

Symmetries of Modules of Differential Operators

H GARGOUBI[†], P MATHONET[‡] and V OVSIENKO^{*}

[†] *I.P.E.I.M., Route de Kairouan, 5019 Monastir, Tunisie*
E-mail: hichem.gargoubi@ipeim.rnu.tn,

[‡] *Institut de Mathématique, Grand traverse 12, B37, 4000 Liège, Belgique*
E-mail: P.mathonet@ulg.ac.be,

^{*} *CNRS, Institut Girard Desargues Université Claude Bernard Lyon 1*
21 Avenue Claude Bernard, 69622 Villeurbanne Cedex, France
E-mail: ovsienko@igd.univ-lyon1.fr

Received June 3, 2004; Accepted December 1, 2004

Abstract

Let $\mathcal{F}_\lambda(S^1)$ be the space of tensor densities of degree (or weight) λ on the circle S^1 . The space $\mathcal{D}_{\lambda,\mu}^k(S^1)$ of k -th order linear differential operators from $\mathcal{F}_\lambda(S^1)$ to $\mathcal{F}_\mu(S^1)$ is a natural module over $\text{Diff}(S^1)$, the diffeomorphism group of S^1 . We determine the algebra of symmetries of the modules $\mathcal{D}_{\lambda,\mu}^k(S^1)$, i.e., the linear maps on $\mathcal{D}_{\lambda,\mu}^k(S^1)$ commuting with the $\text{Diff}(S^1)$ -action. We also solve the same problem in the case of straight line \mathbb{R} (instead of S^1) and compare the results.

1 Introduction

We study the space of linear differential operators acting in the space of tensor densities on S^1 as a module over the group $\text{Diff}(S^1)$ of all diffeomorphisms of S^1 . More precisely, let $\mathcal{D}_{\lambda,\mu}^k(S^1)$ be the space of linear k -th order differential operators

$$A : \mathcal{F}_\lambda(S^1) \longrightarrow \mathcal{F}_\mu(S^1),$$

where $\mathcal{F}_\lambda(S^1)$ and $\mathcal{F}_\mu(S^1)$ are the spaces of tensor densities of degree λ and μ respectively. Recall that a vector field $X = X(x) \frac{d}{dx}$ acts on the space of tensor densities \mathcal{F}_λ by Lie derivative

$$L_X^\lambda(\varphi) = (X(x)\phi'(x) + \lambda X'(x)\phi(x)) (dx)^\lambda.$$

We compute the commutant of the $\text{Diff}(S^1)$ -action on $\mathcal{D}_{\lambda,\mu}^k(S^1)$. This commutant is an associative algebra which we denote $\mathcal{I}_{\lambda,\mu}^k(S^1)$ and call the *algebra of symmetries*.

1.1 This paper is closely related to the classical topic initiated by Veblen [30] in his talk at IMC in 1928, namely the study of *invariant operators* also called *natural operators* (cf. [16]). An operator is called *invariant* if it commutes with the action of the group

of diffeomorphisms. The main two examples are the classical de Rham differential of differential forms

$$d : \Omega_k(M) \longrightarrow \Omega_{k+1}(M),$$

where M is a smooth manifold, and the integral

$$\int : \Omega_n(M) \longrightarrow \mathbb{R}$$

provided M is compact of dimension n .

Usually, one considers differential operators acting on various spaces of tensor fields on a smooth manifold. A famous theorem states that the de Rham differential is, actually, the only invariant *differential* operator in one argument acting on the spaces of tensor fields. This result was conjectured by Schouten and proved independently and using different approaches by Rudakov [27], Kirillov [14] and Terng [29], for a historical account with ramifications to non-tensor settings, see [12].

Many classification results are available now, and it was shown that there are quite few invariant differential operators and most of them are of a great importance. For instance, bilinear invariant differential operators on tensor fields were classified by Grozman [11]. The complete list of such operators is, in a way, very similar to our list; let us remind it for comparizon: There are no invariant operators of order > 3 , operators of orders 2 and 3 are (bar one exception) compositions of 1st order operators; the latter are, up to dualizations, the well-known Lie derivative and various brackets (some well-known, such as the Poisson, Schouten and Nijenhuis brackets, and some new), and one (also discovered by Grozman) exceptional bilinear third-order differential operator

$$G : \mathcal{F}_{-\frac{2}{3}}(S^1) \otimes \mathcal{F}_{-\frac{2}{3}}(S^1) \longrightarrow \mathcal{F}_{\frac{5}{3}}(S^1). \quad (1.1)$$

Note that differential operators invariant with respect of the diffeomorphism groups can be interpreted in terms of the Lie algebras of vector fields. This viewpoint relates the subject with the Gelfand-Fuchs cohomology, see [7] and references therein.

1.2 The main difference of our work from the classic literature is that we consider linear operators acting on differential operators (instead of tensor fields). More precisely, we classify the linear maps

$$T : \mathcal{D}_{\lambda,\mu}^k(S^1) \longrightarrow \mathcal{D}_{\lambda,\mu}^k(S^1) \quad (1.2)$$

commuting with the $\text{Diff}(S^1)$ -action. The module of differential operators $\mathcal{D}_{\lambda,\mu}^k(S^1)$ is not isomorphic to any module of tensor fields (but rather rassembles $\text{gl}(\mathcal{F}_\lambda)$ or $\mathcal{F}_\lambda^* \otimes \mathcal{F}_\mu$). The problem of classification of $\text{Diff}(S^1)$ -invariant operators on $\mathcal{D}_{\lambda,\mu}^k(S^1)$ is, therefore, different from Veblen's problem, although similar.

A well-known example of a map (1.2) is the *conjugation* of differential operators. This map associates to an operator A the *adjoint* operator A^* . If $A \in \mathcal{D}_{\lambda,\mu}^k(S^1)$, then $A^* \in \mathcal{D}_{1-\mu,1-\lambda}^k(S^1)$, so that this map defines a symmetry if and only if $\lambda + \mu = 1$.

Let us emphasize that, unlike Rudakov-Kirillov-Terng, we consider not only differential (or local) symmetries of $\mathcal{D}_{\lambda,\mu}^k(S^1)$ but also *non-local* ones. For instance, we find a version of trace which is an analog of the Adler trace (see [1]).

1.3 The main purpose of this paper is to show that some modules $\mathcal{D}_{\lambda,\mu}^k(S^1)$ are particular and very interesting. It turns out that the algebra of symmetries $\mathcal{I}_{\lambda,\mu}^k(S^1)$ can be quite rich depending on the values of λ and μ as well as on k . This algebra is an important characteristic of the corresponding module which embraces those given in [9, 8, 10].

The first example which is particular (for every k) is the module $\mathcal{D}_{0,1}^k(S^1)$ of operators from the space of functions to the space of 1-forms. The algebra of symmetries in this case is always of maximal dimension.

Another interesting module is $\mathcal{D}_{-\frac{1}{2},\frac{3}{2}}^2(S^1)$. It is related to the famous Virasoro algebra and also to the projective differential geometry, see [13, 25]. This module appears as a particular case in our classification.

Further intriguing examples of modules of differential operators are $\mathcal{D}_{-\frac{2}{3},\frac{5}{3}}^3(S^1)$ and $\mathcal{D}_{-\frac{2}{3},\frac{5}{3}}^4(S^1)$. These modules are related to the Grozman operator (1.1). The algebraic and geometric meaning of these modules is not known.

1.4 We will also consider symmetries of differential operators acting in the space of λ -densities over \mathbb{R} and compare this case with the case of λ -densities over S^1 . The classification of the invariant differential operators (1.2) remains the same as that on S^1 , except that there are no non-local symmetries.

Let us also mention that symmetries of the modules of differential operators in the multi-dimensional case have been classified in [22]. In this case the algebra of symmetries is smaller.

1.5 This paper is organized as follows. In Section 2 we introduce the $\text{Diff}(S^1)$ -modules of differential operators and their symbols. In Section 3 we formulate the classification theorems. In Section 4 we give an explicit construction of all $\text{Diff}(S^1)$ -invariant linear maps on the modules $\mathcal{D}_{\lambda,\mu}^k(S^1)$ in all possible cases. These operators are our main characters; some of them are known and some seem to be new. In Section 5 we calculate the associative algebras of invariant operators and identify them with some associative algebras of matrices. This gives a complete description of the symmetry algebras $\mathcal{I}_{\lambda,\mu}^k(S^1)$. Finally, in Section 6 we prove that there are no other invariant operators on $\mathcal{D}_{\lambda,\mu}^k(S^1)$ than the operators we introduce. This completes the proof of the main theorems.

2 The main definitions

In this section we define the space of differential operators on S^1 acting on the space of densities and the corresponding space of symbols. We pay particular attention to the action of the group of diffeomorphisms $\text{Diff}(S^1)$ and of the Lie algebra of vector fields $\text{Vect}(S^1)$ on these spaces.

2.1 Densities on S^1

Denote by $\mathcal{F}_\lambda(S^1)$, or \mathcal{F}_λ for short, the space of λ -densities on S^1

$$\varphi = \phi(x)(dx)^\lambda,$$

where $\lambda \in \mathbb{C}$ is the *degree* (or *weight*), x is a local coordinate on S^1 and $\phi(x) \in C^\infty(S^1)$. As a vector space, \mathcal{F}_λ is isomorphic to $C^\infty(S^1)$.

The group $\text{Diff}(S^1)$ naturally acts on \mathcal{F}_λ . If $f \in \text{Diff}(S^1)$, then

$$\rho_{f^{-1}}^\lambda : \phi(x)(dx)^\lambda \mapsto (f')^\lambda \phi(f(x))(dx)^\lambda.$$

The $\text{Diff}(S^1)$ -modules \mathcal{F}_λ and \mathcal{F}_μ are not isomorphic unless $\lambda = \mu$ (cf. [7]).

Example 2.1. The space \mathcal{F}_0 is isomorphic to $C^\infty(S^1)$, as a $\text{Diff}(S^1)$ -module; the space \mathcal{F}_1 is nothing but the space of 1-forms (volume forms); the space \mathcal{F}_{-1} is the space of vector fields on S^1 .

The space \mathcal{F}_λ can also be viewed as the space of functions on the cotangent bundle $T^*S^1 \setminus S^1$ (with zero-section removed) homogeneous of degree $-\lambda$. In standard (Darboux) coordinates (x, ξ) on T^*S^1 one writes:

$$\phi(x)(dx)^\lambda \longleftrightarrow \phi(x) \xi^{-\lambda}. \quad (2.1)$$

This identification commutes with the $\text{Diff}(S^1)$ -action.

2.2 Invariant pairing

There is a pairing $\mathcal{F}_\lambda \otimes \mathcal{F}_{1-\lambda} \rightarrow \mathbb{R}$ given by

$$\left\langle \phi(x)(dx)^\lambda, \psi(x)(dx)^{1-\lambda} \right\rangle = \int_{S^1} \phi(x)\psi(x)dx$$

which is $\text{Diff}(S^1)$ -invariant. For instance, the space $\mathcal{F}_{\frac{1}{2}}$ is equipped with a scalar product; this is a natural pre-Hilbert space that is popular in geometric quantization.

2.3 Differential operators on densities

Consider the space of linear differential operators

$$A : \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu$$

with arbitrary $\lambda, \mu \in \mathbb{C}$. This space will be denoted by $\mathcal{D}_{\lambda, \mu}(S^1)$. The subspace of differential operators of order $\leq k$ will be denoted by $\mathcal{D}_{\lambda, \mu}^k(S^1)$.

Fix a (local) coordinate x , a differential operator $A \in \mathcal{D}_{\lambda, \mu}^k(S^1)$ is of the form

$$A = a_k(x) \frac{d^k}{dx^k} + a_{k-1}(x) \frac{d^{k-1}}{dx^{k-1}} + \cdots + a_0(x), \quad (2.2)$$

where $a_i(x)$ are smooth functions. More precisely,

$$A(\varphi) = \left(a_k(x) \frac{d^k \phi(x)}{dx^k} + a_{k-1}(x) \frac{d^{k-1} \phi(x)}{dx^{k-1}} + \cdots + a_0(x) \phi(x) \right) (dx)^\mu,$$

where $\varphi = \phi(x)(dx)^\lambda$.

