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Projective Structures and Contact Forms

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Introduction

In this paper we show that a projective structure on a real simply connected smooth manifold M determines a family of local Lie algebras on M (i.e., a family of Lie brackets on $C^\infty(M)$ that satisfy the localization condition).

As was shown in Kirillov's paper [4], a nondegenerate local Lie algebra is determined by a contact or a symplectic structure on the manifold.

The relation between projective and contact geometries has already been mentioned by Cartan (see [9] and also [2]). Many problems of projective differential geometry lead to contact structures. It turns out that this relationship is absolutely direct. Seemingly, the following elementary (and curious) fact remains unnoticed.

Theorem 1. *Let M be a simply connected manifold (possibly noncompact) of dimension $2k-1$. For a projective structure on M to exist it is necessary and sufficient that there be functions $f_1, \dots, f_{2k} \in C^\infty(M)$ such that the 1-form*

$$\alpha = \sum_{i=1}^{k} (f_{2i-1}\, df_{2i} - f_{2i}\, df_{2i-1}) \tag{1}$$

is contact.

Thus, a locally projective manifold of odd dimension is contact. The corresponding local Lie algebra is nondegenerate and is determined by a Lagrange bracket. The choice of the form (1) is not unique.

An even-dimensional manifold with a projective structure must not be symplectic. However, this holds if the projective structure is affine. From this viewpoint, there is a sharp difference between even-dimensional and odd-dimensional projective structures, while there is a strict analogy between "projective-contact" and "affine-symplectic" geometries.

Theorem 1′. *For a simply connected manifold M of dimension $2k$ to possess an affine structure it is necessary and sufficient that there be functions $f_1, \dots, f_{2k} \in C^\infty(M)$ such that the 2-form*

$$\omega = \sum_{i=1}^{k} df_{2i-1} \wedge df_{2i} \tag{1′}$$

is symplectic.

The following more general fact holds for an arbitrary dimension.

Theorem 1″. *Let M^n be a simply connected manifold. For a projective structure on M to exist it is necessary and sufficient that there be functions $f_1, \dots, f_{n+1} \in C^\infty(M)$ such that the form*

$$\Omega = \sum_{i=1}^{n+1} (-1)^i f_i\, df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_{n+1} \tag{1″}$$

is the volume form on M.

If the dimension is odd then $\Omega = \alpha \wedge d\alpha^{k-1}$; the author learned this fact from S. L. Tabachnikov.

Generically, a local Lie algebra that corresponds to an even-dimensional manifold may degenerate on a geodesic submanifold of codimension 1. It turns out that this submanifold is contact and its complement is symplectic.

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Theorem 2. *On a simply connected manifold M endowed with a projective structure there exists a structure of a local Lie algebra such that*

(A) *if* $\dim M = 2k-1$, *then the Lie algebra* sp_{2k} *can be embedded in* $C^\infty(M)$ *and the action of* sp_{2k} *on* M *is transitive*;

(B) *if* $\dim M = 2k$, *then the affine symplectic Lie algebra* $sp_{2k} \ltimes \mathbb{R}^{2k}$ *can be embedded in* $C^\infty(M)$ *and its action on* $M \setminus \Gamma$ *and* Γ (Γ *denoting a geodesic submanifold of dimension* $2k-1$) *is transitive.*

We propose an analog of the Sturm theorem for projective structures, replacing solutions of a differential equation by geodesic submanifolds with respect to the projective structure.

We also consider relationships between projective structures and invariant differential operators.

Relations of the projective geometry to the symplectic and the contact ones are not understood completely. We indicate two recent papers on this topic [18, 20].

The idea of this paper arose at Kirillov's seminar. It was conjectured by V. I. Arnol'd that a projective structure defines a local Lie algebra on the manifold. I am grateful to them both and also to A. B. Givental, L. Guieu, Ch. Duval, E. Ghys, P. Iglesias, C. Roger, and especially to S. L. Tabachnikov for useful discussions.

§1. Definitions

We say that a manifold M of dimension n is endowed with a *projective structure* if on M there is a fixed atlas with projective coordinate changes. More precisely, a covering (U_i) with a family of local diffeomorphisms $\varphi_i \colon U_i \to \mathbb{R}P^n$ is called a projective atlas if the local transformations $\varphi_j \circ \varphi_i^{-1} \colon \mathbb{R}P^n \to \mathbb{R}P^n$ are projective (i.e., are determined by the action of the group PGL_{n+1} on $\mathbb{R}P^n$). Two projective atlases are said to be equivalent if their union is also a projective atlas. The class of mutually equivalent projective atlases is called a projective structure.

The notion of a projective structure originates from Elie Cartan [9]; it is the flat case of the notion of a projective connection introduced there.

Another way to define a projective structure is connected with the notion of a *geometric structure*, or (X, G)-structure, on a manifold M; this notion was introduced by F. Klein and systematically studied for the first time in Ehresmann [10]. Here G is a Lie group that acts transitively on the "model" manifold X. Such a structure on M is determined by a local (in a neighborhood of each point of M) isomorphism between M and X, which gives a (*local*) *action* of G on M; on the intersections of maps these actions are *conjugate*.

A projective structure on M is a $(PGL_{n+1}, \mathbb{R}P^n)$-structure.

