Vector fields in the presence of a contact structure

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Abstract

We consider the Lie algebra of all vector fields on a contact manifold as a module over the Lie subalgebra of contact vector fields. This module is split into a direct sum of two submodules: the contact algebra itself and the space of tangent vector fields. We study the geometric nature of these two modules.

1 Introduction

Let M be a (real) smooth manifold and $\mathrm{Vect}(M)$ the Lie algebra of all smooth vector fields on M with complex coefficients. We consider the case when M is $(2n+1)$-dimensional and can be equipped with a contact structure. For instance, if $\dim M = 3$, and M is compact and orientable, then the famous theorem of 3-dimensional topology states that there is always a contact structure on M.

Let $\mathrm{CVect}(M)$ be the Lie algebra of smooth vector fields on M preserving the contact structure. This Lie algebra naturally acts on $\mathrm{Vect}(M)$ (by Lie bracket). We will study the structure of $\mathrm{Vect}(M)$ as a $\mathrm{CVect}(M)$-module. First, we observe that $\mathrm{Vect}(M)$ is split, as a $\mathrm{CVect}(M)$-module, into a direct sum of two submodules:

$$\mathrm{Vect}(M) \cong \mathrm{CVect}(M) \oplus \mathrm{TVect}(M)$$

where $\mathrm{TVect}(M)$ is the space of vector fields tangent to the contact distribution. Note that the latter space is a $\mathrm{CVect}(M)$-module but not a Lie subalgebra of $\mathrm{Vect}(M)$.

The main purpose of this paper is to study the two above spaces geometrically. The most important notion for us is that of *invariance*. All the maps and isomorphisms we consider are invariant with respect to the group of contact diffeomorphisms of M. Since we consider only local maps, this is equivalent to the invariance with respect to the action of the Lie algebra $\mathrm{CVect}(M)$.

It is known, see [5, 6], that the adjoint action of $\mathrm{CVect}(M)$ has the following geometric interpretation:

$$\mathrm{CVect}(M) \cong \mathcal{F}_{-\frac{1}{n+1}}(M),$$

where $\mathcal{F}_{-\frac{1}{n+1}}(M)$ is the space of (complex valued) tensor densities of degree $-\frac{1}{n+1}$ on M, that is, of sections of the line bundle

$$\left(\wedge^{2n+1} T^*_{\mathbb{C}} M\right)^{-\frac{1}{n+1}} \to M.$$

In particular, this provides the existence of a nonlinear invariant functional on $\mathrm{CVect}(M)$ defined on the contact vector fields with nonvanishing contact Hamiltonians.

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The analogous interpretation of $\mathrm{TVect}(M)$ is more complicated:

$$\mathrm{TVect}(M) \cong \Omega_0^2(M) \otimes \mathcal{F}_{-\frac{2}{n+1}}(M),$$

where $\Omega_0^2(M)$ is the space of 2-forms on M vanishing on the contact distribution. Here and below the tensor products are defined over $C^\infty(M)$.

We study the relations between $\mathrm{TVect}(M)$ and $\mathrm{CVect}(M)$. We prove the existence of a non-degenerate skew-symmetric invariant bilinear map

$$\mathcal{B} : \mathrm{TVect}(M) \wedge \mathrm{TVect}(M) \to \mathrm{CVect}(M)$$

that measures the non-integrability, i.e., the failure of the Lie bracket of two tangent vector fields to remain tangent.

In order to provide explicit formulæ, we introduce a notion of Heisenberg structure on M. Usually, to write explicit formulæ in contact geometry, one uses the Darboux coordinates. However, this is not the best way to proceed (as already noticed in [4]). The Heisenberg structure provides a universal expression for a contact vector field and its actions.

2 Contact and tangent vector fields

In this section we recall the basic definitions of contact geometry. We then prove our first statement on a decomposition of the Lie algebra of all smooth vector fields viewed as a module over the Lie algebra of contact vector fields.