Example 2.2. The space $\mathcal{D}_{\lambda, \mu}^0(S^1)$ is nothing but $\mathcal{F}_{\mu-\lambda}$. Any zeroth-order differential operator is the operator of multiplication by a $(\mu - \lambda)$ -density:

$$a(x)(dx)^{\mu-\lambda} : \phi(x)(dx)^\lambda \mapsto a(x) \phi(x)(dx)^\mu$$

2.4 Diff(S^1)- and Vect(S^1)-module structure

The space $\mathcal{D}_{\lambda,\mu}(S^1)$ is a Diff(S^1)-module with respect to the action

$$\rho_f^{\lambda,\mu}(A) = \rho_f^\mu \circ A \circ \rho_{f^{-1}}^\lambda,$$

where $f \in \text{Diff}(S^1)$.

We will also consider the Lie algebra of vector fields Vect(S^1) and the natural Vect(S^1)-action on $\mathcal{D}_{\lambda,\mu}^k(S^1)$. The action of Vect(S^1) on the space of differential operators is given by the commutator of A and the Lie derivative:

$$\mathcal{L}_X^{\lambda,\mu}(A) = L_X^\mu \circ A - A \circ L_X^\lambda. \tag{2.3}$$

Note that the above formulæ are independent of the choice of the local coordinate x .

2.4.1 Example: the module $\mathcal{D}_{\lambda,\mu}^1(S^1)$

As a Diff(S^1)- (and Vect(S^1)-) module, the space $\mathcal{D}_{\lambda,\mu}^1(S^1)$ is split into a direct sum

$$\mathcal{D}_{\lambda,\mu}^1(S^1) \cong \mathcal{F}_{\mu-\lambda-1} \oplus \mathcal{F}_{\mu-\lambda}. \tag{2.4}$$

Indeed, every first-order differential operator $A \in \mathcal{D}_{\lambda,\mu}^1(S^1)$

$$A\left(\phi(x)(dx)^\lambda\right) = (a_1(x)\phi'(x) + a_0(x)\phi(x))(dx)^\mu$$

can be rewritten in the form

$$A\left(\phi(dx)^\lambda\right) = (a_1\phi' + \lambda a_1'\phi + (a_0 - \lambda a_1')\phi)(dx)^\mu,$$

and, finally, one obtains an intrinsic expression

$$A(\varphi) = (L_{a_1} + (a_0 - \lambda a_1'))(dx)^{\mu-\lambda}(\varphi),$$

where $a_1 = a_1(x)\frac{d}{dx}$ is understood a vector field and $a_0(x) - \lambda a_1'(x)$ as a function.

Furthermore, using identification (2.1), one can write a more elegant expression of first-order operators:

$$A = \xi^{\mu-\lambda} \left(L_{a_1} + \text{div} a_1 \frac{\partial}{\partial \xi} + a_0 \right).$$

Note that there are no intrinsic formulæ similar to the above ones in the case of modules $\mathcal{D}_{\lambda,\mu}^k(S^1)$ with $k \geq 2$, and, in general, there are no splittings similar to (2.4). The geometric meaning of the modules $\mathcal{D}_{\lambda,\mu}^k(S^1)$ was discussed in [9, 2, 8, 10].

2.5 Space of symbols of differential operators

The filtration

$$\mathcal{D}_{\lambda,\mu}^0(S^1) \subset \mathcal{D}_{\lambda,\mu}^1(S^1) \subset \dots \subset \mathcal{D}_{\lambda,\mu}^k(S^1) \subset \dots$$

is preserved by the Diff(S^1)-action. The graded Diff(S^1)-module $\mathcal{S}_{\lambda,\mu}(S^1) = \text{gr}(\mathcal{D}_{\lambda,\mu}(S^1))$ is called the module of *symbols* of differential operators.

The quotient module $\mathcal{D}_{\lambda,\mu}^k(S^1)/\mathcal{D}_{\lambda,\mu}^{k-1}(S^1)$ is isomorphic to the module of tensor densities $\mathcal{F}_{\mu-\lambda-k}(S^1)$; the isomorphism is provided by the principal symbol. As a $\text{Diff}(S^1)$ -module, the space of symbols depends, therefore, only on the difference

$$\delta = \mu - \lambda,$$

so that $\mathcal{S}_{\lambda,\mu}(S^1)$ can be denoted as $\mathcal{S}_\delta(S^1)$, and finally we have

$$\mathcal{S}_\delta(S^1) = \bigoplus_{i=0}^{\infty} \mathcal{F}_{\delta-i}$$

as $\text{Diff}(S^1)$ -modules.

The space of symbols $\mathcal{S}_\delta(S^1)$ can also be viewed as the space of functions on $T^*S^1 \setminus S^1$. Namely, any k -th order symbol $P \in \mathcal{S}_\delta(S^1)$ can be written in the form

$$P = a_k(x) \xi^{k-\delta} + a_{k-1}(x) \xi^{k-\delta-1} + \cdots + a_0(x) \xi^{-\delta}. \quad (2.5)$$

The natural lift of the action of $\text{Diff}(S^1)$ to T^*S^1 equips the space of functions (2.5) with a structure of $\text{Diff}(S^1)$ -module; this action coincides with the $\text{Diff}(S^1)$ -action on the space $\mathcal{S}_\delta(S^1)$.

Remark 2.3. The spaces $\mathcal{D}_{\lambda,\mu}(S^1)$ and $\mathcal{S}_\delta(S^1)$ are not isomorphic as $\text{Diff}(S^1)$ -modules. There are cohomology classes which are obstructions for existence of such an isomorphism, see [20, 2, 9].

3 The main results

In this section we formulate the main results of this paper and give a complete description of the algebras of symmetries $\mathcal{I}_{\lambda,\mu}^k(S^1)$. We defer the proofs to Sections 4-6.

We will say that the algebra $\mathcal{I}_{\lambda,\mu}^k(S^1)$ is *trivial* if it is generated by the identity map Id , and therefore is isomorphic to \mathbb{R} . Of course, we are interested in the cases when this algebra is non-trivial.

3.1 Introducing four algebras of matrices

We will need the following associative algebras.

1. The commutative algebra $\mathbb{R} \oplus \cdots \oplus \mathbb{R}$ will be denoted by \mathbb{R}^n . Of course, this algebra can be represented by diagonal $n \times n$ -matrices.
2. The algebra of (lower) triangular $(n \times n)$ -matrices will be denoted by \mathfrak{t}_n .
3. The commutative algebra \mathfrak{a} of (2×2) -matrices of the form

$$\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}.$$

4. The algebra \mathfrak{b} of (4×4) -matrices of the form

$$\begin{pmatrix} a & 0 & 0 & d \\ 0 & a & 0 & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

It turns out that the algebras of symmetries $\mathcal{I}_{\lambda,\mu}^k(S^1)$ are always direct sums of the above matrix algebras.

In Appendix we will introduce natural generators of the algebras \mathfrak{a} , \mathfrak{b} and \mathbb{R}^n .

3.2 Stability: the case $k \geq 5$

We start our list of classification theorems with the “stable” case. Namely, if $k \geq 5$, then the algebras $\mathcal{I}_{\lambda,\mu}^k(S^1)$ do not depend on k .

Theorem 3.1. *For $k \geq 5$, the algebra $\mathcal{I}_{\lambda,\mu}^k(S^1)$ is trivial for all (λ, μ) , except*

1. $\mathcal{I}_{\lambda,\mu}^k(S^1) \cong \mathbb{R}^2$, for $\begin{cases} \lambda + \mu = 1, & \lambda \neq 0, \\ \lambda = 0, & \mu \neq 1, 0 \\ \mu = 1, & \lambda \neq 1, 0 \end{cases}$
2. $\mathcal{I}_{0,0}^k(S^1) \cong \mathcal{I}_{1,1}^k(S^1) \cong \mathbb{R}^3$;
3. $\mathcal{I}_{0,1}^k(S^1) \cong \mathfrak{b} \oplus \mathbb{R}^2$.

The exceptional modules $\mathcal{D}_{\lambda,\mu}^k(S^1)$ with $k \geq 5$ are represented in Figure 1.

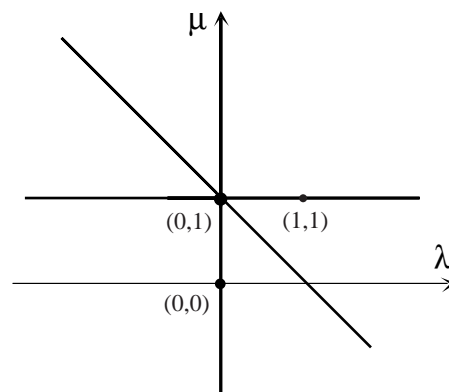


Figure 1. Exceptional modules of higher order operators

We will provide in the sequel a list of generators of the symmetry algebra for each non-trivial case, we will also give its explicit identification with the corresponding algebra of matrices.

3.3 Modules of differential operators of order 4

The result for the modules $\mathcal{D}_{\lambda,\mu}^k(S^1)$ with $k \leq 4$ is different from the above one. (It is interesting to compare this property of differential operators with that in the case of algebraic equations.)

Consider the modules of operators of order $k = 4$. The complete classification of symmetries in this case is given by the following result.

Theorem 3.2. *The algebra $\mathcal{I}_{\lambda,\mu}^4(S^1)$ is trivial for all (λ, μ) except*

1. $\mathcal{I}_{\lambda,\mu}^4(S^1) \cong \mathbb{R}^2$, for $\begin{cases} \lambda + \mu = 1, & \lambda \neq 0, -\frac{2}{3} \\ \lambda = 0, & \mu \neq 3, \frac{5}{4}, 1, 0 \\ \mu = 1, & \lambda \neq 1, 0, -\frac{1}{4}, -2 \end{cases}$
2. $\mathcal{I}_{\lambda,\mu}^4(S^1) \cong \mathbb{R}^3$, for $(\lambda, \mu) = (1, 1), (0, \frac{5}{4}), (0, 0), (-\frac{1}{4}, 1), (-\frac{2}{3}, \frac{5}{3})$;
3. $\mathcal{I}_{0,3}^4(S^1) \cong \mathcal{I}_{-2,1}^4(S^1) \cong \mathfrak{a} \oplus \mathbb{R}$;
4. $\mathcal{I}_{0,1}^4(S^1) \cong \mathfrak{b} \oplus \mathbb{R}^2$.

The exceptional modules $\mathcal{D}_{\lambda,\mu}^4(S^1)$ are represented in Figure 2.

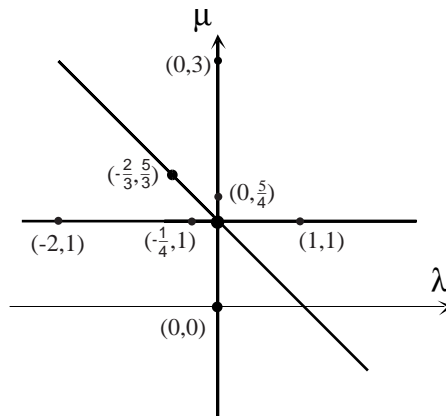


Figure 2. Exceptional modules of 4-th order operators

Remark 3.3. We will show in Section 5.4.1 that the module $\mathcal{D}_{-\frac{2}{3}, \frac{5}{3}}^4(S^1)$ is, indeed, a very special one. This exceptional module is related to the Grozman operator (1.1).

3.4 Modules of differential operators of order 3

Symmetries of the modules of third-order operators are particularly rich.