The *third definition* of a projective structure is in the introduction of a *developing mapping*

$$\varphi \colon \widetilde{M} \to \mathbb{R}P^n \tag{2}$$

defined on the universal covering manifold $\widetilde{M}$ of M. This mapping must have the following properties:

a) φ is a local diffeomorphism;

b) there exists a homomorphism $T \colon \pi_1(M) \to PGL_{n+1}$ that "conjugates" the action of the homotopy group on $\widetilde{M}$, i.e.,

$$\varphi \circ \gamma(m) = T_\gamma \circ \varphi(m). \tag{3}$$

To any projective structure on M there corresponds a developing mapping, which is determined up to a projective transformation of the space $\mathbb{R}P^n$. In turn, the local diffeomorphism (2) with property (3) determines a unique projective structure. The image of the homomorphism T is called a *monodromy group* (or a holonomy one) of the projective structure.

A projective structure is said to be *affine* if the transformations $\varphi_j \circ \varphi_i^{-1}$ are determined by the action of the affine group $GL_n \ltimes \mathbb{R}^n$. In this case the developing mapping of an affine structure takes its values in the affine space, i.e., $\varphi \colon \widetilde{M} \to \mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$.

Remark on the uniqueness. In what follows we assume that the manifold M is *simply connected* ($\pi_1(M) = 0$); however, it may be noncompact. In this case all projective structures on M are diffeomorphic

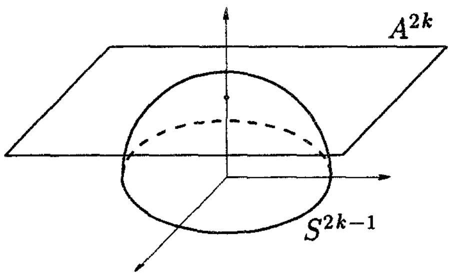

Fig. 1

(e.g., see [10]). In other words, if the projective structure on M exists, then it is *unique* (up to the natural equivalence).

A generalization of Theorems 1 and 2 to manifolds that are not simply connected requires that in this case the monodromy group be conjugate to a subgroup of the projective-symplectic group $PSp_{2k} \subset PGL_{2k}$; we omit the details.

§2. Proof of Theorem 1

Let M^{2k-1} be an oriented simply connected manifold endowed with a projective structure. The developing mapping (2) can be "lifted" (in two ways) to a local diffeomorphism $\widetilde{\varphi}\colon M \to S^{2k-1}$. Consider the standard embedding $S^{2k-1} \subset \mathbb{R}^{2k}$. Let us fix a system of linear coordinates $(x_1, \ldots, x_{2k})$ in $\mathbb{R}^{2k}$. Put $f_i := \widetilde{\varphi}^* x_i$. The standard contact form on S^{2k-1} has the form $\alpha_0 = \sum_{i=1}^{k}(x_{2i-1}\, dx_{2i} - x_{2i}\, dx_{2i-1})$. Thus, the 1-form (1) can be expressed as $\alpha := \widetilde{\varphi}^* \alpha_0$; hence, it is contact.

Conversely, suppose that there is a contact form given by (1) on M^{2k-1}. Let us prove that a projective structure on M exists. Consider the $2k$-dimensional linear space F spanned by the functions f_i: $F = \langle f_1, \ldots, f_{2k}\rangle \cong \mathbb{R}^{2k}$. To any point $m \in M$ we assign the subspace $V_m \subset F$ formed by the functions vanishing at the point m:

$$V_m = \{f \in F \mid f(m) = 0\}.$$

The dimension of the subspace V_m is $2k-1$ (since the form α is nondegenerate). Thus, the mapping $\varphi\colon M \to \mathbb{R}P^{2k-1}$ ($\varphi(m) = V_m$) is constructed. This mapping is determined up to a projective transformation: $\varphi \sim A\varphi$, $A \in PGL_{2k}$ (depending on the choice of a linear isomorphism $F \cong \mathbb{R}^{2k}$).

Lemma 1. *φ is a local diffeomorphism.*

Proof. Since the form (1) is nondegenerate ($\alpha \wedge (d\alpha)^{k-1} \neq 0$), the mapping φ is nondegenerate as well. Indeed, for any point m there is a function in F that is nonzero at m. Without loss of generality we may assume that $f_{2k}(m) \neq 0$. Let us put $t_i = f_i/f_{2k}$, $i = 1, \ldots, 2k-1$, in the neighborhood of m. Then the form $\alpha \wedge (d\alpha)^{k-1}$ is proportional to the form $dt_1 \wedge \cdots \wedge dt_{2k-1} \neq 0$. Thus, the functions $(t_1, \ldots, t_{2k-1})$ form a system of local coordinates on M. The mapping φ^* relates it to a system of affine coordinates on $\mathbb{R}P^{2k-1}$.

Hence, φ is a developing mapping for some projective structure.

It can be easily verified that the correspondence just defined between projective structures and contact forms given by formula (1) (the latter are determined up to a linear transformation of the space F) is bijective. Theorem 1 is proved.

Let us prove Theorem $1'$.

Let $\dim M = 2k$. The developing mapping φ that corresponds to the affine structure on M can be lifted to $\widetilde{\varphi}\colon M \to S^{2k} \setminus S^{2k-1}$. Let us define a projection $M \to A^{2k}$ (see Fig. 1) onto an affine hyperplane $A^{2k} \subset \mathbb{R}^{2k+1}$. Let $(x_1, \ldots, x_{2k+1})$ be coordinates in $\mathbb{R}^{2k+1}$. Suppose that the hyperplane A^{2k} is given by the equation $x_{2k+1} = \text{const}$. Then the symplectic form $(1')$ can be defined as a preimage of the form $\omega_0 = \sum dx_{2i-1} \wedge dx_{2i}$. The necessity is proved.

The sufficiency can be proved as in Theorem 1.

Theorem $1''$ can be proved similarly.