2.1 Main definitions

Let M be a $(2n+1)$-dimensional manifold. A contact structure on M is a codimension 1 distribution ξ which is completely non-integrable. The distribution ξ can be locally defined as the kernel of a differential 1-form α defined up to multiplication by a nonvanishing function. Assume that M is orientable, then the form α can be globally defined on M. Complete non-integrability means that

$$\Omega := \alpha \wedge (d\alpha)^n \neq 0 \tag{1}$$

everywhere on M. In other words, Ω is a volume form. The above condition is also equivalent to the fact that the restriction $d\alpha|_\xi$ to any contact hyperplane is a non-degenerate 2-form. In particular, $\ker d\alpha$ is one-dimensional.

A vector field X on M is a *contact vector field* if it preserves the contact distribution ξ. In terms of contact forms this means that for every contact form α, the Lie derivative of α with respect to X is proportional to α:

$$L_X\alpha = f_X\alpha \tag{2}$$

where $f_X \in C^\infty(M)$. The space of all contact vector fields (with complex coefficients) is a Lie algebra that we denote $\mathrm{CVect}(M)$.

Let us now fix a contact form α. A contact vector field X is called strictly contact if it preserves α, in other words, if $f_X = 0$ everywhere on M. Strictly contact vector fields form a Lie subalgebra of $\mathrm{CVect}(M)$. There is one particular strictly contact vector field Z called the *Reeb field* (or characteristic vector field). It is defined by the following two properties:

$$Z \in \ker d\alpha, \qquad \alpha(Z) \equiv 1.$$

We will also consider the space, $\mathrm{TVect}(M)$, of (complex) vector fields *tangent* to the contact distribution. This space is not a Lie subalgebra of $\mathrm{Vect}(M)$ that follows from non-integrability of the contact distribution.

2.2 The decomposition of $\mathrm{Vect}(M)$

Let $\mathrm{Vect}(M)$ be the Lie algebra of all smooth vector fields (with complex coefficients) on M. The Lie bracket defines a natural action of $\mathrm{CVect}(M)$ on $\mathrm{Vect}(M)$. In particular, the Lie bracket of a contact vector field with a tangent vector field is again a tangent vector field. Therefore, $\mathrm{TVect}(M)$ is a module over $\mathrm{CVect}(M)$.

Proposition 2.1. *The space* $\mathrm{Vect}(M)$ *is split into a direct sum of two* $\mathrm{CVect}(M)$*-modules:*

$$\mathrm{Vect}(M) \cong \mathrm{CVect}(M) \oplus \mathrm{TVect}(M).$$

Proof. Both spaces in the right hand side are $\mathrm{CVect}(M)$-modules. It then remains to check that every vector field can be uniquely decomposed into a sum of a contact vector field and a tangent vector field.

Given a vector field X, there exists a tangent vector field Y such that $X - Y$ is contact. Indeed, set $\beta = L_X\alpha$ and consider the restriction of $\beta|_\xi$ to a contact hyperplane. If Y is a tangent vector field then $L_Y\alpha = i_Y(d\alpha)$. Since $d\alpha$ in non-degenerate on ξ, then for any 1-form β there exists a tangent field Y such that $i_Y(d\alpha)|_\xi = \beta|_\xi$. This means $X - Y$ is contact.

Furthermore, the intersection of $\mathrm{CVect}(M)$ and $\mathrm{TVect}(M)$ is zero. Indeed, let X be a non-zero vector field which is contact and tangent at the same time. Then $L_X\alpha = f\alpha$ for some function f and $L_X\alpha = i_X(d\alpha)$. Since $\ker f\alpha$ contains $\xi = \ker\alpha$ while the restriction $d\alpha|_\xi$ is non-degenerate, this is a contradiction. □

3 The adjoint representation of $\mathrm{CVect}(M)$

In this section we study the action of $\mathrm{CVect}(M)$ on itself.

3.1 Fixing a contact form: contact Hamiltonians

Let M be orientable, fix a contact form α on M. Every contact vector field X is then characterized by a function:

$$H = \alpha(X).$$

This is a one-to-one correspondence between $\mathrm{CVect}(M)$ and the space $C^\infty(M)$ of (complex valued) smooth functions on M, see, e.g., [1]. We can denote the contact vector field corresponding to H by X_H. The function H is called the contact Hamiltonian of X_H.

Example 3.1. The contact Hamiltonian of the Reeb field Z is the constant function $H \equiv 1$. Note also that the function f_X in (2) is given by the derivative $f_{X_H} = Z(H)$.