Theorem 3.4. *The algebra $\mathcal{I}_{\lambda,\mu}^3(S^1)$ is trivial for all (λ, μ) except*

1. $\mathcal{I}_{\lambda,\mu}^3(S^1) \cong \mathbb{R}^2$, for $\begin{cases} \lambda + \mu = 1, & \lambda \neq 0, -\frac{1}{2}, -\frac{2}{3} \\ (3\lambda + 1)(3\mu - 4) = -1, & \lambda \neq 0, -\frac{2}{3} \end{cases}$
2. $\mathcal{I}_{\lambda,\mu}^3(S^1) \cong \mathfrak{a}$, for $\mu - \lambda = 2, \lambda \neq 0, -\frac{1}{2}, -1$;
3. $\mathcal{I}_{\lambda,\mu}^3(S^1) \cong \mathbb{R}^3$, for $\begin{cases} \lambda = 0, & \mu \neq 3, 2, 1 \\ \mu = 1, & \lambda \neq 0, -1, -2 \end{cases}$
4. $\mathcal{I}_{\lambda,\mu}^3(S^1) \cong \mathfrak{a} \oplus \mathbb{R}$, for $(\lambda, \mu) = (0, 3), (0, 2), (-1, 1), (-2, 1)$;
5. $\mathcal{I}_{-\frac{1}{2}, \frac{3}{2}}^3(S^1) \cong \mathfrak{t}_2$;
6. $\mathcal{I}_{-\frac{2}{3}, \frac{5}{3}}^3(S^1) \cong \mathbb{R}^3$;
7. $\mathcal{I}_{0,1}^3(S^1) \cong \mathfrak{b} \oplus \mathbb{R}^2$.

The exceptional modules $\mathcal{D}_{\lambda,\mu}^3(S^1)$ are represented in Figure 3.

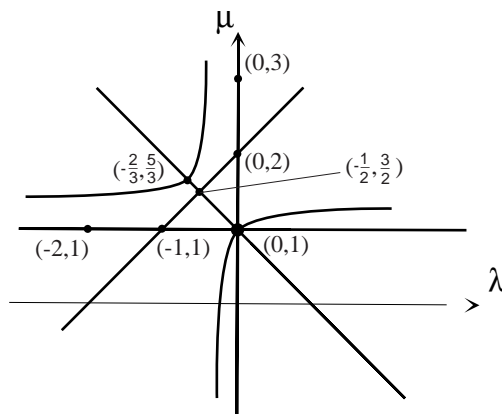


Figure 3. Exceptional modules of third-order operators

3.5 Modules of second-order differential operators

Second-order differential operators are definitely among the most popular objects of mathematics. Their invariants with respect to the action of $\text{Diff}(S^1)$, such as monodromy or the rotation number, were thoroughly studied. Some modules of second-order differential operators on S^1 have geometric meaning, they have been related to various algebraic structures such as the Virasoro algebra and integrable systems.

The result in the second-order case is as follows.

Theorem 3.5. *The algebra $\mathcal{I}_{\lambda,\mu}^2(S^1)$ is isomorphic to \mathbb{R}^2 for all (λ, μ) , except*

1. $\mathcal{I}_{\lambda,\mu}^2(S^1) \cong \mathfrak{a}$, for $\begin{cases} \mu - \lambda = 1, & \lambda \neq 0 \\ \mu - \lambda = 2, & \lambda \neq 0, -\frac{1}{2}, -1 \end{cases}$
2. $\mathcal{I}_{\lambda,\mu}^2(S^1) \cong \mathbb{R}^3$, for $\begin{cases} \lambda = 0, & \mu \neq 2, 1 \\ \mu = 1, & \lambda \neq 0, -1 \end{cases}$
3. $\mathcal{I}_{-\frac{1}{2}, \frac{3}{2}}^2(S^1) \cong \mathfrak{t}_2$;
4. $\mathcal{I}_{0,2}^2(S^1) \cong \mathcal{I}_{-1,1}^2(S^1) \cong \mathfrak{a} \oplus \mathbb{R}$;
5. $\mathcal{I}_{0,1}^2(S^1) \cong \mathfrak{b} \oplus \mathbb{R}$.

The exceptional modules of second-order operators are represented in Figure 4.

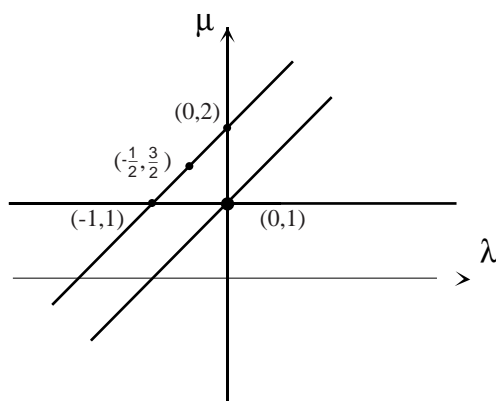


Figure 4. Exceptional modules of second-order operators

3.6 Modules of first-order differential operators

Let us finish this section with the result in the first-order case. The result is less interesting, the only particular module is $\mathcal{D}_{0,1}^1(S^1)$.

Theorem 3.6. *The algebra of $\mathcal{I}_{\lambda,\mu}^1(S^1)$ is isomorphic to \mathbb{R}^2 for all λ, μ , except in the following cases:*

1. $\mathcal{I}_{\lambda,\mu}^1(S^1) \cong \mathfrak{a}$, for $\mu - \lambda = 1, \lambda \neq 0$;
2. $\mathcal{I}_{0,1}^1(S^1) \cong \mathfrak{b}$.

For the sake of completeness, let us also mention the zeroth-order case: the algebra of symmetries $\mathcal{I}_{\lambda,\mu}^0(S^1)$ is trivial. Indeed, the module $\mathcal{D}_{\lambda,\mu}^0(S^1)$ is isomorphic to $\mathcal{F}_{\mu-\lambda}$.

3.7 Non-compact case: differential operators on \mathbb{R}

The algebras of symmetries of the modules of differential operators in the spaces of densities over \mathbb{R} can be different from that on S^1 . This occurs in the “most particular” case $(\lambda, \mu) = (0, 1)$.

Theorem 3.7. (i) If $(\lambda, \mu) \neq (0, 1)$, then the algebra $\mathcal{I}_{\lambda, \mu}^k(\mathbb{R})$ coincides with $\mathcal{I}_{\lambda, \mu}^k(S^1)$.
 (ii) In the exceptional case $(\lambda, \mu) = (0, 1)$ one has:

1. If $k \geq 3$, then $\mathcal{I}_{0,1}^k(\mathbb{R}) \cong \mathfrak{t}_2 \oplus \mathbb{R}^2$;
2. If $k = 2$, then $\mathcal{I}_{0,1}^k(\mathbb{R}) \cong \mathfrak{t}_2 \oplus \mathbb{R}$;
3. If $k = 1$, then $\mathcal{I}_{0,1}^k(\mathbb{R}) \cong \mathfrak{t}_2$.

4 Construction of symmetries

In this section we give an explicit construction of the generators of algebras $\mathcal{I}_{\lambda, \mu}^k(S^1)$ for every case where this algebra is non-trivial. We will prove in Section 6 that our list of invariant differential operators is complete.

4.1 The conjugation

The best known invariant map between the spaces of differential operators is the *conjugation*. It is a linear map

$$C : \mathcal{D}_{\lambda, \mu}^k(S^1) \longrightarrow \mathcal{D}_{1-\mu, 1-\lambda}^k(S^1)$$

that associates to each operator A its adjoint A^* defined by

$$\int_{S^1} A^*(\varphi)\psi = \int_{S^1} \varphi A(\psi)$$

for every $\varphi \in \mathcal{F}_{1-\mu}$ and $\psi \in \mathcal{F}_\lambda$. It follows that the modules $\mathcal{D}_{\lambda, \mu}^k(S^1)$ with λ and μ satisfying the condition

$$\lambda + \mu = 1$$

have non-trivial symmetries.

The conjugation map C is an *involution*; the straight line $\lambda + \mu = 1$ will play a role of symmetry axis in the plane parameterized by (λ, μ) .

For an arbitrary local parameter x on S^1 , the conjugation map is given by the well-known formula

$$C : \sum_{i=0}^k a_i(x) \frac{d^i}{dx^i} \mapsto \sum_{i=0}^k (-1)^i \left(\frac{d}{dx} \right)^i \circ a_i(x) \tag{4.1}$$

that easily follows from the definition.

Remark 4.1. The expression (4.1) is independent from the choice of the parameter x . Indeed, any change of local coordinates is given by a diffeomorphism of S^1 . Note that this fundamental property of coordinate independence is just a different way to express the $\text{Diff}(S^1)$ -equivariance.

4.2 The cases $\lambda = 0$ and $\mu = 1$

We will define a $\text{Diff}(S^1)$ -invariant operator

$$P_0 : \mathcal{D}_{0,\mu}^k(S^1) \longrightarrow \mathcal{D}_{0,\mu}^k(S^1).$$

Let us first consider a $\text{Diff}(S^1)$ -invariant projection $P_0 : \mathcal{D}_{0,\mu}^k(S^1) \longrightarrow \mathcal{F}_\mu$ defined by: $A \mapsto A(1)$, where $1 \in \mathcal{F}_0 \cong C^\infty(M)$ is a constant function on S^1 . In other words,

$$P_0 \left(\sum_{i=0}^k a_i(x) \frac{d^i}{dx^i} \right) = a_0(x) (dx)^\mu. \quad (4.2)$$

Since $\mathcal{F}_\mu \subset \mathcal{D}_{0,\mu}^k(S^1)$, one obtains a non-trivial element of the algebra $\mathcal{I}_{0,\mu}^k(S^1)$.

Thanks to the conjugation map (4.1), one also has a non-trivial symmetry $P_0^* = C \circ P_0 \circ C$

$$P_0^* : \mathcal{D}_{\lambda,1}^k(S^1) \longrightarrow \mathcal{D}_{\lambda,1}^k(S^1).$$

The explicit formula follows from (4.2) and (4.1):

$$P_0^* \left(\sum_{i=0}^k a_i(x) \frac{d^i}{dx^i} \right) = \sum_{i=0}^k (-1)^i a_i(x)^{(i)}. \quad (4.3)$$

The right hand side is understood as a (scalar) differential operator from \mathcal{F}_λ to \mathcal{F}_1 .

4.3 Two additional elements of $\mathcal{I}_{0,1}^k(S^1)$

In the most particular case $(\lambda, \mu) = (0, 1)$, there are two more elements of the symmetry algebra.

- There is a *non-local* element of $\mathcal{I}_{0,1}^1(S^1)$. It is given by the expression

$$L \left(\sum_{i=0}^k a_i(x) \frac{d^i}{dx^i} \right) = \left(\int_{S^1} a_0(x) dx \right) \circ d \quad (4.4)$$

where d is the de Rham differential. Indeed, the projection (4.2) gives a 1-form on S^1 so that the integral above is well-defined; since $d \in \mathcal{D}_{0,1}^1(S^1)$, we can understand L as a linear map $L : \mathcal{D}_{0,1}^k(S^1) \longrightarrow \mathcal{D}_{0,1}^k(S^1)$ and therefore a symmetry.

Remark 4.2. Equation (4.4) is an analog of the well-known Adler trace [1], although the latter is defined on the space of pseudodifferential operators from \mathcal{F}_0 to \mathcal{F}_0 .

- There is one more element of the algebra $\mathcal{I}_{0,1}^1(S^1)$ given by the formula

$$P_1 \left(\sum_{i=0}^k a_i(x) \frac{d^i}{dx^i} \right) = \left(\sum_{i=1}^k (-1)^{i-1} a_i(x)^{(i-1)} \right) \circ d. \quad (4.5)$$

It is easy to check directly that P_1 is $\text{Diff}(S^1)$ -invariant, but its intrinsic form can also be written. In the one-dimensional case, the de Rham differential d is an element of $\mathcal{D}_{0,1}^1(S^1)$. One has a $\text{Diff}(S^1)$ -invariant operator

$$\delta : \mathcal{D}_{1,\mu}^k(S^1) \longrightarrow \mathcal{D}_{0,\mu}^{k+1}(S^1)$$

given by right composition with the de Rham differential: $\delta : A \mapsto A \circ d$. This map is a bijection between $\mathcal{D}_{1,\mu}^k(S^1)$ and $\text{Ker } P_0 \subset \mathcal{D}_{0,\mu}^{k+1}(S^1)$. One has:

$$P_1 = \delta \circ P_0 \circ C \circ \delta^{-1} \circ (\text{Id} - P_0).$$

4.4 Additional elements of $\mathcal{I}_{0,0}^k(S^1)$ and $\mathcal{I}_{1,1}^k(S^1)$

The algebra $\mathcal{I}_{0,0}^k(S^1)$ is generated by the operator P_0 given by (4.2) and

$$S = -C \circ \delta^{-1} \circ (\text{Id} - P_0) \circ C \circ \delta \circ C. \tag{4.6}$$

One can check that the explicit formula for this operator is as follows:

$$S \left(\sum_{i=0}^k a_i(x) \frac{d^i}{dx^i} \right) = \sum_{i=0}^{k-1} (-1)^i \left(\frac{d}{dx} \right)^i \circ (a_i(x) + a'_{i+1}(x)) + (-1)^k \left(\frac{d}{dx} \right)^k \circ a_k(x).$$

The algebra $\mathcal{I}_{1,1}^k(S^1)$ is generated by P_0^* given by (4.3) and the operator $S^* = C \circ S \circ C$.