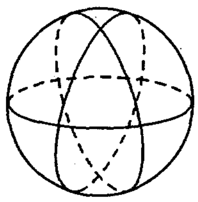

Fig. 2

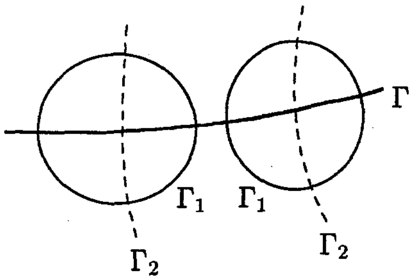

Fig. 3

Examples. a) In the two-dimensional case a form given by ($1''$) can be written as

$$\Omega = f_1\, df_2 \wedge df_3 + f_2\, df_3 \wedge df_1 + f_3\, df_1 \wedge df_2\,;$$

it determines a symplectic structure on M.

b) A four-dimensional projective manifold must not be symplectic (S^4 is a counterexample).

c) The symplectization [1, 2] of a manifold with a contact form given by (1) can be endowed with an affine structure.

§3. Geodesic Submanifolds and the Sturm Theorem on Zeroes

Let M^n be a simply connected manifold endowed with a projective structure. A submanifold $\Gamma^k \subset M$ is called a *k-geodesic* if

$$\Gamma \subset \varphi^{-1}(\mathbb{R}P^k)$$

for some $\mathbb{R}P^k \subset \mathbb{R}P^n$, i.e., Γ can be "developed" into a flat space. A geodesic Γ is called *complete* if $\Gamma = \varphi^{-1}(\mathbb{R}P^k)$.

Theorem A ("on zeroes"). *Let Γ_1, Γ_2, and Γ be different complete geodesics such that* $\dim \Gamma_1 = \dim \Gamma_2 = n-1$ *and* $\dim \Gamma = 1$. *Then between any two points of intersection of the submanifolds Γ and Γ_1 there exists a point of intersection of the submanifolds Γ and Γ_2* (see Figs. 2, 3).

In order to prove this theorem, we establish a relation between geodesics and the functions $f_1, \dots, f_{n+1} \in C^\infty(M)$ from Theorem $1''$. Denote by $F \subset C^\infty(M)$ the subspace generated by these functions.

Lemma 2. (1) *Let $g_1, \dots, g_k \in F$ be linearly independent functions. Then the submanifold*

$$\Gamma_g = \{m \in M \mid g_1(m) = \cdots = g_k(m) = 0\}$$

is a complete $(n-k)$-geodesic.

(2) *For any $(n-k)$-geodesic Γ there exist functions $g_1, \dots, g_k \in F$ such that $\Gamma \subset \Gamma_g$.*

The proof is obvious: it suffices to recall (see §2) that the functions f_i are determined as preimages of the coordinate functions on $\mathbb{R}^{n+1}$ under a developing mapping.

Thus, there exist functions $g_1, g_2 \in F$ such that $\Gamma_1 = \Gamma_{g_1}$ and $\Gamma_2 = \Gamma_{g_2}$. Let us consider the restrictions of these functions to Γ: $\bar g_1 = g_1|_\Gamma$ and $\bar g_2 = g_2|_\Gamma$.

Lemma 3. *The functions $\bar g_1$ and $\bar g_2$ determine a projective structure on Γ, i.e., the 1-form $\alpha_\Gamma = \bar g_1\, d\bar g_2 - \bar g_2\, d\bar g_1$ is nowhere degenerated on Γ.*

Proof. There exist linearly independent functions $g_3, \dots, g_{n+1} \in F$ vanishing on Γ. These functions, together with g_1 and g_2, form a basis of F. The volume form ($1''$) is proportional to the form $\alpha_\Gamma \wedge dg_3 \wedge \cdots \wedge dg_{n+1}$ on Γ. □

This lemma implies our theorem. Indeed, the points of $\Gamma_1 \cap \Gamma$ and $\Gamma_2 \cap \Gamma$ are zeroes for $\bar g_1$ and $\bar g_2$, respectively; thus, they alternate by the ordinary Sturm theorem. The theorem is proved.

We state here an assertion generalizing Lemma 3.

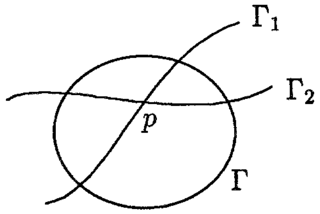

Fig. 4

Proposition 1. *Let $\Gamma \subset M$ be a k-geodesic and let $g_1, \dots, g_{k+1} \in F$ be functions that are not identically zero on Γ. Then the restrictions of these functions to Γ determine a projective structure on Γ.*

The proof is similar.

Let us state another version of the multidimensional Sturm theorem on zeroes.

Theorem B. *Suppose a geodesic $\Gamma^{n-1} \subset M^n$ bounds a compact submanifold $\Delta \subset M$. Then for any complete $(n-1)$-dimensional geodesics $\Gamma_1, \dots, \Gamma_n$ there exists an intersection point $p \in \bigcap_i \Gamma_i$ interior to Δ* (see Fig. 4).

This theorem can be regarded as a multidimensional version of the Sturm theorem on zeroes.

Let us consider the odd-dimensional case separately.

Definition. Let M^{2k-1} be a simply connected manifold with a projective structure. Fix a contact form given by (1) on M. A Legendre submanifold $\Gamma \subset M$ that is geodesic is called a *Legendre geodesic.*

Lemma 4. (1) *Let $L \subset F$ be a Lagrange subspace. Then the submanifold $\Gamma_L = \{m \in M \mid f(m) = 0,\ f \in L\}$ is a Legendre submanifold.*

(2) *Each Legendre geodesic $\Gamma \subset M$ satisfies $\Gamma \subset \Gamma_L$ for some Lagrange subspace $L \subset F$.*

The proof is obvious.