The Lie algebra $\mathrm{CVect}(M)$ is then identified with $C^\infty(M)$ equipped with the *Lagrange bracket* defined by

$$X_{\{H_1,H_2\}} := [X_{H_1}, X_{H_2}].$$

One checks that

$$\{H_1, H_2\} = X_{H_1}(H_2) - Z(H_1)\, H_2. \tag{3}$$

The formula expresses the adjoint representation of $\mathrm{CVect}(M)$ in terms of contact Hamiltonians. The second term in the right hand side shows that this action is different from the natural action of $\mathrm{CVect}(M)$ on $C^\infty(M)$. Let us now clarify the geometric meaning of this action.

3.2 Tensor densities on a contact manifold

Let M be an arbitrary smooth manifold of dimension d. A *tensor density* on M of degree $\lambda \in \mathbb{R}$ is a section of the line bundle $(\wedge^d T^*_{\mathbb{C}} M)^\lambda$. The space of λ-densities is denoted by $\mathcal{F}_\lambda(M)$.

Assume that M is orientable and fix a volume form Ω on M. This is a global section trivializing the above line bundle, so that $\mathcal{F}_\lambda(M)$ can be identified with $C^\infty(M)$. One then represents λ-densities in the form:

$$\varphi = f\,\Omega^\lambda,$$

where f is a function.

Example 3.2. The space $\mathcal{F}_0(M) \cong C^\infty(M)$ while the space $\mathcal{F}_1(M)$ is nothing but the space of differential d-forms.

If M is compact then there is an invariant functional

$$\int_M : \mathcal{F}_1(M) \to \mathbb{C}. \tag{4}$$

More generally, there is an invariant pairing

$$\langle \mathcal{F}_\lambda(M), \mathcal{F}_{1-\lambda}(M) \rangle \to \mathbb{C}$$

given by the integration of the product of tensor densities.

Let now M be a contact manifold of dimension $d = 2n+1$. In this case, there is another way to define tensor densities. Consider the $(2n+2)$-dimensional submanifold S of the cotangent bundle $T^*M \setminus M$ that consists of all non-zero covectors vanishing on the contact distribution ξ. The restriction to S of the canonical symplectic structure on T^*M defines a symplectic structure on S. The manifold S is called the *symplectization* of M (cf. [1, 2]). Clearly S is a line bundle over M, its sections are the 1-forms on M vanishing on ξ. Note that, in the case where M is orientable, S is a trivial line bundle over M.

There is a natural lift of $\mathrm{CVect}(M)$ to S. Indeed, a vector field X on M can be lifted to T^*M, and, if X is contact, then it preserves the subbundle S. The space of sections $\mathrm{Sec}(S)$ is therefore a $\mathrm{CVect}(M)$-module.

The sections of the bundle S can be viewed as tensor densities of degree $\frac{1}{n+1}$ on M.

Proposition 3.3. *There is a natural isomorphism of* $\mathrm{CVect}(M)$*-modules*

$$\mathrm{Sec}(S) \cong \mathcal{F}_{\frac{1}{n+1}}(M).$$

Proof. A section of S is a 1-form on M vanishing on the contact distribution. For every contact vector field X and a volume form Ω as in (1) one has

$$L_X \Omega = (n+1)\, f_X \Omega.$$

The Lie derivative of a tensor density of degree λ is then given by

$$L_X(f\,\Omega^\lambda) = (X(f) + \lambda(n+1) f_X f)\,\Omega^\lambda.$$

The result follows from formula (2). $\square$

One can now represent λ-densities in terms of a contact form: $\varphi = f\,\alpha^{(n+1)\lambda}$.

3.3 Contact Hamiltonian as a tensor density

In this section we identify the algebra $\mathrm{CVect}(M)$ with a space of tensor densities of degree $-\frac{1}{n+1}$ on M; the adjoint action is nothing but a Lie derivative on this space. The result of this section is known (see [5] and [6], Section 7.5) and given here for the sake of completeness.

Let us define a different version of contact Hamiltonian of a contact vector field X as a $-\frac{1}{n+1}$-density on M:

$$\mathcal{H} := \alpha(X)\,\alpha^{-1}.$$

An important feature of this definition is that it is independent of the choice of α. Let us denote $X_{\mathcal{H}}$ the corresponding contact vector field.