4.5 Symmetries and bilinear operators on tensor densities

We now give a general way to construct linear $\text{Diff}(S^1)$ -invariant differential operators on $\mathcal{D}_{\lambda,\mu}^k(S^1)$. Assume there are two $\text{Diff}(S^1)$ -invariant differential operators:

- a bilinear differential operator $J : \mathcal{F}_\nu \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\mu$;
- a linear projection $\pi : \mathcal{D}_{\lambda,\mu}^k(S^1) \longrightarrow \mathcal{F}_\nu$.

We define a linear map $J \circ \pi : \mathcal{D}_{\lambda,\mu}^k(S^1) \longrightarrow \mathcal{D}_{\lambda,\mu}^k(S^1)$ as follows

$$(J \circ \pi)(A)(\cdot) = J(\pi(A), \cdot).$$

This map is obviously $\text{Diff}(S^1)$ -invariant.

This is the way invariant differential operators on the modules $\mathcal{D}_{\lambda,\mu}^k(S^1)$ are related to invariant bilinear differential operators on densities. We give here the complete list of bilinear operators on densities and the complete list of linear projections. We will then specify the generators of the algebras $\mathcal{I}_{\lambda,\mu}^k(S^1)$ that can be obtained by the above construction.

4.5.1 Bilinear invariant differential operators on tensor densities

The classification of invariant bilinear differential operators on tensor fields is due to P. Grozman [11]. His list is particularly interesting in the one-dimensional case (see also [6]).

Let us recall here the complete list.

1. Every zeroth-order operator $\mathcal{F}_\nu \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\nu+\lambda}$ is of the form:

$$\phi(x)(dx)^\lambda \otimes \psi(x)(dx)^\mu \mapsto c\phi(x)\psi(x)(dx)^{\lambda+\mu},$$

where $c \in \mathbb{C}$. From now on we omit a scalar multiple c .

2. Every first order operator $\mathcal{F}_\nu \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\nu+\lambda+1}$ is as follows

$$\left\{ \phi(x)(dx)^\nu, \psi(x)(dx)^\lambda \right\} = (\nu\phi(x)\psi(x)' - \lambda\phi(x)'\psi(x))(dx)^{\nu+\lambda+1}, \quad (4.7)$$

where x is a local coordinate on M and we identify tensor densities with functions. The operator (4.7) is nothing but the Poisson bracket on T^*S^1 (or $T^*\mathbb{R}^1$).

For every $(\nu, \lambda) \neq (0, 0)$, the operator (4.7) is the only $\text{Diff}(S^1)$ - (or $\text{Diff}(\mathbb{R}^1)$ -) invariant operator, otherwise there are two linearly independent operators: $\phi d(\psi)$ and $d(\phi)\psi$, where d is the de Rham differential.

3. There exist second order operators $\mathcal{F}_\nu \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\nu+\lambda+2}$ given by the compositions:

$$\begin{aligned} \phi \otimes \psi &\mapsto \{d\phi, \psi\} && \text{for } \nu = 0, \\ \phi \otimes \psi &\mapsto \{\phi, d\psi\} && \text{for } \lambda = 0, \\ \phi \otimes \psi &\mapsto d\{\phi, \psi\} && \text{for } \nu + \lambda = -1. \end{aligned} \quad (4.8)$$

4. Three third-order bilinear invariant differential operators $\mathcal{F}_\nu \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_{\nu+\lambda+3}$ are also given by compositions:

$$\begin{aligned} \phi \otimes \psi &\mapsto \{d\phi, d\psi\} && \text{for } (\nu, \lambda) = (0, 0), \\ \phi \otimes \psi &\mapsto d\{d\phi, \psi\} && \text{for } (\nu, \lambda) = (0, -2), \\ \phi \otimes \psi &\mapsto d\{\phi, d\psi\} && \text{for } (\nu, \lambda) = (-2, 0). \end{aligned} \quad (4.9)$$

5. The only differential operator of order 3 which is not a composition of the operators of lesser orders is the famous Grozman operator $G : \mathcal{F}_{-\frac{2}{3}}(S^1) \otimes \mathcal{F}_{-\frac{2}{3}}(S^1) \longrightarrow \mathcal{F}_{\frac{5}{3}}(S^1)$ already mentioned in Introduction, see (1.1). It is given by the following expression:

$$\begin{aligned} G\left(\phi(x)(dx)^{-\frac{2}{3}}, \psi(x)(dx)^{-\frac{2}{3}}\right) \\ = \left(2 \begin{vmatrix} \phi(x) & \psi(x) \\ \phi'''(x) & \psi'''(x) \end{vmatrix} + 3 \begin{vmatrix} \phi'(x) & \psi'(x) \\ \phi''(x) & \psi''(x) \end{vmatrix} \right) (dx)^{\frac{5}{3}}. \end{aligned} \quad (4.10)$$

The $\text{Diff}(S^1)$ -invariance of this operator can be easily checked directly.

Remark 4.3. The operator (4.10) remains one of the most mysterious invariant differential operators. Its geometric and algebraic meaning was discussed in [6, 7].

We will use the above bilinear operators to construct the symmetries, but we do not use Grozman’s classification result in our proof.

4.5.2 Invariant projections from $\mathcal{D}_{\lambda,\mu}^k(S^1)$ to \mathcal{F}_ν

Let us now give the list of $\text{Diff}(S^1)$ -invariant linear maps from $\mathcal{D}_{\lambda,\mu}^k(S^1)$ to the space \mathcal{F}_ν .

1. The well-known projection is the *principal symbol* map $\sigma : \mathcal{D}_{\lambda,\mu}^k(S^1) \longrightarrow \mathcal{F}_{\mu-\lambda-k}$ given by the expression

$$\sigma \left(\sum_{i=0}^k a_i(x) \frac{d^i}{dx^i} \right) = a_k(x) (dx)^{\mu-\lambda-k}. \tag{4.11}$$

The map σ is obviously $\text{Diff}(S^1)$ -invariant for all (λ, μ) .

2. For all (λ, μ) , define a linear map

$$V : \mathcal{D}_{\lambda,\mu}^k(S^1) \longrightarrow \mathcal{F}_{\mu-\lambda-k+1}$$

as follows:

$$V(A) = (\alpha a'_k(x) + \beta a_{k-1}(x)) (dx)^{\mu-\lambda-k+1}. \tag{4.12}$$

where

$$\alpha = \lambda k + \frac{k(k-1)}{2}, \quad \beta = \mu - \lambda - k$$

It is easy to check that this map is $\text{Diff}(S^1)$ -invariant. The map (4.12) is a “first-order analog” of the principal symbol.

Remark 4.4. If $\lambda + \mu = 1$, then this map is proportional to the principal symbol of the $(k-1)$ -th order operator $A - (-1)^k A^*$. In other words,

$$V = \sigma \circ (\text{Id} - (-1)^k C)$$

if $\lambda + \mu = 1$.

3. In the particular case,

$$\lambda = \frac{1-k}{2}, \quad \mu = \frac{1+k}{2} \tag{4.13}$$

The map (4.12) vanishes. In this case, there are *two* independent projections onto \mathcal{F}_1 :

$$A \mapsto a'_k(x) dx, \quad A \mapsto a_{k-1}(x) dx$$

which are $\text{Diff}(S^1)$ -invariant.

4. It turns out that for some special values of the parameters λ and μ , there exist second-order analogues of the operators (4.11) and (4.12).

Proposition 4.5. *For every $k \geq 3$ and (λ, μ) satisfying the relation*

$$\left(\lambda + \frac{k-2}{3}\right) \left(\mu - \frac{k+1}{3}\right) + \frac{1}{36}(k+1)(k-2) = 0, \quad (4.14)$$

there exists a $\text{Diff}(S^1)$ -invariant map $W : \mathcal{D}_{\lambda, \mu}^k \longrightarrow \mathcal{F}_{\mu-\lambda-k+2}$ given by

$$W(A) = (\alpha_2 a_k''(x) + \alpha_1 a_{k-1}'(x) + \alpha_0 a_{k-2}(x)) (dx)^{\mu-\lambda-k+2}, \quad (4.15)$$

where the coefficients are defined by

$$\begin{aligned} \alpha_2 &= \frac{2}{3}k(k-1)(k+3\lambda-2)^2 \\ \alpha_1 &= 2(k-1)(k+3\lambda-2)(2-2\lambda-k) \\ \alpha_0 &= 3k^2 + 12\lambda k + 12\lambda^2 - 11k - 24\lambda + 10. \end{aligned} \quad (4.16)$$

Proof. Straightforward. ■

5. There exists one more invariant differential projection $\mathcal{D}_{0,1}^k(S^1) \longrightarrow \mathcal{F}_0(S^1)$ given by the composition

$$\pi_\delta = P_0 \circ C \circ \delta^{-1} \circ (\text{Id} - P_0), \quad (4.17)$$

where P_0 is defined by (4.2) and C is the conjugation.

6. If there is an invariant map from $\mathcal{D}_{\lambda, \mu}^k(S^1)$ to the space of 1-forms \mathcal{F}_1 , then one can integrate the result and obtain a *non-local* (i.e., non-differential) invariant linear map with values in $\mathbb{R} \subset \mathcal{F}_0$. For instance, for $\mathcal{D}_{0,1}^k(S^1)$, the operator P_0 defined by (4.2) satisfies the required condition. One gets a $\text{Diff}(S^1)$ -invariant map:

$$A \mapsto \int_{S^1} a_0(x) dx, \quad A \in \mathcal{D}_{0,1}^k(S^1). \quad (4.18)$$

It was proven in [23] that there are no other $\text{Diff}(S^1)$ -invariant projections $\mathcal{D}_{\lambda, \mu}^k(S^1) \longrightarrow \mathcal{F}_\nu$ than the above ones and their compositions with C and d . We use these operators to construct the generators of the symmetry algebras (but we do not use the classification result of [23] in the proofs of our theorems).

5 Computing the algebras of symmetry

We will now investigate, case by case, the non-trivial algebras $\mathcal{I}_{\lambda, \mu}^k(S^1)$. We will construct the generators of these algebras and calculate the multiplication tables. We then give an explicit identification of the algebras $\mathcal{I}_{\lambda, \mu}^k(S^1)$ with the matrix algebras introduced in Section 3.1. The constructions of this section prove that the algebras $\mathcal{I}_{\lambda, \mu}^k(S^1)$ are *at least* as big as stated in Section 3.

The proof of the second part of our classification theorems, namely that there are *no other* symmetries than we construct and study here, will be given in Section 6.

5.1 The algebra $\mathcal{I}_{0,1}^k(S^1)$

Let us start with the most particular algebra $\mathcal{I}_{0,1}^k(S^1)$ for all k . One has in this case

$$\mathcal{I}_{0,1}^k(S^1) = \text{Span}(\text{Id}, C, P_0, P_0^*, P_1, L)$$

where the generators are defined by (4.1)-(4.5).

Let us now calculate the relations between these generators.