Thus, any Legendre geodesic is determined by a Lagrange subspace of the space F.

§4. Local Lie Algebras Determined by Projective Structures

In this section we define a family of local Lie algebras on a simply connected manifold endowed with a projective structure and prove Theorem 2. Let us recall the definitions (see [4]).

We say that the space $C^\infty(M)$ is endowed with the structure of a local Lie algebra if on $C^\infty(M)$ there is an operation $[\ ,\ ]$ given by a smooth differential operation on M and satisfying the Jacobi identity.

As was shown in [4], the local Lie algebra structure on $C^\infty(M)$ is determined by a pair (a, c), where $a = \sum a^k(x)\partial_k$ is a vector field and $c = \sum_{i,j} c^{ij}\partial_i \wedge \partial_j$ is a bivector field on M. In this notation, the commutator can be expressed explicitly:

$$[F, G] = FaG - GaF + \langle c, dF \wedge dG\rangle .$$

The Jacobi identity for this operation is equivalent to the following relation (see [4]):

$$a(c) = 0, \qquad \delta c \wedge c = a \wedge c,$$

where $a(c)$ is the Lie derivative of the bivector c along the vector field a and $\delta c = \sum_{i,j} \partial_i c^{ij}(x)\partial_j$.

To any function G we assign the vector field $\xi_G = Ga + \frac{1}{2}\langle c, dG\rangle$; then we have $[G, H] = \xi_G H - \xi_H G$.

A local Lie algebra is said to be *transitive* if for any point $m \in M$ the tangent space $T_m M$ is spanned by the vectors $\xi_G(m)$. As was shown in Kirillov [4], any transitive Lie algebra is determined by some structure that is either contact or symplectic.

A. $\dim M = 2k - 1$. The projective structure on M determines a family of contact forms given by (1). A contact form determines a transitive local Lie algebra on M called a *Lagrange bracket* (e.g., see [1, 2, 4]). Let us indicate the explicit formula in local coordinates which are related to the projective structure. To this end, put $x_i = f_{2i-1}/f_{2k}$, $y_i = f_{2i}/f_{2k}$, $i = 1, \dots, k-1$, and $z = f_{2k-1}/f_{2k}$. Then the Lagrange bracket on $C^\infty(M)$ has the form

$$[F,G]_\alpha = \{F,G\}_0 + (2F - \mathcal{E}F)G_z - F_z(2G - \mathcal{E}G), \tag{4}$$

where

$$\{F,G\}_0 = \sum_{i=1}^{k-1}(F_{x_i}G_{y_i} - F_{y_i}G_{x_i}), \qquad \mathcal{E} = \sum_{i=1}^{k-1}(x_i\partial_{x_i} + y_i\partial_{y_i}).$$

In this case we have $(a, c) = (2\partial_z, \sum \partial_{x_i} \wedge \partial_{y_i} + \partial_z \wedge \mathcal{E})$. The quadratic polynomials in $f_1, \dots, f_{2k}$ form a subalgebra $sp_{2k} \subset C^\infty(M)$, and the corresponding vector fields determine an action of sp_{2k} on M.

B. $\dim M = 2k$. The product $M \times \mathbb{R}$ can be endowed with the projective structure (induced from M). We fix a contact form given by (1) on $M \times \mathbb{R}$ and the corresponding Lagrange bracket.

Lemma 5. *The natural embedding $C^\infty(M) \hookrightarrow C^\infty(M \times \mathbb{R})$ endows M with the structure of a local Lie algebra.*

Proof. Let t be a coordinate on $\mathbb{R}$; let us define an embedding $C^\infty(M) \subset C^\infty(M \times \mathbb{R})$ by identifying $C^\infty(M)$ with the space of functions on $M \times \mathbb{R}$ that do not depend on t. Let us rewrite the form (1):

$$\alpha = t\, df_{2k+1} - f_{2k+1}\, dt + \sum_{i=1}^{k}(f_{2i-1}\, df_{2i} - f_{2i}\, df_{2i-1}), \tag{5}$$

where $f_1, \dots, f_{2k-1}$ is an arbitrary basis of the space $F \subset C^\infty(M)$ (F is determined by the projective structure). The space $C^\infty(M) \subset C^\infty(M \times \mathbb{R})$ is a Lie subalgebra, because the Lagrange bracket of two functions that do not depend on t does not depend on t as well. The locality condition is clear.

Thus, a projective structure on M determines a family of local Lie algebra structures on $C^\infty(M)$. In what follows, a local Lie algebra $C^\infty(M)$ means a structure from this family.

Lemma 6. *Fix a projective structure on M^{2k}. Then a local Lie algebra from the above family is determined by an embedding of the affine algebra $sp_{2k} \ltimes \mathbb{R}^{2k}$ in sl_{2k+1}.*

Proof. The form (5) is determined by this embedding, and quadratic polynomials in $f_1, \dots, f_{2k}, f_{2k+1}$ generate the Lie algebra $sp_{2k} \ltimes \mathbb{R}^{2k}$.

Generally, local Lie algebras defined above are not transitive (e.g., this is the case if M is not symplectic). Let us recall that the submanifold $\Gamma_{f_{2k+1}} \subset M$, on which the function f_{2k+1} vanishes, is itself a complete geodesic (Lemma 2). Let us show that a local algebra $C^\infty(M)$ decomposes into transitive algebras on $M \setminus \Gamma_{f_{2k+1}} \sqcup \Gamma_{f_{2k+1}}$.