The space $\mathcal{F}_{-\frac{1}{n+1}}(M)$ is now identified with $\mathrm{CVect}(M)$. Moreover, the Lie bracket of contact vector fields corresponds to the Lie derivative.

Proposition 3.4. *The adjoint representation of* $\mathrm{CVect}(M)$ *is isomorphic to* $\mathcal{F}_{-\frac{1}{n+1}}(M)$.

Proof. The Lagrange bracket coincides with a Lie derivative:

$$\{\mathcal{H}_1, \mathcal{H}_2\} = L_{X_{\mathcal{H}_1}}(\mathcal{H}_2). \tag{5}$$

This formula is equivalent to (3). □

Geometrically speaking, a contact Hamiltonian is not a function but rather a tensor density of degree $-\frac{1}{n+1}$.

3.4 Invariant functional on $\mathrm{CVect}(M)$

Assume M is compact and orientable, fix a contact form α and the corresponding volume form $\Omega = \alpha \wedge d\alpha^n$. The geometric interpretation of the adjoint action of $\mathrm{CVect}(M)$ implies the existence of an invariant (non-linear) functional on $\mathrm{CVect}(M)$.

Let $\mathrm{CVect}^*(M)$ be the set of contact vector fields with nonvanishing contact Hamiltonians, this is an invariant open subset of $\mathrm{CVect}(M)$.

Corollary 3.5. *The functional on* $\mathrm{CVect}^*(M)$ *defined by*

$$\mathcal{I} : X_H \mapsto \int_M H^{-(n+1)}\,\Omega$$

is invariant. This functional is independent of the choice of the contact form.

Proof. Consider is a contact vector field X_F, then according to (3), one has

$$L_{X_F}\,(H^{-(n+1)}) = X_F\,(H^{-(n+1)}) + (n+1)\,Z(F)$$

so that the quantity $H^{-(n+1)}\,\Omega$ is a well defined element of the space $\mathcal{F}_1(M)$. The functional $\mathcal{I}$ is then given by the invariant functional (4).

Furthermore, choose a different contact form $\alpha' = f\,\alpha$ and the corresponding volume form $\Omega' = f^{n+1}\,\Omega$. The contact Hamiltonian of the vector field X_H with respect to the contact form α' is the function $H' = \alpha'(X_H) = f\,H$. Hence, $H'^{-(n+1)}\,\Omega' = H^{-(n+1)}\,\Omega$ so that the functional $\mathcal{I}$ is, indeed, independent of the choice of the contact form. □

4 The structure of $\mathrm{TVect}(M)$

In this section we study the structure of the space of tangent vector fields $\mathrm{TVect}(M)$ viewed as a $\mathrm{CVect}(M)$-module.

4.1 A geometric realization

Let us start with a geometric realization of the $\mathrm{CVect}(M)$-module structure on $\mathrm{TVect}(M)$ which if quite similar to that of Section 3.3.

Let $\Omega^2_0(M)$ be the space of 2-forms on M vanishing on the contact distribution. In other words, elements of $\Omega^2_0(M)$ are proportional to α:

$$\omega = \alpha \wedge \beta,$$

where β is an arbitrary 1-form.

The following statement is similar to Proposition 3.4.

Theorem 4.1. *There is an isomorphism of* $\mathrm{CVect}(M)$*-modules*

$$\mathrm{TVect}(M) \cong \Omega^2_0(M) \otimes \mathcal{F}_{-\frac{2}{n+1}}(M),$$

where the tensor product is defined over $C^\infty(M)$.

Proof. Let M be orientable, fix a contact form α on M. Consider a linear map from $\mathrm{TVect}(M)$ to the space $\Omega^2_0(M)$ that associates to a tangent vector field X the 2-form

$$\langle X, \alpha \wedge d\alpha \rangle = -\alpha \wedge i_X d\alpha.$$

This map is bijective since the restriction $d\alpha|_\xi$ of the 2-form $d\alpha$ to the contact hyperplane ξ is non-degenerate.

