subbundle E_1 which satisfies the following conditions:

1) the restriction of the scalar product to E_1 is nondegenerate and indefinite;

2) the Peterson form of the subbundle TM^n effects an isomorphism of the tangent bundle $Hom(E_1, E/E_1)$, all of whose sections have common kernel.

Here the Weyl tensor of the metric $g(X, Y) = \langle \nabla_X s, \nabla_Y s \rangle$ (where s is an arbitrary section of E_1) coincides with the curvature of the connection, projected to the orthogonal complement to E_1 (after identifying the latter with an $End(TM^n)$-valued 2-form).

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KORTEWEG–DE VRIES SUPEREQUATION AS AN EULER EQUATION

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It is known that the Korteweg–de Vries (KdV) equation is associated with the Virasoro algebra (see [2; 3]). In [6] (see also [5; 7]) the Korteweg–de Vries superequation (sKdV) was proposed, corresponding to the simplest superanalogues of the Virasoro algebra, i.e., the Neveu–Schwarz and the Ramond superalgebras. The present note concerns one geometric aspect of this connection. Its goal is to show that (s)KdV is the Euler equation on the corresponding groups, i.e., the equation of the geodesics of some one-sidely invariant metrics.

1. Recall the well-known definitions from mechanics (see [1]). Let $\mathfrak{G}$ be a Lie (super)algebra. The (right-)invariant metric on the corresponding group is uniquely defined by symmetric operator $A: \mathfrak{G} \to \mathfrak{G}^*$, which is called the inertia operator of an extended rigid body. It is given by the conveyance over the group of (right) shifts of the scalar product on $\mathfrak{G}$:

$$(\xi, \eta) = \langle A\xi, \eta \rangle, \quad \text{where } \xi, \eta \in \mathfrak{G}.$$

Let $g(t)$ be a geodesic of the right-invariant metric on the group. An element $\omega = R_{g^{-1}}\dot{g}$ of the Lie algebra is called the angular velocity of the body. The element $M = A\omega$ of $\mathfrak{G}^*$ is called the kinetic moment with respect to the body.

The moment vector with respect to the body satisfies equation $dM/dt = ad^*_\omega M$ which is called the Euler equation.

On the dual space to the Lie (super)algebra there exists a natural Poisson–Lie–Berezin–Kirillov bracket. Let F and G be functions on $\mathfrak{G}^*$. Then

$$\{F, G\}(M) = \langle M, [dF, dG] \rangle \tag{1}$$

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(the differentials dF and dG, taken at point M, lie in the closure of $\mathfrak{G}$ and their commutator is defined).

The Euler equation preserves the orbits of the coadjoint representation of $\mathfrak{G}$ and is a Hamiltonian equation with Hamiltonian $H(M) = \langle M, A^{-1}M\rangle$, which is called the energy.

2. Recall that the Virasoro algebra V is the unique nontrivial central extension by means of R of the Lie algebra Vect S^1 (of vector fields of the circle). Its elements can be identified with the pairs (2π-periodic function, number). Then a commutator in V takes the form

$$[(f(x), a), (g(x), b)] = (f(x)g'(x) - g(x)f'(x), \textstyle\int f'(x)g''(x)\,dx)$$

(here and below the integration is over interval [0, 2π]).

Space V* can be identified with pairs (2π-periodic function, number). Bracket (1) on functions on V* is given by the formula

$$\{F, G\}(u(x), c) = \int \left[\left(\frac{\delta F}{\delta u}\left(\frac{\delta G}{\delta u}\right)' - \frac{\delta G}{\delta u}\left(\frac{\delta F}{\delta u}\right)'\right)u + c\left(\frac{\delta F}{\delta u}\right)'\left(\frac{\delta G}{\delta u}\right)''\right]dx \tag{1'}$$

(as functions on V* it is sufficient to consider integrals of differential polynomials (see [2])), $\delta F/\delta u(x)$ is defined by the equation

$$d/d\varepsilon F(u + \varepsilon v, c)|_{\varepsilon=0} = \int \frac{\delta F}{\delta u}(x)v(x)\,dx.$$

The Hamiltonian equation with Hamiltonian F takes the form

$$\dot{u} = 2\left(\frac{\delta F}{\delta u}\right)'u + \frac{\delta F}{\delta u}u' - c\left(\frac{\delta F}{\delta u}\right)''', \quad \dot{c} = 0. \tag{2'}$$

Consider inertia operator A such that $A(f, a) = (f, a) \in V^*$. It defines a scalar product on V: $((f, a), (g, b)) = \int fg\,dx + ab$.