Lemma 7. *A local Lie algebra $C^\infty(M)$ is nondegenerate on $M \setminus \Gamma_{f_{2k+1}}$.*

Proof. Let $m \in M$ be a point for which $f_{2k+1} \neq 0$. In a neighborhood of m the functions $x_i = f_{2k-1}/f_{2k+1}$ and $y_i = f_{2i}/f_{2k+1}$, $i = 1, \dots, k$, are coordinates on M. In these coordinates, a local Lie algebra $C^\infty(M \setminus \Gamma_{f_{2k+1}})$ is determined by the standard Poisson bracket

$$\{F,G\} = \sum_i (F_{x_i}G_{y_i} - F_{y_i}G_{x_i})$$

and is transitive. $\square$

Lemma 8. *The submanifold $\Gamma_{f_{2k+1}}$ is invariant. The restriction of a local Lie algebra $C^\infty(M)$ to this submanifold is determined by a canonical contact structure given by* (1).

Proof. Suppose that $f_{2k} \neq 0$ in a neighborhood of a point $m \in \Gamma_\varphi$. Then the functions $x_i = f_{2i-1}/f_{2k}$, $y_i = f_{2i}/f_{2k}$, $i = 1, \dots, k-1$, $z = f_{2k-1}/f_{2k}$, and $s = f_{2k+1}/f_{2k}$ are local coordinates on M. It is easy to calculate that in these coordinates the Lie bracket of a local Lie algebra $C^\infty(M)$ has the form

$$[F,G] = [F,G]_\alpha + s(F_zG_s - F_sG_z), \tag{6}$$

where $[F, G]_\alpha$ is given by formula (4). Thus, for $s = 0$, i.e., on the submanifold $\Gamma_{f_{2k+1}}$, we obtain the standard contact Lagrange bracket $[F, G]_\alpha$. $\square$

Thus, the local Lie algebra structure defined above is determined by a symplectic structure on $M \setminus \Gamma_{f_{2k+1}}$. This structure degenerates on the geodesic submanifold $\Gamma_{f_{2k+1}} \subset M$, on which this structure determines the standard contact Lagrange bracket.

Theorem 2 is proved.

The pair (a, c) (a is a vector field and c is a bivector) corresponding to our local Lie algebra can be easily calculated in the coordinates (x_i, y_i, z, s), $i = 1, \dots, k-1$:

$$a = 2\partial_z, \qquad c = \sum_{i=1}^{k-1} \partial_{x_i} \wedge \partial_{y_i} + \partial_z \wedge \mathcal{E},$$

where $\mathcal{E} = \sum_i (x_i \partial_{x_i} + y_i \partial_{y_i})$ is the Euler field.

Remark. If the projective structure on M^{2k} is affine, then there exists a regular structure among corresponding local Lie algebra structures on $C^\infty(M)$, given by a symplectic form (1′). (It corresponds to the choice of a function f_{2k+1} which is nowhere zero.)

Example. The local Lie algebra $C^\infty(S^{2k})$ corresponding to the standard projective structure on S^{2k} degenerates on the sphere $S^{2k-1} \subset S^{2k}$ and determines the standard contact structure on the sphere.

§5. Two Invariant Differential Operators

Here we define two invariant differential operators on an arbitrary manifold. Their relationship to projective structures will be discussed in the next section.

We will not give the general definition of invariant differential operators (see [6]) and restrict ourselves to a particular case.

Definition. By a tensorial density of degree λ on M we mean a section of the bundle $(\Lambda^n T^* M)^{\otimes \lambda}$ over M. In local coordinates, it can be expressed in the form

$$f = f(x_1, \dots, x_n)(dx_1 \wedge \dots \wedge dx_n)^\lambda.$$

We denote by $\mathcal{F}_\lambda$ the space of all the tensorial densities of degree λ.

A differential q-linear operator $A\colon \mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_q} \to \mathcal{F}_\mu$ is said to be *invariant* if it commutes with the action of the diffeomorphism group of M.

On any n-dimensional manifold there is a $(n+1)$-linear invariant skew-symmetric differential operator

$$W\colon \Lambda^{n+1} \mathcal{F}_\lambda \to \mathcal{F}_{(n+1)\lambda+1}$$

(the "Wronskian"). In local coordinates this operator is given by the formula

$$W(f_1, \dots, f_{n+1}) = \begin{vmatrix} f_1 & \dots & f_{n+1} \\ \partial_1 f_1 & \dots & \partial_1 f_{n+1} \\ \vdots & \ddots & \vdots \\ \partial_n f_1 & \dots & \partial_n f_{n+1} \end{vmatrix} (dx_1 \wedge \dots \wedge dx_n)^{(n+1)\lambda+1}, \tag{7}$$

where $\partial_i f = \partial f / \partial x_i$. The fact that the operator W is invariant means that formula (7) does not depend on the choice of local coordinates. Let us call W the *Wronski operator.*

Note that for $\lambda = -1/(n+1)$ the operator W takes values in the space $\mathcal{F}_0 = C^\infty(M)$.