Proposition 5.1. *The multiplication table for the associative algebra $\mathcal{I}_{0,1}^k(S^1)$ is as follows:*

	Id	P_0	C	P_0^*	P_1	L	
Id	Id	P_0	C	P_0^*	P_1	L	
P_0	P_0	P_0	P_0^*	P_0^*	0	0	
C	C	P_0	Id	P_0^*	$P_0^* - P_1 - P_0$	$-L$	(5.1)
P_0^*	P_0^*	P_0	P_0	P_0^*	$P_0^* - P_0$	0	
P_1	P_1	0	$-P_1$	0	P_1	L	
L	L	L	L	L	0	0	

Proof. First, consider the product of P_1 and C . From the definition (4.1) one obtains

$$\begin{aligned} (P_1 C)(A) &= P_1 \left(\sum_{i=0}^k (-1)^i \left(\frac{d}{dx} \right)^i \circ a_i(x) \right) \\ &= P_1 \left(\sum_{i=0}^k \sum_{j=0}^i (-1)^i \binom{i}{j} a_i^{(i-j)}(x) \frac{d^j}{dx^j} \right) \\ &= P_1 \left(\sum_{j=0}^k \sum_{i=j}^k (-1)^i \binom{i}{j} a_i^{(i-j)}(x) \frac{d^j}{dx^j} \right) \end{aligned}$$

and then from (4.5) it follows that

$$\begin{aligned} &= \left(\sum_{j=1}^k \sum_{i=j}^k (-1)^{i+j-1} \binom{i}{j} a_i^{(i-1)}(x) \right) \circ d \\ &= \left(\sum_{i=1}^k \sum_{j=1}^i (-1)^{i+j-1} \binom{i}{j} a_i^{(i-1)}(x) \right) \circ d \\ &= \left(\sum_{i=1}^k (-1)^i a_i^{(i-1)}(x) \right) \circ d \\ &= -P_1(A) \end{aligned}$$

as presented in table (5.1).

Now, consider the product $C P_1$. One then has from (4.1), (4.5) and (4.3):

$$\begin{aligned}(C P_1)(A) &= \sum_{i=1}^k (-1)^i \left(a_i^{(i-1)}(x) \frac{d}{dx} + a_i^{(i)}(x) \right) \\ &= P_0^* - P_1 - P_0.\end{aligned}$$

Furthermore, one obtains $P_0^* P_1 = (P_0 C P_1)(A) = P_0^* - P_0$.

For other products of the the generators the results given in the table immediately follow from the definition. ■

Let us finally give an explicit isomorphism between the algebra $\mathcal{I}_{0,1}^k(S^1)$ and the matrix algebra $\mathfrak{b} \oplus \mathbb{R}^2$ described in Section 3.1.

One checks using the multiplication table (5.1) that the formulæ

$$\begin{aligned}\bar{a} &= \frac{1}{2}(2P_1 + P_0 - P_0^*), \\ \bar{b} &= \frac{1}{2}(P_0 + P_0^*), \\ \bar{c} &= \frac{1}{2}(P_0 - P_0^*), \\ \bar{d} &= L,\end{aligned}$$

where $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are the generators of \mathfrak{b} , see Appendix 8, define an isomorphism of the associative algebras

$$\text{Span}(P_0, P_0^*, P_1, L) \cong \mathfrak{b}.$$

The two more generators

$$\begin{aligned}z_1 &= \text{Id} + C - P_0 - P_0^* \\ z_2 &= \text{Id} - C - P_0 + P_0^* - 2P_1\end{aligned}$$

are in the center and span the second summand \mathbb{R}^2 .

The above generators are linearly independent if $k \geq 3$ and span the algebra

$$\mathcal{I}_{0,1}^k(S^1) \cong \mathfrak{b} \oplus \mathbb{R}^2,$$

in accordance with Theorem 3.1, 3, Theorem 3.2, 4 and Theorem 3.4, 7.

If $k = 2$, then $z_2 = 0$ so that $\mathcal{I}_{0,1}^2(S^1) \cong \mathfrak{b} \oplus \mathbb{R}$, see Theorem 3.5, 5. Finally, if $k = 1$, then $z_1 = z_2 = 0$ and one has $\mathcal{I}_{0,1}^1(S^1) \cong \mathfrak{b}$ as stated Theorem 3.6, 2.

5.2 Algebras of symmetry in order $k \geq 5$

Assume that $k \geq 5$. We already investigated the algebra $\mathcal{I}_{0,1}^k(S^1)$. There are two more non-trivial algebras in this case, namely the algebra $\mathcal{I}_{0,0}^k(S^1)$ and the algebra $\mathcal{I}_{1,1}^k(S^1)$ which is isomorphic to $\mathcal{I}_{0,0}^k(S^1)$ by conjugation.

The algebra of symmetry $\mathcal{I}_{0,0}^k(S^1)$ is as follows

$$\mathcal{I}_{0,0}^k(S^1) = \text{Span}(\text{Id}, P_0, S),$$

where P_0 and S are as in (4.2) and (4.6), respectively. These operators are independent for $k \geq 4$. We have:

$$P_0 S = S P_0 = P_0, \quad P_0^2 = P_0, \quad S^2 = \text{Id}.$$

Indeed, the first two relations are due to the fact that the scalar term of $S(A)$ is equal to $a_0(x)$, cf. eq. (4.6) and the explicit expression for S . Put

$$1 = \text{Id}, \quad \bar{a}_1 = P_0 \quad \text{and} \quad \bar{a}_2 = \frac{1}{\sqrt{2}}(\text{Id} - S).$$

One obtains the generators of the algebra \mathbb{R}^3 , cf. Appendix 8. Finally, one has

$$\mathcal{I}_{0,0}^k(S^1) \cong \mathbb{R}^3,$$

as stated in Theorem 3.1, 2.

The algebras $\mathcal{I}_{0,\mu}^k(S^1)$ corresponding to the generic values of μ have only two generators: Id and P_0 . These algebras are obviously isomorphic to \mathbb{R}^2 .

5.3 Algebras of symmetry in order 4

Consider the modules of differential operators of order $k = 4$:

$$A = a_4(x) \frac{d^4}{dx^4} + a_3(x) \frac{d^3}{dx^3} + a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x).$$

We will study all the exceptional modules systematically and investigate every non-trivial algebra of symmetry.

The generators of the algebras $\mathcal{I}_{0,1}^4(S^1)$ and $\mathcal{I}_{0,0}^4(S^1) \cong \mathcal{I}_{1,1}^4(S^1)$ are the same as for $k = 5$. Let us consider other interesting cases.

5.3.1 The algebras $\mathcal{I}_{0,\frac{5}{4}}^4(S^1)$ and $\mathcal{I}_{-\frac{1}{4},1}^4(S^1)$

We already constructed two generators of the algebra $\mathcal{I}_{0,\frac{5}{4}}^4(S^1)$, namely Id and P_0 . One extra generator is obtained by the following procedure.

The values $(\lambda, \mu) = (0, \frac{5}{4})$ satisfy the relation (4.14), so that the map

$$W : \mathcal{D}_{0,\frac{5}{4}}^4(S^1) \longrightarrow \mathcal{F}_{-\frac{3}{4}}$$

defined by (4.15) is $\text{Diff}(S^1)$ -invariant. There is a second-order bilinear differential operator

$$J : \mathcal{F}_{-\frac{3}{4}} \otimes \mathcal{F}_0 \longrightarrow \mathcal{F}_{\frac{5}{4}}$$

defined by the second formula in (4.8). Applying the construction of Section 4.5 we consider the composition $J \circ W$ defined as in Section 4.5 to obtain an element of algebra $\mathcal{I}_{0,\frac{5}{4}}^4(S^1)$.

Let us now compute the relations between the generators. The constructed map is given by the following explicit formula

$$(J \circ W)(A) = \left(\frac{16}{7}a_4''(x) - \frac{12}{7}a_3'(x) + a_2(x) \right) \frac{d^2}{dx^2} + \frac{4}{3} \left(\frac{12}{7}a_4'''(x) - \frac{9}{7}a_3''(x) + \frac{3}{4}a_2'(x) \right) \frac{d}{dx}. \quad (5.2)$$

Note that the right hand side is understood as an element of $\mathcal{D}_{0, \frac{5}{4}}^4(S^1)$. Since $P_0(A) = a_0(x)$, the product of $J \circ W$ and P_0 vanishes:

$$(J \circ W) P_0 = P_0 (J \circ W) = 0.$$

One also has the relations

$$P_0^2 = P_0 \quad \text{and} \quad (J \circ W)^2 = J \circ W.$$

Finally, one gets the following answer:

$$\mathcal{I}_{0, \frac{5}{4}}^4(S^1) = \text{Span}(\text{Id}, P_0, J \circ W) \cong \mathbb{R}^3,$$

as stated by Theorem 3.2, 2.

The conjugation establishes an isomorphism between the algebras $\mathcal{I}_{-\frac{1}{4}, 1}^4(S^1)$ and $\mathcal{I}_{0, \frac{5}{4}}^4(S^1)$.

5.3.2 The algebras $\mathcal{I}_{0,3}^4(S^1)$ and $\mathcal{I}_{-2,1}^4(S^1)$

The algebra $\mathcal{I}_{0,3}^4(S^1)$ has the generators Id , P_0 and the following one constructed in Section 4.5.

Consider the projection $V : \mathcal{D}_{0,3}^4(S^1) \rightarrow \mathcal{F}_0$ defined by formula (4.12) and the third-order bilinear map $J : \mathcal{F}_0 \otimes \mathcal{F}_0 \rightarrow \mathcal{F}_3$, namely the first of the three operators (4.9). Their composition $J \circ V$ is an element of $\mathcal{I}_{0,3}^4(S^1)$:

$$(J \circ V)(A) = (6a_4''(x) - a_3'(x)) \frac{d^2}{dx^2} - (6a_4'''(x) - a_3''(x)) \frac{d}{dx}, \quad (5.3)$$

where the right hand side is understood as an element of $\mathcal{D}_{0,3}^4(S^1)$.

Finally, the algebra $\mathcal{I}_{0,3}^4(S^1)$ is of the form

$$\mathcal{I}_{0,3}^4(S^1) = \text{Span}(\text{Id}, P_0, J \circ V).$$

To obtain the isomorphism $\mathcal{I}_{0,3}^4(S^1) \cong \mathfrak{a} \oplus \mathbb{R}$ (see Theorem 3.2 part 3), one checks the following relations

$$P_0 (J \circ V) = (J \circ V) P_0 = (J \circ V)^2 = 0.$$

Then the standard generators of $\mathfrak{a} \oplus \mathbb{R}$ correspond to $\{\text{Id} - P_0, J \circ V, P_0\}$.

The algebra $\mathcal{I}_{-2,1}^4(S^1)$ is isomorphic to $\mathcal{I}_{0,3}^4(S^1)$ by conjugation.

5.3.3 The algebra $\mathcal{I}^4_{-\frac{2}{3}, \frac{5}{3}}(S^1)$ and the Grozman operator

The conjugation map C and Id are, of course, generators of symmetry of the module $\mathcal{D}^4_{-\frac{2}{3}, \frac{5}{3}}(S^1)$. One extra generator can be obtained as follows.

Consider the operator (4.12)

$$V : \mathcal{D}^4_{-\frac{2}{3}, \frac{5}{3}}(S^1) \longrightarrow \mathcal{F}_{-\frac{2}{3}}$$

and compose it with the Grozman operator G given by (4.10); we obtain (up to a constant) the following operator:

$$\begin{aligned} (G \circ V)(A) = & (a_3(x) - 2a'_4(x)) \frac{d^3}{dx^3} + \left(\frac{3}{2} a'_3(x) - 3a''_4(x) \right) \frac{d^2}{dx^2} \\ & - \left(\frac{3}{2} a''_3(x) - 3a'''_4(x) \right) \frac{d}{dx} - (a'''_3(x) - 2a^{(IV)}_4(x)) \end{aligned} \tag{5.4}$$

which is a generator of $\mathcal{I}^4_{-\frac{2}{3}, \frac{5}{3}}(S^1)$.

The relations between the conjugation map and the above operator are:

$$(G \circ V) C = C (G \circ V) = -G \circ V.$$

One also has:

$$(G \circ V)^2 = G \circ V.$$

One easily deduces from the above relations that the algebra

$$\mathcal{I}^4_{-\frac{2}{3}, \frac{5}{3}}(S^1) = \text{Span}(\text{Id}, C, G \circ V)$$

is, indeed, isomorphic to \mathbb{R}^3 , cf. Theorem 3.2, 2.

5.4 Algebras of symmetry in order 3

Consider the differential operators of order $k = 3$:

$$A = a_3(x) \frac{d^3}{dx^3} + a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x).$$

We will describe all the non-trivial algebras of symmetry.