However, the above map depends on the choice of the contact form and, therefore, cannot be $\mathrm{CVect}(M)$-invariant. In order to make this map independent of the choice of α, one defines the following map

$$X \mapsto \langle X, \alpha \wedge d\alpha \rangle \otimes \alpha^{-2} \tag{6}$$

with values in $\Omega^2_0(M) \otimes \mathcal{F}_{-\frac{2}{n+1}}(M)$. Note that the term α^{-2} in the right-hand-side is a well defined element of the space of tensor densities $\mathcal{F}_{-\frac{2}{n+1}}(M)$, see Section 3.2.

It remains to check the $\mathrm{CVect}(M)$-invariance of the map (6). Let X_H be a contact vector field, one has

$$\begin{aligned} L_{X_H}\left(\langle X, \alpha \wedge d\alpha \rangle \otimes \alpha^{-2}\right) &= \langle [X_H, X], \alpha \wedge d\alpha \rangle \otimes \alpha^{-2} \\ &\quad + \langle X, f_X \alpha \wedge d\alpha + \alpha \wedge df_X \alpha \rangle \otimes \alpha^{-2} - \langle X, \alpha \wedge d\alpha \rangle \otimes (2 f_X \alpha^{-2}) \\ &= \langle [X_H, X], \alpha \wedge d\alpha \rangle \otimes \alpha^{-2}. \end{aligned}$$

Hence the result. $\square$

The isomorphism (6) identifies the $\mathrm{CVect}(M)$-action on $\mathrm{TVect}(M)$ by Lie bracket with the usual Lie derivative. It is natural to say that this map defines an analog of contact Hamiltonian of a tangent vector field.

4.2 A skew-symmetric pairing on $\mathrm{TVect}(M)$ over $\mathrm{CVect}(M)$

There exists an invariant skew-symmetric bilinear map from $\mathrm{TVect}(M)$ to $\mathrm{CVect}(M)$ that can be understood as a "symplectic structure" on the space $\mathrm{TVect}(M)$ over $\mathrm{CVect}(M)$.

Theorem 4.2. *There exists a non-degenerate skew-symmetric invariant bilinear map*

$$\mathcal{B} : \mathrm{TVect}(M) \wedge \mathrm{TVect}(M) \to \mathrm{CVect}(M),$$

where the $\wedge$-product is defined over $C^\infty(M)$.

Proof. Assume first that M is orientable and fix the contact form α. Given 2 tangent vector fields X and Y, consider the function

$$H_{X,Y} = \langle X \wedge Y \,,\, d\alpha \rangle \,.$$

Define first a $(2n)$-linear map B from $\mathrm{TVect}(M)$ to $C^\infty(M)$ by

$$B_\alpha : X \wedge Y \mapsto H_{X,Y}. \tag{7}$$

The definition of the function $H_{X,Y}$ and thus of the map B_α depends on the choice of α. Our task is to understand it as a map with values in $\mathrm{CVect}(M)$ which is independent of the choice of the contact form. This will, in particular, extend the definition to the case where M is not orientable.

It turns out that the above function $H_{X,Y}$ is a well-defined contact Hamiltonian.

Lemma 4.3. *Choose a different contact form $\alpha' = f\,\alpha$, then $H'_{X,Y} = f\,H_{X,Y}$.*

Proof. By definition,

$$H'_{X,Y} = \left\langle X \wedge Y, d\alpha' \right\rangle = f\,\langle X \wedge Y, d\alpha\rangle + \langle X \wedge Y, df \wedge \alpha\rangle = f\,H_{X,Y}$$

since the second term vanishes. ☐

We observe that the function $H_{X,Y}$ depends on the choice of α precisely in the same way as a contact Hamiltonian (cf. Section 3.1). It follows that the bilinear map

$$\mathcal{B} : X \wedge Y \mapsto H_{X,Y}\,\alpha^{-1} \tag{8}$$

with values in $\mathcal{F}_{-\frac{1}{n+1}} \cong \mathrm{CVect}(M)$ (cf. Section 3.3) is well-defined and independent of the choice of α.