Let $\operatorname{Vect} M$ be the space of vector fields on M. Define an invariant differential operator

$$L\colon \mathcal{F}_{\lambda_1} \otimes \dots \otimes \mathcal{F}_{\lambda_n} \to \mathcal{F}_{\lambda_1 + \dots + \lambda_n - 1} \otimes_{C^\infty(M)} \operatorname{Vect} M$$

by the formula

$$L(f_1,\dots,f_n)=(dx_1\wedge\dots\wedge dx_n)^{\lambda_1+\dots+\lambda_n+1}\otimes\sum_{i=1}^{n}(-1)^i\begin{vmatrix}\lambda_1 f_1 & \dots & \lambda_n f_n\\ \partial_1 f_1 & \dots & \partial_1 f_n\\ \vdots & \ddots & \vdots\\ \widehat{\partial_i f_1} & \dots & \widehat{\partial_i f_n}\\ \vdots & \ddots & \vdots\\ \partial_n f_1 & \dots & \partial_n f_n\end{vmatrix}\frac{\partial}{\partial x_i}. \tag{8}$$

Lemma 9. *The operator L is invariant.*

Proof. It can be checked directly that formula (8) does not depend on the choice of local coordinates.

Example. Let $f\in\mathcal{F}_{-2/3}$ and $g\in\mathcal{F}_{-1/3}$, $n=2$. Then $L(f,g)$ is the vector field of the form

$$L(f,g)=\begin{vmatrix}2f & g\\ f_x & g_x\end{vmatrix}\partial_y-\begin{vmatrix}2f & g\\ f_y & g_y\end{vmatrix}\partial_x.$$

The operator L will be used in the following particular case. Let $\lambda=-1/(n+1)$. Consider the composition of the operator L with the multiplication by tensorial densities. Let us define the operator

$$\overline{L}(f\,;f_1,\dots,f_n):=fL(f_1,\dots,f_n) \tag{9}$$

on the space $\otimes^{n+1}\mathcal{F}_{-1/(n+1)}$ with values in $\operatorname{Vect}M$.

The following identity connects the operators $\overline{L}$ and W.

Proposition 2. *Let $f,f_1,\dots,f_n,g\in\mathcal{F}_{-1/(n+1)}$. Then the following identity holds:*

$$\overline{L}(f\,;f_1,\dots,f_n)(g)=fW(f_1,\dots,f_n,g)-\frac{1}{n+1}gW(f,f_1,\dots,f_n), \tag{10}$$

where on the left-hand side we have a Lie derivative along a vector field.

Proof. Direct calculations.

§6. Projective Structures and Tensorial Densities

A. Theorem. *Let M^n be a simply connected manifold.*

1) *To any projective structure $\mathfrak{P}$ on M there corresponds a $(n+1)$-dimensional subspace $\mathcal{F}=\mathcal{F}_{\mathfrak{P}}\subset\mathcal{F}_{-1/(n+1)}$ such that*

$$W(\bar f_1,\dots,\bar f_{n+1})=\text{const} \tag{11}$$

for each $\bar f_1,\dots,\bar f_{n+1}\in\mathcal{F}$; W determines a nondegenerate volume form on $\mathcal{F}$.

2) *The correspondence $\mathfrak{P}\to\mathcal{F}_{\mathfrak{P}}$ is bijective and invariant.*

3) *The operator $\overline{L}$: $\otimes^{n+1}\mathcal{F}_{\mathfrak{P}}\to\operatorname{Vect}M$ determines an action of sl_{n+1} on M.*

Proof. The first assertion follows from Theorem 1″; it suffices to put $\bar f_i=f_i\Omega^{-1/(n+1)}$.

The second assertion can be derived from Theorem 1″ and the fact that relation (11) is invariant.

The third assertion follows from identity (10).

Indeed, consider the space $\mathfrak{a}=\mathfrak{a}_{\mathfrak{P}}\subset\operatorname{Vect}M$ spanned by the vector fields $\overline{L}(\bar f_1\,;\bar f_2,\dots,\bar f_{n+1})$, where $\bar f_i\in\mathcal{F}$ (thus, $\mathfrak{a}$ is the image of $\mathcal{F}$ under the operator $\overline{L}$). From (10) it follows that

a) $\mathfrak{a}$ is a Lie subalgebra;

b) $\mathfrak{a}$ acts on $\mathcal{F}$ (by the Lie derivative).

This action preserves the operator W (because W is invariant), i.e., preserves the volume form on $\mathcal{F}$. This defines a homomorphism

$$\mathfrak{a}\to sl_{n+1}.$$

The image of this mapping is sl_{n+1} (see below). On the other hand, $\dim \mathfrak{a} \le (n+1)^2 - 1$. Indeed, the operator $\bar{L}$ has a nonzero kernel on the space $\mathcal{F}$: if $\bar{f}_1, \dots, \bar{f}_{n+1} \in \mathcal{F}$, then

$$\mathrm{Alt}_{1,\dots,n+1} \bar{L}(\bar{f}_1\, ;\, \bar{f}_2, \dots, \bar{f}_{n+1}) = 0$$

(explicit calculations are given below). The theorem is proved. □

Remark. In the one-dimensional case ($n = 1$) it is well known that the space of all projective structures is naturally isomorphic to the space of Sturm–Liouville operators on M (e.g., see [5, 15])

$$A = \partial_x^2 + u(x).$$

In this case the aforementioned two-dimensional space of tensorial densities of degree $-1/2$ is the space of solutions of the equation $A\bar{f} = 0$. In the one-dimensional case the operator $\bar{L}$ is the ordinary multiplication: $\bar{L}(\bar{f}_1\, ;\, \bar{f}_2) = \bar{f}_1 \cdot \bar{f}_2$. The third assertion of our theorem coincides (for $n = 1$) with the following remark [5]: if the action of the Lie algebra sl_2 preserves the operator A, then it is determined by the product of solutions of the equation $A\bar{f} = 0$.