5.4.1 The algebra $\mathcal{I}^3_{-\frac{2}{3}, \frac{5}{3}}(S^1)$

The conjugation map C , as well as the identity Id , are, of course, generators of the symmetry algebra $\mathcal{I}^3_{-\frac{2}{3}, \frac{5}{3}}(S^1)$. Let us construct one more generator. The principal symbol map is of the form:

$$\sigma : \mathcal{D}^3_{-\frac{2}{3}, \frac{5}{3}}(S^1) \longrightarrow \mathcal{F}_{-\frac{2}{3}}.$$

We compose it with the Grozman operator to obtain a new generator $G \circ \sigma$. This operator is given by the same formula (5.4) as above, but with $a_4(x) \equiv 0$. The relations between the generators are also the same as above, so that the symmetry algebra is $\mathcal{I}^3_{-\frac{2}{3}, \frac{5}{3}}(S^1) = \mathbb{R}^3$.

5.4.2 The hyperbola $(3\lambda + 1)(3\mu - 4) = -1$

Consider the class of modules $\mathcal{D}_{\lambda,\mu}^3(S^1)$ with (λ, μ) satisfying the quadratic relation

$$(3\lambda + 1)(3\mu - 4) = -1, \quad (5.5)$$

see Theorem 3.4, 1.

First of all, we observe that this relation is precisely the relation (4.14) specified for $k = 3$. The operator $W : \mathcal{D}_{\lambda,\mu}^3(S^1) \rightarrow \mathcal{F}_{\mu-\lambda-1}$ is then well-defined. Composing this operator with the Poisson bracket (4.7)

$$\{\cdot, \cdot\} : \mathcal{F}_{\mu-\lambda-1} \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu,$$

one obtains a generator of the algebra $\mathcal{I}_{\lambda,\mu}^3(S^1)$. Let us denote this generator by \mathcal{W} :

$$\begin{aligned} \mathcal{W}(A) = & (\mu - \lambda - 1) (\alpha_2 a_3''(x) + \alpha_1 a_2'(x) + \alpha_0 a_1(x)) \frac{d}{dx} \\ & - \lambda (\alpha_2 a_3'''(x) + \alpha_1 a_2''(x) + \alpha_0 a_1'(x)), \end{aligned} \quad (5.6)$$

where according to (4.16)

$$\begin{aligned} \alpha_2 &= (3\lambda + 1)^2, \\ \alpha_1 &= -(3\lambda + 1)(1 - 2\lambda), \\ \alpha_0 &= 3\lambda^2 + 3\lambda + 1. \end{aligned}$$

In the generic case, $(\lambda, \mu) \neq (0, 1)$ or $(-\frac{2}{3}, \frac{5}{3})$ the symmetry algebra has two generators: $\mathcal{I}_{\lambda,\mu}^3(S^1) = \text{Span}(\text{Id}, S)$. The generator \mathcal{W} satisfies the relation

$$\mathcal{W}^2 = \alpha_0(\mu - \lambda - 1)\mathcal{W}.$$

But $\alpha_0 \neq 0$ and if $\mu - \lambda - 1 = 0$, then (5.5) implies $(\lambda, \mu) = (0, 1)$. Finally, in the generic case, one obtains:

$$\mathcal{I}_{\lambda,\mu}^3(S^1) \cong \mathbb{R}^2.$$

Remark 5.2. The module $\mathcal{D}_{-\frac{2}{3}, \frac{5}{3}}^3(S^1)$ belongs to the family (5.5). We have already considered this module separately, see Section 5.4.1. In this case, we have three generators: $\mathcal{I}_{-\frac{2}{3}, \frac{5}{3}}^3(S^1) = \text{Span}(\text{Id}, C, \mathcal{W})$ which are different from $G \circ \sigma$. One checks, however, that the generator $G \circ \sigma$ can be expressed in terms of the above ones:

$$G \circ \sigma = \frac{1}{2}(\text{Id} - C) - \frac{9}{4}\mathcal{W}.$$

5.4.3 The line $\mu - \lambda = 2$

Consider the family of modules $\mathcal{D}_{\lambda,\mu}^3(S^1)$ satisfying the property $\mu - \lambda = 2$ as in Theorem 3.4, 2.

The operator (4.12) is, in this case, $V : \mathcal{D}_{\lambda,\mu}^3(S^1) \rightarrow \mathcal{F}_0$. Consider its composition with the second-order bilinear operator $J : \mathcal{F}_0 \otimes \mathcal{F}_\lambda \rightarrow \mathcal{F}_\mu$ given by the first formula in (4.8). In the generic case, that is, where

$$(\lambda, \mu) \neq (0, 2), \left(-\frac{1}{2}, \frac{3}{2}\right), (-1, 1),$$

one has $\mathcal{I}_{\lambda,\mu}^3(S^1) = \text{Span}(\text{Id}, J \circ V)$.

One obtains the following explicit formula for the constructed generator:

$$(J \circ V)(A) = (3(\lambda + 1)a_3''(x) - a_2'(x)) \frac{d}{dx} - \lambda(3(\lambda + 1)a_3'''(x) - a_2''(x)) \tag{5.7}$$

and immediately gets the following relation:

$$(J \circ V)^2 = 0.$$

This implies $\mathcal{I}_{\lambda,\mu}^3(S^1) \cong \mathfrak{a}$ in the generic case.

5.4.4 The modules $\mathcal{D}_{0,2}^3(S^1)$, $\mathcal{D}_{-1,1}^3(S^1)$ and $\mathcal{D}_{-\frac{1}{2},\frac{3}{2}}^3(S^1)$

Let us consider some exceptional modules still satisfying $\mu - \lambda = 2$.

In the case, $(\lambda, \mu) = (0, 2)$, one has an extra generator of symmetry, as compared with the preceding section. It is given by the operator P_0 as in (4.2). One then has from (5.7)

$$(J \circ V)P_0 = 0, \quad P_0(J \circ V) = 0.$$

The isomorphism

$$\mathcal{I}_{0,2}^3(S^1) = \text{Span}(\text{Id}, J \circ V, P_0) \cong \mathfrak{a} \oplus \mathbb{R}$$

is then obvious in accordance with Theorem 3.4, 4.

The conjugation map C establishes an isomorphism $\mathcal{I}_{-1,1}^3(S^1) \cong \mathcal{I}_{0,2}^3(S^1)$, so that the algebra $\mathcal{I}_{-1,1}^3(S^1)$ is also isomorphic to $\mathfrak{a} \oplus \mathbb{R}$.

In the interesting case $\mathcal{D}_{-\frac{1}{2},\frac{3}{2}}^3(S^1)$, the extra generator is given by the conjugation map C , so that $\mathcal{I}_{-\frac{1}{2},\frac{3}{2}}^3(S^1) = \text{Span}(\text{Id}, J \circ V, C)$. The relations between the generators are

$$(J \circ V)C = J \circ V, \quad C(J \circ V) = -J \circ V$$

as follows from (5.7) and (4.1). One obtains

$$\mathcal{I}_{-\frac{1}{2},\frac{3}{2}}^3(S^1) \cong \mathfrak{t}_2,$$

as stated by Theorem 3.4, 5.

5.4.5 The modules $\mathcal{D}_{0,3}^3(S^1)$ and $\mathcal{D}_{-2,1}^3(S^1)$

The principal symbol map (4.11) is as follows $\sigma : \mathcal{D}_{0,3}^3(S^1) \rightarrow \mathcal{F}_0$. Compose this map with the third-order bilinear operator $J : \mathcal{F}_0 \otimes \mathcal{F}_0 \rightarrow \mathcal{F}_3$ defined by the first equation in (4.9). The explicit expression of the constructed generator is

$$(J \circ \sigma)(A) = a_3'(x) \frac{d^2}{dx^2} - a_3''(x) \frac{d}{dx}. \tag{5.8}$$

The symmetry algebra is then $\mathcal{I}_{0,3}^3(S^1) = \text{Span}(\text{Id}, J \circ \sigma, P_0)$. One easily gets the relations

$$(J \circ \sigma)P_0 = P_0(J \circ \sigma) = (J \circ \sigma)^2 = 0$$

and, finally, $\mathcal{I}_{0,3}^3(S^1) \cong \mathfrak{a} \oplus \mathbb{R}$, see Theorem 3.4, 4. The algebra $\mathcal{I}_{-2,1}^3(S^1)$ is isomorphic to the above one by conjugation.

5.5 Algebras of symmetry in orders 2

Consider now the modules of differential operators of order 2.

For all (λ, μ) , there is a generator of the algebra $\mathcal{I}_{\lambda, \mu}^2(S^1)$ given by the composition of the projection (4.12)

$$V : \mathcal{D}_{\lambda, \mu}^2(S^1) \longrightarrow \mathcal{F}_{\mu - \lambda - 1}$$

and the Poisson bracket

$$\{\cdot, \cdot\} : \mathcal{F}_{\mu - \lambda - 1} \otimes \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\mu.$$

The explicit formula for this generator is as follows:

$$\begin{aligned} \mathcal{V}(A) = & (\mu - \lambda - 1) \left((2\lambda + 1) a_2'(x) + (\mu - \lambda - 2) a_1(x) \right) \frac{d}{dx} \\ & - \lambda \left((2\lambda + 1) a_2''(x) + (\mu - \lambda - 2) a_1'(x) \right). \end{aligned} \quad (5.9)$$

This generator satisfies the following relation:

$$\mathcal{V}^2 = (\mu - \lambda - 1)(\mu - \lambda - 2) \mathcal{V}.$$

In the generic case, the algebra of symmetry is $\mathcal{I}_{\lambda, \mu}^2(S^1) = \text{Span}(\text{Id}, \mathcal{V})$. It is obviously isomorphic to \mathbb{R}^2 .

If either $\mu - \lambda = 1$ or $\mu - \lambda = 2$, then $\mathcal{V}^2 = 0$. In these cases, $\mathcal{I}_{\lambda, \mu}^2(S^1) = \text{Span}(\text{Id}, \mathcal{V}) \cong \mathfrak{a}$, see Theorem 3.5, 1.

Of course, if $\lambda = 0$, then there is one more generator, namely P_0 . Then one has

$$P_0 \mathcal{V} = \mathcal{V} P_0 = 0$$

and so $\mathcal{I}_{0, \mu}^2(S^1) = \text{Span}(\text{Id}, \mathcal{V}, P_0)$ which is isomorphic to \mathbb{R}^3 for generic μ while

$$\mathcal{I}_{0, 2}^2(S^1) \cong \mathcal{I}_{-1, 1}^2(S^1) \cong \mathfrak{a} \oplus \mathbb{R}$$

see Theorem 3.5, 2 and 3.5, 4.

In the case where $\lambda + \mu = 1$, the conjugation map is well-defined. One checks that in this case the generator (5.9) is a linear combination of Id and C :

$$\mathcal{V} = \lambda(2\lambda + 1)(C - \text{Id})$$

so that $\mathcal{I}_{\lambda, \mu}^2(S^1) \cong \mathbb{R}^2$.

The exceptional case $(\lambda, \mu) = (-\frac{1}{2}, \frac{3}{2})$ corresponds to (4.13). There are two invariant projections from $\mathcal{D}_{-\frac{1}{2}, \frac{3}{2}}^2(S^1)$ to \mathcal{F}_1 in this case. Composing one of them with the Poisson bracket

$$\{\cdot, \cdot\} : \mathcal{F}_1 \otimes \mathcal{F}_{-\frac{1}{2}} \longrightarrow \mathcal{F}_{\frac{3}{2}}$$

one obtains a generator

$$a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x) \mapsto a_2'(x) \frac{d}{dx} + \frac{1}{2} a_2''(x)$$

independent of Id and C . One easily gets an isomorphism $\mathcal{I}_{-\frac{1}{2}, \frac{3}{2}}^2(S^1) \cong \mathfrak{t}_2$.

6 Proof of the main theorems

In this section we prove that there are no other symmetries of the modules $\mathcal{D}_{\lambda,\mu}^k(S^1)$ than those constructed above. In other words, we give here a complete classification of symmetries.

Let T be a linear map (1.2) commuting with the $\text{Diff}(S^1)$ -action. There are two cases:

1. The map T is *local*, that is, one has $\text{Supp}(T(A)) \subset \text{Supp}(A)$ for all $A \in \mathcal{D}_{\lambda,\mu}^k(S^1)$. In this case, the famous Peetre theorem (see [26]) guarantees that T is a *differential operator* in coefficients of A .
2. The map T is *non-local*, that is, for some $A \in \mathcal{D}_{\lambda,\mu}^k(S^1)$ vanishing in an open subset $U \subset S^1$, the operator $T(A)$ does not vanish on U .