Proposition 1. The Euler equation corresponding to inertia operator A coincides with the KdV equation.

Proof. The energy on V* equals $H(u, c) = {}^1/_2 \int u^2(x)dx + c^2/2$. The Hamiltonian equation

$$\dot{u} = 3u'u - cu''' \tag{3}$$

corresponds to it.

3. The Neveu–Schwarz (NS) and Ramond (R) superalgebras are the simplest superanalogues of the Virasoro algebra. They enter into a number of so-called Lie superalgebras of string theories [4]. The even parts of NS and R coincide with the Virasoro algebra, and the odd parts can be identified with functions of one variable such as $\psi(x + 2\pi) = -\psi(x)$ for NS and $\psi(x + 2\pi) = \psi(x)$ for R. A commutator in NS and R takes the form

$$[(f, \varphi, a), (g, \psi, b)] = (fg' - gf' + \varphi\psi/2, f\psi' - \psi f'/2 - g\varphi' + \varphi g'/2, \textstyle\int (f'g'' + \varphi'\psi'/2)).$$

The dual spaces NS* and R* can be identified with a set of triples $(u(x), \xi(x), c)$, where $u(x)$ is a function on S^1 with values in the even part, and $\xi(x)$ is a function such that $\xi(x + 2\pi) = -\xi(x)$ (respectively, $\xi(x)$) with values in the odd part of some supercommutative ring.

Bracket (1) on NS* and R* (see [5; 6]) is defined by operator

$$P(u, \xi, c) = \begin{pmatrix} c\partial^3 - u\partial - \partial u & -\partial\xi/2 - \xi\partial \\ -\partial\xi - \xi\partial/2 & (c\partial^2 - u)/2 \end{pmatrix},$$

where $(u, \xi, c) \in NS^*(R^*)$, $\partial = d/dx$, and takes the form

$$\{F, G\}(u, \xi, c) = ((\delta F/\delta u, \delta F/\delta\xi), P\begin{pmatrix}\delta G/\delta u \\ \delta G/\delta\xi\end{pmatrix}). \tag{1''}$$

The Hamiltonian equation with Hamiltonian F is defined by formula

$$\begin{pmatrix} \dot{u} \\ \dot{\xi} \end{pmatrix} = -P \begin{pmatrix} \delta F/\delta u \\ \delta F/\delta \xi \end{pmatrix}. \tag{2''}$$

Consider inertia operator A_S: $NS \to NS^*$ $(R \to R^*)$:

$$A_s(f(x), \varphi(x), a) = (f(x), {}^1/_4\partial^{-1}\varphi(x), c).$$

In the case of the NS superalgebra, it is uniquely defined and is nondegenerate, since integration operator ∂^{-1} acts in the space of functions with null average. For the R superalgebra, ∂^{-1} is defined by the formula $(\partial^{-1}u)(x) = \int_0^x \left(u - \int u\right) dy - \int\int_0^x \left(u - \int u\right) dy$. The corresponding metric proves to be degenerate.

Proposition 2. The Euler equation corresponding to inertia operator A_S coincides with the Korteweg–de Vries superequation from [6].

Proof. The energy equals $H(u, \xi, c) = {}^1/_2\int(u^2(x) = 4\xi'(x)\xi(x))dx + c^2/2$. By formula (2''), the Hamiltonian equation with Hamiltonian H has the form

$$\dot{u} = 3u'u - cu''' - 6\xi''\xi,$$
$$\dot{\xi} = 3u\xi' + 3u'\xi/2 - 2c\xi'''.$$

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