Thus, the space of *tensorial densities* $\mathcal{F}$ *is similar to the space of solutions of a Sturm–Liouville equation.*

B. Let us consider an odd-dimensional manifold M^{2k-1} endowed with a projective structure $\mathfrak{P}$.

We fix a canonical contact form given by (1) on M. (The choice of a contact form is equivalent to the choice of a Lie subalgebra $sp_{2k} \subset sl_{2k}$.)

Let us define a symplectic structure on $\mathcal{F}_{\mathfrak{P}}$. We recall some general properties of tensorial densities on a contact manifold.

Definition (for details see [7, 16]). Let (M^{2k-1}, α) be a contact manifold. A *contact-tensorial density* of degree λ on M is an object of the form

$$\varphi = \varphi(x_1, \dots, x_{2k-1})\alpha^{\lambda}.$$

We denote by $\mathcal{K}_\lambda$ the space of all contact-tensorial densities of degree λ.

Assertion 1 (see [7, 16]). (1) *There is a natural isomorphism*

$$\mathcal{K}_\lambda \cong \mathcal{F}_{\lambda/2k}.$$

(2) *The space* $\mathcal{K}_\lambda$ *is isomorphic to the space of homogeneous functions of degree* $-\lambda$ *on the symplectization* $S \to M$.

Proof. Fix a volume form $\Omega = \alpha \wedge (d\alpha)^{k-1}$. Let $G\colon M \to M$ be a contact diffeomorphism. The tensorial density $\varphi = \varphi(x)\Omega^{\mu}$ is transformed under G in the same way as $\widetilde{\varphi} = \varphi(x)\alpha^{k\mu}$. Indeed, if $G^*\alpha = g\cdot\alpha$, where $g \in C^\infty(M)$, then $G^*\Omega = g\alpha \wedge (dg\alpha)^{k-1} = g^k\Omega$. The second assertion is evident.

Corollary. $\mathcal{F}_{\mathfrak{P}} \subset \mathcal{K}_{-1/2}$.

Definition (for details see [7, 16]). A *generalized Lagrange bracket* on M is a Lie algebra structure

$$\{\ ,\ \}\colon \mathcal{K}_\lambda \otimes \mathcal{K}_\mu \to \mathcal{K}_{\lambda+\mu+1}$$

determined by a Poisson bracket on the symplectization S (see §2).

We obtain the following "refinement" of our theorem in the contact case.

Assertion 2 (see [16]). *Let* $\bar{f}_1, \bar{f}_2 \in \mathcal{F}_{\mathfrak{P}}$.

(1) $\{\bar{f}_1, \bar{f}_2\} = \mathrm{const}$ *and the Lagrange bracket determines a nondegenerate symplectic form on* $\mathcal{F}_{\mathfrak{P}}$; *the form* (6) *is the corresponding volume form.*

(2) *Let* $Q = q^{ij}\bar{f}_i\bar{f}_j$ *be a homogeneous polynomial of degree* 2 *and* $\bar{f}_i, \bar{f}_j \in \mathcal{F}_{\mathfrak{P}}$. *Let* ξ_Q *be the contact vector field corresponding to the contact Hamiltonian* Q. *Then* $\xi_Q \in sp_{2k} \subset \mathfrak{a}_{\mathfrak{P}}$.

Proof. This immediately follows from the definition of the space $\mathcal{F}$ given in the proof of Theorem 1.

Remark. The space $\mathcal{F}_{\mathfrak{P}}$ and the Lie subalgebra $sp_{2k} \subset \mathfrak{a}$ form the Lie superalgebra $osp(1|2k)$ (see [5]).

Now we describe canonical coordinates.

For projective structures the following "Darboux theorem" holds.

Lemma 10. *Let M be a manifold with a fixed projective structure. Then in a neighborhood of any point of M there exist coordinates such that:*

1) *the functions $\{1, x_1, \dots, x_n\}$ form a basis of the space $\mathcal{F}$ (we omitted the factor $(dx_1 \wedge \dots \wedge dx_n)^{-1/(n+1)}$);*

2) *the action of sl_{n+1} is determined by the vector fields $\{\partial_{x_i}, x_j\partial_{x_i}, x_i\mathcal{E}\}$, where $\mathcal{E} = \sum_{i=1}^n x_i\partial_{x_i}$.*

Proof. For any point $m \in M$ there exists $\bar{f} \in \mathcal{F}$ such that $\bar{f}(m) \neq 0$. We may assume (without loss of generality) that $\bar{f} = \bar{f}_{n+1}$. Then the functions $x_i = \bar{f}_i/\bar{f}_{n+1}$ ($\bar{f}_i$ is a basis in $\mathcal{F}$) are canonical coordinates.

In the odd-dimensional case the action of the Lie algebra sp_{2k} can be easily calculated as well.

§7. Two Problems

A. Nonstandard contact structures. There exists an interesting example of a globally defined contact form that determines a contact structure not diffeomorphic to the structure given by (1). In the elegant papers [12–14] contact forms given by the formula

$$\beta = g_1(f_1\,df_2 - f_2\,df_1) + g_2(f_3\,df_4 - f_4\,df_3) \tag{12}$$

were considered on a three-dimensional manifold, where $f_i, g_i \in C^\infty(M)$. The submanifolds $(g_1 = 0)$ and $(g_2 = 0)$ do not intersect. Scaling the form β properly, we can assume that $g_1 = \sin\varphi$ and $g_2 = \cos\varphi$, where $\varphi \in C^\infty(M)$.