These two cases are completely different and should be treated separately.

6.1 The identification

Let us fix a parameter x on S^1 and the corresponding coordinate ξ on the fibers of T^*S^1 .

For our computations, we will need to identify the spaces $\mathcal{D}_{\lambda,\mu}(S^1)$ and $\mathcal{S}_\delta(S^1)$ using the map

$$\sigma_{\text{tot}} : \mathcal{D}_{\lambda,\mu}(S^1) \longrightarrow \mathcal{S}_\delta(S^1) \tag{6.1}$$

assigns to an operator (2.2) the polynomial on T^*S^1 given by (2.5).

The map σ_{tot} is an isomorphism of vector spaces but *not* an isomorphism of $\text{Diff}(S^1)$ -modules. It will, nevertheless, allow us to compare the $\text{Diff}(S^1)$ -action on both spaces.

6.2 The affine Lie algebra

We introduce our main tool that will allow us to use the results of the classic invariant theory.

Let x be an affine parameter on S^1 , the Lie algebra aff of affine transformations is the two-dimensional Lie algebra generated by the translations and linear vector fields:

$$\text{aff} = \text{Span} \left(\frac{d}{dx}, x \frac{d}{dx} \right). \tag{6.2}$$

Since aff is a subalgebra of $\text{Vect}(S^1)$, every $\text{Diff}(S^1)$ -invariant map has to commute with the aff -action.

Proposition 6.1. *The aff-action on $\mathcal{D}_{\lambda,\mu}^k(S^1)$ depends only on the difference $\mu - \lambda$ and coincides with the action on $\mathcal{S}_\delta(S^1)$ after identification (6.1).*

Proof. Straightforward. ■

The well-known result of invariant theory states that the associative algebra of differential operators on T^*S^1 commuting with the aff -action is generated by

$$E = \xi \frac{\partial}{\partial \xi} \quad \text{and} \quad D = \frac{\partial}{\partial x} \frac{\partial}{\partial \xi}, \tag{6.3}$$

see [31]. The operator E is called the *Euler field* and the operator D the *divergence*.

Every differential $\text{Diff}(M)$ -invariant operator (1.2) can therefore be expressed in terms of these operators. In local coordinates, any $\text{Diff}(M)$ -invariant map (1.2) is therefore of the form

$$T = T(E, D).$$

Example 6.2. The expression

$$C = \exp(D) \circ \exp(i\pi E)$$

for the conjugation map (4.1) is worth mentioning for the aesthetic reasons.

6.3 The local case: invariant differential operators

We consider the algebra $\mathcal{I}_{\lambda, \mu}^k \text{loc}(S^1)$ of local (and thus differential) $\text{Diff}(S^1)$ -invariant linear maps (1.2).

Let us restrict the map T to the homogeneous component $\mathcal{F}_{k-\delta}$ in (2.5). Since the Euler operator E reduces to a constant, one has

$$T|_{\mathcal{F}_{\delta-k}} = \sum_{\ell=0}^k T_{k,\ell} D^\ell \tag{6.4}$$

where $T_{k,\ell}$ are some constants.

The operator T has to commute with the $\text{Vect}(S^1)$ -action on $\mathcal{D}_{\lambda, \mu}^k(S^1)$. Consider the Lie subalgebra $\mathfrak{sl}(2) \subset \text{Vect}(S^1)$ generated by three vector fields:

$$\mathfrak{sl}(2) = \text{Span} \left(\frac{d}{dx}, x \frac{d}{dx}, x^2 \frac{d}{dx} \right). \tag{6.5}$$

Remark 6.3. Assume that x is an *affine parameter* on S^1 , that is, we identify S^1 with \mathbb{RP}^1 with homogeneous coordinates $(x_1 : x_2)$ and choose $x = x_1/x_2$. The vector fields (6.5) are then globally defined and correspond to the standard projective structure on \mathbb{RP}^1 , see, e.g., [13].

Proposition 6.4. *A linear map (1.2) written in the form (6.4) is $\mathfrak{sl}(2, \mathbb{R})$ -invariant if and only if it satisfies the recurrence relation*

$$\begin{aligned} & (k + 2\lambda - 1) T_{k-1, \ell-1} \\ & - (k + 2\lambda - \ell) T_{k, \ell-1} \\ & - \ell (2(\mu - \lambda) - 2k + \ell - 1) T_{k, \ell} = 0. \end{aligned} \tag{6.6}$$

Proof. The form (6.4) is already invariant with respect to the affine subalgebra (6.2) of $\mathfrak{sl}(2, \mathbb{R})$. It remains to impose the equivariance condition with respect to the vector field $X = x^2 \frac{d}{dx}$. Let us compute the Lie derivative (2.3) along this vector field. Again, we use the identification (6.1) and express it in terms of symbols. One has

$$\mathcal{L}_X^{\lambda, \mu} = L_X^{\mu-\lambda} - (2\lambda + E) \frac{\partial}{\partial \xi},$$

where

$$L_X^{\mu-\lambda} = x^2 \frac{\partial}{\partial x} - 2x\xi \frac{\partial}{\partial \xi} + 2(\mu - \lambda)x.$$

The equivariance condition $[T, \mathcal{L}_X^{\lambda,\mu}] = 0$ readily leads to the relation (6.6). ■

Let us now impose the equivariance condition with respect to the vector field

$$X = x^3 \frac{d}{dx}.$$

This vector field is not globally defined on S^1 and has a singularity at $x = \infty$. Indeed, choose the coordinate $z = \frac{1}{x}$ in a vicinity of the point $z = 0$, the vector field X is written $X = -\frac{1}{z} \frac{d}{dz}$, cf. Remark 6.3. However, every $\text{Diff}(S^1)$ -invariant differential operator T has to commute with this vector field everywhere for $x \neq \infty$.

Proposition 6.5. *A differential operator T commutes with the action of $X = x^3 \frac{d}{dx}$ if and only if T satisfies the relation (6.6) together with the relation*

$$(6\lambda + 3k - 3) T_{k-1,\ell-1} + \ell(3(\mu - \lambda) - 3k + \ell - 2) T_{k,\ell} = 0. \tag{6.7}$$

Proof is similar to that of Proposition 6.4.

Let us now calculate the dimensions of the algebras $\mathcal{I}_{\lambda,\mu}^{k \text{ loc}}(S^1)$.

Theorem 6.6. *The dimensions of the algebras $\mathcal{I}_{\lambda,\mu}^{k \text{ loc}}(S^1)$ are given in the following table*

k	0	1	2	3	4	≥ 5
(λ, μ) generic	1	2	2	1	1	1
$\lambda = 0$, or $\mu = 1$, generic	1	2	3	3	2	2
$\lambda + \mu = 1$, generic	1	2	2	2	2	2
$(3\lambda + 1)(3\mu - 4) = -1$, or $\mu - \lambda = 2$, generic	1	2	2	2	1	1
$(\lambda, \mu) = (-\frac{1}{4}, 1), (-2, 1), (0, \frac{5}{4}), (0, 3)$	1	2	3	3	3	2
$(\lambda, \mu) = (0, 0), (1, 1)$	1	2	3	3	3	3
$(\lambda, \mu) = (-\frac{2}{3}, \frac{5}{3})$	1	2	2	3	3	2
$(\lambda, \mu) = (-\frac{1}{2}, \frac{3}{2})$	1	2	3	3	2	2
$(\lambda, \mu) = (0, 1)$	1	3	4	5	5	5

Proof. We will use the following result (which is similar to that of [18]).

Proposition 6.7. *Every $\text{Diff}(S^1)$ -invariant differential operator T on $\mathcal{D}_{\lambda,\mu}^k(S^1)$ with $k \geq 3$ is completely determined by its restriction to the subspace of third-order operators $T|_{\mathcal{D}_{\lambda,\mu}^3(S^1)}$.*

Proof. Assume that $k \geq 4$ and $T|_{\mathcal{D}_{\lambda,\mu}^3(S^1)} = 0$ that is, $T_{r,\ell} = 0$ for all ℓ and all $r \leq 3$. Then the system of equations (6.6,6.7) readily leads to $T_{r,\ell} = 0$ for all (r, ℓ) . ■

The end of the proof of Theorem 6.6 is as follows.

We solve the system (6.6), (6.7) explicitly for $k \leq 5$ (we omit here the tedious computations) and obtain the result in this case.

It follows from Proposition 6.7 that the dimension of the algebras $\mathcal{I}_{\lambda,\mu}^k \text{loc}(S^1)$ of local symmetries with $k \geq 3$ can only decrease as k becomes $k+1$. On the other hand, we have already constructed a set of generators of the algebra $\mathcal{I}_{\lambda,\mu}^k(S^1)$ that gives a lower bound for the dimension. We thus conclude, by Proposition 6.7, that the constructed generators span the algebras $\mathcal{I}_{\lambda,\mu}^k(S^1)$ for all k .

Theorem 6.6 is proved. ■

6.4 Non-local operators

Consider now the non-local case. We already constructed an example of a non-local linear map (1.2) commuting with the $\text{Vect}(S^1)$ -action, see formula (4.4). Let us show that there are no other such maps.

Let T be a non-local linear map (1.2) commuting with the $\text{Vect}(S^1)$ -action. Assume that $A \in \mathcal{D}_{\lambda,\mu}^k(S^1)$ vanishes on an open subset $U \subset S^1$, but $T(A) \in \mathcal{D}_{\lambda,\mu}^k(S^1)$ does not vanish on U . Let X be a vector field with support in U , then $\mathcal{L}_X^{\lambda,\mu}(A) = 0$. Since T satisfies the relation of equivariance

$$\mathcal{L}_X^{\lambda,\mu}(T(A)) = T(\mathcal{L}_X^{\lambda,\mu}(A)),$$

one gets

$$\mathcal{L}_X^{\lambda,\mu}(T(A)) = 0.$$

Consider the restriction $T(A)|_U$, this is an element of $\mathcal{D}_{\lambda,\mu}^k(U)$. We just proved that $T(A)|_U$ is a $\text{Vect}(S^1)$ - and thus a $\text{Diff}(S^1)$ -invariant differential operator $\mathcal{F}_\lambda(U) \rightarrow \mathcal{F}_\mu(U)$.

The classification of such invariant differential operators (on any manifold) is well known (see, e.g., [7, 11] and [17] for proofs); the answer is as follows. There exists a unique non-trivial invariant differential operator, namely the de Rham differential. Therefore $T(A)|_U$ is proportional to one of the operators:

- $\text{Id} \in \mathcal{D}_{\lambda,\lambda}^k(U)$, so that $\mu = \lambda$, or
- $d \in \mathcal{D}_{0,1}^k(U)$, so that $(\lambda, \mu) = (0, 1)$.

In each case, one gets an invariant linear functional, namely, the operator T is of the form

- $T = t \text{Id}$, where $t : \mathcal{D}_{\lambda,\lambda}^k(U) \rightarrow \mathbb{R}$, or
- $T = t d$, where $t : \mathcal{D}_{0,1}^k(U) \rightarrow \mathbb{R}$,

respectively. The linear functional t has to be $\text{Diff}(S^1)$ -invariant.

The space of symbols corresponding to the above modules are

$$\mathcal{F}_0 \oplus \cdots \oplus \mathcal{F}_{-k} \quad \text{and} \quad \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_{1-k},$$

respectively. Projecting the functional t to these modules of symbols one obtains a linear functional which is again $\text{Diff}(S^1)$ -invariant. Moreover, this functional is non-zero if and only if t is non-zero.

Lemma 6.8. *There exists a unique (up to a multiplicative constant) non-trivial invariant functional $\mathcal{F}_\lambda(S^1) \rightarrow \mathbb{R}$ if and only if $\lambda = 1$. This functional is the integral of 1-forms*

$$\int : \mathcal{F}_1(S^1) \rightarrow \mathbb{C}$$

Proof. The statement follows from the fact that any module of tensor fields on a manifold M is irreducible except the modules of differential forms $\Omega^k(M)$, see [27]. However, in the one-dimensional case, one can give simple direct arguments.

