with functional coefficients.

The relationship between the Lie algebras g_5, g_7 and the Moyal bracket is based on the following realization of tensor-densities by homogeneous functions on $\mathbf{R}^2$.

3.1. Tensor-densities on S^1 and homogeneous functions on $\mathbf{R}^2$

Consider homogeneous functions on $\mathbf{R}^2 \setminus \{0\}$ (with singularities in the origin): $F(\kappa q, \kappa p) = \kappa^\lambda F(q, p)$, where $\kappa > 0$.

Lemma 1. *(i) The space of homogeneous functions of degree 2 on $\mathbf{R}^2 \setminus \{0\}$ is a subalgebra of the Poisson Lie algebra $C^\infty(\mathbf{R}^2 \setminus \{0\})$*
(ii) This subalgebra is isomorphic to $\mathrm{Vect}(S^1)$.

Proof. The isomorphism is given by: $f = f(x)d/dx \leftrightarrow F = r^2 f(\phi)$ where r, ϕ are the polar coordinates.

Moreover, the Poisson bracket defines a structure of $\mathrm{Vect}(S^1)$-*module* on the space of homogeneous functions. Let F and G be homogeneous functions of degree 2 and λ respectively. Then, their Poisson bracket $\{F, G\}$ is again a homogeneous function of degree λ.

Lemma 2. *There exists an isomorphism of* $\mathrm{Vect}(S^1)$*-modules: the space of homogeneous functions of degree λ on $\mathbf{R}^2 \setminus \{0\}$ and the space of tensor-densities $\mathcal{F}_{-\frac{\lambda}{2}}$.*

Proof. Take the mapping given by:

$$f = f(x)(dx)^{-\frac{\lambda}{2}} \longleftrightarrow F = r^\lambda f(\phi) \tag{23}$$

It is easy to verify that the Poisson bracket corresponds to the Lie derivative by this mapping: $L_f g \longleftrightarrow \{F, G\}$.

Corollary. *(i) There exists a series of SL_2-invariant operations*

$$\{.,.\}_n : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+n}$$

on the space of tensor-densities on S^1.

(ii) Moreover, there exists a Lie structure given by the Moyal bracket (21) on the space of tensor-densities on S^1.

Remark. The isomorphism (23) is in fact, much more general. It is valid in the case on an arbitrary contact manifold (see [OR]).

3.2. Cocycle c_5

Let us substitute two vector fields $f = f(x)/dx$, $g = g(x)d/dx$ (corresponding to homogeneous functions of degree 2 on $\mathbf{R}^2 \setminus \{0\}$) to the Moyal bracket. We get a formal series in t with coefficients in the space of tensor-densities.

Lemma 3. *(i) If $f, g \in \mathrm{Vect}(S^1)$, then $\{f, g\}_3 = \{f, g\}_5 \equiv 0$.*
(ii) The first non-zero term: $\{f, g\}_7 \in \mathcal{F}_5$ is proportional to c_5.

Proof. Let F, G be two functions on $\mathbf{R}^2 \setminus \{0\}$, then $\{F, G\}_7$ is a homogeneous function of degree -10. Thus, $\{f, g\}_7 \in \mathcal{F}_5$. It is easy to verify that $\{f, g\}_7 = 20160 c_5$.

Consequently, c_5 is a 2-cocycle. Indeed, the Jacobi identity for the Moyal bracket implies that the first non-zero term in this series is a 2-cocycle on $\mathrm{Vect}(S^1)$.

3.3. Cocycle c_7

Lemma 4. *The second non-zero term of the Moyal bracket: $\{f, g\}_9 \in \mathcal{F}_7$ is proportional to c_7.*

Proof: straightforward.