It remains to check that the constructed map (8) is $\mathrm{CVect}(M)$-invariant. This can be done directly but also follows from

Proposition 4.4. *The Lie bracket of two tangent vector fields $X, Y \in \mathrm{TVect}(M)$ is of the form*

$$[X, Y] = \mathcal{B}(X, Y) \quad + \quad \textit{(tangent vector field)} \tag{9}$$

Proof. Consider the decomposition from Proposition 2.1 applied to the Lie bracket $[X, Y]$. The "non-tangent" component of $[X, Y]$ is a contact vector field with contact Hamiltonian $\alpha([X, Y])$. One has

$$i_{[X,Y]}\alpha = (L_X\, i_Y - i_Y\, L_X)\,\alpha = -i_Y\, L_X\,\alpha = -i_Y\, i_X\, d\alpha = H_{X,Y}$$

The result follows. ☐

Theorem 4.2 is proved. ☐

Proposition 4.4 is an alternative definition of $\mathcal{B}$: the map $\mathcal{B}$ measures the failure of the Lie bracket of two tangent vector fields to remain tangent.

5 Heisenberg structures

In order to investigate the structure of $\mathrm{TVect}(M)$ as a $\mathrm{CVect}(M)$-module in more details, we will write explicit formulæ for the $\mathrm{CVect}(M)$-action.

We assume that there is an action of the Heisenberg Lie algebra $\mathfrak{h}_n$ on M, such that the center acts by the Reeb field while the generators are tangent to the contact structure. We then say that M is equipped with the Heisenberg structure. Existence of a globally defined Heisenberg structure is a strong condition on M, however, locally such structure always exists.

5.1 Definition of a Heisenberg structure

Recall that the Heisenberg Lie algebra $\mathfrak{h}_n$ is a nilpotent Lie algebra of dimension $2n+1$ with the basis $\{a_1, \ldots, a_n, b_1, \ldots, b_n, z\}$ and the commutation relations

$$[a_i, b_j] = \delta_{ij}\, z, \qquad [a_i, a_j] = [b_i, b_j] = [a_i, z] = [b_i, z] = 0, \qquad i, j = 1, \ldots, n.$$

The element z spans the one-dimensional center of $\mathfrak{h}_n$.

Remark 5.1. The algebra $\mathfrak{h}_n$ naturally appears in the context of symplectic geometry as a Poisson algebra of linear functions on the standard $2n$-dimensional symplectic space.

We say that M is equipped with a *Heisenberg structure* if one fixes a contact form α on M and a $\mathfrak{h}_n$-action spanned by $2n+1$ vector fields $\{A_1, \ldots, A_n, B_1, \ldots, B_n, Z\}$, such that the $2n$ vector fields A_i, B_j are independent at any point and tangent to the contact structure:

$$i_{A_i}\, \alpha = i_{B_j}\, \alpha = 0$$

and $[A_i, B_i] = Z$, where Z is the Reeb field, while the other Lie brackets are zero.

5.2 Example: the local Heisenberg structure

The Darboux theorem states that locally contact manifolds are diffeomorphic to each other. An effective way to formulate this theorem is to say that in a neighborhood of any point of M there is a system of local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ such that the contact structure ξ is given by the 1-form

$$\alpha = \sum_{i=1}^{n} \frac{x_i\, dy_i - y_i\, dx_i}{2} + dz.$$

These coordinates are called the Darboux coordinates.

Proposition 5.2. *The vector fields*

$$A_i = \frac{\partial}{\partial x_i} + \frac{y_i}{2}\frac{\partial}{\partial z}, \qquad B_i = -\frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial z}, \qquad Z = \frac{\partial}{\partial z}, \tag{10}$$

where $i = 1, \ldots, n$, define a Heisenberg structure on $\mathbb{R}^{2n+1}$.

Proof. One readily checks that A_i, B_j are tangent and

$$[A_i, B_j] = \delta_{ij}\, Z$$

while other commutation relations are zero. The vector field Z is nothing but the Reeb field. □

There is a well-known formula for a contact vector field in the Darboux coordinates (see, e.g., [1, 2, 4]). We will not use this formula since the expression in terms of the Heisenberg structure is much simpler.

5.3 Contact vector fields and Heisenberg structure

Assume that M is equipped with an arbitrary Heisenberg structure. It turns out that every contact vector fields can be expressed in terms of the basis of the $\mathfrak{h}_n$-action by a universal formula.