Let $\tau : \mathcal{F}_\lambda(S^1) \rightarrow \mathbb{R}$ be a $\text{Diff}(S^1)$ -invariant linear functional. Then $\ker \tau \subset \mathcal{F}_\lambda(S^1)$ is a submodule. For every vector field $X = X(x)\frac{d}{dx}$ and every λ -density $\varphi = \phi(x)(dx)^\lambda$, one has

$$\tau(L_X\varphi) = 0.$$

Consider the function $\psi(x)$ defined by the expression

$$\phi(x) = X(x)\phi'(x) + \lambda X'(x)\phi(x).$$

It is easy to check that, in the case $\lambda \neq 1$, for every $\psi(x)$ there exist $X(x)$ and $\phi(x)$ such that the above equation is satisfied. If $\lambda = 1$ then $\psi(x)$ has to satisfy the property $\int \psi(x) dx = 0$. ■

The lemma implies that the functional t can be non-zero only in the second case and has to be proportional to (4.18). We conclude that every non-local $\text{Diff}(S^1)$ -invariant operator T has to be proportional to the operator L given by (4.4).

Theorems 3.1-3.6 are proved.

7 Conclusion and outlook

The classification theorems of Section 3 provide a number of particular examples of modules of differential operators. Some of these modules are known (however, the precise values of parameters (λ, μ) are often implicit), other ones are new.

7.1 Known modules of differential operators

There are particular examples of modules of differential operators on S^1 that have been known for a long time. Some of these modules appear in our classification. Let us briefly mention here some interesting cases.

The family of modules $\mathcal{D}_{\lambda,\mu}^k(S^1)$ with (λ, μ) satisfying the condition $\lambda + \mu = 1$ is, of course, the best known class. This is the only case when one can speak of conjugation and split a given differential operator into the sum of a symmetric and a skew-symmetric operators.

The module $\mathcal{D}_{-\frac{1}{2}, \frac{3}{2}}^2(S^1)$ is a well-known module of second-order differential operators. It contains a submodule of Sturm-Liouville operators

$$A = \frac{d^2}{dx^2} + a(x)$$

which is related to the Virasoro algebra (see [13]). This module also has a very interesting geometric meaning: it is isomorphic to the space of projective structures on S^1 (see [13] and also [25]).

The value $\lambda = -\frac{1}{2}$ is also related to the Lie *superalgebras* of Neveu-Schwarz and Ramond (see [15]). More precisely, the odd parts of these superalgebras consist of $-\frac{1}{2}$ -densities.

The above module is a particular case of the following series of modules.

The modules $\mathcal{D}_{\frac{1-k}{2}, \frac{1+k}{2}}^k(S^1)$, see formula (4.13), also have interesting geometric and algebraic meaning. These modules have already been studied in [32], it turns out that they are related to the space of curves in the projective space \mathbb{P}^{k-1} (see, e.g., [25]). These modules are also related to so-called Adler-Gelfand-Dickey bracket (see, e.g., [24] and references therein).

7.2 New examples of modules of differential operators

Some of the particular modules $\mathcal{D}_{\lambda, \mu}^k(S^1)$ provided by our classification theorems seem to be new.

The modules (4.14) can be characterized as the modules that are “abnormally close” to the corresponding $\text{Diff}(S^1)$ -modules of symbols. More precisely, in this case, the quotient-module

$$\mathcal{D}_{\lambda, \mu}^k(S^1)/\mathcal{D}_{\lambda, \mu}^{k-3}(S^1) \cong \mathcal{F}_{\delta-k} \oplus \mathcal{F}_{\delta-k+1} \oplus \mathcal{F}_{\delta-k+2}$$

where $\delta = \mu - \lambda$. The operator (4.15) has been known to the classics in some particular cases (cf. [32]).

An interesting particular case that belongs to the above family is the module $\mathcal{D}_{-\frac{2}{3}, \frac{5}{3}}^3(S^1)$. It can be characterized by three conditions: $k = 3$ together with $\lambda + \mu = 1$ and (4.14). This module is related to the Grozman operator (4.10).

The module $\mathcal{D}_{-\frac{2}{3}, \frac{5}{3}}^4(S^1)$ is also related to the Grozman operator. We strongly believe that this exceptional module has an interesting geometric and algebraic meaning.

Other examples of exceptional modules of Theorems 3.2 and 3.4, such as $\mathcal{D}_{-\frac{1}{4}, 1}^4(S^1)$, $\mathcal{D}_{-2, 1}^4(S^1)$ or $\mathcal{D}_{-1, 1}^3(S^1)$, etc. are even more mysterious.

7.3 Results in the multi-dimensional case

For the sake of completeness, let us mention the results in the multi-dimensional case. Let M be a smooth manifold, $\dim M \geq 2$; the classification of invariant operators on $\mathcal{D}_{\lambda, \mu}^k(M)$ was obtained in [22]. The algebra of symmetries $\mathcal{I}_{\lambda, \mu}^k(M)$ does not depend on the topology of M . This algebra is trivial for all (λ, μ) , except for the cases $\lambda + \mu = 1$ and $\lambda = 0$ or $\mu = 1$. The generators are C, P_0 and Id .

8 Appendix: generators of the algebras \mathfrak{a} , \mathfrak{b} and \mathbb{R}^n

Let us introduce the generators of the matrix algebras \mathfrak{a} , \mathfrak{b} and \mathbb{R}^n that are useful in order to establish the isomorphisms with the algebras of symmetry $\mathcal{I}_{\lambda, \mu}^k(S^1)$.

In the case of the algebra \mathfrak{a} , we take the matrices

$$\tilde{a} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The relations between the generators are:

$$\tilde{a}^2 = \tilde{a}, \quad \tilde{a}\tilde{b} = \tilde{b}\tilde{a} = \tilde{b}, \quad \tilde{b}^2 = 0.$$

For the algebra \mathfrak{b} , we consider

$$\bar{a} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{d} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The multiplication table for this algebra is

		\bar{a}	\bar{b}	\bar{c}	\bar{d}
\bar{a}		\bar{a}	0	0	\bar{d}
\bar{b}		0	\bar{b}	\bar{c}	0
\bar{c}		\bar{c}	0	0	0
\bar{d}		0	\bar{d}	0	0

In the case of the algebra \mathbb{R}^n , one can choose the generators $1, \bar{a}_1, \dots, \bar{a}_{n-1}$ with relations:

$$1 \bar{a}_i = \bar{a}_i 1 = \bar{a}_i, \quad \bar{a}_i \bar{a}_j = \delta_{ij}.$$

Acknowledgments: We are pleased to thank D. Leites for his help. We are grateful to C. Duval, A. El Gradechi, P. Lecomte and S. Parmentier for enlightening discussions and their interest in this work. The third author is grateful to Swiss National Science Foundation for its support and to A. Alekseev for hospitality.

References

- [1] Adler M, On a trace functional for formal pseudo differential operators and the symplectic structure of the Korteweg-de Vries type equations, *Invent. Math.* **50** (1978/79), 219–248.
- [2] Bouarroudj S, Ovsienko V, Three cocycles on $\text{Diff}(S^1)$ generalizing the Schwarzian derivative, *Internat. Math. Res. Notices* bf 1 (1998), 25–39.
- [3] Cartan E, *Leçons sur la théorie des espaces à connexion projective*, Gauthier -Villars, Paris, 1937.
- [4] Duval C, Ovsienko V, Space of second order linear differential operators as a module over the Lie algebra of vector fields, *Adv. Math.* **132:2** (1997), 316–333.
- [5] Feigin B L, Fuchs D B, Homology of the Lie algebra of vector fields on the line, *Func. Anal. Appl.* **14** (1980), 201–212.
- [6] Feigin B L, Fuchs D B, Skew-symmetric invariant differential operators on the line and Verma modules over the Virasoro algebra. (Russian) *Func. Anal. Appl* **16:2** (1982), 47–63.

-
- [7] Fuchs D B, Cohomology of infinite-dimensional Lie algebras, Consultants Bureau, New York, 1987.
- [8] Gargoubi H, Sur la géométrie de l'espace des opérateurs différentiels linéaires sur \mathbf{R} , *Bull. Soc. Roy. Sci. Liège* **69:1** (2000), 21–47.
- [9] Gargoubi H, Ovsienko V, Space of linear differential operators on the real line as a module over the Lie algebra of vector fields, *Int. Math. Res. Not.* **5** (1996), 235–251.
- [10] Gargoubi H, Ovsienko V, Modules of differential operators on the real line, *Funct. Anal. Appl.* **35:1** (2001), 13–18.
- [11] Grozman P Ya, Classification of bilinear invariant operators over tensor fields, *Functional Anal. Appl.* **14:2** (1980), 127–128.
id Invariant bilinear differential operators, Preprint ESI 1114 (2001) (www.esi.ac.at)
- [12] Grozman P, Leites D, Shchepochkina I, Invariant operators on supermanifolds and standard models, in: Multiple facets of quantization and supersymmetry, 508–555, M. Olshanetski, A. Vainstein (eds.), World Sci. Publishing, 2002, math.RT/0202193.
- [13] Kirillov A A, Infinite dimensional Lie groups : their orbits, invariants and representations. The geometry of moments, Lect. Notes in Math., **970** Springer-Verlag (1982), 101–123.
- [14] Kirillov A A, Invariant operators over geometric quantities (Russian), in: Current Problems in Mathematics, **16**, 3–29, Akad. Nauk SSSR, VINITI, Moscow, 1980; [English translation: J. Sov. Math **18:1** (1982), 1–21].
- [15] Kirillov A A, The orbits of the group of diffeomorphisms of the circle, and local Lie superalgebras, *Funct. Anal. Appl.* **15:2** (1981), 75–76.
- [16] Kolár I, Michor P, Slovák J, Natural operations in differential geometry. Springer-Verlag, Berlin, 1993.
- [17] Lebedev A, Leites D, Shereshevskii I, Lie superalgebra structures in cohomology spaces of Lie algebras with coefficients in the adjoint representation, *math.KT/0404139*.
- [18] Lecomte P B A, Mathonet P, Tousset E, Comparison of some modules of the Lie algebra of vector fields, *Indag. Mathem., N.S.* **7:4** (1996), 461–471.
- [19] Lecomte P B A, Ovsienko V, Projectively invariant symbol calculus, *Lett. Math. Phys.* **49:3** (1999), 173–196.
- [20] Lecomte P B A, Ovsienko V, Cohomology of the vector fields Lie algebra and modules of differential operators on a smooth manifold, *Compos. Math.* **124:1** (2000), 95–110.
- [21] Martin C, Piard A, Classification of the indecomposable bounded admissible modules over the Virasoro Lie algebra with weightspaces of dimension not exceeding two, *Comm. Math. Phys.* **150:3** (1992), 465–493.

-
- [22] Mathonet P, Intertwining operators between some spaces of differential operators on a manifold, *Comm. Algebra* **27:2** (1999), 755–776.
- [23] Mathonet P, Geometric quantities associated to differential operators, *Comm. Algebra* **28:2** (2000), 699–718.
- [24] Ovsienko O, Ovsienko V, Lie derivatives of order n on the line. Tensor meaning of the Gelfand-Dikii bracket, *Adv. Soviet Math.* **2** (1991), 221–231.
- [25] Ovsienko V, Tabachnikov S, Projective differential geometry old and new: from the Schwarzian derivative to cohomology of diffeomorphism groups, Cambridge University Press, 2004.
- [26] Peetre J, Une caractérisation abstraite des opérateurs différentiels, *Math. Scand.* **7** (1959), 211–218 and **8** (1960), 116–120.
- [27] Rudakov A N, Irreducible representations of infinite-dimensional Lie algebras of Cartan type, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 835–866.
- [28] Segal G B, Unitary representations of some infinite dimensional groups, *Comm. Math. Phys.* **80:3** (1981), 301–342.
- [29] Terng C L, Natural vector bundles and natural differential operators, *Amer. J. Math.* **100:4** (1978), 775–828.
- [30] Veblen O, Differential invariants and geometry, Atti del Congresso Internazionale dei Matematici, Bologna, 1928.
- [31] Weyl H, *The Classical Groups*, Princeton University Press, 1946.
- [32] Wilczynski, E J Projective differential geometry of curves and ruled surfaces, Leipzig - Teubner, 1906.