Let us prove that $\{f, g\}_9$ is again a 2-cocycle on $\mathrm{Vect}(S^1)$. The Jacobi identity for the Moyal bracket $\{f, g\}_t$ implies:

$$\{f, \{g, h\}_9\}_1 + \{f, \{g, h\}_1\}_9 + \{f, \{g, h\}_7\}_3 \ (+cycle) = 0$$

for any $f, g, h \in \mathrm{Vect}(S^1)$. One checks that the expression $\{f, \{g, h\}_7\}_3$ is proportional to $f'''(g'''h^{IV} - g^{IV}h''')$, and so one gets: $\{f, \{g, h\}_7\}_3$ $(+\,cycle) = 0$. We obtain the following relation:

$$\{f, \{g, h\}_9\}_1 + \{f, \{g, h\}_1\}_9 \ (+cycle) = 0$$

which means that $\{f, g\}_9$ is a 2-cocycle. Indeed, recall that for any tensor-density a, $\{f, a\}_1 = L_f a$. Therefore, this relation coincides with:

$$L_f\{g, h\}_9 + \{f, \{g, h\}_1\}_9 \ (+cycle) = 0$$

which is exactly the relation $d\{.,.\}_9 = 0$.

3.4. Group cocycle B_7

Let us recall that the mapping $s : f(x)d/dx \to f'''(x)(dx)^2$ is a 2-cocycle (an algebraic analogue of the Schwarzian derivative) generating the cohomology group $H^1(\mathrm{Vect}(S^1); \mathcal{F}_2)$. It is easy to check, that the following relation is satisfied:

$$\{f, g\}_9 = 504\{f''', g'''\}_3$$

Thus, it is natural to look for a group version of the cocycle c_7 in the following form: $B(\Phi, \Psi) = \{\Psi^* S(\Phi), S(\Psi)\}_3$. However, this formula does not define a cocycle on the group $\mathrm{Diff}^+(S^1)$ since the operation $A \otimes B \mapsto \{A, B\}_3$ is not invariant.

Lemma 5. *Let $a, b \in \mathcal{F}_2$, Φ is a diffeomorphism, then*

$$\{\Phi^* a, \Phi^* b\}_3 = \Phi^*\{a, b\}_3 + 480 S(\Phi)\Phi^*\{a, b\}_1$$

where $S(\Phi)$ is the Schwarzian derivative.

Proof: straightforward.
Let us verify now that B_7 is a 2-cocycle. One must show that

$$dB_7(\Phi, \Psi, \Xi) = B_7(\Phi, \Psi \circ \Xi) - B_7(\Phi \circ \Psi, \Xi) + B_7(\Psi, \Xi) - \Xi^* B_7(\Phi, \Psi) = 0$$

for any $\Phi, \Psi, \Xi \in \mathrm{Diff}^+(S^1)$.

Let us take the expression: $\bar{B}(\Phi, \Psi) = \{\Psi^* S(\Phi), S(\Psi)\}_3$. Lemma 5 implies: $d\bar{B}(\Phi, \Psi, \Xi) = 480 S(\Xi) \cdot \Xi^*\{\Psi^* S(\Phi), S(\Psi)\}_1$. Consider another expression $\tilde{B}(\Phi, \Psi) = (S(\Psi) + S(\Phi \circ \Psi))\{S(\Phi), S(\Psi)\}_1$. A simple computation gives: $d\tilde{B}(\Phi, \Psi, \Xi) = 3S(\Xi) \cdot \Xi^*\{\Psi^* S(\Phi), S(\Psi)\}_1$. Thus, the expression

$$B(\Phi, \Psi) = \{S\Phi, S\Psi\}_3 - 160(S\Psi + S\Phi \circ \Psi)\{S\Phi, S\Psi\}_1$$

is a 2-cocycle. One checks that this formula is proportional to the formula (8).

4. PROOFS OF MAIN THEOREMS

The proofs are quite simple but they use high technique of cohomology of infinite-dimensional Lie algebras. All the necessary details can be found in the book of D.B. Fuchs.