Proposition 5.3. *Given an arbitrary Heisenberg structure on M, a contact vector field with a contact Hamiltonian H is given by the formula*

$$X_H = H\,Z - \sum_{i=1}^{n} \left(A_i(H)\,B_i - B_i(H)\,A_i\right). \tag{11}$$

Proof. Let us first check that the vector field (11) is, indeed, contact. If X be as the right-hand-side of (11), then the Lie derivative $L_X\alpha := (d\circ i_X + i_X\circ d)\,\alpha$ is given by

$$L_X\alpha = dH - \sum_{i=1}^{n} \left(A_i(H)\,i_{B_i} - B_i(H)\,i_{A_i}\right) d\alpha.$$

To show that the 1-form $L_X\alpha$ is proportional to α, it suffice to check that

$$i_{A_i}\,(L_X\alpha) = i_{B_j}\,(L_X\alpha) = 0 \qquad \text{for all} \quad i,j = 1,\dots n.$$

The first relation is a consequence of the formulæ $i_{A_i}\,(dH) = A_i(H)$ together with

$$i_{A_i}i_{B_j}\,d\alpha = i_{A_i}\left(L_{B_j}\alpha\right) = i_{[A_i,B_j]}\alpha = \delta_{ij}\,i_Z\,\alpha = \delta_{ij}, \qquad i_{A_i}i_{A_j}\,d\alpha = i_{B_i}i_{B_j}\,d\alpha = 0. \tag{12}$$

The second one follows from the similar relations for i_{B_j}.

Second, observe that, if X be as in (11), then $i_X\,\alpha = H$. This means that the contact Hamiltonian of the contact vector field (11) is precisely H. □

Note that a formula similar to (11) was used in [4] to define a contact structure.

5.4 The action of $\mathrm{CVect}(M)$ on $\mathrm{TVect}(M)$

Since $2n$ vector fields A_i and B_j are linearly independent at any point, they form a basis of $\mathrm{TVect}(M)$ over $C^\infty(M)$. Therefore, an arbitrary tangent vector field X has a unique decomposition

$$X = \sum_{i=1}^{n} \left(F_i\,A_i + G_i\,B_i\right), \tag{13}$$

where (F_i, G_j) in an $2n$-tuple of smooth functions on M. The space $\mathrm{TVect}(M)$ is now identified with the direct sum

$$\mathrm{TVect}(M) \cong \underbrace{C^\infty(M)\oplus\cdots\oplus C^\infty(M)}_{2n \text{ times}},$$

Let us calculate explicitly the action of $\mathrm{CVect}(M)$ on $\mathrm{TVect}(M)$.

Proposition 5.4. *The action of* $\mathrm{CVect}(M)$ *on* $\mathrm{TVect}(M)$ *is given by the first-order* $(2n\times 2n)$-*matrix differential operator*

$$X_H \begin{pmatrix} F \\ G \end{pmatrix} = \left(X_H\cdot\mathbf{1} - \begin{pmatrix} AB(H) & BB(H) \\ -AA(H) & -BA(H) \end{pmatrix}\right)\begin{pmatrix} F \\ G \end{pmatrix} \tag{14}$$

where F and G are n-vector functions, $\mathbf{1}$ is the unit $(2n \times 2n)$-matrix, $AA(H), AB(H), BA(H)$ and $BB(H)$ are $(n \times n)$-matrices, namely

$$AA(H)_{ij} = A_i A_j(H),$$

the three other expressions are similar.

Proof. Straightforward from (11) and (13). □

Proposition 5.5. *The bilinear map (7) has the following explicit expression:*

$$H_{X,\widetilde{X}} = \sum_{i=1}^{n} \begin{vmatrix} F_i & \widetilde{F}_i \\ G_i & \widetilde{G}_i \end{vmatrix},$$

where $X = \sum_{i=1}^{n}(F_i\, A_i + G_i\, B_i)$, and $\widetilde{X} = \sum_{j=1}^{n}(\widetilde{F}_j\, A_j + \widetilde{G}_j\, B_j)$.

Proof. This follows from the definition (7) and formula (12). □

Note that formula (14) implies that $H_{X,\widetilde{X}}$ transforms as a contact Hamiltonian according to (3) since the partial traces of the $(2n \times 2n)$-matrix in (14) are $A_iB_i(H) - B_iA_i(H) = Z(H)$.

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