Let $M = S^3$. Connected components of the submanifolds $(g_1 = 0)$ and $(g_2 = 0)$ are isomorphic to $\mathbb{T}^2$ (see [12–14]). The form β is equivalent to one of the forms

$$\beta_n = \sin(\pi/4 + n(x_1^2 + x_2^2))(x_1\,dx_2 - x_2\,dx_1) + \cos(\pi/4 + n(x_1^2 + x_2^2))(x_3\,dx_4 - x_4\,dx_3),$$

where x_i are the coordinates on the space $\mathbb{R}^4 \supset S^3$. If $n \neq 0$, then the contact structure given by the form β_n is not equivalent to the standard one [8] (although these structures are homotopic in the class of 2-distributions). All structures β_n are equivalent for $n \neq 0$.

We pose the following problem: *Does a form given by* (12) *determine a geometric structure on M?*

B. The main problem in the theory of geometric structures is the **existence problem.**

Under what conditions on the manifold M a projective structure on M exists?

This problem remains open and is very complicated.

For the odd-dimensional case Theorem 1 gives a (rather weak) restriction: a locally projective manifold is contact. In particular, its third Stiefel–Whitney class vanishes (see [8]).

Any two-dimensional surface can be endowed with a projective structure (for details and numerous examples see [11, 17, 19]).

In the three-dimensional case the existence problem is of particular interest. There is a bold conjecture that such a structure always exists (see [17]); this conjecture is not withdrawn. Three-dimensional manifolds with a hyperbolic metric yield numerous examples (see [19]). The affirmative solution of the existence problem in the three-dimensional case would imply the proof of the Poincaré conjecture. (Indeed, a projective structure on a compact simply connected manifold of dimension 3 determines a transitive action of the group $SL(4, \mathbb{R})$ on this manifold. This immediately implies that this manifold is diffeomorphic to S^3.)

We conclude this paper with the following remark. In order to prove the Poincaré conjecture it is sufficient to show that on a three-dimensional simply connected manifold M there exists a (globally defined) contact form given by the formula

$$\alpha = f\,dg - g\,df + h\,dk - k\,dh, \qquad f, g, h, k \in C^\infty(M).$$

The remark follows from Theorem 1.

References

1. V. I. Arnol'd, Mathematical Methods of Classical Mechanics [in Russian], Nauka, Moscow (1989).
2. V. I. Arnol'd, Supplementary Chapters to the Theory of Ordinary Differential Equations [in Russian], Nauka, Moscow (1979).
3. V. I. Arnol'd and A. B. Givental, "Symplectic Geometry," Contemporary Problems in Mathematics. Fundamental Directions [in Russian], Vol. 4, 5–139, Itogi Nauki i Tekhniki, VINITI, Moscow (1985).
4. A. A. Kirillov, "Local Lie algebras," Usp. Mat. Nauk, **31**, No. 4, 57–76 (1976).
5. A. A. Kirillov, "The orbits of the group of diffeomorphisms of the circle, and local Lie superalgebras," Funkts. Anal. Prilozhen., **15**, No. 2, 75–76 (1981).
6. A. A. Kirillov, "Invariant operators over geometric quantities," Contemporary Problems in Mathematics [in Russian], Vol. 16, 3–29, VINITI, Moscow (1980).
7. V. Yu. Ovsienko and C. Roger, "Deformations of Poisson brackets and extensions of Lie algebra of contact vector fields," Usp. Mat. Nauk, **47**, No. 6, 141–194 (1992).
8. D. Bennequin, "Entrelacements et équations de Pfaff," Third Schnepfenried Geometry Conference, Vol. 1 (Schnepfenried, 1982), p. 87–161; Asterisque, **107–108** (1983).
9. E. Cartan, "Sur les variétés a connexion projective," Bull. Soc. Math. France, **52**, 205–241 (1924).
10. Ch. Ehresmann, "Sur les espaces localement homogénes," Enseign. Math., **35**, 317–333 (1936).
11. W. M. Goldman, "Convex real projective structures on compact surfaces," J. Diff. Geom., **31**, 791–845 (1990).
12. J. Gomez and F. Varela, "Sur certaines expressions globales d'une forme volume," Lect. Notes in Math., Vol. 1251, Springer-Verlag, Berlin–New York (1987).
13. J. Gonzalo, "Un modèle global pour quelques formes de contact en dimension trois," C. R. Acad. Sci. Paris Sér. I, **279**, 125–128.
14. J. Gonzalo and F. Varela, "Modéles globaux des variétés de contact," Asterisque, **107–108**, 163–168 (1983).
15. A. A. Kirillov, "Infinite-dimensional Lie groups: their orbits, invariants and representations," In: Geometry of Moments, Lect. Notes in Math., Vol. 970 (1982), pp. 101–123.
16. V. Yu. Ovsienko and O. D. Ovsienko, "Projective structures and infinite-dimensional Lie algebras associated with a contact manifold," Adv. in Soviet Math. (D. B. Fuchs, ed.), to appear.
17. D. Sullivan and W. Thurston, "Manifolds with canonical coordinate charts: some examples," Enseign. Math., **29**, 15–25 (1983).
18. S. Tabachnikov, "Projective structures and group Vey cocycle," Preprint ENS de Lyon (1992).
19. W. Thurston, "Three-dimensional manifolds, Kleinian groups and hyperbolic geometry," Bull. Am. Math. Soc., **6**, No. 3, 357–381 (1982).
20. V. Yu. Ovsienko, "Lagrange Schwarzian derivative and symplectic Sturm theory," Ann. Fac. Sci. Toulouse Math., No. 2, to appear.

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