Proof of Proposition 1. Cohomology groups of Lie algebras of smooth vector fields with coefficients in modules of tensor fields, where calculated essentially in [GF 2], [T] (see also [F], p. 147). Let us recall here the answer.
The cohomology group

$$H^q(\mathrm{Vect}(S^1); \mathcal{F}_{\frac{3r^2 \pm r}{2}}) = H^{q-r}(Y(S^1); \mathbf{R})$$

where $Y(S^1) \simeq S^1 \times S^1 \times \Omega S^3$, and $H^q(\mathrm{Vect}(S^1); \mathcal{F}_\lambda = 0$ if $\lambda \neq 0, 1, 2, 5, 7, \ldots, \frac{3r^2 \pm r}{2}, \ldots$. The cohomology ring $H^*(Y(S^1); \mathbf{R})$ is a free anti-commutative algebra with generators in dimensions 1,1,2.

One obtains immediately the proof of Proposition 1.

Proof of Theorem 3. The cohomology ring $H^2(\mathrm{Vect}(S^1); C^\infty(S^1))$ is generated by three cocycles:

$$f(x)\frac{d}{dx} \mapsto f(x), \quad f(x)\frac{d}{dx} \mapsto f'(x), \quad \omega(f, g)$$

(cf. Theorem 2.4.12 of [F]). Thus, one has two non trivial cocycles (9), (10).

$H^2(\mathrm{Vect}(S^1); \mathcal{F}_1)$ is a free $H^2(\mathrm{Vect}(S^1); C^\infty(S^1))$-module. This fact implies formulae (11), (12) for the generating cocycles.

The cocycles (13) and (14) can be obtained from the isomorphism: $\mathcal{F}_2 \simeq \mathcal{F}_1 \otimes \mathcal{F}_1$.

The proof that formulae (15) and (16) define 2-cocycles on $\mathrm{Vect}(S^1)$ is given in the sect. 3. (For the cocycle (15) it follows also from the formula (7) for the group cocycle). These cocycles define non-trivial cohomology classes. Indeed, one can check, that the Lie algebras g_5 and g_7 do not verify the same identities as $\mathrm{Vect}(S^1)$.

Proof of Theorem 1. The Van Est cohomology ring for the Lie group $\mathrm{Diff}^+(S^1)$ is defined by the following isomorphism (cf [F], p.244):

$$H_c^*(\mathrm{Diff}^+(S^1); \mathcal{F}_\lambda) \simeq H^*(\mathrm{Vect}(S^1), S0(2); \mathcal{F}_\lambda)$$

(where $SO(2) \subset \mathrm{Diff}^+(S^1)$ is the maximal compact subgroup of 'rotations' of S^1). Thus, $H_c^*(\mathrm{Diff}^+(S^1); \mathcal{F}_\lambda) = 0$ if $\lambda \neq 0, 1, 2, 5, 7$.
The cohomology ring $H^*(\mathrm{Vect}(S^1), SO(2); \mathcal{F}_\lambda)$ is defined by cochains which are identically zero on the subalgebra $so \subset \mathrm{Vect}(S^1)$ (that means, by cochains given by differential operators without zero order terms). To prove that the cocycles $\bar{c}_0$, $\bar{c}_1$, $\bar{c}_2$ can not be integrated to $\mathrm{Diff}^+(S^1)$, one must show now that the cohomology classes of these cocycles can not be represented by such cocycles.

Suppose, that there exists a cocycle $\bar{c}_0'$ cohomological to $\bar{c}_0$ such that $\bar{c}_0'(f, g) = \sum_{i,j\geq 1} c_{ij} f^{(i)} g^{(j)}$. Then $\bar{c}_0 - \bar{c}_0'$ is a differential of some 1-cochain σ. But it is easy to verify that the expression $d\sigma(f, g)$ depends only on the derivatives of f and g. The contradiction means that there is no cohomology class in $H^*(\mathrm{Vect}(S^1), SO(2); \mathcal{F}_0)$ corresponding to the cocycle $\bar{c}_0$. Thus, $H^*(\mathrm{Vect}(S^1), SO(2); \mathcal{F}_0) = 0$. Analogous arguments are valid for $\bar{c}_1$, $\bar{c}_2$.

To finish the proof of the theorem, one should show that the cocycles $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_5$, $\bar{c}_7$ correspond to some group cocycles, but it follows from formulae given by Theorem 2.

Proof of Theorem 2. It follows from the construction, that the mappings (4)-(7) are cocycles on $\mathrm{Diff}(S^1)$. It was proved in the sect. 3 that the formula (8) defines a cocycle. The Lie algebra cocycles associated with the group cocycles (4)-(8), are $\bar{c}_0$, $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_5$, $\bar{c}_7$ correspondingly. This proves that the cocycles (4)-(8) represent non-trivial cohomology classes.

Proof of Propositions 2 and 3. In general, let L be a Lie algebra and M an L-module, consider a Lie structur L_M on $L \oplus M$ with the commutator:

$$[(X, \xi)(Y, \eta)] = ([X, Y], X(\eta) - Y(\xi) + \alpha(X, Y))$$

where α is a 2-cocycle: $\alpha \in Z^2(L; M)$. Let c be a 2-cocycle on L_M with scalar values. Then:

(1) The restriction $c\,|_{L\otimes L}$ is a 2 cocycle on L.

(2) The condition $dc(X, Y, Z) = 0$ implies: $c(\alpha(X, Y), Z)\ (+cycle) = 0$.

(3) $dc(X, a, b) = 0$ implies: $c(X(a), b) + c(a, X(b)) = 0$. That is, the restriction $\tilde{c} = c\,|_{M\otimes M}$ is L-invariant.

(4) $dc(X, Y, a) = 0$ means:

$$c(\alpha(f, g), a) = c(X(a), Y) - c(Y(a), X) - c([X, Y], a)$$

Thus, the linear mapping $\bar{c} : L \to M^*$ given by the relation: $\bar{c}(X)a = c(X, a)$ satisfies the following condition:

$$d\bar{c} = \tilde{c} \circ \alpha$$

Let us apply these facts to the case: $L = \mathrm{Vect}(S^1)$, $M = \mathcal{F}_\lambda$.

First of all, the restriction $c\,|_{L\otimes L}$ is proportional to the Gelfand-Fuchs cocycle.

The unique invariant bilinear mapping $\tilde{c} : \mathcal{F}_\lambda \wedge \mathcal{F}_\lambda \to \mathbf{R}$ is

$$\tilde{c}(a, b) = \int_{S^1} (adb - bda)$$

if $\lambda = 0$. One obtains the cocycle (20) for the semi-direct product. This cocycle can not be extended on the Lie algebra g_0, since the property $d\bar{c} = \tilde{c} \circ \alpha$ is not satisfied. If $\lambda \neq 0$, then $\bar{c}$ is 1-cocycle.

The dual module $\mathcal{F}_\lambda^*$ is isomorphic to $\mathcal{F}_{1-\lambda}$. Consider the cohomology group: $H^1(\mathrm{Vect}(S^1); \mathcal{F}_{1-\lambda})$, where $\lambda = 0, 1, 2, 5, 7$. It is not trivial in two cases: $\lambda = 0, 1$. Otherwise, there is no non-trivial extensions which are not equivalent to the Virasoro one. The group $H^1(\mathrm{Vect}(S^1); \mathcal{F}_0)$ is generated by the following two elements:

$$f(x)d/dx \mapsto f(x) \quad f(x)d/dx \mapsto f'(x)$$

(the first part of Theorem 2.4.12 of [F]). One obtains the cocycles (17) and (18) which satisfy the condition 2). The group $H^1(\mathrm{Vect}(S^1); \mathcal{F}_1)$ has one generator

$$f(x)d/dx \mapsto f''(x)dx$$

(not $f \mapsto f'dx$: there is a misprint in [F] here). One gets the cocycle (19).

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